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NON-UNIPOTENT REPRESENTATIONS AND CATEGORICAL CENTRES

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Abstract

Let G be a connected reductive group defined over a finite field F_q. We give a parametrization of the irreducible representations of $G(F_q)$ in terms of (twisted) categorical centres of various monoidal categories associated to G. Results of this type were known earlier for unipotent representations and also for character sheaves.

0. Introduction

0.1. Let $\mathbf{k}$ be an algebraic closure of the finite field with p elements. Let G be a connected reductive group over $\mathbf{k}$. We denote by F_q the subfield of $\mathbf{k}$ with exactly q elements; here q is a power of p. Let $F : G \to G$ be the Frobenius map for an F_q-rational structure on G. We fix a prime number l different from p. Let $\mathrm{Irr}(G^F)$ be the set of isomorphism classes of irreducible representations (over $\bar{\mathbf{Q}}_l$) of the finite group $G^F = \{g \in G; F(g) = g\} = G(F_q)$. In [7] I gave a parametrization of $\mathrm{Irr}(G^F)$ in terms of the group of type dual to that of G. (For "most" representations in $\mathrm{Irr}(G^F)$ this has been already done in [3].) For the part of $\mathrm{Irr}(G^F)$ consisting of unipotent representations in a fixed two-sided cell of W (with G assumed to be F_q-split) the parametrization was in terms of a set $M(\Gamma)$ where Γ is a certain finite group associated to the two-sided cell and $M(\Gamma)$ is the set of simple objects (up to isomorphism) of the category $Vec_\Gamma(\Gamma)$ of Γ-equivariant vector bundles on Γ

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(here Γ acts on Γ by conjugation). In the early 1990's, Drinfeld pointed out to me that the category $Vec_{\Gamma}(\Gamma)$ can be interpreted as the categorical centre of the monoidal category of finite dimensional representations of Γ. (The notion of categorical centre of a monoidal category is due to Joyal, Street, Majid and Drinfeld.) This suggested that one should be able to reformulate the parametrization of $\mathrm{Irr}(G^F)$ in terms of categorical centres of suitable monoidal categories associated with G. This is achieved in the present paper, except that we must allow certain twisted categorical centres instead of usual categorical centres. Note that in our approach the representation theory of $G(F_q)$ cannot be separated from the theory of character sheaves on G which appears as the limit of the first theory when q tends to 1; in particular we also obtain the parametrization of character sheaves on G in terms of categorical centres (no twisting needed in this case).

Earlier results of this type were known in the following cases:

(i) the case [2] of character sheaves on G (with centre assumed to be connected and with $\mathbf{k}$ replaced by $\mathbf{C}$);

(ii) the case [19] of unipotent character sheaves on G;

(iii) the case [20] of unipotent representations of G^F;

(iv) the case [21] of not necessarily unipotent character sheaves on G.

The papers [20], [21] were generalizations of [19] in different directions; the present paper is a common generalization of [20], [21]; the methods used in (ii), (iii), (iv) and the present paper are quite different from those used in (i) which relied on techniques not available in positive characteristic.

Let $\mathbf{B}$ be a Borel subgroup of G and let $\mathbf{T}$ be a maximal torus of $\mathbf{B}$. In this subsection we assume that $F(\mathbf{B}) = \mathbf{B}$, $F(\mathbf{T}) = \mathbf{T}$. Let W be the Weyl group of G with respect to $\mathbf{T}$. Let $\mathfrak{s}$ be an indexing set for the isomorphism classes of Kummer local systems (over $\bar{\mathbf{Q}}_l$); note that W acts naturally on $\mathfrak{s}$.

Let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. A key role in this paper is played by an $\mathcal{A}$-algebra $\mathbf{H}$ (without 1 in general) which has $\mathcal{A}$-basis $\{T_w 1_\lambda; w \in W, \lambda \in \mathfrak{s}\}$ and multiplication defined in 1.5 (see also [14, 31.2]). This is a monodromic version of the usual Hecke algebra of W, closely related to an algebra defined in [23]; it contains the usual Hecke algebra as a subalgebra. Now $\mathbf{H}$ has a canonical basis, two-sided cells and an asymptotic version H^∞ (introduced in [15], [21]) which generalize the analogous notions for the usual Hecke algebra, see [5], [8]; the two-sided cells form a partition of $W \times \mathfrak{s}$ and we have $H^\infty = \oplus_{\mathbf{c}} H^\infty_{\mathbf{c}}$ as rings ($\mathbf{c}$ runs over the two-sided cells and each $H^\infty_{\mathbf{c}}$ is a ring with 1). For any $\mathbf{c}$, $H^\infty_{\mathbf{c}}$ admits a category version (for which H^∞ is the Grothendieck group) which is a semisimple monoidal category $\mathcal{C}^{\mathbf{c}}$ with finitely many simple objects (up to isomorphism) indexed by the elements of $\mathbf{c}$, see §5. (In the case where $\mathbf{c} \subset W \times \{1\}$, this reduces to the monoidal category defined is [12].) Now $\mathcal{C}^{\mathbf{c}}$ has a well defined categorical centre which is again a semisimple abelian category. Note that F acts naturally on $\mathfrak{s}$ and on W hence on $W \times \mathfrak{s}$; this induces an action of F on the set of two-sided cells. If $\mathbf{c}$ is a two-sided cell such that $F(\mathbf{c}) = \mathbf{c}$ then F defines an equivalence of categories $\mathcal{C}^{\mathbf{c}} \to \mathcal{C}^{\mathbf{c}}$ and one can define the notion of F-centre of $\mathcal{C}^{\mathbf{c}}$ (see 5.5) which is a twisted version of the usual centre; it is a semisimple abelian category. We denote by $[\mathbf{c}]$ the set of isomorphism classes of simple objects of this category (a finite set).

Our main result is that $\mathrm{Irr}(G^F)$ is in natural bijection with $\sqcup_{\mathbf{c}}[\mathbf{c}]$ (disjoint union over all F-stable two-sided cells $\mathbf{c}$). (See Theorem 7.3.) In the case where $\mathbf{c} \subset W \times \{1\}$, this reduces to the main result in [20].

The fact that the asymptotic Hecke algebra $\mathbf{H}^\infty$ plays a role in the classification is perhaps not surprising since its non-monodromic versions appeared implicitly in the arguments of [6], through the traces of their canonical basis elements in their various simple modules (the algebras themselves were not defined at the time where [6] was written).

Many arguments in this paper follow very closely the arguments in [21]; we generalize them by taking into account also the arguments in [20]. We have written the proofs in such a way that they apply at the same time in the case of character sheaves on a connected component of a possibly disconnected algebraic group with identity component G. In this case, the classification involves twisted categorical centers, unlike that for the character sheaves on G.

We plan to show elsewhere that the parametrization of $\mathrm{Irr}(G^F)$ given in [7] can be deduced from the main result of this paper.

0.2. Notation. Let $\mathbf{N}^* = \{n \in \mathbf{Z} - p\mathbf{Z}; n \geq 1\}$. Let T be a torus over $\mathbf{k}$. For $n \in \mathbf{N}^*$ let $T_n = \{t \in T; t^n = 1\}$; we have $\sharp(T_n) = n^{\dim T}$. For n, n' in $\mathbf{N}^*$ such that $n'/n \in \mathbf{Z}$ we have a surjective homomorphism $N^{n'}_n : T_{n'} \to T_n$, $t \mapsto t^{n'/n}$. Hence we can form the projective limit T^∞ of the groups T_n with

$n \in \mathbf{N}^*$ (a profinite abelian group). Then for any $n \in \mathbf{N}^*$, T_n is naturally a quotient of T^∞.

All algebraic varieties are over $\mathbf{k}$. We denote by $\mathbf{p}$ the algebraic variety consisting of a single point. For an algebraic variety X we write $\mathcal{D}(X)$ for the bounded derived category of constructible $\bar{\mathbf{Q}}_l$-sheaves on X. Let $\mathcal{M}(X)$ be the subcategory of $\mathcal{D}(X)$ consisting of perverse sheaves on X. For $K \in \mathcal{D}(X)$ and $i \in \mathbf{Z}$ let $\mathcal{H}^i K$ be the i-th cohomology sheaf of K and let K^i be the i-th perverse cohomology sheaf of K. Let $\mathfrak{D}(K)$ be the Verdier dual of K. For any constructible sheaf $\mathcal{E}$ on X let $\mathcal{E}_x$ be the stalk of $\mathcal{E}$ at $x \in X$. If X has a fixed F_q-structure X_0, we denote by $\mathcal{D}_m(X)$ what in [1, 5.1.5] is denoted by $\mathcal{D}^b_m(X_0, \bar{\mathbf{Q}}_l)$; let $\mathcal{M}_m(X)$ be the corresponding category of mixed perverse sheaves. In this paper we often encounter maps of algebraic varieties which are not morphisms but only quasi-morphisms (as in [20, 0.3]). For such maps the usual operations with derived categories are defined as in [20, 0.3].

Note that if $K \in \mathcal{D}_m(X)$ then K can be viewed as an object of $\mathcal{D}(X)$ denoted again by K. If $K \in \mathcal{M}_m(X)$ and $h \in \mathbf{Z}$, we denote by $gr_h(K)$ the subquotient of pure weight h of the weight filtration of K. If $K \in \mathcal{D}_m(X)$ and $i \in \mathbf{Z}$ we write $K\langle i\rangle = K[i](i/2)$ where $[i]$ is a shift and $(i/2)$ is a Tate twist; we write $K^{\{i\}} = gr_i(K^i)(i/2)$. If K is a perverse sheaf on X and A is a simple perverse sheaf on X we write $(A : K)$ for the multiplicity of A in a Jordan-Hölder series of K. If $C \in \mathcal{D}_m(X)$ and $\{C_i; i \in I\}$ is a family of objects of $\mathcal{D}_m(X)$ then the relation $C \approx \{C_i; i \in I\}$ is as in [21, 0.2].

Let $^-: \mathcal{A} \to \mathcal{A}$ be the ring homomorphism such that $\overline{v^m} = v^{-m}$ for any $m \in \mathbf{Z}$. If $f \in \mathbf{Q}[v, v^{-1}]$ and $j \in \mathbf{Z}$ we write $(j; f)$ for the coefficient of v^j in f.

Let $\mathcal{B}$ be the variety of Borel subgroups of G. For any $B \in \mathcal{B}$ let U_B be the unipotent radical of B. In this paper we fix a Borel subgroup $\mathbf{B}$ of G and a maximal torus $\mathbf{T}$ of $\mathbf{B}$. Let $\mathbf{U} = U_{\mathbf{B}}$. Let $\nu = \dim \mathbf{U} = \dim \mathcal{B}$, $\rho = \dim \mathbf{T}$, $\Delta = \dim G = 2\nu + \rho$.

For any algebraic variety X let $\mathfrak{L} = \mathfrak{L}_X = \alpha_! \bar{\mathbf{Q}}_l \in \mathcal{D}(X)$ where $\alpha : X \times \mathbf{T} \to X$ is the obvious projection. When X and T are defined over $\mathbf{F}_q$, $\mathfrak{L}$ is naturally an object of $\mathcal{D}_m(X)$.

Unless otherwise specified, all vector spaces are over $\bar{\mathbf{Q}}_l$; in particular, all representations of finite groups are assumed to be in (finite dimensional) $\bar{\mathbf{Q}}_l$-vector spaces.

1. The Monodromic Hecke Algebra and Its Asymptotic Version

1.1. Let $N\mathbf{T}$ be the normalizer of $\mathbf{T}$ in G, let $W = N\mathbf{T}/\mathbf{T}$ be the Weyl group and let $\kappa : N\mathbf{T} \to W$ be the obvious homomorphism. For $w \in W$ we set $G_w = \mathbf{U}\kappa^{-1}(w)\mathbf{U}$ so that $G = \sqcup_{w\in W} G_w$; let $\mathcal{O}_w = \{(x\mathbf{B}x^{-1}, y\mathbf{B}y^{-1}); x \in G, y \in G, x^{-1}y \in G_w\}$ so that $\mathcal{B} \times \mathcal{B} = \sqcup_{w\in W} \mathcal{O}_w$. For $w \in W$ let $\bar{G}_w$ be the closure of G_w in G; we have $\bar{G}_w = \cup_{y\le w} G_y$ for a well defined partial order $\le$ on W. Let $\bar{\mathcal{O}}_w$ be the closure of $\mathcal{O}_w$ in $\mathcal{B} \times \mathcal{B}$. Now W is a (finite) Coxeter group with length function $w \mapsto |w| = \dim \mathcal{O}_w - \nu$ and with set of generators $S = \{\sigma \in W; |\sigma| = 1\}$; it acts on $\mathbf{T}$ by $w : t \mapsto w(t) = \omega t \omega^{-1}$ where $\omega \in \kappa^{-1}(w)$.

1.2. Let $R \subset \mathrm{Hom}(\mathbf{T}, \mathbf{k}^*)$ be the set of roots of G with respect to $\mathbf{T}$. Now W acts on R by $w : \alpha \mapsto w(\alpha)$ where $(w(\alpha))(t) = \alpha(w^{-1}(t))$ for $t \in \mathbf{T}$. Let R^+ be the set of $\alpha \in R$ such that the corresponding root subgroup is contained in $\mathbf{U}$. For $\alpha : \mathbf{T} \to \mathbf{k}^*$ we denote by $\check{\alpha} : \mathbf{k}^* \to \mathbf{T}$ the corresponding coroot and by σ_α the corresponding reflection in W. For any $\sigma \in S$ let $\mathbf{U}_\sigma$ be the unique root subgroup of $\mathbf{U}$ with respect to $\mathbf{T}$ such that $\mathbf{U}_\sigma^- := \omega \mathbf{U}_\sigma \omega^{-1} \not\subset \mathbf{U}$ for some/any $\omega \in \kappa^{-1}(\sigma)$. Let $\alpha_\sigma : \mathbf{T} \to \mathbf{k}^*$ be the root corresponding to $\mathbf{U}_\sigma$; then the coroot $\check{\alpha}_\sigma : \mathbf{k}^* \to \mathbf{T}$ is well defined.

For any $\sigma \in S$ we fix an element $\xi_\sigma \in \mathbf{U}_\sigma - \{1\}$; there is a unique $\xi'_\sigma \in \mathbf{U}_\sigma^- - \{1\}$ such that $\xi_\sigma \xi'_\sigma \xi_\sigma = \xi'_\sigma \xi_\sigma \xi'_\sigma \in \kappa^{-1}(\sigma) \subset N\mathbf{T}$; the two sides of the last equality are denoted by $\dot{\sigma}$. We have $\kappa(\dot{\sigma}) = \sigma$ and $\dot{\sigma}^2 = \check{\alpha}_\sigma(-1)$. For any $w \in W$ we define $\dot{w} \in N\mathbf{T}$ by $\dot{w} = \dot{\sigma}_1 \dot{\sigma}_2 \dots \dot{\sigma}_r$ where $w = \sigma_1 \sigma_2 \dots \sigma_r$ with $r = |w|, \sigma_j \in S$; note that, by a result of Tits, $\dot{w}$ is well defined. Let $N_0\mathbf{T}$ be the subgroup of $N\mathbf{T}$ generated by $\{\dot{\sigma}; \sigma \in S\}$. This is a finite subgroup of $N\mathbf{T}$ containing $\dot{w}$ for any $w \in W$. Let $\kappa_0 : N_0\mathbf{T} \to W$ be the restriction of $\kappa : N\mathbf{T} \to W$.

1.3. For $n \in \mathbf{N}^*$ let $\mathfrak{s}_n = \mathrm{Hom}(\mathbf{T}_n, \bar{\mathbf{Q}}_l^*)$; we have $\sharp(\mathfrak{s}_n) = n^\rho$. For n, n' in $\mathbf{N}^*$ such that $n'/n \in \mathbf{Z}$, the surjective homomorphism $N_n^{n'} : \mathbf{T}_{n'} \to \mathbf{T}_n$, $t \mapsto t^{n'/n}$ induces an imbedding $\mathfrak{s}_n \subset \mathfrak{s}_{n'}$, $\lambda \mapsto \lambda N_n^{n'}$. Hence we can form the union $\mathfrak{s}_\infty = \cup_{n\in\mathbf{N}^*} \mathfrak{s}_n$ (a countable abelian group). Then for any $n \in \mathbf{N}^*$,

$\mathfrak{s}_n$ is a subgroup of $\mathfrak{s}_\infty$. Note also that $\mathfrak{s}_\infty$ is the group of homomorphisms $\mathbf{T}^\infty \to \bar{\mathbf{Q}}_l^*$ which factor through $\mathbf{T}_n$ for some $n \in \mathbf{N}^*$. For any $\lambda \in \mathfrak{s}_\infty$ there is a well defined local system L_λ on $\mathbf{T}$ such that for some/any $n \in \mathbf{N}^*$ for which $\lambda \in \mathfrak{s}_n$, L_λ is equivariant for the $\mathbf{T}$-action $t_1 : t \mapsto t_1^n t$ on $\mathbf{T}$ and the natural $\mathbf{T}_n$ action on the stalk of L_λ at 1 is through the character λ. For $\lambda, \lambda' \in \mathfrak{s}_\infty$ we have canonically $L_\lambda \otimes L_{\lambda'} = L_{\lambda\lambda'}$; for $\lambda \in \mathfrak{s}_\infty$ we have canonically $L_\lambda^* = L_{\lambda^{-1}}$; here $()^*$ denotes the dual local system.

The W-action on $\mathbf{T}$ restricts to a W-action on $\mathbf{T}_n$ for any $n \in \mathbf{N}^*$. This induces a W-action on $\mathbf{T}^\infty$, a W-action on $\mathfrak{s}_n$ for any $n \in \mathbf{N}^*$; for $\lambda \in \mathfrak{s}_n$, $w \in W$ and $t \in \mathbf{T}_n$ we have $(w(\lambda))(t) = \lambda(w^{-1}(t))$. There is a unique W-action of $\mathfrak{s}_\infty$ which for any $n \in \mathbf{N}^*$ restricts to the W-action on $\mathfrak{s}_n$ just described. We set $I = W \times \mathfrak{s}_\infty$; for $w \in W, \lambda \in \mathfrak{s}_\infty$ we write $w \cdot \lambda$ instead of (w, λ).

1.4. If $\alpha \in R$, the coroot $\check{\alpha} : \mathbf{k}^* \to \mathbf{T}$ restricts to a homomorphism $\mathbf{k}_n^* \to \mathbf{T}_n$ for any $n \in \mathbf{N}^*$ and by passage to projective limits, this induces a homomorphism $\check{\alpha}^\infty : \mathbf{k}^\infty \to \mathbf{T}^\infty$ (notation of 0.2). Let $\lambda \in \mathfrak{s}_\infty$. We say that $\alpha \in R_\lambda$ if the composition $\mathbf{k}^\infty \xrightarrow{\check{\alpha}^\infty} \mathbf{T}^\infty \xrightarrow{\lambda} \bar{\mathbf{Q}}_l^*$ is identically 1 or equivalently if $\check{\alpha}^* L_\lambda \cong \bar{\mathbf{Q}}_l$ as local systems on $\mathbf{k}^*$. Note that for $w \in W$ we have $w(R_\lambda) = R_{w(\lambda)}$. Let $R_\lambda^+ = R_\lambda \cap R^+$, $R_\lambda^- = R_\lambda - R_\lambda^+$. Let W_λ be the subgroup of W generated by $\{\sigma_\alpha; \alpha \in R_\lambda\}$. We have $W_\lambda = W_{\lambda^{-1}}$. Let $W'_\lambda = \{w \in W; w(\lambda) = \lambda\}$. We have $W_\lambda \subset W'_\lambda$. As in [9, 5.3], there is a unique Coxeter group structure on W_λ with length function $W_\lambda \to \mathbf{N}$, $w \mapsto |w|_\lambda = \sharp\{\alpha \in R_\lambda^+; w(\alpha) \in R_\lambda^-\}$; note that, if $w \in W_\lambda$ and $w = \sigma_1\sigma_2 \ldots \sigma_r$ is any reduced expression of w in W, then

$$|w|_\lambda = \text{card}\{i \in [1, r]; \sigma_r \ldots \sigma_{i+1}\sigma_i\sigma_{i+1} \ldots \sigma_r \in W_\lambda\}.$$

1.5. For $n \in \mathbf{N}^*$ we set $I_n = \{w \cdot \lambda \in I; \lambda \in \mathfrak{s}_n\}$. As in [14, 31.2], let $\mathbf{H}_n$ be the associative $\mathcal{A}$-algebra with generators $T_w (w \in W)$, $1_\lambda (\lambda \in \mathfrak{s}_n)$ and relations:

$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda$ for $\lambda, \lambda' \in \mathfrak{s}_n$;

$T_w T_{w'} = T_{ww'}$ if $w, w' \in W$ and $|ww'| = |w| + |w'|$;

$T_w 1_\lambda = 1_{w(\lambda)} T_w$ for $w \in W, \lambda \in \mathfrak{s}_n$;

$T_\sigma^2 = v^2 T_1 + (v^2 - 1) \sum_{\lambda \in \mathfrak{s}_n; \sigma \in W_\lambda} T_\sigma 1_\lambda$ for $\sigma \in S$;

$T_1 = \sum_{\lambda\in\mathfrak{s}_n} 1_\lambda$.

The algebra $\mathbf{H}_n$ is closely related to the algebra introduced by Yokonuma [23]. (It specializes to it under $v = \sqrt{q}, n = q-1$ where q is a power of a prime; this is shown in [15, Sec.35].) Note that T_1 is the unit element of $\mathbf{H}_n$. In [15, 31.2] it is shown that $\{T_w 1_\lambda; w\cdot\lambda \in I_n\}$ is an $\mathcal{A}$-basis of $\mathbf{H}_n$. (In [21, 1.7] we write $\mathbf{H}$ instead of $\mathbf{H}_n$, but here we shall not do so.)

Now, for $\sigma \in S$, T_σ is invertible in $\mathbf{H}_n$; indeed, we have

$$T_\sigma^{-1} = v^{-2}T_\sigma + (1-v^{-2})(\sum_{\lambda\in\mathfrak{s}_n;\sigma\in W_\lambda} 1_\lambda).$$

It follows that T_w is invertible in $\mathbf{H}_n$ for any $w \in W$. As shown in [14, 31.3], there is a unique ring homomorphism $\mathbf{H}_n \to \mathbf{H}_n$, $h \mapsto \bar{h}$ such that $\overline{T_w} = T_{w^{-1}}^{-1}$ for any $w \in W$ and $\overline{f1_\lambda} = \bar{f}1_\lambda$ for any $f \in \mathcal{A}$, $\lambda \in \mathfrak{s}_n$. It is an involution called the *bar involution.*

If $n, n' \in \mathbf{N}^*$ and $n'/n \in \mathbf{Z}$, then $I_n \subset I_{n'}$ and the $\mathcal{A}$-linear map $j_{n,n'}$: $\mathbf{H}_n \to \mathbf{H}_{n'}$ given by $T_w 1_\lambda \mapsto T_w 1_\lambda$ for $w\cdot\lambda \in I_n$ is an $\mathcal{A}$-algebra imbedding which does not necessarily preserve the unit element. Let $\mathbf{H}$ be the union of all $\mathbf{H}_n$ for various $n \in \mathbf{N}^*$ according to the imbeddings $j_{n,n'}$ above. Then $\mathbf{H}$ is an $\mathcal{A}$-algebra without 1 in general; it has an $\mathcal{A}$-basis $\{T_w 1_\lambda = 1_{w(\lambda)}T_w; w\cdot\lambda \in I\}$. If $n \in \mathbf{N}^*$, then $\mathbf{H}_n$ is the $\mathcal{A}$-submodule of $\mathbf{H}$ with basis $\{T_w 1_\lambda; w\cdot\lambda \in I_n\}$; it is an $\mathcal{A}$-subalgebra of $\mathbf{H}$. The algebra $\mathbf{H}_n$ has been studied in [15] and [21, 1.7]. We shall often refer to *loc.cit.* for properties of $\mathbf{H}$ which in *loc.cit.* are stated for $\mathbf{H}_n$ with n fixed and which imply immediately the corresponding properties of $\mathbf{H}$.

We show that, if $n, n' \in \mathbf{N}^*$ and $n'/n \in \mathbf{Z}$, then $j_{n,n'} : \mathbf{H}_n \to \mathbf{H}_{n'}$ is compatible with the bar-involution on $\mathbf{H}_n$ and $\mathbf{H}_{n'}$. It is enough to show that $j_{n,n'}(\bar{\xi}) = \overline{j_{n,n'}(\xi)}$ for $\xi = 1_\lambda, \lambda \in \mathfrak{s}_n$ or $\xi = T_\sigma$, $\sigma \in S$. The case where $\xi = 1_\lambda, \lambda \in \mathfrak{s}_n$ is immediate. For $\sigma \in S$ we have $j_{n,n'}(T_\sigma) = T_\sigma \sum_{\lambda\in\mathfrak{s}_n} 1_\lambda$, hence

$$\begin{aligned} j_{n,n'}(\overline{T_\sigma}) &= j_{n,n'}(v^{-2}T_\sigma + (1-v^{-2})(\sum_{\lambda\in\mathfrak{s}_n;\sigma\in W_\lambda} 1_\lambda)) \\ &= v^{-2}T_\sigma \sum_{\lambda\in\mathfrak{s}_n} 1_\lambda + (1-v^{-2})(\sum_{\lambda\in\mathfrak{s}_n;\sigma\in W_\lambda} 1_\lambda) = T_\sigma^{-1}\sum_{\lambda\in\mathfrak{s}_n} 1_\lambda = \overline{j_{n,n'}(T_\sigma)}, \end{aligned}$$

as desired. It follows that there is a unique ring homomorphism $\mathbf{H} \to \mathbf{H}$, $h \mapsto \bar{h}$, whose restriction to $\mathbf{H}_n$ (for any $n \in \mathbf{N}^*$) is the bar involution. This has square 1 and is again called the bar involution.

The $\mathcal{A}$-linear map $\mathbf{H} \to \mathbf{H}$, $h \mapsto \tilde{h}$ given by $T_w 1_\lambda \mapsto T_w 1_{\lambda^{-1}}$ for $w \cdot \lambda \in I$ is an algebra involution. The $\mathcal{A}$-linear map $\mathbf{H} \to \mathbf{H}$, $h \mapsto h^\flat$, given by $T_w 1_\lambda \mapsto 1_\lambda T_{w^{-1}}$ is an involutive algebra antiautomorphism. (See [15, 32.19].)

1.6. As in [15, 34.4], for any $w \cdot \lambda \in I$ there is a unique element $c_{w\cdot\lambda} \in \mathbf{H}$ such that

$$c_{w\cdot\lambda} = \sum_{y \in W} p_{y\cdot\lambda, w\cdot\lambda} v^{-|y|} T_y 1_\lambda$$

where $p_{y\cdot\lambda,w\cdot\lambda} \in v^{-1}\mathbf{Z}[v^{-1}]$ if $y \neq w$, $p_{w\cdot\lambda,w\cdot\lambda} = 1$ and $\overline{c_{w\cdot\lambda}} = c_{w\cdot\lambda}$. For $\lambda \in \mathfrak{s}_\infty$, y', w' in W_λ let $P^\lambda_{y',w'}$ be the polynomial defined in [5] in terms of the Coxeter group W_λ; let

$$p^\lambda_{y',w'} = v^{-|w'|_\lambda + |y'|_\lambda} P^\lambda_{y',w'}(v^2) \in \mathbf{Z}[v^{-1}].$$

Let $w \cdot \lambda \in I$. From [6, 1.9(i)] we see that wW_λ contains a unique element z such that $|z|$ is minimum; we write $z = \min(wW_\lambda)$; we have $w = zw'$ with $w' \in W_\lambda$. We have

(a) $$c_{w\cdot\lambda} = \sum_{y' \in W_\lambda} p^\lambda_{y',w'} v^{-|zy'|} T_{zy'} 1_\lambda.$$

See [21, 1.8(a)]. From (a) we see that

$$p_{y\cdot\lambda, zw'\cdot\lambda} = p^\lambda_{y',w'}(v^2) \text{ if } y = zy', y' \in W_\lambda,$$

$$p_{y\cdot\lambda, zw'\cdot\lambda} = 0 \text{ if } y \notin zW_\lambda.$$

In particular we have $p_{y\cdot\lambda,w\cdot\lambda} \in \mathbf{N}[v^{-1}]$. From [21, 1.8] for $w \cdot \lambda \in I$ we have

$$\widetilde{c_{w\cdot\lambda}} = c_{w\cdot\lambda^{-1}}, c^\flat_{w\cdot\lambda} = c_{w^{-1}\cdot w(\lambda)}.$$

1.7. Now $\mathbf{H}$ can be regarded as a two-sided ideal in an $\mathcal{A}$-algebra $\mathbf{H}'$ with 1 as follows.

Let $[\mathfrak{s}_\infty]$ be the set of formal $\mathcal{A}$-linear combinations $\sum_{\lambda\in\mathfrak{s}_\infty} c_\lambda 1_\lambda$ with $c_\lambda \in \mathcal{A}$; this is an $\mathcal{A}$-module in an obvious way. We regard $[\mathfrak{s}_\infty]$ as a (commutative) $\mathcal{A}$-algebra with multiplication

$$(\sum_{\lambda\in\mathfrak{s}_\infty} c_\lambda 1_\lambda)(\sum_{\lambda\in\mathfrak{s}_\infty} c'_\lambda 1_\lambda) = \sum_{\lambda\in\mathfrak{s}_\infty} c_\lambda c'_\lambda 1_\lambda.$$

This algebra has a unit element $1 = \sum_{\lambda\in\mathfrak{s}_\infty} 1_\lambda$.

Let $\mathbf{H}'$ be the $\mathcal{A}$-algebra with generators $T_w (w \in W)$ and $\phi \in [\mathfrak{s}_\infty]$ and relations:

$T_w T_{w'} = T_{ww'}$ if $w, w' \in W$ and $|ww'| = |w| + |w'|$;

$T_\sigma^2 = v^2 T_1 + (v^2-1)T_\sigma(\sum_{\lambda\in\mathfrak{s}_\infty;\sigma\in W_\lambda} 1_\lambda)$ for $\sigma \in S$;

$T_w\phi = \phi' T_w$ for $\phi = \sum_{\lambda\in\mathfrak{s}_\infty} c_\lambda 1_\lambda, \phi' = \sum_{\lambda\in\mathfrak{s}_\infty} c_{w^{-1}(\lambda)} 1_\lambda$ in $[\mathfrak{s}_\infty]$, $w \in W$;

the map $[\mathfrak{s}_\infty] \to \mathbf{H}'$, $\xi \mapsto \xi$ respects the algebra structures.

It follows that $\mathbf{H}'$ is a free left $[\mathfrak{s}_\infty]$-module with basis $\{T_w; w \in W\}$ and a right free $[\mathfrak{s}_\infty]$-module with basis $\{T_w; w \in W\}$. Note that the algebra $\mathbf{H}'$ has a unit element $\sum_{\lambda\in\mathfrak{s}_\infty} 1_\lambda$. Now $\mathbf{H}$ can be identified with the two-sided ideal of $\mathbf{H}'$ which as an $\mathcal{A}$-submodule is free with basis $\{T_w 1_\lambda = 1_{w(\lambda)} T_w; w\cdot\lambda \in I\}$.

1.8. Let $W\backslash\mathfrak{s}_\infty$ be the set of W-orbits on $\mathfrak{s}_\infty$. For any $\mathfrak{o} \in W\backslash\mathfrak{s}_\infty$ we set $I_\mathfrak{o} = \{w\cdot\lambda \in I; \lambda \in \mathfrak{o}\}$. This is a finite set. We have $I = \sqcup_\mathfrak{o} I_\mathfrak{o}$, $\mathbf{H} = \oplus_\mathfrak{o} \mathbf{H}_\mathfrak{o}$ where $\mathbf{H}_\mathfrak{o}$ is the $\mathcal{A}$-submodule of $\mathbf{H}$ spanned by $\{T_w 1_\lambda = 1_{w(\lambda)} T_w; w\cdot\lambda \in I_\mathfrak{o}\}$ (thus, $H_\mathfrak{o}$ is a free $\mathcal{A}$-module of finite rank). If $\mathfrak{o}, \mathfrak{o}'$ are distinct in $W\backslash\mathfrak{s}_\infty$, then clearly $\mathbf{H}_\mathfrak{o}\mathbf{H}_{\mathfrak{o}'} = 0$. Thus, each $\mathbf{H}_\mathfrak{o}$ is a subalgebra of $\mathbf{H}$; unlike $\mathbf{H}$, it has a unit element $\sum_{\lambda\in\mathfrak{o}} 1_\lambda$. It is stable under $h \mapsto \bar{h}$ and under $h \mapsto h^\flat$. Moreover, $h \mapsto \tilde{h}$ is an isomorphism of $\mathbf{H}_\mathfrak{o}$ onto $\mathbf{H}_{\mathfrak{o}^{-1}}$. For any $w\cdot\lambda \in I_\mathfrak{o}$ we have $c_{w\cdot\lambda} \in \mathbf{H}_\mathfrak{o}$; moreover, $\{c_{w\cdot\lambda}; w\cdot\lambda \in I_\mathfrak{o}\}$ is an $\mathcal{A}$-basis of $\mathbf{H}_\mathfrak{o}$.

1.9. For i, i' in I we write $c_i c_{i'} = \sum_{j\in I} h_{i,i',j} c_j$ (product in $\mathbf{H}$) where $h_{i,i',j} \in \mathcal{A}$. Let $j \underset{\text{left}}{\preceq} i$ (resp. $j \preceq i$) be the preorder on I generated by the relations $h_{i',i,j} \neq 0$ for some $i' \in I$, resp. by the relations

$$h_{i,i',j} \neq 0 \text{ or } h_{i',i,j} \neq 0 \text{ for some } i' \in I.$$

We say that $i \underset{\text{left}}{\sim} j$ (resp. $i \sim j$) if $i \underset{\text{left}}{\preceq} j$ and $j \underset{\text{left}}{\preceq} i$ (resp. $i \preceq j$ and $j \preceq i$). This is an equivalence relation on I; the equivalence classes are called left cells (resp. two-sided cells). Note that any two-sided cell is a union of left cells. Since for $\mathfrak{o} \in W\backslash\mathfrak{s}_\infty$, $\mathbf{H}_\mathfrak{o}$ is closed under left and right multiplication by elements in $\mathbf{H}$, we see that

$$h_{i,i',j} \neq 0, i \in I_\mathfrak{o} \textit{ implies } i', j \in I_\mathfrak{o}; \ h_{i,i',j} \neq 0, i' \in I_\mathfrak{o} \textit{ implies } i, j \in I_\mathfrak{o}.$$

It follows that $j \preceq i, i \in I_\mathfrak{o}$ implies $j \in I_\mathfrak{o}$. In particular, $j \sim i, i \in I_\mathfrak{o}$ implies $j \in I_\mathfrak{o}$. Thus any two-sided cell is contained in $I_\mathfrak{o}$ for a unique $\mathfrak{o}$.

For $i = w \cdot \lambda \in I$ we set

$$i^! = w^{-1} \cdot w(\lambda) \in I.$$

Note that $i \mapsto i^!$ is an involution of I preserving $I_\mathfrak{o}$ for any $\mathfrak{o}$.

If $\mathbf{c}$ is a two-sided cell and $i \in I$, we write $i \preceq \mathbf{c}$ (resp. $\mathbf{c} \preceq i$) if $i \preceq i'$ (resp. $i' \preceq i$) for some $i' \in \mathbf{c}$; we write $i \prec \mathbf{c}$ (resp. $\mathbf{c} \prec i$) if $i \preceq \mathbf{c}$ (resp. $\mathbf{c} \preceq i$) and $i \notin \mathbf{c}$. If $\mathbf{c}, \mathbf{c}'$ are two-sided cells, we write $\mathbf{c} \preceq \mathbf{c}'$ (resp. $\mathbf{c} \prec \mathbf{c}'$) if $i \preceq i'$ (resp. $i \preceq i'$ and $i \not\sim i'$) for some $i \in \mathbf{c}, i' \in \mathbf{c}'$.

Let $j \in I$. We can find an integer $m \geq 0$ such that $h_{i,i',j} \in v^{-m}\mathbf{Z}[v]$ for all i, i'; let $a(j)$ be the smallest such m. For i, i', j in I there is a well defined integer $h^*_{i,i',j}$ such that

$$h_{i,i',j^!} = h^*_{i,i',j} v^{-a(j^!)} + \text{ higher powers of } v.$$

Note that

$$h^*_{i,i',j} \neq 0, i \in I_\mathfrak{o} \textit{ implies } i', j \in I_\mathfrak{o}; \ h^*_{i,i',j} \neq 0, i' \in I_\mathfrak{o} \textit{ implies } i, j \in I_\mathfrak{o}.$$

Let $\mathbf{D}$ be the set of all $w \cdot \lambda \in I$ where w is a distinguished involution of the Coxeter group W_λ, see [8]. We have $\mathbf{D} = \sqcup_\mathfrak{o}(\mathbf{D} \cap \mathfrak{o})$.

By [21, 1.11], the following properties hold:

Q1. If $j \in \mathbf{D}$ and $i, i' \in I$ satisfy $h^*_{i,i',j} \neq 0$ then $i' = i^*$.

Q2. If $i \in I$, there exists a unique $j \in \mathbf{D}$ such that $h^*_{i^!,i,j} \neq 0$.

Q3. If $i' \preceq i$ then $a(i') \geq a(i)$. Hence if $i' \sim i$ then $a(i') = a(i)$.

Q4. If $j \in \mathbf{D}$, $i \in I$ and $h^*_{i^!,i,j} \neq 0$ then $h^*_{i^!,i,j} = 1$.

Q5. For any i, j, k in I we have $h^*_{i,j,k} = h^*_{j,k,i}$.

Q6. Let i, j, k in I be such that $h^*_{i,j,k} \neq 0$. Then $i \underset{\text{left}}{\sim} j^!$, $j \underset{\text{left}}{\sim} k^!$, $k \underset{\text{left}}{\sim} i^!$.

Q7. If $i' \underset{\text{left}}{\preceq} i$ and $a(i') = a(i)$ then $i' \underset{\text{left}}{\sim} i$.

Q8. If $i' \preceq i$ and $a(i') = a(i)$ then $i' \sim i$.

Q9. Any left cell Γ of I contains a unique element of $j \in \mathbf{D}$. We have $h^*_{i^!,i,j} = 1$ for all $i \in \Gamma$.

Q10. For any $i \in I$ we have $i \sim i^!$.

Note that $h^*_{i,j,k} \in \mathbf{N}$ for all i, j, k in I, see [21, 1.11].

Let $\mathbf{H}^\infty$ be the free abelian group with basis $\{t_i; i \in I\}$. We define a $\mathbf{Z}$-bilinear multiplication $\mathfrak{A}^\infty \times \mathfrak{A}^\infty \to \mathfrak{A}^\infty$ by

$$t_i t_{i'} = \sum_{j \in I} h^*_{i,i',j^!} t_j.$$

For any $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$ let $\mathbf{H}^\infty_\mathfrak{o}$ be the free abelian subgroup of $\mathbf{H}^\infty$ with basis $\{t_i; i \in I_\mathfrak{o}\}$. We have $\mathbf{H}^\infty = \oplus_\mathfrak{o} \mathbf{H}^\infty_\mathfrak{o}$; moreover, if $\mathfrak{o}, \mathfrak{o}'$ are distinct in $W \backslash \mathfrak{s}_\infty$, then $\mathbf{H}^\infty_\mathfrak{o} \mathbf{H}^\infty_{\mathfrak{o}'} = 0$. Thus each $\mathbf{H}^\infty_\mathfrak{o}$ is a subalgebra of $\mathbf{H}$; unlike $\mathbf{H}^\infty$, $\mathbf{H}^\infty_\mathfrak{o}$ has a unit element $\sum_{i \in \mathbf{D} \cap \mathfrak{o}} t_i$. The $\mathbf{Z}$-linear map $\mathbf{H}^\infty \to \mathbf{H}^\infty$, $h \mapsto h^\flat$ defined by $t_i^\flat = t_{i^!}$ for all $i \in I$ is a ring antiautomorphism preserving each $\mathbf{H}^\infty_\mathfrak{o}$. We define an $\mathcal{A}$-linear map $\psi : \mathbf{H} \to \mathcal{A} \otimes \mathbf{H}^\infty$ by

$$\psi(c_i) = \sum_{i' \in I, j \in \mathbf{D}; i' \sim j} h_{i,j,i'} t_{i'} \text{ for all } i \in I.$$

(This last sum is finite. We have $i \in I_\mathfrak{o}$ for some $\mathfrak{o}$. If $h_{i,j,i'} \neq 0$ then we have $i' \in \mathfrak{o}, j \in \mathfrak{o}$. Thus i', j run through a finite set.) By [21, 1.9, 1.11(vi)], ψ is a homomorphism of $\mathcal{A}$-algebras. For any $\mathfrak{o}$, ψ restricts to a homomorphism of $\mathcal{A}$-algebras $\psi_\mathfrak{o} : \mathbf{H}_\mathfrak{o} \to \mathcal{A} \otimes \mathbf{H}^\infty_\mathfrak{o}$ which takes 1 to 1.

We set $\mathbf{H}^v = \mathbf{Q}(v) \otimes_\mathcal{A} \mathbf{H}$, $\mathbf{J} = \mathbf{Q} \otimes \mathbf{H}^\infty$; for any $\mathfrak{o}$ we set $\mathbf{H}^v_\mathfrak{o} = \mathbf{Q}(v) \otimes_\mathcal{A} \mathbf{H}_\mathfrak{o}$, $\mathbf{J}_\mathfrak{o} = \mathbf{Q} \otimes_\mathcal{A} \mathbf{H}^\infty_\mathfrak{o}$. For any $\mathfrak{o}$, ψ induces an algebra isomorphism $\psi^v_\mathfrak{o} : \mathbf{H}^v_\mathfrak{o} \xrightarrow{\sim} \bar{\mathbf{Q}}_l(v) \otimes \mathbf{J}_\mathfrak{o}$; hence ψ induces an algebra isomorphism $\psi^v : \mathbf{H}^v \xrightarrow{\sim} \bar{\mathbf{Q}}_l(v) \otimes \mathbf{J}$.

We define a group homomorphism $\mathbf{t} : \mathbf{H}^\infty \to \mathbf{Z}$ by $\mathbf{t}(t_i) = 1$ if $i \in \mathbf{D}$, $\mathbf{t}(t_i) = 0$ if $i \in I - \mathbf{D}$. As in [21, 1.9(a)], the following can be deduced from Q1,Q2,Q4.

(a) *For $i, j \in I$ we have $\mathbf{t}(t_i t_j) = 1$ if $j = i^!$ and $\mathbf{t}(t_i t_j) = 0$ if $j \neq i^!$.*

1.10. For $n \in \mathbf{N}^*$ we set $\mathbf{H}_n^1 = \bar{\mathbf{Q}}_l \otimes_{\mathcal{A}} \mathbf{H}_n$; this is a $\bar{\mathbf{Q}}_l$-algebra with 1. It is the algebra with generators $T_w (w \in W)$, $1_\lambda (\lambda \in \mathfrak{s}_n)$ and relations:

$1_\lambda 1_{\lambda'} = \delta_{\lambda,\lambda'} 1_\lambda$ for $\lambda, \lambda' \in \mathfrak{s}_n$;

$T_w T_{w'} = T_{ww'}$ for $w, w' \in W$;

$T_w 1_\lambda = 1_{w(\lambda)} T_w$ for $w \in W, \lambda \in \mathfrak{s}_n$;

$T_1 = \sum_{\lambda \in \mathfrak{s}_n} 1_\lambda$.

It has a basis $\{T_w 1_\lambda; w \cdot \lambda \in I_n\}$. Let $\mathbf{H}^1 = \bar{\mathbf{Q}}_l \otimes_{\mathcal{A}} \mathbf{H}$. This is a $\bar{\mathbf{Q}}_l$-algebra without 1 in general. As a vector space it has basis $\{T_w 1_\lambda, w \cdot \lambda \in I\}$. It contains naturally $\mathbf{H}_n^1$ as a subalgebra for any $n \in \mathbf{N}^*$. For any $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$ we set $\mathbf{H}_{\mathfrak{o}}^1 = \bar{\mathbf{Q}}_l \otimes_{\mathcal{A}} \mathbf{H}_{\mathfrak{o}}$; this is a $\bar{\mathbf{Q}}_l$-algebra with 1. It has a basis $\{T_w 1_\lambda; w \cdot \lambda \in I_{\mathfrak{o}}\}$. We have $\mathbf{H}^1 = \oplus_{\mathfrak{o}} \mathbf{H}_{\mathfrak{o}}^1$. Now ψ in 1.9 induces an algebra isomorphism $\psi^1 : \mathbf{H}^1 \xrightarrow{\sim} \mathbf{J}$; for any $\mathfrak{o}$, $\psi_{\mathfrak{o}}$ in 1.9 induces an algebra isomorphism $\psi_{\mathfrak{o}}^1 : \mathbf{H}_{\mathfrak{o}}^1 \xrightarrow{\sim} \mathbf{J}_{\mathfrak{o}}$ taking 1 to 1.

1.11. Let $n \in \mathbf{N}^*$. Consider the group algebra $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$ where $W\mathbf{T}_n$ is the semidirect product of W and $\mathbf{T}_n$ with $\mathbf{T}_n$ normal and W acting on $\mathbf{T}_n$ by $w : t \mapsto w(t)$. Now $w(t) \mapsto \sum_{\lambda \in \mathfrak{s}_n} \lambda(t) T_w 1_\lambda$ defines a $\bar{\mathbf{Q}}_l$-linear isomorphism $u_n : \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \xrightarrow{\sim} \mathbf{H}_n^1$ which is in fact an algebra isomorphism taking 1 to 1.

Now let $n, n' \in \mathbf{N}^*$ be such that $n'/n \in \mathbf{Z}$. We define a $\bar{\mathbf{Q}}_l$-linear imbedding $h_{n,n'} : \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \to \bar{\mathbf{Q}}_l[W\mathbf{T}_{n'}]$ by

$$h_{n,n'}(wt) = (n/n')^\rho \sum_{t' \in \mathbf{T}_{n'}; t'^{n'/n} = t} wt'.$$

We show that $h_{n,n'}$ is compatible with multiplication, that is, for w, w' in W and t, t' in $\mathbf{T}_n$ we have

$$\begin{aligned} &((n/n')^\rho \sum_{\tilde{t} \in \mathbf{T}_{n'}; \tilde{t}^{n'/n} = t} w\tilde{t})((n/n')^\rho \sum_{\tilde{t}' \in \mathbf{T}_{n'}; \tilde{t}'^{n'/n} = t'} w'\tilde{t}') \\ &= (n/n')^\rho \sum_{\tilde{t}'' \in \mathbf{T}_{n'}; \tilde{t}''^{n'/n} = w'^{-1}(t)t'} ww'\tilde{t}'', \end{aligned}$$

or equivalently

$$\left((n/n')^\rho \sum_{\tilde{t}, \tilde{t}' \in \mathbf{T}_{n'}; \tilde{t}^{n'/n} = t, \tilde{t}'^{n'/n} = t'} w'^{-1}(\tilde{t})\tilde{t}' \sum_{\tilde{t}'' \in \mathbf{T}_{n'}; \tilde{t}''^{n'/n} = w'^{-1}(t)t'} \tilde{t}'' \right),$$

which is easily verified.

Let $j^1_{n,n'} : \mathbf{H}^1_n \xrightarrow{\sim} \mathbf{H}^1_{n'}$ be the specialization of $j_{n,n'}$ (see 1.5) at $v = 1$. We have $u_{n'}h_{n,n'} = j_{n,n'}u_n$; equivalently for $w \in W, t \in \mathbf{T}_n$, we have

$$(n/n')^\rho \sum_{t' \in \mathbf{T}_{n'}; t'^{n'/n} = t} \sum_{\lambda \in \mathfrak{s}_{n'}} \lambda(t')T_w 1_\lambda = \sum_{\lambda \in \mathfrak{s}_n} \lambda(t) T_w 1_\lambda.$$

(It is enough to show that for any $\lambda \in \mathfrak{s}_{n'}$,

$$(n/n')^\rho \sum_{t' \in \mathbf{T}_{n'}; t'^{n'/n} = t} \lambda(t') = \lambda(t).$$

is equal to $\lambda(t)$ if $\lambda \in \mathfrak{s}_n$ and to 0 if $\lambda \notin \mathfrak{s}_n$. This is immediate: we use that the kernel of the surjective homomorphism $\mathbf{T}_{n'} \to \mathbf{T}_n$, $t' \mapsto t'^{n'/n}$ has exactly $(n'/n)^\rho$ elements.)

We can form the union $\cup_{n \in \mathbf{N}^*} \bar{\mathbf{Q}}_l[W\mathbf{T}_n]$ over all imbeddings $h_{n,n'}$ as above. This union has an algebra structure whose restriction to $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$ (for any $n \in \mathbf{N}^*$) is the algebra structure of $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$. Moreover, there is a unique isomorphism of algebras $\cup_{n \in \mathbf{N}^*} \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \xrightarrow{\sim} \mathbf{H}^1$ whose restriction to $\bar{\mathbf{Q}}_l[W\mathbf{T}_n]$ (for any $n \in \mathbf{N}^*$) is $u_n : \bar{\mathbf{Q}}_l[W\mathbf{T}_n] \xrightarrow{\sim} \mathbf{H}^1_n$.

1.12. For $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$, $\mathbf{H}^1_\mathfrak{o}$ is a semisimple $\bar{\mathbf{Q}}_l$-algebra. Let $\mathrm{Irr}(H^1_\mathfrak{o})$ be a set of representatives for the isomorphism classes of simple $\mathbf{H}^1_\mathfrak{o}$-modules.

1.13. We have $\mathbf{H}^\infty = \oplus_\mathbf{c} \mathbf{H}^\infty_\mathbf{c}$, $\mathbf{J} = \oplus_\mathbf{c} \mathbf{J}_\mathbf{c}$, where $\mathbf{c}$ runs over the two-sided cells in I, $\mathbf{H}^\infty_\mathbf{c}$ is the $\mathcal{A}$-submodule of $\mathbf{H}^\infty$ with basis $\{t_i; i \in \mathbf{c}\}$ and $\mathbf{J}_\mathbf{c}$ is the $\bar{\mathbf{Q}}_l$-subspace of $\mathbf{J}$ with basis $\{t_i; i \in \mathbf{c}\}$. Each $\mathbf{H}^\infty_\mathbf{c}$ is an $\mathcal{A}$-subalgebra of $\mathbf{H}^\infty$ with unit $\sum_{i \in \mathbf{D_c}} t_i$ where $\mathbf{D_c} = \mathbf{D} \cap \mathbf{c}$. Each $\mathbf{J}_\mathbf{c}$ is a $\bar{\mathbf{Q}}_l$-subalgebra of $\mathbf{J}$ with the same unit as $\mathbf{H}^\infty_\mathbf{c}$. Moreover if $\mathbf{c}, \mathbf{c}'$ are distinct two-sided cells in I we have $\mathbf{J}_\mathbf{c}\mathbf{J}_{\mathbf{c}'} = 0$. Recall from 1.9 that any two-sided cell in I is contained in $I_\mathfrak{o}$ for a unique $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$. It follows that for any $\mathfrak{o} \in W \backslash \mathfrak{s}_\infty$ we have $\mathbf{J}_\mathfrak{o} = \oplus_{\mathbf{c} \subset I_\mathfrak{o}} \mathbf{J}_\mathbf{c}$. Hence, if $E \in \mathrm{Irr}(H^1_\mathfrak{o})$ then there is a unique two-sided cell $\mathbf{c}_E$ such that $\mathbf{J}_\mathbf{c}$ acts as zero on E^∞ for any $\mathbf{c} \subset I_\mathfrak{o}$ with $\mathbf{c} \neq \mathbf{c}_E$. Thus E^∞ can be viewed as a simple $\mathbf{J}_{\mathbf{c}_E}$-module. We define $a_E \in \mathbf{N}$ to be the constant value of the restriction of $a : I \to \mathbf{N}$ to $\mathbf{c}_E$.

1.14. If $\mathbf{c}$ is a two-sided cell of I then its image $\widetilde{\mathbf{c}}$ under $I \to I$, $w{\cdot}\lambda \mapsto w{\cdot}\lambda^{-1}$ is a two-sided cell of I. (See [21, 1.14]) As noted in 1.9, we have $\mathbf{c} \subset I_\mathfrak{o}$ for a unique $\mathfrak{o}$; from the definitions we have $\widetilde{\mathbf{c}} \subset I_{\mathfrak{o}^{-1}}$. Moreover, the value of the a-function on $\widetilde{\mathbf{c}}$ is equal to the value of the a-function on $\mathbf{c}$. From Q3,Q10 in 1.9, we see that $a(i^!) = a(i)$ for $i \in I$.

1.15. For i, i' in I we show:

(a) *If* $i \underset{\text{left}}{\sim} i'$, *then for some* $u \in I$, $t_{i'}$ *appears with* $\neq 0$ *coefficient in* $t_u t_i$.

(b) *If* $i^! \underset{\text{left}}{\sim} i'^!$, *then for some* $u \in I$, $t_{i'}$ *appears with* $\neq 0$ *coefficient in* $t_i t_u$.

(c) *If* $i \sim i'$, *then for some* u, u' *in* I, $t_{i'}$ *appears with nonzero coefficient in* $t_u t_i t'_u$.

(d) *If* $i \sim i'$, *then* $t_i t_j t_{i'} \neq 0$ *for some* $j \in I$.

The proof is along the lines of that of [13, 18.4]. Let $J^+ = \sum_{k \in I} \mathbf{N} t_k$. We will use repeatedly that $J^+ J^+ \subset J^+$.

Let i, i' be as in (a). Let $d, d' \in \mathbf{D}$ be such that $h^*_{i^!,i,d} \neq 0$ and $h^*_{i'^!,i',d'} \neq 0$. Then $i \underset{\text{left}}{\sim} d$, $i' \underset{\text{left}}{\sim} d'$. Hence $d \underset{\text{left}}{\sim} d'$. By Q9 in 1.9 we have $d = d'$ and $h^*_{i^!,i,d} = 1$, $h^*_{i'^!,i',d} = 1$. Hence $t_{i^!} t_i = t_d + J^+$, $t_{i'^!} t_{i'} = t_d + J^+$, $t_d t_d = t_d$; it follows that $t_{i^!} t_i t_{i'^!} t_{i'} \in t_d t_d + J^+ = t_d + J^+$. In particular, $t_i t_{i'^!} \neq 0$. Thus, $h^*_{i,i'^!,u} \neq 0$ for some $u \in I$. Using Q5 in 1.9 we deduce that $h^*_{u,i,i'^*} \neq 0$ hence $t_{i'}$ appears with $\neq 0$ coefficient in $t_u t_i$. This proves (a). Now (b) follows from (a) using the antiautomorphism of $\mathbf{H}^\infty$ such that $t_u \mapsto t_{u^!}$ for all $u \in I$.

Let i_1, i_2, i_3 in I be such that $i_1 \sim i_2 \sim i_3$. If the conclusion of (c) holds for $(i, i') = (i_1, i_2)$ and for $(i, i') = (i_2, i_3)$ then clearly it holds for $(i, i') = (i_1, i_3)$. Applying this repeatedly, we see that it is enough to prove (c) in the case where i, i' satisfy either $i \underset{\text{left}}{\sim} i'$ or $i^! \underset{\text{left}}{\sim} i'^!$. In these cases the desired result follows from (a),(b).

Let i, i' be as in (d). Then $i \sim i'^!$. By (c), we have $t_{u'} t_i t_u \in a t_{i'^!} + J^+$ for some $u, u' \in I$ and some $a \in \mathbf{Z}_{>0}$. Hence $t_{u'} t_i t_u t_{i'} \in a t_{i'^!} t_{i'} + J^+$. Since $t_{i'^!} t_{i'}$ has some coefficient 1 and the other coefficients are ≥ 0, it follows that $t_{u'} t_i t_u t_{i'} \neq 0$. Thus, $t_i t_u t_{i'} \neq 0$. This proves (d).

2. The Group $\tilde{G}$

2.1. In this paper (except in 2.2) we fix a group $\tilde{G}$ containing G as a subgroup, such that $\tilde{G}/G$ is cyclic of order $\mathbf{m} \leq \infty$ with a fixed generator.

For $s \in \mathbf{Z}$ let $\tilde{G}_s$ be the inverse image of the s-th power of this generator under the obvious map $\tilde{G} \to \tilde{G}/G$. For $\gamma \in \tilde{G}$, the map $G \to G$, $g \mapsto \gamma g \gamma^{-1}$ is denoted by $\mathrm{Ad}(\gamma)$.

We shall always assume that we are in one of the two cases below (later referred to as case A and case B).

(A) We have $\mathbf{m} = \infty$ and one of the following two equivalent conditions are satisfied (q denotes a fixed power of p):

(i) for some $\gamma \in \tilde{G}_1$, $\mathrm{Ad}(\gamma) : G \to G$ is the Frobenius map for an F_q-rational rational structure on G;

(ii) for any $s > 0$ and any $\gamma \in \tilde{G}_s$, $\mathrm{Ad}(\gamma) : G \to G$ is the Frobenius map for an F_{q^s}-rational rational structure on G.

(B) $\mathbf{m} < \infty$ and $\tilde{G}$ is an algebraic group with identity component G.

We show the equivalence of (i), (ii) in case A. Clearly, if (ii) holds then (i) holds. Conversely, assume that (i) holds for $\gamma \in \tilde{G}_1$. If $\gamma' \in \tilde{G}_s$ with $s > 0$, then we have $\gamma' = g_1\gamma^s$ where $g_1 \in G$. By Lang's theorem applied to $\mathrm{Ad}(\gamma^s) : G \to G$, which is the Frobenius map for an F_{q^s}-rational structure on G, we have $g_1 = g_2^{-1}\mathrm{Ad}(\gamma^s)(g_2)$ for some $g_2 \in G$ hence $\gamma' = g_2^{-1}\mathrm{Ad}(\gamma^s)(g_2)\gamma^s = g_2^{-1}\gamma^s g_2$ and $\mathrm{Ad}(\gamma') = \mathrm{Ad}(g_2)^{-1}\mathrm{Ad}(\gamma^s)\mathrm{Ad}(g_2)$. Since $\mathrm{Ad}(g_2) : G \to G$ is an isomorphism of algebraic varieties, it follows that $\mathrm{Ad}(\gamma') : G \to G$ is the Frobenius map for an F_{q^s}-rational structure on G. Thus (ii) holds.

Let $s \in \mathbf{Z}$. In case B, $\tilde{G}_s$ is naturally an algebraic variety. In case A, we view $\tilde{G}_s$ as an algebraic variety using the bijection $g \mapsto g\gamma$ where γ is fixed in $\tilde{G}_s$; this algebraic structure on $\tilde{G}_s$ is independent of the choice of γ. For $s = 0$ this gives the usual structure of algebraic variety of G. For $s \in \mathbf{Z}, s' \in \mathbf{Z}$, the multiplication $\tilde{G}_s \times \tilde{G}_{s'} \to \tilde{G}_{s+s'}$ is obviously a morphism of algebraic varieties in case B, but is only a quasi-morphism in the sense of [20, 0.3] in case A. Similarly, for $s \in \mathbf{Z}$, $\tilde{G}_s \to \tilde{G}_{-s}$, $\gamma \mapsto \gamma^{-1}$ is a morphism of algebraic varieties in case B, but is only a quasi-morphism in case A.

Note that in case A with $s \neq 0$, the conjugation action of G on $\tilde{G}_s$ is transitive. (If $s > 0$, this follows from as above using Lang's theorem, while if $s < 0$ this follows using the bijection $\tilde{G}_s \to \tilde{G}_{-s}$, $\gamma \mapsto \gamma^{-1}$, which commutes with the G-actions.) Moreover in this case for any $\gamma \in \tilde{G}_s$, the stabilizer of γ for this G-action is finite. (This stabilizer is the fixed point set of $\mathrm{Ad}(\gamma) : G \to G$ which is a Frobenius map relative to an F_{q^s}-structure if $s > 0$ or the inverse of a Frobenius map if $s < 0$.)

We show:

(a) *If* $\gamma \in \tilde G_s$ *and* $B \in \mathcal{B}$ *then* $\mathrm{Ad}(\gamma)(B) \in \mathcal{B}$, $\mathrm{Ad}(\gamma)(U_B) = U_{\mathrm{Ad}(\gamma)B}$ *and* $\mathrm{Ad}(\gamma) : \mathcal{B} \to \mathcal{B}$ *is a bijection.*

In case A with $s = 0$ and in case B, (a) is obvious. In case A with $s > 0$, (a) follows from (ii); in case A with $s < 0$, (a) follows from (ii) applied to γ^{-1}.

2.2. Here are some examples in case A.

(i) Let $F : G \to G$ be the Frobenius map for an F_q-rational structure on G. Let $\tilde G = G \times \mathbf{Z}$ regarded as a group with multiplication $(g, s)(g', s') = (gF^s(g'), s + s')$. Define a homomorphism $\tilde G \to \mathbf{Z}$ by $(g, s) \mapsto s$. Its kernel $\{(g, s) \in \tilde G; s = 0\}$ can be identified with G. Note that $\tilde G$ and $\tilde G \to \mathbf{Z}$ are as in case A; we have $(1, 1) \in \tilde G_1$ and $\mathrm{Ad}(1, 1) : G \to G$ is just $F : G \to G$. Moreover, any $\tilde G$ and $\tilde G \to \mathbf{Z}$ as in case A is obtained by the procedure above.

(ii) In the case where G is adjoint we define $\tilde G_s$ for $s \in \mathbf{Z}_{<0}$ to be the set of Frobenius maps $G \to G$ with respect to various split F_{q^s}-rational structures on G; we define $\tilde G_s$ for $s \in \mathbf{Z}_{<0}$ to be the set of maps $G \to G$ whose inverse is in $\tilde G_{-s}$ and we set $\tilde G_0 = G$. Then $\tilde G = \sqcup_{s \in \mathbf{Z}} \tilde G_s$ is as in case A. (This case has been considered in [20].)

(iii) Let V be a finite dimensional $\mathbf{k}$-vector space. For any $s \in \mathbf{Z}$ let $\widetilde{GL(V)}_s$ be the set of all group isomorphisms $T : V \to V$ such that $T(zx) = z^{q^s} T(x)$ for all $z \in \mathbf{k}, x \in V$; in particular we have $\widetilde{GL(V)}_0 = GL(V)$. Then $\widetilde{GL(V)} := \sqcup_{s \in \mathbf{Z}} \widetilde{GL(V)}_s$ is a group under composition of maps; it is of the form $\tilde G$ (as in case A) where $G = GL(V)$.

(iv) Let V be a finite dimensional $\mathbf{k}$-vector space with a nondegenerate symplectic form $(,) : V \times V \to \mathbf{k}$. For any $s \in \mathbf{Z}$ let $\widetilde{Sp(V)}_s$ be the set of all $T \in \widetilde{GL(V)}_s$ such that $(T(x), T(x')) = (x, x')^{q^s}$ for all x, x' in V; in particular we have $\widetilde{Sp(V)}_0 = Sp(V)$. Then $\widetilde{Sp(V)} := \sqcup_{s \in \mathbf{Z}} \widetilde{Sp(V)}_s$ is a group under composition of maps; it is of the form $\tilde G$ (as in case A) where $G = Sp(V)$.

2.3. *In the rest of this paper we fix* $\tau \in \tilde{G}_1$ *such that* $\tau \mathbf{B} \tau^{-1} = \mathbf{B}, \tau \mathbf{T} \tau^{-1} = \mathbf{T}$*. and such that for any* $\sigma \in S$*,* $\mathrm{Ad}(\tau)$ *carries* $\xi_\sigma \in \mathbf{U}_\sigma - \{1\}$ *to* $\xi_{\sigma'} \in \mathbf{U}_{\sigma'} - \{1\}$ *for some* $\sigma' \in S$*.*

Note that such τ exists.

We define a group homomorphism $\mathbf{e} : \tilde{G} \to \tilde{G}$ by $\mathbf{e}(\gamma) = \tau\gamma\tau^{-1}$. We have $\mathbf{e}(\tilde{G}_s) = \tilde{G}_s$ for all $s \in \mathbf{Z}$, $\mathbf{e}(\mathbf{T}) = \mathbf{T}$, $\mathbf{e}(\mathbf{B}) = \mathbf{B}$ (hence $\mathbf{e}(\mathbf{U}) = \mathbf{U}$), $\mathbf{e}(N\mathbf{T}) = N\mathbf{T}$; thus $\mathbf{e}$ induces an automorphism of W denoted again by $\mathbf{e}$ which preserves the Coxeter group structure. If $B \in \mathcal{B}$ then $\mathbf{e}(B) \in \mathcal{B}$ and $B \mapsto \mathbf{e}(B)$, $\mathcal{B} \to \mathcal{B}$ is an automorphism in case B and is the Frobenius map for an $\mathbf{F}_q$-rational structure on $\mathcal{B}$ in case A. We define $\mathbf{e} : \mathcal{B} \times \mathcal{B} \to \mathcal{B} \times \mathcal{B}$ by $\mathbf{e}(B, B') = (\mathbf{e}(B), \mathbf{e}(B'))$. For $w \in W$ we have $\mathbf{e}(G_w) = G_{\mathbf{e}(w)}$ and $\mathbf{e}(\mathcal{O}_w) = \mathcal{O}_{\mathbf{e}(w)}$.

The set $\{\dot{\sigma}; \sigma \in S\}$ of $N\mathbf{T}$ is stable under $\mathbf{e} : N\mathbf{T} \to N\mathbf{T}$. For $w \in W$ we have $(\mathbf{e}(w))^{\cdot} = \mathbf{e}(\dot{w})$. Hence $N_0\mathbf{T}$ is stable under $\mathbf{e} : N\mathbf{T} \to N\mathbf{T}$.

Now for $n \in \mathbf{N}^*$, $\mathbf{e} : \mathbf{T} \to \mathbf{T}$ restricts to an isomorphism $\mathbf{e} : \mathbf{T}_n \to \mathbf{T}_n$ and this induces an isomorphism $\mathbf{e} : \mathfrak{s}_n \to \mathfrak{s}_n$ by $\lambda \mapsto \mathbf{e}(\lambda)$ where $(\mathbf{e}(\lambda))(t) = \lambda(\mathbf{e}^{-1}(t))$ for $t \in \mathbf{T}_n$. Let $\mathbf{e} : \mathfrak{s}_\infty \to \mathfrak{s}_\infty$ be the isomorphism whose restriction to $\mathfrak{s}_n$ is $\mathbf{e} : \mathfrak{s}_n \to \mathfrak{s}_n$ as above for any $n \in \mathbf{N}^*$.

We shall fix a Frobenius map $\Psi : G \to G$ relative to some sufficiently large finite subfield $F_{q'}$ of $\mathbf{k}$ such that $\mathbf{B}, \mathbf{T}$ are Ψ-stable, Ψ acts on t by $t \mapsto t^{q'}$ (hence it acts as the identity on W) and such that $\Psi\mathbf{e} = \mathbf{e}\Psi : G \to G$ and $\Psi(\omega) = \omega$ for any $\omega \in N_0\mathbf{T}$; in case B we also require that $\Psi(\tau^{\mathbf{m}}) = \tau^{\mathbf{m}}$.

For any $s \in \mathbf{Z}$ we define an $F_{q'}$-rational structure on $\tilde{G}_s$ with Frobenius map $\Psi : \tilde{G}_s \to \tilde{G}_s$ by the requirement that $\Psi(g\tau^s) = \Psi(g)\tau^s$ for any $g \in G$; in case B, this rational structure depends only on $\tilde{G}_s$ not on s.

Now for any $n \in \mathbf{N}^*$ we have $\Psi(\mathbf{T}_n) = \mathbf{T}_n$; hence we can define $\Psi : \mathfrak{s}_n \xrightarrow{\sim} \mathfrak{s}_n$ by $(\Psi\lambda)(t) = \lambda(\Psi^{-1}(t))$ for $t \in \mathbf{T}_n$, $\lambda \in \mathfrak{s}_n$. There is a unique bijection $\Psi : \mathfrak{s}_\infty \to \mathfrak{s}_\infty$ whose restriction to $\mathfrak{s}_n$ is as above for any $n \in \mathbf{N}^*$. Now Ψ induces $F_{q'}$-rational structures on various varieties that will appear in the sequel. When we consider $\mathcal{D}_m()$ or $\mathcal{M}_m()$ for such varieties, we will refer to these specific $F_{q'}$-structures.

2.4. We define a bijection $\mathbf{e} : I \to I$ by $\mathbf{e}(w \cdot \lambda) = \mathbf{e}(w) \cdot \mathbf{e}(\lambda)$. The $\mathcal{A}$-linear map $\mathbf{e} : \mathbf{H} \to \mathbf{H}$ defined by $\mathbf{e}(T_w 1_\lambda) = T_{\mathbf{e}(w)} 1_{\mathbf{e}(\lambda)}$ for $w \cdot \lambda \in I$ is an algebra isomorphism commuting with $^-$: $\mathbf{H} \to \mathbf{H}$. It follows that $\mathbf{e}(c_i) = c_{\mathbf{e}(i)}$ for all $i \in I$ and that $\mathbf{e} : I \to I$ maps any left (resp. two-sided) cell of I onto a left (resp. two-sided) cell of I. It also maps any W-orbit in $\mathfrak{s}_\infty$ onto a W-orbit in $\mathfrak{s}_\infty$.

Let $\mathfrak{o} \in \mathfrak{s}_\infty$ and $s \in \mathbf{Z}$ be such that $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$. The $\mathcal{A}$-linear map $\mathbf{e}^s : \mathbf{H} \to \mathbf{H}$ restricts to an $\mathcal{A}$-algebra isomorphism $\mathbf{e}^s : \mathbf{H}_\mathfrak{o} \to \mathbf{H}_\mathfrak{o}$; this gives rise by extension of scalars to a $\bar{\mathbf{Q}}_l$-algebra isomorphism $\mathbf{e}^s : \mathbf{H}^1_\mathfrak{o} \to \mathbf{H}^1_\mathfrak{o}$ and to a $\bar{\mathbf{Q}}_l(v)$-algebra isomorphism $\mathbf{e} : \mathbf{H}^v_\mathfrak{o} \to \mathbf{H}^v_\mathfrak{o}$; moreover the $\bar{\mathbf{Q}}_l$-linear map $\mathbf{e}^s : \mathbf{J}_\mathfrak{o} \to \mathbf{J}_\mathfrak{o}$ given by $t_i \mapsto t_{\mathbf{e}^s(i)}$ for $i \in I_\mathfrak{o}$ is an algebra isomorphism and $\psi^v_\mathfrak{o} : \mathbf{H}^v_\mathfrak{o} \xrightarrow{\sim} \bar{\mathbf{Q}}_l(v) \otimes \mathbf{J}_\mathfrak{o}$, $\psi^1_\mathfrak{o} : \mathbf{H}^1_\mathfrak{o} \xrightarrow{\sim} \mathbf{J}_\mathfrak{o}$ are compatible with the action of $\mathbf{e}^s$.

Let $\mathrm{Irr}_s(\mathbf{H}^1_\mathfrak{o})$ be the set of all $E \in \mathrm{Irr}(\mathbf{H}^1_\mathfrak{o})$ with the following property: there exists a linear isomorphism $\mathbf{e}_s : E \to E$ such that for any $w \cdot \lambda \in I_\mathfrak{o}$ and any $e \in E$ we have

$$\mathbf{e}_s((T_w 1_\lambda)(e))) = (T_{\mathbf{e}^s(w)} 1_{\mathbf{e}^s(\lambda)})(\mathbf{e}_s(e)).$$

(Such $\mathbf{e}_s$ is clearly unique up to a nonzero scalar, if it exists.) We assume that for any $E \in \mathrm{Irr}_s(\mathbf{H}^1_\mathfrak{o})$, an $\mathbf{e}_s$ as above has been chosen; we can assume that $\mathbf{e}_s$ has finite order (since $\mathbf{e}^s : I_\mathfrak{o} \to I_\mathfrak{o}$ has finite order); moreover, when $s = 0$ we have $\mathrm{Irr}_s(\mathbf{H}^1_\mathfrak{o}) = \mathrm{Irr}(\mathbf{H}^1_\mathfrak{o})$ and for any E in this set we can take $\mathbf{e}_s = 1$. If $E \in \mathrm{Irr}(H^1_\mathfrak{o})$ we can view E as a simple $\mathbf{J}_\mathfrak{o}$-module via $\psi^1_\mathfrak{o}$; we denote this $\mathbf{J}_\mathfrak{o}$-module by E^∞. Moreover we can view $\bar{\mathbf{Q}}_l(v) \otimes E^\infty$ as a simple $\mathbf{H}^v_\mathfrak{o}$-module via $\psi^v_\mathfrak{o}$; we denote this $\mathbf{H}^v_\mathfrak{o}$-module by E^v. If in addition we have $E \in \mathrm{Irr}_s(H^1_\mathfrak{o})$, then $\mathbf{e}_s$ can be viewed as a $\bar{\mathbf{Q}}_l$-linear isomorphism $E^\infty \to E^\infty$ (denoted again by $\mathbf{e}_s$) and as a $\bar{\mathbf{Q}}_l(v)$-linear isomorphism $E^v \to E^v$ (denoted again by $\mathbf{e}_s$).

Note that for any $\xi \in \mathbf{J}_\mathfrak{o}$, $e \in E^\infty$ we have $\mathbf{e}_s(\xi(e)) = \mathbf{e}^s(\xi)(\mathbf{e}_s(e))$; for any $\xi' \in \mathbf{H}_\mathfrak{o}$, $e' \in E^v$ we have $\mathbf{e}_s(\xi'(e')) = \mathbf{e}^s(\xi')(\mathbf{e}_s(e'))$.

2.5. For $s \in \mathbf{Z}$ let

$$I^s = \{w \cdot \lambda \in I; w(\lambda) = \mathbf{e}^{-s}(\lambda)\}.$$

For any two-sided cell $\mathbf{c}$ of I we set

$$\mathbf{c}^s = I^s \cap \mathbf{c}.$$

We show:

(a) *If* $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$ *and* $i \in \mathbf{c}$, $j \in I$ *satisfy* $t_{i^!} t_j t_{\mathbf{e}^s(i)} \neq 0$, *then* $j \in \mathbf{c}^s$.

(b) *If* $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$, *then* $\mathbf{c}^s \neq \emptyset$.

We prove (a). Let $i = w \cdot \lambda$, $j = z \cdot \lambda'$. From our assumption we have $t_{z\cdot\lambda'} t_{\mathbf{e}^s(w)\cdot\mathbf{e}^s(\lambda)} \neq 0$ (which implies $\lambda' = \mathbf{e}^s(w(\lambda)))$ and $t_{w^{-1}\cdot w(\lambda)} t_{z\cdot\lambda'} \neq 0$ (which implies $w(\lambda) = z(\lambda')$). We deduce that $z(\lambda') = \mathbf{e}^{-s}(\lambda')$ so that $j \in I^s$. Since $t_{i^!} t_j \neq 0$ and $i^! \in \mathbf{c}$ we must have $j \in \mathbf{c}$. Thus we have $j \in I^s \cap \mathbf{c}$ and (a) is proved.

We prove (b). Let $i \in \mathbf{c}$. By assumption we have $\mathbf{e}^s(i) \in \mathbf{c}$; by Q10 in 1.9 we have $i^! \in \mathbf{c}$. Using 1.15(d) with i, i' replaced by $i^!, \mathbf{e}^s(i)$ we see that for some $j = z \cdot \lambda' \in I$ we have $t_{i^!} t_j t_{\mathbf{e}^s(i)} \neq 0$. Using (a) we deduce that $j \in \mathbf{c}^s$ and (b) is proved.

3. Sheaves on $\tilde{\mathcal{B}}^2$

3.1. Let $\tilde{\mathcal{B}} = G/\mathbf{U}$. We have $\tilde{\mathcal{B}}^2 = \sqcup_{w\in W} \tilde{\mathcal{O}}_w$ where

$$\tilde{\mathcal{O}}_w = \{(x\mathbf{U}, y\mathbf{U}) \in \tilde{\mathcal{B}}^2; x^{-1}y \in G_w\}.$$

The closure of $\tilde{\mathcal{O}}_w$ in $\tilde{\mathcal{B}}^2$ is $\bar{\tilde{\mathcal{O}}}_w = \cup_{y\in W; y\le w} \tilde{\mathcal{O}}_y$. For $w \in W$ and $\omega \in \kappa_0^{-1}(w)$ we define $G_w \to \mathbf{T}$ by $g \mapsto g_\omega$ where $g \in \mathbf{U}\omega g_\omega \mathbf{U}$, $g_\omega \in \mathbf{T}$. We define $j^\omega : \tilde{\mathcal{O}}_w \to \mathbf{T}$ by $j^\omega(x\mathbf{U}, y\mathbf{U}) = (x^{-1}y)_\omega$. For $\lambda \in \mathfrak{s}_\infty$ we set $L^\omega_\lambda = (j^\omega)^* L_\lambda$, a local system on $\tilde{\mathcal{O}}_w$. Let $L^{\omega\sharp}_\lambda$ be its extension to an intersection cohomology complex on $\bar{\tilde{\mathcal{O}}}_w$ viewed as a complex on $\tilde{\mathcal{B}}^2$, equal to 0 on $\tilde{\mathcal{B}}^2 - \bar{\tilde{\mathcal{O}}}_w$. We shall view L^ω_λ as a constructible sheaf on $\tilde{\mathcal{B}}^2$ which is 0 on $\tilde{\mathcal{B}}^2 - \tilde{\mathcal{O}}_w$. Let $\mathbf{L}^\omega_\lambda = L^{\omega\sharp}_\lambda \langle |w| + \nu + 2\rho\rangle$, a simple perverse sheaf on $\tilde{\mathcal{B}}^2$.

(a) *In the remainder of this section we fix a two-sided cell* $\mathbf{c}$ *of* I *and we set* $a = a(i)$ *for some/any* $i \in \mathbf{c}$. *We define* $\mathfrak{o} \in W\backslash\mathfrak{s}_\infty$ *by* $\mathbf{c} \subset I_\mathfrak{o}$. *We denote by* n *the smallest integer in* $\mathbf{N}^*$ *such that* $\mathfrak{o} \subset \mathfrak{s}_n$. *We shall assume that* Ψ *in 2.3 acts as* 1 *on the finite subset* $\{t \in \mathbf{T}; t^n \in \mathbf{T} \cap N_0\mathbf{T}\}$ *of* $\mathbf{T}$.

In particular, $\Psi(t) = t$ for any $t \in \mathbf{T}_n$ (hence $\Psi(\lambda) = \lambda$ for any $\lambda \in \mathfrak{s}_n$).

Now, if $w \in W, \omega \in \kappa_0^{-1}(w), \lambda \in \mathfrak{s}_n$, then $L^\omega_\lambda|_{\tilde{\mathcal{O}}_w}$, $L^{\omega\sharp}_\lambda$ and $\mathbf{L}^\omega_\lambda$ can be regarded naturally as objects in the mixed derived category of pure weight zero. Moreover, $L^\omega_\lambda|_{\tilde{\mathcal{O}}_w}$ (resp. $L^{\omega\sharp}_\lambda$, $\mathbf{L}^\omega_\lambda$) is (noncanonically) isomorphic to

$L^{\dot w}_\lambda|_{\tilde{\mathcal{O}}_w}$ (resp. $L^{\dot w\sharp}_\lambda$, $\mathbf{L}^{\dot w}_\lambda$) in the mixed derived category. (It is enough to show that if $t,t'\in\mathbf{T}, t^n=t'=\dot w\omega^{-1}$ and $h_{t'}:\mathbf{T}\to\mathbf{T}$ is translation by t', then t defines an isomorphism $h^*_{t'}L_\lambda\to L_\lambda$; see [21, 1.15])

We define $\tilde{\mathfrak{h}}:\tilde{\mathcal{B}}^2\to\tilde{\mathcal{B}}^2$ by $(x\mathbf{U},y\mathbf{U})\mapsto(y\mathbf{U},x\mathbf{U})$.

We define an action of $G\times\mathbf{T}^2$ on $\tilde{\mathcal{B}}^2$ (resp. on $\mathbf{T}$) by

$$(g,t_1,t_2):(x\mathbf{U},y\mathbf{U})\mapsto(gxt_1^n\mathbf{U},gyt_2^n\mathbf{U})$$

(resp. by $(g,t_1,t_2):t\mapsto w^{-1}(t_1)^{-n}tt_2^n$). For any $w\in W$, the $G\times\mathbf{T}^2$-action leaves stable $\tilde{\mathcal{O}}_w$ and its restriction to $\tilde{\mathcal{O}}_w$ is transitive; moreover, j^ω is compatible with actions of $G\times\mathbf{T}^2$ on $\tilde{\mathcal{O}}_w$ and $\mathbf{T}$.

If $\lambda\in\mathfrak{s}_n$ then L_λ is a $G\times\mathbf{T}^2$-equivariant local system on $\mathbf{T}$ hence L^λ_w is a $G\times\mathbf{T}^2$-equivariant local system on $\tilde{\mathcal{O}}_w$. By [21, 2.1], the following holds.

(c) *For fixed $w\in W,\omega\in\kappa_0^{-1}(w)$, the local systems L^ω_λ with $\lambda\in\mathfrak{s}_n$ form a set of representatives for the isomorphism classes of irreducible $G\times\mathbf{T}^2$-equivariant local systems on $\tilde{\mathcal{O}}_w$.*

3.2. We define $p_{01}:\tilde{\mathcal{B}}^3\to\tilde{\mathcal{B}}^2$, $p_{12}:\tilde{\mathcal{B}}^3\to\tilde{\mathcal{B}}^2$, $p_{02}:\tilde{\mathcal{B}}^3\to\tilde{\mathcal{B}}^2$ by

$$\begin{aligned}p_{01}(x\mathbf{U},y\mathbf{U},z\mathbf{U}) &= (x\mathbf{U},y\mathbf{U}),p_{12}(x\mathbf{U},y\mathbf{U},z\mathbf{U})=(y\mathbf{U},z\mathbf{U}),\\ p_{02}(x\mathbf{U},y\mathbf{U},z\mathbf{U}) &= (x\mathbf{U},z\mathbf{U}).\end{aligned}$$

For any $L\in\mathcal{D}(\tilde{\mathcal{B}}^2)$, $L'\in\mathcal{D}(\tilde{\mathcal{B}}^2)$, we set

$$L\circ L'=p_{02!}(p_{01}^*L\otimes p_{12}^*L')\in\mathcal{D}(\tilde{\mathcal{B}}^2).$$

This defines a monoidal structure on $\mathcal{D}(\tilde{\mathcal{B}}^2)$. Thus, if ${}^iL\in\mathcal{D}(\tilde{\mathcal{B}})$ for $i=1,\dots,k$, then ${}^1L\circ{}^2L\circ\dots\circ{}^kL\in\mathcal{D}(\tilde{\mathcal{B}})$ is well defined. Note that, if $L\in\mathcal{D}_m(\tilde{\mathcal{B}}^2)$, $L'\in\mathcal{D}_m(\tilde{\mathcal{B}}^2)$ then $L\circ L'$ is naturally in $\mathcal{D}_m(\tilde{\mathcal{B}}^2)$.

3.3. Now assume that $w,w'\in W$, $\omega\in\kappa_0^{-1}(w),\omega'\in\kappa_0^{-1}(w')$, $\lambda,\lambda'\in\mathfrak{s}_\infty$. From [21, 2.3] we see that:

(a) *if $w'(\lambda')\neq\lambda$, then $L^\omega_\lambda\circ L^{\omega'}_{\lambda'}=0$.*

3.4. Now assume that $w,w'\in W$, $\omega\in\kappa_0^{-1}(w),\omega'\in\kappa_0^{-1}(w')$, $\lambda,\lambda'\in\mathfrak{s}_\infty$. Let Ξ be the set of all $(x\mathbf{U},y\mathbf{U},z\mathbf{U})\in\tilde{\mathcal{B}}^3$ such that $x^{-1}y\in\mathbf{U}\omega t\mathbf{U}$,

$y^{-1}z \in \mathbf{U}\omega' t'\mathbf{U}$ for some t, t' in $\mathbf{T}$ (which are in fact uniquely determined). Define $c : \Xi \to \mathbf{T} \times \mathbf{T}$ by $c(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) = (t, t')$ where $x^{-1}y \in \mathbf{U}\omega t\mathbf{U}$, $y^{-1}z \in \mathbf{U}\omega' t'\mathbf{U}$. Define $p'_{02} : \Xi \to \tilde{\mathcal{B}}^2$ by $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \mapsto (x\mathbf{U}, z\mathbf{U})$. From the definitions we see that

(a) $$L^{\omega}_{\lambda} \circ L^{\omega'}_{\lambda'} = p'_{02!}(c^*(L_\lambda \boxtimes L_{\lambda'})).$$

We show:

(b) *If $w'(\lambda') = \lambda$ and $|ww'| = |w| + |w'|$, then we have canonically $L^{\omega}_{\lambda} \circ L^{\omega'}_{\lambda'} = L^{\omega\omega'}_{\lambda'} \otimes \mathfrak{L}$, with $\mathfrak{L}$ as in* 0.2.

Let $Y = \{(x\mathbf{U}, z\mathbf{U}, t, t') \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T}; x^{-1}z \in \mathbf{U}\omega t\mathbf{U}\omega' t'\mathbf{U}\}$. We define $\Xi \to Y$ by $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \mapsto (x\mathbf{U}, z\mathbf{U}, t, t')$ where t, t' in $\mathbf{T}$ are given by $x^{-1}y \in \mathbf{U}\omega t\mathbf{U}$, $y^{-1}z \in \mathbf{U}\omega' t'\mathbf{U}$. This is an isomorphism since $|ww'| = |w| + |w'|$. We identify $\Xi = Y$ through this isomorphism. Then $c : \Xi \to \mathbf{T} \times \mathbf{T}$ becomes $c : Y \to \mathbf{T} \times \mathbf{T}$, $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (t, t')$. We define $h : \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ by $h(t, t') = w'^{-1}(t)t'$. We have

$$Y = \{(x\mathbf{U}, z\mathbf{U}, t, t') \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T}; x^{-1}z \in \mathbf{U}\omega\omega' h(t, t')\mathbf{U}\}.$$

Define $j : Y \to \tilde{\mathcal{O}}_{ww'}$ by $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (x\mathbf{U}, z\mathbf{U})$. Let $j' = j^{\omega\omega'} : \tilde{\mathcal{O}}_{ww'} \to \mathbf{T}$. Using (a) and the cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & \mathbf{T} \times \mathbf{T} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle h} \\ \tilde{\mathcal{O}}_{ww'} & \xrightarrow{j'} & \mathbf{T} \end{array}$$

we see that

$$L^{\omega}_{\lambda} \circ L^{\omega'}_{\lambda'} = j_! c^*(L_\lambda \boxtimes L_{\lambda'}) = j'^* h_!(L_\lambda \boxtimes L_{\lambda'}).$$

Since $L^{\omega\omega'}_{\lambda'} \otimes \mathfrak{L} = j'^*(L_{\lambda'} \otimes \mathfrak{L})$, we see that to prove (b) it is enough to show that $h_!(L_\lambda \boxtimes L_{\lambda'}) = L_{\lambda'} \otimes \mathfrak{L}$ (assuming that $w'(\lambda') = \lambda$). This is proved as in the last paragraph of [21, 2.4].

3.5. Let $\sigma \in S$ and let $\omega \in \kappa_0^{-1}(\sigma)$, $\lambda' \in \mathfrak{s}_\infty$. Define $\delta_\omega : \mathbf{U}_\sigma - \{1\} \to \mathbf{T}$ by $\xi \mapsto t_\xi^{-1}$ where $t_\xi \in \mathbf{T}$ is given by $\omega^{-1}\xi^{-1}\omega \in \mathbf{U}\omega^{-1}t_\xi\mathbf{U}$; let $\mathcal{E} = \delta_\omega^* L^*_{\lambda'}$. Let $\delta' : \mathbf{U}_\sigma - \{1\} \to \mathbf{p}$ be the obvious map. From the definitions we see that:

(a) $\delta'_!\mathcal{E}=0$ *if* $\sigma\notin W_{\lambda'}$; $\delta'_!\mathcal{E}\approx\{\bar{\mathbf{Q}}_l\langle -2\rangle,\bar{\mathbf{Q}}_l[-1]\}$ *if* $\sigma\in W_{\lambda'}$.

Consider the diagram $\mathbf{T}\xleftarrow{\tilde{k}}\mathbf{T}\times(\mathbf{U}_\sigma-\{1\})\xrightarrow{\tilde{h}}\mathbf{T}$ where $\tilde{k}:(t,\xi)\mapsto t_\xi^{-1}$ and $\tilde{h}:(t,\xi)\mapsto tt_\xi^{-1}$. We show:

(b) *Let* $\lambda'\in\mathfrak{s}_\infty$. *If* $\sigma\notin W_{\lambda'}$, *then* $\tilde{h}_!\tilde{k}^*L_{\lambda'}=0$. *If* $\sigma\in W_{\lambda'}$ *then* $\tilde{h}_!\tilde{k}^*L^*_{\lambda'}\approx\{\bar{\mathbf{Q}}_l\langle -2\rangle,\bar{\mathbf{Q}}_l[-1]\}$.

We have $\tilde{k}^*L^*_{\lambda'}=\bar{\mathbf{Q}}_l\boxtimes\mathcal{E}$. Now $\tilde{h}=\tilde{h}'y$ where $y:\mathbf{T}\times(\mathbf{U}_\sigma-\{1\})\to\mathbf{T}\times(\mathbf{U}_\sigma-\{1\})$ is $(t,\xi)\mapsto(tt_\xi^{-1},\xi)$ and $\tilde{h}':\mathbf{T}\times(\mathbf{U}_\sigma-\{1\})\to\mathbf{T}$ is $(t,\xi)\mapsto t$. Clearly, $y_!(\bar{\mathbf{Q}}_l\boxtimes\mathcal{E})=\bar{\mathbf{Q}}_l\boxtimes\mathcal{E}$. It remains to note that $\tilde{h}_!(\bar{\mathbf{Q}}_l\boxtimes\mathcal{E})$ is 0 if $\sigma\notin W_{\lambda'}$ and is $\approx\{\bar{\mathbf{Q}}_l\langle -2\rangle,\bar{\mathbf{Q}}_l[-1]\}$ if $\sigma\in W_\lambda$. (This follows from (a).)

We show:

(c) *Assume that* $\lambda\in\mathfrak{s}_\infty$ *satisfies* $\sigma\in W_\lambda$ *and that* $\omega\in\{\dot{\sigma},\dot{\sigma}^{-1}\}$. *Then we have canonically* $L^\omega_\lambda=L^{\omega^{-1}}_\lambda$.

Define $\zeta:\mathbf{T}\to\mathbf{T}$ by $t\mapsto\omega^2t$. It is enough to show that $\zeta^*L_\lambda=L_\lambda$ canonically. For $t\in\mathbf{T}$ we have $(\zeta^*L_\lambda)_t=(L_\lambda)_{\omega^2t}=(L_\lambda)_{\check{\alpha}_\sigma(-1)}\otimes(L_\lambda)_t$. Hence it is enough to show that we have canonically $(L_\lambda)_{\check{\alpha}_\sigma(-1)}=\bar{\mathbf{Q}}_l$. It is also enough to show that $\check{\alpha}^*_\sigma L_\lambda=\bar{\mathbf{Q}}_l$. This follows from $\alpha_\sigma\in R_\lambda$.

3.6. Now assume that $w=w'=\sigma\in S$, $\omega\in\kappa_0^{-1}(\sigma)$, $\lambda,\lambda'\in\mathfrak{s}_\infty$ are such that $\sigma(\lambda')=\lambda$. In this subsection we show:

(a) *If* $\sigma\notin W_\lambda$, *then* $L^\omega_\lambda\circ L^{\omega^{-1}}_{\lambda'}=L^1_{\lambda'}\langle -2\rangle\otimes\mathfrak{L}$.

(b) *If* $\sigma\in W_\lambda$, *then*

$$L^\omega_\lambda\circ L^{\omega^{-1}}_{\lambda'}\approx\{L^1_{\lambda'}\langle -2\rangle\otimes\mathfrak{L},L^\omega_{\lambda'}\langle -2\rangle\otimes\mathfrak{L},L^\omega_{\lambda'}[-1]\otimes\mathfrak{L}\}.$$

(Note that the conditions $\sigma\in W_\lambda$ and $\sigma\in W_{\lambda'}$ are equivalent.) With the notation of 3.4, we have

$$\Xi=\{(x\mathbf{U},y\mathbf{U},z\mathbf{U})\in\tilde{\mathcal{B}}^3;x^{-1}y\in\mathbf{U}\omega t\mathbf{U},y^{-1}z\in\mathbf{U}\omega^{-1}t'\mathbf{U}\text{ for some }t,t'\text{ in }\mathbf{T}\}.$$

If $(x\mathbf{U},y\mathbf{U},z\mathbf{U})\in\Xi$ then $x^{-1}z\in\mathbf{U}\omega\mathbf{U}\omega^{-1}w'^{-1}(t)t'\mathbf{U}$; in particular we have $x^{-1}z\in\mathbf{B}\cup\mathbf{B}\omega\mathbf{B}$. Thus, Ξ can be partitioned as $\tilde{\mathcal{B}}^I\cup\tilde{\mathcal{B}}^{II}$ where

$$\tilde{\mathcal{B}}^I=\{(x\mathbf{U},y\mathbf{U},z\mathbf{U})\in\Xi;x^{-1}z\in\mathbf{B}\}$$

is a closed subset and

$$\tilde{\mathcal{B}}^{II} = \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \Xi; x^{-1}z \in \mathbf{B}\omega\mathbf{B}\}$$

is an open subset. The map $p'_{02} : \Xi \to \tilde{\mathcal{B}}^2$ (see 3.4) restricts to maps

$$p^I_{02} : \tilde{\mathcal{B}}^I \to \tilde{\mathcal{O}}_1, p^{II}_{02} : \tilde{\mathcal{B}}^{II} \to \tilde{\mathcal{O}}_\sigma;$$

using 3.4(a) we deduce

$$L^\omega_\lambda \circ L^{\omega^{-1}}_{\lambda'} \Leftrightarrow \{p^I_{02!}(c^*(L_\lambda \boxtimes L_{\lambda'})), \quad p^{II}_{02!}(c^*(L_\lambda \boxtimes L_{\lambda'}))\}.$$

We show:

(c) $$p^I_{02!}(c^*(L_\lambda \boxtimes L_{\lambda'})) = L^1_{\lambda'} \otimes \mathfrak{L}\langle -2\rangle .$$

We have

$$\begin{aligned}\tilde{\mathcal{B}}^I &= \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}z \in \mathbf{U}\omega^{-1}t'\mathbf{U} \\ &\quad \text{for some } t, t' \text{ in } \mathbf{T}, x^{-1}z \in \mathbf{B}\},\end{aligned}$$

or equivalently

$$\tilde{\mathcal{B}}^I = \{(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, x^{-1}z \in \mathbf{U}\sigma(t)t'\mathbf{U} \text{ for some } t, t' \text{ in } \mathbf{T}\}.$$

Let $Y = \{(x\mathbf{U}, z\mathbf{U}, t, t') \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T}; x^{-1}z \in \mathbf{U}\sigma(t)t'\mathbf{U}\}$. We define $d : \tilde{\mathcal{B}}^I \to Y$ by $(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \mapsto (x\mathbf{U}, z\mathbf{U}, t, t')$ where t, t' in $\mathbf{T}$ are as in the last formula for $\tilde{\mathcal{B}}^I$. The fibre of d at $(x\mathbf{U}, z\mathbf{U}, t, t') \in Y$ can be identified with $\{y\mathbf{U}; y \in x\mathbf{U}\omega t\mathbf{U}\}$, an affine line. Thus, d is an affine line bundle. We have a cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{c^I} & \mathbf{T} \times \mathbf{T} \\ {\scriptstyle j^I}\downarrow & & \downarrow{\scriptstyle h} \\ \tilde{\mathcal{O}}_1 & \xrightarrow{\tilde{j}^I} & \mathbf{T} \end{array}$$

where $c^I : Y \to \mathbf{T}\times\mathbf{T}$ is $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (t, t')$, $j^I : Y \to \tilde{\mathcal{O}}_1$ is $(x\mathbf{U}, z\mathbf{U}, t, t') \mapsto (x\mathbf{U}, z\mathbf{U})$, $\tilde{j}^I = j^1 : \tilde{\mathcal{O}}_1 \to \mathbf{T}$, $h : \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ is $(t, t') \mapsto \sigma(t)t'$. As in 3.4

we have $h_!(L_\lambda \boxtimes L_{\lambda'}) = L_{\lambda'} \otimes \mathfrak{L}$ (since $\sigma(\lambda') = \lambda$). It follows that

$$(j^I)!(c^I)^*(L_\lambda \boxtimes L_{\lambda'}) = (\tilde{j}^I)^* h_!(L_\lambda \boxtimes L_{\lambda'}) = (\tilde{j}^I)^* L_{\lambda'} \otimes \mathfrak{L}.$$

Hence

$$\begin{aligned} p^I_{02!}(c^*(L_\lambda \boxtimes L_{\lambda'})) &= (j^I)!d_!d^*(c^I)^*(L_\lambda \boxtimes L_{\lambda'}) = (j^I)!(c^I)^*(L_\lambda \boxtimes L_{\lambda'}) \langle -2\rangle \\ &= (\tilde{j}^I)^* L_{\lambda'} \otimes \mathfrak{L} \langle -2\rangle = L^1_{\lambda'} \otimes \mathfrak{L} \langle -2\rangle . \end{aligned}$$

This proves (c). Next we show that

(d) $p^{II}_{02!}(c^*(L_\lambda \boxtimes L_{\lambda'}))$ is 0 if $\sigma \notin W_{\lambda'}$ and is $\Leftrightarrow \{L^\omega_{\lambda'} \langle -2\rangle , L^\omega_{\lambda'}[-1]\}$ if $\sigma \in W_{\lambda'}$.

We have

$$\begin{aligned} \tilde{\mathcal{B}}^{II} = \{&(x\mathbf{U}, y\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{B}}^3; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}z \in \mathbf{U}\omega^{-1}t'\mathbf{U} \\ &\text{for some } t, t' \text{ in } \mathbf{T}, x^{-1}z \in \mathbf{U}\omega t_1\mathbf{U} \text{ for some } t_1 \in \mathbf{T}\}. \end{aligned}$$

Let $(x\mathbf{U}, z\mathbf{U}) \in \tilde{\mathcal{O}}_\sigma$. We can write uniquely $z = x\xi_0\omega t_1 u_1$ where $\xi_0 \in \mathbf{U}_\sigma$, $t_1 \in \mathbf{T}$, $u_1 \in \mathbf{U}$. The fibre Φ of p^{II}_{02} at $(x\mathbf{U}, z\mathbf{U})$ can be identified with

$$\begin{aligned} &\{y\mathbf{U} \in G/UU; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}z \in \mathbf{U}\omega^{-1}t'\mathbf{U}\} \\ &= \{y\mathbf{U} \in G/UU; x^{-1}y \in \mathbf{U}\omega t\mathbf{U}, y^{-1}x\xi_0\omega t_1u_1 \in \mathbf{U}\omega^{-1}t'\mathbf{U}\}. \end{aligned}$$

Setting $x^{-1}y = \xi\omega tu'$ where $\xi \in \mathbf{U}_\sigma$, we can identify

$$\begin{aligned} \Phi &= \{(t, t', \xi) \in \mathbf{T} \times \mathbf{T} \times \mathbf{U}_\sigma; u'^{-1}t^{-1}\omega^{-1}\xi^{-1}\xi_0\omega t_1 \in \mathbf{U}\omega^{-1}t'\mathbf{U}\} \\ &= \{(t, t', \xi) \in \mathbf{T} \times \mathbf{T} \times \mathbf{U}_\sigma; \omega^{-1}\xi^{-1}\xi_0\omega \in \mathbf{U}\omega^{-1}\sigma(t)t't_1^{-1}\mathbf{U}\} \\ &= \{(t, t', \xi) \in \mathbf{T} \times \mathbf{T} \times (\mathbf{U}_\sigma - \{\xi_0\}); t_{\xi^{-1}\xi_0} = \sigma(t)t't_1^{-1}\} \end{aligned}$$

where for $\xi_1 \in \mathbf{U}_s - \{1\}$ we define $t_{\xi_1} \in \mathbf{T}$ by $\omega^{-1}\xi_1^{-1}\omega \in \mathbf{U}\omega^{-1}t_{\xi_1}\mathbf{U}$. Let

$$\begin{aligned} Y' &= \{(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}); \\ &\qquad x^{-1}z \in \mathbf{U}_\sigma\omega\sigma(t)t't_{\xi_1}^{-1}\mathbf{U}\}, \\ Y'_1 &= \{(x\mathbf{U}, z\mathbf{U}, t'_1, \xi_1) \in \tilde{\mathcal{B}} \times \tilde{\mathcal{B}} \times \mathbf{T} \times (\mathbf{U}_\sigma - \{1\}); x^{-1}z \in \mathbf{U}_\sigma\omega t'_1 t_{\xi_1}^{-1}\mathbf{U}\}. \end{aligned}$$

We see that $\tilde{\mathcal{B}}^{II}$ may be identified with Y'. (The identification is via

$$(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \mapsto (x\mathbf{U}, x\xi_0\xi_1^{-1}\omega t\mathbf{U}, z\mathbf{U})$$

where $\xi_0 \in \mathbf{U}_\sigma$ is given by $x^{-1}z \in \xi_0 \omega \mathbf{T}\mathbf{U}$.) Under this identification, p_{02}^{II} becomes the composition fj^{II} where $j^{II} : Y' \to Y_1'$ is

$$(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \mapsto (x\mathbf{U}, z\mathbf{U}, s(t)t', \xi_1)$$

and $f : Y_1' \to \tilde{\mathcal{O}}_\sigma$ is

$$(x\mathbf{U}, z\mathbf{U}, t_1', \xi_1) \mapsto (x\mathbf{U}, z\mathbf{U});$$

moreover, the local system $c^*(L_\lambda \boxtimes L_{\lambda'})$ on $\tilde{\mathcal{B}}^{II}$ becomes the local system $(c^{II})^*(L_\lambda \boxtimes L_{\lambda'})$ on Y' where $c^{II} : Y' \to \mathbf{T} \times \mathbf{T}$ is $(x\mathbf{U}, z\mathbf{U}, t, t', \xi_1) \mapsto (t, t')$. We have a diagram with cartesian squares

$$\begin{array}{ccccc}
 & & Y' & \xrightarrow{c^{II}} & \mathbf{T} \times \mathbf{T} \\
 & & {\scriptstyle j^{II}}\downarrow & & {\scriptstyle h}\downarrow \\
\mathbf{T} \times (\mathbf{U}_\sigma - \{1\}) & \xleftarrow{\tilde{j}'} & Y_1' & \xrightarrow{\tilde{j}^{II}} & \mathbf{T} \\
{\scriptstyle \tilde{h}}\downarrow & & {\scriptstyle f}\downarrow & & \\
\mathbf{T} & \xleftarrow{j'} & \tilde{\mathcal{O}}_s & &
\end{array}$$

where $\tilde{j}^{II} : Y_1' \to \mathbf{T}$ is $(x\mathbf{U}, z\mathbf{U}, t_1', \xi_1) \mapsto t_1'$, $j' : \tilde{\mathcal{O}}_\sigma \to \mathbf{T}$ is j^ω, $\tilde{j}' : Y_1' \to \mathbf{T} \times (\mathbf{U}_\sigma - \{1\})$ is $(x\mathbf{U}, z\mathbf{U}, t_1', \xi_1) \mapsto (t_1', \xi_1)$, $h : \mathbf{T} \times \mathbf{T} \to \mathbf{T}$ is $(t, t') \mapsto \sigma(t)t'$ and $\tilde{h}'$ is as in 3.5.

Let $L' = (\tilde{j}^{II})^* L_{\lambda'}$ (a local system on Y_1'). Let $L'' = j'^* L_{\lambda'} = L_{\lambda'}^\omega$ (a local system on $\tilde{\mathcal{O}}_\sigma$). Define $\tilde{f} : Y_1' \to \mathbf{T}$ by $(x\mathbf{U}, z\mathbf{U}, t_1', \xi_1) \mapsto t_{\xi_1}^{-1}$. Let $\tilde{L} = \tilde{f}^* L_{\lambda'}$ (a local system on Y_1'). The stalk of L' at $(x\mathbf{U}, z\mathbf{U}, t_1', \xi_1) \in Y_1'$ is $(L_{\lambda'})_{t_1'}$. The stalk of f^*L'' at $(x\mathbf{U}, z\mathbf{U}, t_1', \xi_1) \in Y_1'$ is $(L_{\lambda'})_{t_1' t_{\xi_1}^{-1}} = (L_{\lambda'})_{t_1'} \otimes (L_{\lambda'})_{t_{\xi_1}^{-1}}$. Thus we have $L' = f^*L'' \otimes \tilde{L}^*$.

As in 3.4 we have $h_!(L_\lambda \boxtimes L_{\lambda'}) = L_{\lambda'} \otimes \mathfrak{L}$ (since $\sigma(\lambda') = \lambda$). Using the cartesian diagrams above, we see that

$$\begin{aligned}
p_{02!}^{II}(c^*(L_\lambda \boxtimes L_{\lambda'})) &= f_! j_!^{II} (c^{II})^*(L_\lambda \boxtimes L_{\lambda'}) = f_! j_!^{II} (c^{II})^*(L_\lambda \boxtimes L_{\lambda'}) \\
&= f_! (\tilde{j}^{II})^* h_! (L_\lambda \boxtimes L_{\lambda'}) = f_! (\tilde{j}^{II})^* (L_{\lambda'} \otimes \mathfrak{L}) \\
&= f_!(L') \otimes \mathfrak{L} = f_!(f^*L'' \otimes \tilde{L}^*) \otimes \mathfrak{L} = L'' \otimes f_!(\tilde{L}^*) \otimes \mathfrak{L} \\
&= L'' \otimes f_! \tilde{j}'^* \tilde{k}^* (L_{\lambda'}^*) = L'' \otimes f_! \tilde{j}'^* \tilde{k}^* (L_{\lambda'}^*) \\
&= L'' \otimes j'^* \tilde{h}_! \tilde{k}^* (L_{\lambda'}^*) = L'' \otimes j'^* \tilde{h}_! \tilde{k}^* (L_{\lambda'}^*).
\end{aligned}$$

Here $\tilde{k}$ is as in 3.5. Using 3.5(b) we see that this is 0 if $\sigma \notin W_{\lambda'}$ and is $\Bumpeq \{L'' \langle -2 \rangle, L''[-1]\}$ if $\sigma \in W_{\lambda'}$. This proves (d). Now (a),(b) follow from (c),(d).

3.7. Now assume that $w \in W$, $\sigma \in S$, $\omega \in \{\dot{\sigma}, \dot{\sigma}^{-1}\}, \omega' \in \kappa_0^{-1}(w)$, $\lambda, \lambda' \in \mathfrak{s}_\infty$ are such that $w(\lambda') = \lambda$, $|\sigma w| < |w|$. We show:

(a) *If $\sigma \notin W_\lambda$, then $L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}$.*

(b) *If $\sigma \in W_\lambda$, then*

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} \Bumpeq \{L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'}[-1] \otimes \mathfrak{L} \otimes \mathfrak{L}\}.$$

Using 3.4(b), we have $L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_{(\sigma w)(\lambda')}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'}$. Hence $L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_\lambda^\omega \circ L_{(\sigma w)(\lambda')}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'}$. We now apply 3.6(a),(b) to describe $L_\lambda^\omega \circ L_{(\sigma w)(\lambda')}^{\omega^{-1}}$. If $\sigma \notin W_\lambda$, we obtain

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} = L_{(\sigma w)(\lambda')}^{1} \circ L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L}.$$

By 3.4(b) this equals $L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{l}^{\otimes 2}$, proving (a). If $\sigma \in W_\lambda$, we obtain

$$\begin{aligned} L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} \Bumpeq \{ & L_{(\sigma w)\lambda'}^{1} \circ L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L}, \\ & L_{(\sigma w)\lambda'}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L}, L_{(\sigma w)\lambda'}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'}[-1] \otimes \mathfrak{L}\}. \end{aligned}$$

(We have used that $L_{(\sigma w)\lambda'}^{\omega} = L_{(\sigma w)\lambda'}^{\omega^{-1}}$, see 3.5(c).) We now substitute

$$L_{(\sigma w)\lambda'}^{1} \circ L_{\lambda'}^{\omega\omega'} = L_{\lambda'}^{\omega\omega'} \otimes \mathfrak{L}, \; L_{(\sigma w)\lambda'}^{\omega^{-1}} \circ L_{\lambda'}^{\omega\omega'} = L_{\lambda'}^{\omega'} \otimes \mathfrak{L},$$

see 3.4(b); we obtain

$$L_\lambda^\omega \circ L_{\lambda'}^{\omega'} \otimes \mathfrak{L} \Bumpeq \{L_{\lambda'}^{\omega\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'} \langle -2 \rangle \otimes \mathfrak{L} \otimes \mathfrak{L}, L_{\lambda'}^{\omega'}[-1] \otimes \mathfrak{L} \otimes \mathfrak{L}\}.$$

This proves (b).

3.8. Let $\mathcal{D}^\spadesuit \tilde{\mathcal{B}}^2$ be the subcategory of $\mathcal{D}(\tilde{\mathcal{B}}^2)$ consisting of objects which are restrictions of objects in the $G \times \mathbf{T}^2$-equivariant derived category. Let $\mathcal{M}^\spadesuit \tilde{\mathcal{B}}^2$ be the subcategory of $\mathcal{D}^\spadesuit \tilde{\mathcal{B}}^2$ consisting of objects which are perverse sheaves. Let $\mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$ (resp. $\mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$) be the subcategory of $\mathcal{M}^\spadesuit \tilde{\mathcal{B}}^2$ whose objects are perverse sheaves L such that any composition factor of L is of the form $\mathbf{L}_\lambda^{\dot{w}}$ for some $w \cdot \lambda \preceq \mathbf{c}$ (resp. $w \cdot \lambda \prec \mathbf{c}$). Let $\mathcal{D}^{\preceq} \tilde{\mathcal{B}}^2$ (resp. $\mathcal{D}^{\prec} \tilde{\mathcal{B}}^2$) be the subcategory of $\mathcal{D}^\spadesuit \tilde{\mathcal{B}}^2$ whose objects are complexes L such that L^j is in $\mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$ (resp.

$\mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$) for any j. We write $\mathcal{D}_m()$ or $\mathcal{M}_m()$ for the mixed version of any of the categories above. Let $\mathcal{C}^{\spadesuit}\tilde{\mathcal{B}}^2$ be the subcategory of $\mathcal{M}^{\spadesuit}\tilde{\mathcal{B}}^2$ consisting of semisimple objects. Let $\mathcal{C}_0^{\spadesuit}\tilde{\mathcal{B}}^2$ be the subcategory of $\mathcal{M}_m^{\spadesuit}\tilde{\mathcal{B}}^2$ consisting of objects of pure of weight zero. Let $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ be the subcategory of $\mathcal{M}^{\spadesuit}\tilde{\mathcal{B}}^2$ consisting of objects which are direct sums of objects of the form $\mathbf{L}_{\lambda}^{\dot{w}}$ with $w\cdot\lambda \in \mathbf{c}$. Let $\mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2$ be the subcategory of $\mathcal{C}_0^{\spadesuit}\tilde{\mathcal{B}}^2$ consisting of those $L \in \mathcal{C}_0^{\spadesuit}\tilde{\mathcal{B}}^2$ such that, as an object of $\mathcal{C}^{\spadesuit}\tilde{\mathcal{B}}^2$, L belongs to $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$. For $L \in \mathcal{C}_0^{\spadesuit}\tilde{\mathcal{B}}^2$ let $\underline{L}$ be the largest subobject of L such that as an object of $\mathcal{C}^{\spadesuit}\tilde{\mathcal{B}}^2$, we have $\underline{L} \in \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$.

3.9. Let $r \geq 1$. Let $\mathbf{w} = (w_1, \ldots, w_r) \in W^r$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_r)$ be such that $\omega_i \in \kappa_0^{-1}(w_i)$ for $i = 1, \ldots, r$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathfrak{s}_n^r$. We set

$$|\mathbf{w}| = |w_1| + |w_2| + \cdots + |w_r|.$$

For $J \subset [1, r]$, let

$$\begin{aligned}\tilde{\mathcal{O}}_{\mathbf{w}}^J &= \{(x_0\mathbf{U}, x_1\mathbf{U}, \ldots, x_r\mathbf{U}) \in \tilde{\mathcal{B}}^{r+1};\\ &\quad x_{i-1}^{-1}x_i\mathbf{U} \in \bar{G}_{w_i} \forall i \in J, x_{i-1}^{-1}x_i \in G_{w_i} \forall i \in [1, r] - J\}.\end{aligned}$$

Define $c : \tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset} \to \mathbf{T}^r$ by

$$c(x_0\mathbf{U}, x_1\mathbf{U}, \ldots, x_r\mathbf{U}) = ((x_0^{-1}x_1)_{\omega_1}, (x_1^{-1}x_2)_{\omega_2}, \ldots, (x_{r-1}^{-1}x_r)_{\omega_r}).$$

Let $M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega}} \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1})$ be the local system $c^*(L_{\lambda_1} \boxtimes \ldots \boxtimes L_{\lambda_r})$ on $\tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset}$ extended by 0 on $\tilde{\mathcal{B}}^{r+1} - \tilde{\mathcal{O}}_{\mathbf{w}}^{\emptyset}$. For $J \subset [1, r]$ we set

$$\begin{aligned}M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},J} &= p_{01}^{*}\,{}^1L \otimes p_{12}^{*}\,{}^2L \otimes \ldots \otimes p_{r-1,r}^{*}\,{}^rL \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1}),\\ L_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},J} &= p_{0r!}M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},J}\langle|\mathbf{w}|\rangle = {}^1L \circ {}^2L \circ \ldots \circ {}^rL\langle|\mathbf{w}|\rangle \in \mathcal{D}_m(\tilde{\mathcal{B}}^2),\end{aligned}$$

where ${}^iL$ is $L_{\lambda_i}^{\omega_i\sharp}$ for $i \in J$ and $L_{\lambda_i}^{\omega_i}$ for $i \notin J$. Note that $M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},\emptyset} = M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega}}$. Moreover, from [21, 2.15] we have:

(a) $M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},J}$ *is the intersection cohomology complex of* $\tilde{\mathcal{O}}_{\mathbf{w}}^J$ *with coefficients in* $M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega}}$.

Consider the free $\mathbf{T}^{r-1}$-action on $\tilde{\mathcal{B}}^{r+1}$ given by

$$\begin{aligned}&(\tau_1, \tau_2, \ldots, \tau_{r-1}) : (x_0\mathbf{U}, x_1\mathbf{U}, \ldots, x_{r-1}\mathbf{U}, x_r\mathbf{U}) \mapsto\\ &(x_0\mathbf{U}, x_1\tau_1\mathbf{U}, \ldots, x_{r-1}\tau_{r-1}\mathbf{U}, x_r\mathbf{U}).\end{aligned}$$

Note that $\tilde{\mathcal{O}}^J_{\mathbf{w}}$ is stable under this $\mathbf{T}^{r-1}$-action. We also have a free $\mathbf{T}^{r-1}$-action on $\mathbf{T}^r$ given by

$$(\tau_1,\tau_2,\dots,\tau_{r-1}):(t_1,t_2,\dots,t_r)\mapsto$$
$$(t_1\tau_1, w_2^{-1}(\tau_1^{-1})t_2\tau_2, w_3^{-1}(\tau_2^{-1})t_3\tau_3,\dots,w_{r-1}^{-1}(\tau_{r-2}^{-1})t_{r-1}\tau_{r-1}, w_r^{-1}(\tau_{r-1}^{-1})t_r).$$

Let ${}'\tilde{\mathcal{B}}^{r+1}=\mathbf{T}^{r-1}\backslash\tilde{\mathcal{B}}^{r+1}$. Let ${}'\tilde{\mathcal{O}}^J_{\mathbf{w}}=\mathbf{T}^{r-1}\backslash\tilde{\mathcal{O}}^J_{\mathbf{w}}$ (a locally closed subvariety of ${}'\tilde{\mathcal{B}}^{r+1}$). Let ${}'\mathbf{T}^r=\mathbf{T}^{r-1}\backslash\mathbf{T}^r$. Note that ${}'\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}}=\mathbf{T}^{r-1}\backslash\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}}$ is an open dense smooth irreducible subvariety of ${}'\tilde{\mathcal{O}}^J_{\mathbf{w}}$. Now $c:\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}}\to\mathbf{T}^r$ is compatible with the $\mathbf{T}^{r-1}$-actions on $\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}},\mathbf{T}^r$ hence it induces a map ${}'c:{}'\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}}\to{}'\mathbf{T}^r$. The homomorphism $c':\mathbf{T}^r\to\mathbf{T}$ given by

$$(t_1,t_2,\dots,t_r)\mapsto t_1w_2(t_2)w_2w_3(t_3)\dots w_2w_3\dots w_r(t_r)$$

is constant on each orbit of the $\mathbf{T}^{r-1}$-action on $\mathbf{T}^r$ hence it induces a morphism ${}'\mathbf{T}^r\to\mathbf{T}$ whose composition with ${}'c$ is denoted by $\bar{c}:{}'\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}}\to\mathbf{T}$. Let ${}'M^{\boldsymbol{\omega},\emptyset}_{\boldsymbol{\lambda}}$ be the local system $\bar{c}^*L_{\lambda_1}$ on ${}'\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}}$ extended by 0 on ${}'\tilde{\mathcal{B}}^{r+1}-{}'\tilde{\mathcal{O}}^\emptyset_{\mathbf{w}}$. Let ${}'M^{\boldsymbol{\omega},J}_{\boldsymbol{\lambda}}\in\mathcal{D}_m({}'\tilde{\mathcal{B}}^{r+1})$ be the intersection cohomology complex of ${}'\tilde{\mathcal{O}}^J_{\mathbf{w}}$ with coefficients in ${}'M^{\boldsymbol{\omega},\emptyset}_{\boldsymbol{\lambda}}$ extended by 0 on ${}'\tilde{\mathcal{B}}^{r+1}-{}'\tilde{\mathcal{O}}^J_{\mathbf{w}}$. Let $\bar{p}_{0r}:{}'\tilde{\mathcal{O}}^J_{\mathbf{w}}\to\tilde{\mathcal{B}}^2$ be the map induced by $p_{0r}:\tilde{\mathcal{O}}^J_{\mathbf{w}}\to\tilde{\mathcal{B}}^2$. We define ${}'L^{\boldsymbol{\omega},J}_{\boldsymbol{\lambda}}\in\mathcal{D}^{\spadesuit}_m\tilde{\mathcal{B}}^2$ as follows:

if $\lambda_k=w_{k+1}(\lambda_{k+1})$ for $k=1,2,\dots,r-1$, we set ${}'L^{\boldsymbol{\omega},J}_{\boldsymbol{\lambda}}=\bar{p}_{0r!}{}'M^{\boldsymbol{\omega},J}_{\boldsymbol{\lambda}}\langle|\mathbf{w}|\rangle$;

otherwise, we set ${}'L^{\boldsymbol{\omega},J}_{\boldsymbol{\lambda}}=0$.

3.10. For $L,L'\in\mathcal{C}^{\mathbf{c}}_0\tilde{\mathcal{B}}^2$ we set

$$L\underline{\circ}L'=\underline{(L\circ L')^{\{a-\nu\}}}\in\mathcal{C}^{\mathbf{c}}_0\tilde{\mathcal{B}}^2.$$

(For the notation ${}^{\{i\}}$ see 0.2.) By [21, 2.24], $L,L'\mapsto L\underline{\circ}L'$ defines a monoidal structure on $\mathcal{C}^{\mathbf{c}}_0\tilde{\mathcal{B}}^2$. Hence if $L,{}^2L,\dots,{}^rL$ are in $\mathcal{C}^{\mathbf{c}}_0\tilde{\mathcal{B}}^2$ then ${}^1L\underline{\circ}{}^2L\underline{\circ}\dots\underline{\circ}{}^rL\in\mathcal{C}^{\mathbf{c}}_0\tilde{\mathcal{B}}^2$ is well defined.

3.11. Let $w\cdot\lambda\in I_n$ and let $\omega\in\kappa^{-1}(w), s\in\mathbf{Z}$. We show that we have canonically:

(a) $$(\mathbf{e}^s)^*L^\omega_\lambda=L^{\mathbf{e}^{-s}(\omega)}_{\mathbf{e}^{-s}(\lambda)},\ (\mathbf{e}^s)^*\mathbf{L}^\omega_\lambda=\mathbf{L}^{\mathbf{e}^{-s}(\omega)}_{\mathbf{e}^{-s}(\lambda)}.$$

It is enough to prove the first of these equalities. Let $\xi=(x\mathbf{U},y\mathbf{U})\in\tilde{\mathcal{B}}^2$. We have $x^{-1}y\in\mathbf{U}\mathbf{e}^{-s}(\omega)t\mathbf{U}$ with $t\in\mathbf{T}$ hence $\mathbf{e}^s(x)^{-1}\mathbf{e}^s(y)\in\mathbf{U}\omega\mathbf{e}^s(t)\mathbf{U}$.

The stalk of $(\mathbf{e}^s)^* L_\lambda^\omega$ at ξ is equal to the stalk of L_λ at $\mathbf{e}^s(t)$ hence to the stalk of $(\mathbf{e}^s)^* L_\lambda$ at t. The stalk of $L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\omega)}$ at ξ is equal to the stalk of $L_{\mathbf{e}^{-s}(\lambda)}$ at t. It remains to show that $(\mathbf{e}^s)^* L_\lambda = L_{\mathbf{e}^{-s}(\lambda)}$. This follows from the definitions.

4. Sheaves on Z_s

4.1. *In this section we fix $s \in \mathbf{Z}$.*

Now $\mathbf{T}$ acts on $\tilde{\mathcal{B}}^2$ by $t : (x\mathbf{U}, y\mathbf{U}) \mapsto (xt\mathbf{U}, y\mathbf{e}^s(t)\mathbf{U})$. Let $\mathbf{T}\backslash_s\tilde{\mathcal{B}}^2$ be the set of orbits. Let

$$Z_s = \{(B, B', \gamma U_B); B \in \mathcal{B}, B' \in \mathcal{B}, \gamma U_B \in \tilde{G}_s/U_B; \gamma B \gamma^{-1} = B'\}.$$

We define $\epsilon_s : \tilde{\mathcal{B}}^2 \to Z_s$ by $\epsilon_s : (x\mathbf{U}, y\mathbf{U}) \mapsto (x\mathbf{B}x^{-1}, y\mathbf{B}y^{-1}, y\tau^s\mathbf{U}x^{-1})$. Clearly, ϵ_s induces a map $\mathbf{T}\backslash_s\tilde{\mathcal{B}}^2 \to Z_s$. We show:

(a) *ϵ_s induces an isomorphism $\mathbf{T}\backslash_s\tilde{\mathcal{B}}^2 \to Z_s$.*

We show only that our map is bijective. Let $(B, B', \gamma) \in \mathcal{B} \times \mathcal{B} \times \tilde{G}_s$ be such that $\gamma B \gamma^{-1} = B'$. We can find $x \in G$ such that $B = x\mathbf{B}x^{-1}$. We set $y = \gamma x \tau^{-s} \in G$. Then ϵ_s carries the $\mathbf{T}$-orbit of $(x\mathbf{U}, y\mathbf{U})$ to $(B, \gamma B \gamma^{-1}, \gamma x \mathbf{U} x^{-1}) = (B, B', \gamma U_B)$; thus our map is surjective. Now assume that x, x', y, y' in G are such that

$$(x\mathbf{B}x^{-1}, y\mathbf{B}y^{-1}, y\tau^s\mathbf{U}x^{-1}) = (x'\mathbf{B}x'^{-1}, y'\mathbf{B}y'^{-1}, y'\tau^s\mathbf{U}x'^{-1}).$$

To complete the proof of (a) it is enough to show that $x' = xtu$, $y' = y\mathbf{e}^s(t)u'$ for some u, u' in $\mathbf{U}$ and some $t \in \mathbf{T}$. Since $x^{-1}x' \in \mathbf{B}$ we have $x' = xtu$ for some $u \in \mathbf{U}$ and some $t \in \mathbf{T}$. We have $y'\tau^s\mathbf{U}u^{-1}t^{-1}x^{-1} = y\tau^s\mathbf{U}x^{-1}$ hence $y' = y\mathbf{e}^s(t)u'$ for some $u' \in \mathbf{U}$. This completes the proof of (a).

For $w \in W$ let $Z_s^w = \{(B, B', \gamma U_B) \in Z_s; (B, B') \in \mathcal{O}_w\}$. The closure of Z_s^w in Z_s is $\bar{Z}_s^w = \{(B, B', gU_B); (B, B') \in \bar{\mathcal{O}}_w, g \in G, gBg^{-1} = B'\}$. We have $\epsilon_s^{-1}(Z_s^w) = \tilde{\mathcal{O}}_w$, $\epsilon_s^{-1}(\bar{Z}_s^w) = \bar{\tilde{\mathcal{O}}}_w$.

Let $\omega \in \kappa_0^{-1}(w)$ and let $\lambda \in \mathfrak{s}_\infty$ be such that $w \cdot \lambda \in I^s$. We have a diagram $\mathbf{T} \xrightarrow{j^\omega} \tilde{\mathcal{B}}_w^2 \xrightarrow{\epsilon_s^w} Z_s^w$ where ϵ_s^w is the restriction of ϵ_s and j^ω is as in 3.1. The $\mathbf{T}$-action on $\tilde{\mathcal{B}}^2$ described above is compatible under j^ω with the $\mathbf{T}$-action on $\mathbf{T}$ given by $t : t' \mapsto w^{-1}(t^{-1})t'\mathbf{e}^s(t)$. From [14, 28.2] we see that L_λ

is equivariant for the $\mathbf{T}$-action on $\mathbf{T}$ given by $t : t' \mapsto w^{-1}(\mathbf{e}^{-s}(t_1))t't_1^{-1}$. (We use that $w{\cdot}\lambda \in I^s$.) Using the change of variable $t_1 = \mathbf{e}^s(t)^{-1}$, we deduce that L_λ is also equivariant for the $\mathbf{T}$-action on $\mathbf{T}$ given by $t : t' \mapsto w^{-1}(t^{-1})t'\mathbf{e}^s(t)$. It follows that $(j^\omega)^* L_\lambda$ is $\mathbf{T}$-equivariant, so that there is a well defined local system $\mathcal{L}^\omega_{\lambda,s}$ of rank 1 on Z^w_s such that $(\epsilon^w_s)^*\mathcal{L}^\omega_{\lambda,s} = (j^\omega)^* L_\lambda = L^\omega_\lambda$. Let $\mathcal{L}^{\omega\sharp}_{\lambda,s}$ be its extension to an intersection cohomology complex of $\bar{Z}^w_s$, viewed as a complex on Z_s, equal to 0 on $Z_s - \bar{Z}^w_s$. We shall view $\mathcal{L}^\omega_{\lambda,s}$ as a constructible sheaf on Z_s which is 0 on $Z_s - Z^w_s$. Let

$$\mathbb{L}^\omega_{\lambda,s} = \mathcal{L}^{\omega\sharp}_{\lambda,s}\,\langle |w| + \nu + \rho\rangle\,,$$

a simple perverse sheaf on Z_s.

In the remainder of this subsection we assume that $s \neq 0$ *and that we are in case A.*

Let $w \in W$ and let $X^w_s = \{B \in \mathcal{B}; (B, \mathbf{e}^s(B)) \in \mathcal{O}_w\}$. When $s > 0$, X^w_s coincides with the variety X_w defined in [3] in terms of the Frobenius map $\mathbf{e}^s : G \to G$; when $s < 0$, X^w_s can be identified with the variety $X_{\mathbf{e}^{-s}(w^{-1})}$ defined in [3] in terms of the Frobenius map $\mathbf{e}^{-s} : G \to G$. Note that the finite group $G^{\mathbf{e}^s} = \{g \in G; \mathbf{e}^s(g) = g\}$ acts by conjugation on X^w_s.

Let $\tilde{X}^w_s = \{x\mathbf{U} \in G/\mathbf{U}; x^{-1}\mathbf{e}^s(x) \in G_w\}$. We define $\phi : \tilde{X}^w_s \to X^w_s$ by $x\mathbf{U} \mapsto x\mathbf{B}x^{-1}$. This is a principal $\mathbf{T}$-bundle with $\mathbf{T}$ acting on $\tilde{X}^w_s$ by $t : x\mathbf{U} \mapsto xt\mathbf{U}$. We define $j'_{\dot{w}} : \tilde{X}^w_s \to \mathbf{T}$ by $j'_{\dot{w}}(x\mathbf{U}) = (x^{-1}\mathbf{e}^s(x))_{\dot{w}}$. Now let $\lambda \in \mathfrak{s}_\infty$ be such that $w \cdot \lambda \in I^s$. Then there is a well defined local system $\mathcal{F}^{\dot{w}}_{\lambda,s}$ on X^w_s such that $\phi^*\mathcal{F}^{\dot{w}}_{\lambda,s} = (j'_{\dot{w}})^* L_\lambda$. (This is in fact the restriction of $\mathcal{L}^{\dot{w}}_{\lambda,s}$ to X^w_s under the imbedding $X^w_s \to Z^w_s$, $x\mathbf{B}x^{-1} \mapsto (x\mathbf{B}x^{-1}, \mathbf{e}^s(x)\mathbf{B}\mathbf{e}^s(x^{-1}), \tau^s x\mathbf{U}x^{-1})$.) The local system $\mathcal{F}^{\dot{w}}_{\lambda,s}$ on X^w_s is of the type considered in [3]. Note also that $\mathcal{F}^{\dot{w}}_{\lambda,s}$ has a natural $G^{\mathbf{e}^s}$-equivariant structure. (It is the restriction of the G-equivariant structure of $\mathcal{L}^{\dot{w}}_{\lambda,s}$.) It follows that for $j \in \mathbf{Z}$, $H^j_c(X^w_s, \mathcal{F}^{\dot{w}}_{\lambda,s})$ is naturally a $G^{\mathbf{e}^s}$-module. (This representation of $G^{\mathbf{e}^s}$ is one of the main themes of [3].) Let $\bar{X}^w_s = \{B \in \mathcal{B}; (B, \mathbf{e}^s(B)) \in \bar{\mathcal{O}}_w\}$. Then X^w_s is open dense smooth in $\bar{X}^w_s$ and $G^{\mathbf{e}^s}$ acts by conjugation on $\bar{X}^w_s$. Hence for $j \in \mathbf{Z}$, the intersection cohomology space $IH^j(\bar{X}^w_s, \mathcal{F}^{\dot{w}}_{\lambda,s})$ is naturally a $G^{\mathbf{e}^s}$-module.

If $\mathbf{r}, \mathbf{r}'$ are $G^{\mathbf{e}^s}$-modules and $\mathbf{r}$ is irreducible we denote by $(\mathbf{r} : \mathbf{r}')$ the multiplicity of $\mathbf{r}$ in $\mathbf{r}'$. Let $\mathrm{Irr}(G^{\mathbf{e}^s})$ be the set of isomorphism classes of irreducible representations of $G^{\mathbf{e}^s}$. From [3, 7.7] it is known that for any $\mathbf{r} \in \mathrm{Irr}(G^{\mathbf{e}^s})$

(i) there exists $w \cdot \lambda \in I^s$ such that $(\mathbf{r} : \oplus_j H_c^j(X_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$.

From [6, 2.8] we see using (i) that for any $\mathbf{r} \in \mathrm{Irr}(G^{\mathbf{e}^s})$

(ii) there exists $w \cdot \lambda \in I^s$ such that $(\mathbf{r} : \oplus_j IH^j(X_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$.

By [3, 6.3], any $\mathbf{r} \in \mathrm{Irr}(G^{\mathbf{e}^s})$ determines a W-orbit $\mathfrak{o}$ on $\mathfrak{s}_\infty$: the set of all $\lambda \in \mathfrak{s}_\infty$ such that $(\mathbf{r} : \oplus_j H_c^j(X_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$ for some $w \in W$ with $w \cdot \lambda \in I^s$ or equivalently (see [6, 2.8]) such that $(r : \oplus_j IH^j(\bar{X}_s^w, \mathcal{F}_{\lambda,s}^{\dot{w}})) \neq 0$ for some $w \in W$ with $w \cdot \lambda \in I^s$; we have necessarily $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$. For any $\mathfrak{o} \in W\backslash\mathfrak{s}_\infty$ such that $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$, let $\mathrm{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s})$ be the set of all $\mathbf{r} \in \mathrm{Irr}(G^{\mathbf{e}^s})$ such that the W-orbit on $\mathfrak{s}_\infty$ determined by $\mathbf{r}$ is $\mathfrak{o}$. With notation in 1.14 we have the following result:

(b) *There exists a pairing* $\mathrm{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s}) \times \mathrm{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1) \to \bar{\mathbf{Q}}_l$, $(\mathbf{r}, E) \mapsto b_{\mathbf{r},E}$ *such that for any* $\mathbf{r} \in \mathrm{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s})$, *any* $z \cdot \lambda \in I^s \cap I_{\mathfrak{o}}$ *and any* $j \in \mathbf{Z}$ *we have*

$$(\mathbf{r} : IH^j(\bar{X}_s^z, \mathcal{F}_{\lambda,s}^{\dot{z}})) = (-1)^j (j - |z| : \sum_{E \in \mathrm{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1} b_{\mathbf{r},E} \mathrm{tr}(\mathbf{e}_s c_{z\cdot\lambda}, E^v).$$

In the case where G has connected centre, (b) is just a reformulation on [6, 3.8(ii)]. A proof similar to that in *loc.cit.* applies without the hypothesis on the centre.

4.2. *In the remainder of this section let* $\mathbf{c}, a, \mathfrak{o}, n, \Psi$ *be as in 3.1(a).*

The $G \times \mathbf{T}^2$-action on $\tilde{\mathcal{B}}^2$ defined in 3.1 commutes with the $\mathbf{T}$-action on $\tilde{\mathcal{B}}^2$ in 4.1; hence it induces a $G \times \mathbf{T}^2$-action on $\mathbf{T}\backslash_s\tilde{\mathcal{B}}^2$. We define a $G \times \mathbf{T}^2$-action on Z_s by

$$(g, t_1, t_2) : (B, B', \gamma U_B) \mapsto (gBg^{-1}, gB'g^{-1}, g\gamma x_0 \mathbf{e}^s(t_2^{-n}) t_1^n x_0^{-1} g^{-1} U_{gBg^{-1}})$$

where x_0 is any element of G such that $x_0 \mathbf{B} x_0^{-1} = B$. (The choice of x_0 does not matter; to see this, it is enough to show that for $b \in B$ we have

$$\gamma x_0 \mathbf{e}^s(t_2^{-n}) t_1^n x_0^{-1} U_B = \gamma x_0 b \mathbf{e}^s(t_2^{-n}) t_1^n b^{-1} x_0^{-1} U_B$$

which is immediate.) In this $G \times \mathbf{T}^2$ action, the subgroup $\{(1, t_1, t_2) \in G \times \mathbf{T}^2; t_1 = \mathbf{e}^s(t_2)\}$ acts trivially. Note that the bijection $\mathbf{T}\backslash_s \tilde{\mathcal{B}}^2 \to Z_s$ in 4.1(a) is compatible with the $G \times \mathbf{T}^2$-actions.

Let $w \in W, \omega \in \kappa_0^{-1}(w)$. Since the $G \times \mathbf{T}^2$-action on $\tilde{\mathcal{O}}_w$ is transitive, it follows that the $G \times \mathbf{T}^2$-action on Z_s^w is transitive. We show :

(a) *Let $\mathcal{L}$ be an irreducible $G \times \mathbf{T}^2$-equivariant local system on Z_s^w. Then $\mathcal{L}$ is isomorphic to $\mathcal{L}_{\lambda,s}^\omega$ for a unique $\lambda \in \mathfrak{s}_n$ such that $w \cdot \lambda \in I^s$.*

The local system $(\epsilon_s^w)^* \mathcal{L}$ on $\tilde{\mathcal{O}}_w$ is irreducible and $G \times \mathbf{T}^2$-equivariant hence, by 3.1(c), is isomorphic to L_λ^ω for a well defined $\lambda \in \mathfrak{s}_n$. Now the restriction of $(\epsilon_s^w)^* \mathcal{L}$ to any fibre of ϵ_s^w is $\bar{\mathbf{Q}}_l$. On the other hand, the restriction of L_λ^ω to the fibre of ϵ_s^w passing through $(\mathbf{U}, \omega\mathbf{U})$ is (under an obvious identification with $\mathbf{T}$) the inverse image of L_λ under the map $\mathbf{T} \to \mathbf{T}$, $t \mapsto w^{-1}(t^{-1})\mathbf{e}^s(t)$, hence it is $L_{w(\lambda^{-1})\mathbf{e}^{-s}(\lambda)}$ which is $\bar{\mathbf{Q}}_l$ if and only if $w(\lambda) = \mathbf{e}^{-s}\lambda$. We see that we must have $w(\lambda) = \mathbf{e}^{-s}(\lambda)$. We have $(\epsilon_s^w)^* \mathcal{L} \cong (\epsilon_s^w)^* \mathcal{L}_{\lambda,s}^\omega$ (both are L_λ^ω) hence $\mathcal{L} \cong \mathcal{L}_{\lambda,s}^\omega$. This proves (a).

We define $\mathfrak{h} : Z_s \to Z_{-s}$ by $(B, B', gU_B) \mapsto (B', B, g^{-1}U_{B'})$. Note that $\mathfrak{h}\epsilon_s = \epsilon_{-s}\tilde{\mathfrak{h}} : \tilde{\mathcal{B}}^2 \to Z_{-s}$ with $\tilde{\mathfrak{h}}$ as in 3.1. For $L \in \mathcal{D}_m(Z_{-s})$ we set $L^\dagger = \mathfrak{h}^* L$.

4.3. Let

$$I_n^s = I_n \cap I^s.$$

Note that if $w \cdot \lambda \in I_n^s$ and $\omega \in \kappa_0^{-1}(w)$, then $\mathcal{L}_{\lambda,s}^\omega|_{Z_s^w}$, $\mathbb{L}_{\lambda,s}^\omega$ can be regarded naturally as objects in the mixed derived category of pure weight zero. Moreover, $\mathcal{L}_{\lambda,s}^\omega|_{Z_s^w}$ (resp. $\mathbb{L}_{\lambda,s}^\omega$) is (noncanonically) isomorphic to $\mathcal{L}_{\lambda,s}^{\dot{w}}|_{Z_s^w}$ (resp. $\mathbb{L}_{\lambda,s}^{\dot{w}}$) in the mixed derived category.

We define $\tilde{\epsilon}_s : \mathcal{D}(Z_s) \to \mathcal{D}(\tilde{\mathcal{B}}^2)$, $\tilde{\epsilon}_s : \mathcal{D}_m(Z_s) \to \mathcal{D}_m(\tilde{\mathcal{B}}^2)$ by

$$\tilde{\epsilon}_s(L) = \epsilon_s^*(L) \langle \rho \rangle .$$

From the definition we have

$$\epsilon_s^* \mathcal{L}_{\lambda,s}^{\omega\sharp} = L_\lambda^{\omega\sharp}, \quad \tilde{\epsilon}_s \mathbb{L}_{\lambda,s}^\omega = \mathbf{L}_\lambda^\omega.$$

Let $\mathcal{D}^\spadesuit Z_s$ be the subcategory of $\mathcal{D}(Z_s)$ consisting of objects which are restrictions of objects in the $G \times \mathbf{T}^2$-equivariant derived category. Let $\mathcal{M}^\spadesuit Z_s$ be the subcategory of $\mathcal{D}^\spadesuit Z_s$ consisting of objects which are perverse sheaves.

Let $\mathcal{M}^{\preceq}Z_s$ (resp. $\mathcal{M}^{\prec}Z_s$) be the subcategory of $\mathcal{D}^{\spadesuit}Z_s$ whose objects are perverse sheaves L such that any composition factor of L is of the form $\mathbb{L}_{\lambda,s}^{\dot{w}}$ for some $w \cdot \lambda \in I_n^s$ such that $w \cdot \lambda \preceq \mathbf{c}$ (resp. $w \cdot \lambda \prec \mathbf{c}$). Let $\mathcal{D}^{\preceq}Z_s$ (resp. $\mathcal{D}^{\prec}Z_s$) be the subcategory of $\mathcal{D}^{\spadesuit}Z_s$ whose objects are complexes L such that L^j is in $\mathcal{M}^{\preceq}Z_s$ (resp. $\mathcal{M}^{\prec}Z_s$) for any j. We write $\mathcal{D}_m()$ or $\mathcal{M}_m()$ for the mixed version of any of the categories above.

Let $\mathcal{C}^{\spadesuit}Z_s$ be the subcategory of $\mathcal{M}^{\spadesuit}Z_s$ consisting of semisimple objects. Let $\mathcal{C}_0^{\spadesuit}Z_s$ be the subcategory of $\mathcal{M}_m^{\spadesuit}Z_s$ consisting of objects of pure of weight zero. Let $\mathcal{C}^{\mathbf{c}}Z_s$ be the subcategory of $\mathcal{M}^{\spadesuit}Z_s$ consisting of objects which are direct sums of objects of the form $\mathbb{L}_{\lambda,s}^{\dot{w}}$ with $w \cdot \lambda \in \mathbf{c}^s$. Let $\mathcal{C}_0^{\mathbf{c}}Z_s$ be the subcategory of $\mathcal{C}_0^{\spadesuit}Z_s$ consisting of those $L \in \mathcal{C}_0^{\spadesuit}Z_s$ such that, as an object of $\mathcal{C}^{\spadesuit}Z_s$, L belongs to $\mathcal{C}^{\mathbf{c}}Z_s$. For $L \in \mathcal{C}_0^{\spadesuit}Z_s$ let $\underline{L}$ be the largest subobject of L such that as an object of $\mathcal{C}^{\spadesuit}Z_s$, we have $\underline{L} \in \mathcal{C}^{\mathbf{c}}Z_s$.

From 4.2(a) we see that, if $M \in \mathcal{M}^{\spadesuit}Z_s$, then any composition factor of M is of the form $\mathbb{L}_{\lambda,s}^{\dot{w}}$ for some $w \cdot \lambda \in I_n^s$. From the definitions we see that $M \mapsto \tilde{\epsilon}_s M$ is a functor $\mathcal{D}^{\spadesuit}Z_s \to \mathcal{D}^{\spadesuit}\tilde{\mathcal{B}}^2$ and also $\mathcal{D}_m^{\spadesuit}Z_s \to \mathcal{D}_m^{\spadesuit}\tilde{\mathcal{B}}^2$; moreover, it is a functor $\mathcal{M}^{\spadesuit}Z_s \to \mathcal{M}^{\spadesuit}\tilde{\mathcal{B}}^2$ and also $\mathcal{M}_m^{\spadesuit}Z_s \to \mathcal{M}_m^{\spadesuit}\tilde{\mathcal{B}}^2$. From the definitions we see that for $M \in \mathcal{M}^{\spadesuit}Z_s$

(a) *we have $M \in \mathcal{M}^{\preceq}Z_s$ if and only if $\tilde{\epsilon}_s M \in \mathcal{M}^{\preceq}\tilde{\mathcal{B}}^2$; we have $M \in \mathcal{M}^{\prec}Z_s$ if and only if $\tilde{\epsilon}_s M \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$.*

Note that if $X \in \mathcal{D}(Z_s)$ and $j \in \mathbf{Z}$, then

(b) $$(\epsilon_s^* X)^{j+\rho} = \epsilon_s^*(X^j)[\rho].$$

Moreover, if $Y \in \mathcal{M}_m(Z_s)$ and $j' \in \mathbf{Z}$ then

(c) $$gr_{j'}(\tilde{\epsilon}_s Y) = \tilde{\epsilon}_s(gr_{j'}Y).$$

For $w \cdot \lambda \in I_n$ we show:

(d) *We have $w \cdot \lambda \in I_n^s$ if and only if $w^{-1} \cdot w(\lambda^{-1}) \in I_n^{-s}$.*

We must show that we have $w(\lambda) = \mathbf{e}^{-s}(\lambda)$ if and only if $\lambda^{-1} = \mathbf{e}^s(w(\lambda^{-1}))$. In other words, we must show that $\lambda(w^{-1}(t)) = \lambda(\tau^s t \tau^{-s})$ for all $t \in \mathbf{T}_n$ if and only if $\lambda(t') = \lambda(w^{-1}(\tau^{-s}t'\tau^s))$ for all $t' \in \mathbf{T}_n$. Setting $t' = \tau^s t \tau^{-s}$, we have $w^{-1}(t) = w^{-1}(\tau^{-s}t'\tau^s)$ and it remains to use that $t \mapsto \tau^s t\tau^{-s}$ is a bijection $\mathbf{T}_n \to \mathbf{T}_n$.

For $w \cdot \lambda \in I_n^s$ we show:

(e) *Let* $\omega \in \kappa_0^{-1}(w)$. *We have canonically* $(\mathbb{L}_{\lambda,s}^{\omega})^\dagger = \mathbb{L}_{w(\lambda^{-1}),-s}^{\omega^{-1}}$.

(The equality in (e) makes sense in view of (d).) By [21, 2.2(a)] and with notation of 3.1 we have canonically $\tilde{\mathfrak{h}}^*\mathbf{L}_\lambda^\omega = \mathbf{L}_{w(\lambda^{-1})}^{\omega^{-1}}$. Hence $\epsilon_{-s}^*\mathbf{L}_{w(\lambda^{-1})}^{\omega^{-1}} = \epsilon_{-s}^*\tilde{\mathfrak{h}}^*\mathbf{L}_\lambda^\omega = \mathfrak{h}^*\epsilon_s^*\mathbf{L}_\lambda^\omega$ so that $\tilde{\epsilon}_{-s}\mathbf{L}_{w(\lambda^{-1})}^{\omega^{-1}} = \mathfrak{h}^*\tilde{\epsilon}_s\mathbf{L}_\lambda^\omega$ and (e) follows.

4.4. Let r, f be integers such that $0 \le f \le r-3$. Let

$$\mathcal{Y} = \{((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^{r+1} \times \tilde{G}_s; \gamma \in x_{f+3}\mathbf{U}\tau^s x_f^{-1}, \\ \gamma \in x_{f+2}\mathbf{B}\tau^s x_{f+1}^{-1}\}.$$

Define $\vartheta : \mathcal{Y} \to \tilde{\mathcal{B}}^{r+1}$ by $((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}), \gamma) \mapsto (x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U})$. For $y', y'' \in W$ let

$$\tilde{\mathcal{B}}_{[y',y'']}^{r+1} = \{(x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}) \in \tilde{\mathcal{B}}^{r+1}; x_f^{-1}x_{f+1} \in G_{y'}, x_{f+2}^{-1}x_{f+3} \in G_{y''^{-1}}\}.$$

We show:

(a) *Let* $\xi \in \tilde{\mathcal{B}}_{[y',y'']}^{r+1}$. *If* $\mathbf{e}^s(y') \ne y''$ *then* $\vartheta^{-1}(\xi) = \emptyset$. *If* $\mathbf{e}^s(y') = y''$ *then* $\vartheta^{-1}(\xi) \cong \mathbf{k}^{\nu-|y'|}$.

We set $\xi = (x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U})$. If $\vartheta^{-1}(\xi) \ne \emptyset$ then $x_f^{-1}x_{f+1} \in G_{y'}$, $x_{f+2}^{-1}x_{f+3} \in G_{y''^{-1}}$ and $(x_{f+3}\mathbf{U}\tau^s x_f^{-1}) \cap (x_{f+2}\mathbf{B}\tau^s x_{f+1}^{-1}) \ne \emptyset$. Hence for some $u \in \mathbf{U}$, $b \in \mathbf{B}$ we have

$$u\tau^s x_f^{-1}x_{f+1} = x_{f+3}^{-1}x_{f+2}b\tau^s \in \tau^s G_{y'} \cap G_{y''}\tau^s$$

so that $\mathbf{e}^s(y') = y''$. If we assume that $\mathbf{e}^s(y') = y''$, then $\vartheta^{-1}(\xi)$ can be identified with

$$\{\gamma \in \tilde{G}_s; \gamma \in x_{f+3}\mathbf{U}\tau^s x_f^{-1}, \gamma \in x_{f+2}\mathbf{B}\tau^s x_{f+1}^{-1}\}$$

hence with

$$\{(u, b) \in \mathbf{U} \times \mathbf{B}; u\tau^s x_f^{-1}x_{f+1} = x_{f+3}^{-1}x_{f+2}b\tau^s\}.$$

We substitute $x_{f+3}^{-1}x_{f+2} = u_0\mathbf{e}^s(\dot{y}')t_0u_0'$, $x_f^{-1}x_{f+1} = u_1\dot{y}'t_1u_1'$, where $t_0 \in \mathbf{T}$, $u_0, u_0', u_1, u_1' \in \mathbf{U}$. Then $\vartheta^{-1}(\xi)$ is identified with $\{(u,b) \in \mathbf{U}\times\mathbf{B}; u\tau^s u_1\dot{y}'t_1u_1' = u_0\mathbf{e}^s(\dot{y}')t_0u_0'b\tau^s\}$. The map $(u,b) \mapsto u_0^{-1}u\mathbf{e}^s(u_1)$ identifies this variety with $\mathbf{U} \cap \mathbf{e}^s(\dot{y}')\mathbf{B}\mathbf{e}^s(\dot{y}')^{-1} \cong \mathbf{k}^{\nu-|y'|}$. This proves (a).

Now $\mathbf{T}^2$ acts freely on $\mathcal{Y}$ by

$$(t_1,t_2) \;:\; ((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}), \gamma) \mapsto ((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_f\mathbf{U}, x_{f+1}t_1\mathbf{U}, x_{f+2}t_2\mathbf{U}, x_{f+3}\mathbf{U}, \dots, x_r\mathbf{U}), \gamma).$$

Let

$$^!\mathcal{Y} = \mathbf{T}\backslash\{((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^{r+1} \times \tilde{G}_s; \gamma \in x_{f+3}\mathbf{U}\tau^s x_f^{-1}, \gamma \in x_{f+2}\mathbf{U}\tau^s x_{f+1}^{-1}\}$$

where $\mathbf{T}$ acts freely by

$$t \;:\; ((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}), \gamma) \mapsto ((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_f\mathbf{U}, x_{f+1}\mathbf{e}^{-s}(t)\mathbf{U}, x_{f+2}t\mathbf{U}, x_{f+3}\mathbf{U}, \dots, x_r\mathbf{U}), \gamma).$$

Note that the obvious map $\beta : {}^!\mathcal{Y} \to \mathbf{T}^2\backslash\mathcal{Y}$ is an isomorphism. We define ${}^!\eta : {}^!\mathcal{Y} \to Z_s$ by

$$((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}), \gamma) \mapsto \epsilon_s(x_{f+1}\mathbf{U}, x_{f+2}\mathbf{U}).$$

We define $\boldsymbol{\tau} : \mathcal{Y} \to {}^!\mathcal{Y}$ as the composition of the obvious map $\mathcal{Y} \to \mathbf{T}^2\backslash\mathcal{Y}$ with β^{-1}. Let $\eta = {}^!\eta\boldsymbol{\tau} : \mathcal{Y} \to Z_s$. We have

$$\eta((x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}), \gamma) = \epsilon_s(x_{f+1}t^{-1}\mathbf{U}, x_{f+2}t'^{-1}\mathbf{U})$$

where t, t' in $\mathbf{T}$ are such that $\gamma \in x_{f+2}t'^{-1}\mathbf{U}\tau^s t x_{f+1}^{-1}$.

4.5. Let $z \cdot \lambda \in I_n^s$. Let $P = \eta^*\mathcal{L}_{\lambda,s}^{\dot{z}\sharp}$. Let $p_{ij} : \tilde{\mathcal{B}}^{r+1} \to \tilde{\mathcal{B}}^2$ be the projection to the ij coordinates. We have the following result:

(a) $\vartheta_!P \approx \{p_{f,f+1}^* L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{f+1,f+2}^* L_\lambda^{\dot{z}\sharp} \otimes p_{f+2,f+3}^* L_{y(\lambda)}^{\dot{y}^{-1}} \langle 2|y| - 2\nu\rangle \,; y \in W\}$.

Define $e : \tilde{\mathcal{B}}^{r+1} \to \tilde{\mathcal{B}}^4$ by

$$(x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U}) \mapsto (x_f\mathbf{U}, x_{f+1}\mathbf{U}, x_{f+2}\mathbf{U}, x_{f+3}\mathbf{U}).$$

Then (a) is obtained by applying e^* to the statement similar to (a) in which $\{0, 1, \dots, r\}$ is replaced by $\{f, f+1, f+2, f+3\}$. Thus it is enough to prove (a) in the special case where $r = 3, f = 0$. In the remainder of the proof we assume that $r = 3, f = 0$.

For any y', y'' in W let $\vartheta_{y',y''} : \vartheta^{-1}(\tilde{\mathcal{B}}^4_{[y',y'']}) \to \tilde{\mathcal{B}}^4$ be the restriction of ϑ. Let $P^{y',y''}$ be the restriction of P to $\vartheta^{-1}(\tilde{\mathcal{B}}^4)_{[y',y'']}$. Clearly, we have

$$\vartheta_! P \approx \{(\vartheta_{y',y''})_! P^{y',y''}; (y', y'') \in W^2\}.$$

Since $\vartheta^{-1}(\tilde{\mathcal{B}}^{r+1}_{[y',y'']}) = \emptyset$ when $\mathbf{e}^s(y') \neq y''$, see 4.4(a), we deduce that

$$\vartheta_! P \approx \{(\vartheta_{\mathbf{e}^{-s}(y),y^{-1}})_! P^{\mathbf{e}^{-s}(y),y^{-1}}; y \in W\}.$$

Hence to prove (a) it is enough to show for any $y \in W$ that

$$\vartheta_{y!} P_y = p_{01}^* L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_\lambda^{\dot{z}\sharp} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}} \langle 2|y| - 2\nu \rangle\,,$$

where we write ϑ_y, P_y instead of $\vartheta_{\mathbf{e}^{-s}(y),y^{-1}}, P^{\mathbf{e}^{-s}(y),y^{-1}}$. Using $z(\lambda) = \mathbf{e}^{-s}(\lambda)$ we can replace $p_{01}^* L_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})}$ by $p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})}$. Thus it is enough to show for any $y \in W$ that

(b) $$\vartheta_{y!} P_y = p_{01}^* L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^* L_\lambda^{\dot{z}\sharp} \otimes p_{23}^* L_{y(\lambda)}^{\dot{y}^{-1}} \langle 2|y| - 2\nu \rangle\,.$$

We have a cartesian diagram

$$\begin{array}{ccc} \tilde{V}_y & \xrightarrow{\tilde{b}} & \tilde{\mathcal{V}}_y \\ \downarrow & & \downarrow \\ V_y & \xrightarrow{b} & \mathcal{V}_y \end{array}$$

where

$$V_y = \{(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}) \in \tilde{\mathcal{B}}^4; x_0^{-1}x_1 \in G_{\mathbf{e}^{-s}(y)}, x_1^{-1}x_2 \in G_z, x_2^{-1}x_3 \in G_{y^{-1}}\},$$

$$\mathcal{V}_y = \mathbf{T}\backslash\{(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}) \in \tilde{\mathcal{B}}^4; x_0^{-1}x_1 \in G_{\mathbf{e}^{-s}(y)}, x_1^{-1}x_2 \in G_z, \\ x_2^{-1}x_3 \in G_{y^{-1}}, \mathbf{e}^s((x_0^{-1}x_1)_{\mathbf{e}^{-s}(\dot{y})}) = (x_3^{-1}x_2)_{\dot{y}}\}$$

with $\mathbf{T}$ acting (freely) by

$$t : (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}) \mapsto (x_0\mathbf{U}, x_1\mathbf{e}^{-s}(t)\mathbf{U}, x_2t\mathbf{U}, x_3\mathbf{U}),$$

$\tilde{V}_y = \vartheta^{-1}(V_y)$ and

$$\tilde{\mathcal{V}}_y = \mathbf{T}\backslash\{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^4 \times \tilde{G}_s; x_0^{-1}x_1 \in G_{\mathbf{e}^{-s}(y)}, x_1^{-1}x_2 \in G_z, \\ x_2^{-1}x_3 \in G_{y^{-1}}, \gamma \in x_3\mathbf{U}\tau^s x_0^{-1}, \gamma \in x_2\mathbf{U}\tau^s x_1^{-1}\}$$

with $\mathbf{T}$ acting (freely) by

$$t : ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}).\gamma) \mapsto ((x_0\mathbf{U}, x_1\mathbf{e}^{-s}(t)\mathbf{U}, x_2t\mathbf{U}, x_3\mathbf{U}), \gamma);$$

we have

$$b(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}) = \mathbf{T} - \text{orbit of } (x_0\mathbf{U}, x_1t\mathbf{U}, x_2t'\mathbf{U}, x_3\mathbf{U})$$

where t, t' in $\mathbf{T}$ are such that $\mathbf{e}^s((x_0^{-1}x_1t)_{\mathbf{e}^{-s}(\dot{y})}) = (x_3^{-1}x_2t')_{\dot{y}}$,

$$\tilde{b}((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) = \mathbf{T} - \text{orbit of } ((x_0\mathbf{U}, x_1t\mathbf{U}, x_2t'\mathbf{U}, x_3\mathbf{U}), \gamma)$$

where t, t' in $\mathbf{T}$ are such that $\gamma \in x_2t'\mathbf{U}\tau^s t^{-1}x_1^{-1}$; the vertical maps are the obvious ones. We also have a cartesian diagram

$$\begin{array}{ccc} \tilde{V}'_y & \xrightarrow{\tilde{b}'} & \tilde{\mathcal{V}}'_y \\ \downarrow & & \downarrow \\ V'_y & \xrightarrow{b'} & \mathcal{V}'_y \end{array}$$

where $\tilde{V}'_y, \tilde{\mathcal{V}}'_y, V'_y, \mathcal{V}'_y$ are defined in the same way as $\tilde{V}_y, \tilde{\mathcal{V}}_y, V_y, \mathcal{V}_y$ but the condition $x_1^{-1}x_2 \in G_z$ is replaced by the condition $x_1^{-1}x_2 \in \bar{G}_z$; the maps $\tilde{b}', b'$ are given by the same formulas as $\tilde{b}, b$; the vertical maps are the obvious ones.

Let $j : V'_y \to \tilde{\mathcal{B}}^4$ be the inclusion. It is enough to show that

$$j^*\vartheta_{y!}P_y = j^*(p_{01}^*L_{z(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \otimes p_{12}^*L_\lambda^{\dot{z}\sharp} \otimes p_{23}^*L_{y(\lambda)}^{\dot{y}^{-1}}) \langle 2|y| - 2\nu \rangle .$$

By definition, $P|_{\tilde{V}'_y}$ is the inverse image of $\mathcal{L}^{\dot{z}\sharp}_{\lambda,s}$ under the composition of $\tilde{b}'$ with $\tilde{\mathcal{V}}'_y \to \mathcal{V}'_y \xrightarrow{{}^!\eta_y} Z_s$ where the first map is the obvious one and

$${}^!\eta_y(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}) = \epsilon_s(x_1\mathbf{U}, x_2\mathbf{U}).$$

Hence $P|_{\tilde{V}'_y}$ is the inverse image of $\mathcal{L}^{\dot{z}\sharp}_{\lambda,s}$ under the composition of $\eta_y := {}^!\eta_y b'$ with the obvious map $\vartheta'_y : \tilde{V}'_y \to V'_y$. Since ϑ_y is an affine space bundle with fibres of dimension $\nu - |y|$, it follows that $j^*\vartheta_{y!}P_y = \eta_y^*\mathcal{L}^{\dot{z}\sharp}_{\lambda,s}\langle 2|y| - 2\nu\rangle$. Thus it is enough to show that

$$\eta_y^*\mathcal{L}^{\dot{z}\sharp}_{\lambda,s} = j^*(p_{01}^* L^{\mathbf{e}^{-s}(\dot{y})}_{z(\lambda)} \otimes p_{12}^* L^{\dot{z}\sharp}_{\lambda} \otimes p_{23}^* L^{\dot{y}^{-1}}_{y(\lambda)}).$$

Since η_y is smooth as a map to $\bar{Z}^z_s$, we see that $\eta_y^*\mathcal{L}^{\dot{z}\sharp}_{\lambda,s}$ is the intersection cohomology complex of V'_y with coefficients in the local system $(\eta_y^0)^*\mathcal{L}^{\dot{z}}_{\lambda,s}$ on V_y; here $\eta_y^0 : V_y \to Z^z_s$ is the restriction of $\eta_y : V'_y \to \bar{Z}^z_s$. By 3.9(a),

$$j^*(p_{01}^* L^{\mathbf{e}^{-s}(\dot{y})}_{z(\lambda)} \otimes p_{12}^* L^{\dot{z}\sharp}_{\lambda} \otimes p_{23}^* L^{\dot{y}^{-1}}_{y(\lambda)})$$

is the intersection cohomology complex of V'_y with coefficients in the local system

$$\tilde{L} = j^*(p_{01}^* L^{\mathbf{e}^{-s}(\dot{y})}_{z(\lambda)} \otimes p_{12}^* L^{\dot{z}}_{\lambda} \otimes p_{23}^* L^{\dot{y}^{-1}}_{y(\lambda)})$$

on V_y. It is then enough to show that $\tilde{L} = (\eta_y^0)^*\mathcal{L}^{\dot{z}}_{\lambda,s}$.

Let $\xi = (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}) \in V_y$. From the definition of η_y^0 we see that the stalk $((\eta_y^0)^*\mathcal{L}^{\dot{z}}_{\lambda,s})_\xi$ is equal to

$$(\mathcal{L}^{\dot{z}}_{\lambda,s})_{\epsilon_s(x_1 t_1^{-1}, x_2 t_2^{-1})} = (L_\lambda)_{t_0}$$

where $t_0 \in \mathbf{T}$, $t_1 \in \mathbf{T}$, $t_2 \in \mathbf{T}$ are such that $t_0 = (t_1 x_1^{-1} x_2 t_2^{-1})_{\dot{z}}$,

$$\mathbf{e}^s((x_0^{-1} x_1 t_1^{-1})_{\mathbf{e}^{-s}(\dot{y})}) = (x_3^{-1} x_2 t_2^{-1})_{\dot{y}},$$

We can choose t_1, t_2 so that

$$(x_0^{-1} x_1 t_1^{-1})_{\mathbf{e}^{-s}(\dot{y})} = 1, (x_3^{-1} x_2 t_2^{-1})_{\dot{y}} = 1;$$

thus we can assume that $t_1 = (x_0^{-1} x_1)_{\mathbf{e}^{-s}(\dot{y})}$, $t_2 = (x_3^{-1} x_2)_{\dot{y}} = 1$.

The stalk $\tilde{L}_\xi$ is $(L_{z(\lambda)})_{t'_1} \otimes (L_\lambda)_{t'_2} \otimes (L_{y(\lambda)})_{t'_3}$ where

$$t'_1 = (x_0^{-1}x_1)_{\mathbf{e}^{-s}(\dot{y})} \in \mathbf{T}, t'_2 = (x_1^{-1}x_2)_{\dot{z}} \in \mathbf{T}, t'_3 = (x_2^{-1}x_3)_{\dot{y}^{-1}} \in \mathbf{T}.$$

It is enough to show that $(\eta_y^* \mathcal{L}_{\lambda,s}^{\dot{z}})_\xi = \tilde{L}_\xi$, or that

$$(t_1 x_1^{-1} x_2 t_2^{-1})_{\dot{z}} = z^{-1}(t'_1) t'_2 y^{-1}(t'_3)$$

where $t_1, t_2, t'_1, t'_2, t'_3$ are as above. We have $t_1 = t'_1$ and $x_3^{-1}x_2 \in \mathbf{U}\dot{y}t_2\mathbf{U}$, hence

$$x_2^{-1}x_3 \in \mathbf{U}t_2^{-1}\dot{y}^{-1}\mathbf{U} = \mathbf{U}\dot{y}^{-1}y(t_2^{-1})\mathbf{U},$$

so that $t'_3 = y(t_2^{-1})$ and $t_2^{-1} = y^{-1}(t'_3)$. We have

$$t_1 x_1^{-1} x_2 t_2^{-1} \in t_1 \mathbf{U}\dot{z}t'_2\mathbf{U}t_2^{-1} = \mathbf{U}\dot{z}z^{-1}(t_1)t'_2 t_2^{-1}\mathbf{U},$$

so that

$$(t_1 x_1^{-1} x_2 t_2^{-1})_{\dot{z}} = z^{-1}(t_1)t'_2 t_2^{-1} = z^{-1}(t'_1)t'_2 y^{-1}(t'_3),$$

as required. This completes the proof of (b) hence that of (a).

4.6. Let

$$(w_1, w_2, \dots, w_f, w_{f+2}, w_{f+4}, \dots, w_r) \in W^{r-2},$$

$$(\lambda_1, \lambda_2, \dots, \lambda_f, \lambda_{f+2}, \lambda_{f+4}, \dots, \lambda_r) \in \mathfrak{s}_n^{r-2}.$$

We set $z = w_{f+2}, \lambda = \lambda_{f+2}$. We assume that $z(\lambda) = \mathbf{e}^{-s}(\lambda)$. Let P be as in 4.5. Let

$$P' = \otimes_{i \in [1,r] - \{f+1, f+2, f+3\}} p_{i-1,i}^* L_{\lambda_i}^{\dot{w}_i \sharp} \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1}),$$

$\tilde{P} = P \otimes \vartheta^* P' \in \mathcal{D}_m(\mathcal{Y})$. For any $y \in W$ we set

$$\mathbf{w}_y = (w_1, w_2, \dots, w_f, \mathbf{e}^{-s}(y), w_{f+2}, y^{-1}, w_{f+4}, \dots, w_r) \in W^r,$$

$$\boldsymbol{\omega}_y = (\dot{w}_1, \dot{w}_2, \dots, \dot{w}_f, \mathbf{e}^{-s}(\dot{y}), \dot{w}_{f+2}, \dot{y}^{-1}, \dot{w}_{f+4}, \dots, \dot{w}_r),$$

$$\boldsymbol{\lambda}_y = (\lambda_1, \lambda_2, \dots, \lambda_f, \mathbf{e}^{-s}(\lambda_{f+2}), \lambda_{f+2}, y(\lambda_{f+2}), \lambda_{f+4}, \dots, \lambda_r) \in \mathfrak{s}_n^r.$$

We set $\Xi = \vartheta_! \tilde{P}$. We have:

(a) $$\Xi \,\widehat{=}\, \{M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y,[1,r]-\{f+1,f+3\}} \langle 2|y| - 2\nu\rangle \, ; y \in W\}$$

in $\mathcal{D}_m(\tilde{\mathcal{B}}^{r+1})$. This follows immediately from 4.5(a) since $\Xi = P' \otimes \vartheta_!(P)$.

4.7. We preserve the setup of 4.6. Let $\mathcal{S} = \sqcup_{\mathbf{w}'} \tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}'}$ where the union is over all $\mathbf{w}' = (w'_1, \dots, w'_r) \in W^r$ such that $w'_i = w_i$ for $i \notin \{f+1, f+3\}$. This is a locally closed subvariety of $\tilde{\mathcal{B}}^{r+1}$. For $y \in W$ let R_y be the restriction of $M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y,\emptyset}$ to $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}$ extended by 0 on $\mathcal{S} - \tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}$ (a constructible sheaf on $\mathcal{S}$). From the definitions we have

$$M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y,[1,r]-\{f+1,f+3\}}|_{\mathcal{S}} = R_y.$$

From 4.6(a) we deduce $\Xi|_{\mathcal{S}} \,\widehat{=}\, \{R_y \langle 2|y| - 2\nu\rangle \, ; y \in W\}$. We now restrict further to $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}$ (for $y \in W$); we obtain

$$\Xi|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}} \,\widehat{=}\, \{R_{y'} \left\langle 2|y'| - 2\nu\right\rangle |_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}} ; y' \in W\}.$$

In the right hand side we have $R_{y'} \langle 2|y'| - 2\nu\rangle |_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}} = 0$ if $y' \neq y$. It follows that $\Xi|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_\lambda}} = R_y \langle 2|y| - 2\nu\rangle |_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}}$. Since $R_y|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}}$ is a local system we deduce for $y \in W$ the following result.

(a) *Let $h \in \mathbf{Z}$. If $h = 2\nu - 2|y|$ then $\mathcal{H}^h \Xi|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}} = R_y|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}}(|y| - \nu)$. If $h \neq 2\nu - 2|y|$, then $\mathcal{H}^h \Xi|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_y}} = 0$.*

4.8. We preserve the setup of 4.6. We set

(a) $$k = 3\nu + (r+1)\rho + \sum_{i \in [1,r]-\{f+1,f+3\}} |w_i|.$$

For $y \in W$ we set

$$K_y = M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y,[1,r]-\{f+1,f+3\}} \langle |\mathbf{w}_y| + \nu + (r+1)\rho\rangle \,,$$

$$\tilde{K}_y = M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y,[1,r]} \langle |\mathbf{w}_y| + \nu + (r+1)\rho\rangle \,.$$

From 4.6(a) we deduce:

(b) $$\Xi\langle k\rangle \approx \{K_y; y \in W\}.$$

We show:

(c) *For any $j > 0$ we have $(\Xi\langle k\rangle)^j = 0$. Equivalently, $\Xi^j = 0$ for any $j > k$.*

Using (b) we see that it is enough to show that for any $y \in W$ we have $(K_y)^j = 0$ for any $j > 0$. Now $\tilde{K}_y$ is a (simple) perverse sheaf hence for any j we have $\dim \mathrm{supp}\mathcal{H}^j\tilde{K}_y \le -j$. Moreover K_y is obtained by restricting $\tilde{K}_y$ to an open subset of its support and then extending the result (by zero) on the complement of this subset in $\tilde{\mathcal{B}}^{r+1}$. Hence $\mathrm{supp}\mathcal{H}^jK_y \subset \mathrm{supp}\mathcal{H}^i\tilde{K}_y$ so that $\dim\mathrm{supp}\mathcal{H}^iK_y \le -j$. Since this holds for any j we see that $(K_y)^j = 0$ for any $j > 0$.

4.9. We preserve the notation of 4.6. We show:

(a) *Let $j \in \mathbf{Z}$ and let X be a composition factor of Ξ^j. Then $X \cong M_{\boldsymbol{\lambda}'}^{\boldsymbol{\omega}',[1,r]}\langle|\mathbf{w}'| + \nu + (r+1)\rho\rangle$ for some*

$$\mathbf{w}' = (w'_1, w'_2, \dots, w'_r) \in W^r, \boldsymbol{\lambda}' = (\lambda'_1, \lambda'_2, \dots, \lambda'_r) \in \mathfrak{s}_n^r$$

such that $w'_i = w_i$, $\lambda'_i = \lambda_i$ for $i \in [1,r] - \{f+1, f+3\}$ and such that

$$\lambda'_{f+1} = w'_{f+2}(\lambda'_{f+2}), \lambda'_{f+2} = w'_{f+3}(\lambda'_{f+3}).$$

Here $\boldsymbol{\omega}' = (\dot{w}'_1, \dot{w}'_2, \dots, \dot{w}'_r)$.

From 4.6(a) we see that, for some $y \in W$, X is a composition factor of

$$(M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y,[1,r]-\{f+1,f+3\}}\langle 2|y| - 2\nu\rangle)^j$$

where $\boldsymbol{\omega}_y, \boldsymbol{\lambda}_y$ are as in 4.6. Using this and [21, 2.18(b)] we see that

$$X \cong M_{\boldsymbol{\lambda}'}^{\boldsymbol{\omega}',[1,r]}\langle|\mathbf{w}'| + \nu + (r+1)\rho\rangle$$

for some

$$\mathbf{w}' = (w'_1, w'_2, \dots, w'_r) \in W^r, \boldsymbol{\lambda}' = (\lambda'_1, \lambda'_2, \dots, \lambda'_r) \in \mathfrak{s}_n^r$$

such that $w'_i = w_i$, $\lambda'_i = \lambda_i$ for $i \in [1,r] - \{f+1, f+3\}$; here $\boldsymbol{\omega}' = (\dot w'_1, \dot w'_2, \dots, \dot w'_r)$. It remains to show that we have automatically

$$\lambda'_{f+1} = w'_{f+2}(\lambda'_{f+2}), \lambda'_{f+2} = w'_{f+3}(\lambda'_{f+3}).$$

To see this we note that $(M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y, [1,r]-\{f+1,f+3\}}\langle 2|y| - 2\nu\rangle)^j$ is equivariant for the $\mathbf{T}^2$-action

$$\begin{aligned}&(t_1, t_2) : (x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_r\mathbf{U})\\ &\mapsto (x_0\mathbf{U}, x_1\mathbf{U}, \dots, x_f\mathbf{U}, x_{f+1}t_1\mathbf{U}, x_{f+2}t_2\mathbf{U}, x_{f+3}\mathbf{U}, \dots, x_r\mathbf{U})\end{aligned}$$

hence so are its composition factors and this implies that the equalities above for $\lambda'_{f+1}, \lambda'_{f+2}$ do hold.

4.10. From 4.8(c) we see that we have a distinguished triangle $(\Xi', \Xi, \Xi^k[-k])$ where $\Xi' \in \mathcal{D}_m(\tilde{\mathcal{B}}^{r+1})$ satisfies $(\Xi')^j = 0$ for all $j \geq k$. We show:

(a) *Let $j \in \mathbf{Z}$ and let K be one of Ξ, Ξ^j, Ξ'. For any $\mathbf{w}' \in W^r$ and any $h \in \mathbf{Z}$, $\mathcal{H}^h K|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}'}}$ is a local system.*

We prove (a) for $K = \Xi$ or $K = \Xi^j$. Using 4.6(a), we see that it is enough to show that $\mathcal{H}^h(M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y, [1,r]-\{f+1,f+3\}})|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}'}}$ is a local system for any h and that $\mathcal{H}^h((M_{\boldsymbol{\lambda}_y}^{\boldsymbol{\omega}_y, [1,r]-\{f+1,f+3\}})^j)|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}'}}$ is a local system for any h and any j. This follows by an argument entirely similar to that in the proof of [21, 3.10].

Now (a) for $K = \Xi'$ follows from (a) for Ξ and $\Xi^k[-k]$ using the long exact sequence for cohomology sheaves of $(\Xi', \Xi, \Xi^k[-k])$ restricted to $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}'}$.

We show:

(b) *Let $(y', y'') \in W^2$, $j = 2\nu - |y'| - |y''|$. Let*

$$\mathbf{w}_{y',y''} = (w_1, w_2, \dots, w_f, y', w_{f+2}, y''^{-1}, w_{f+3}, \dots, w_r) \in W^r.$$

The induced homomorphism $\mathcal{H}^j\Xi|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}} \to \mathcal{H}^{j-k}(\Xi^k)|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}}$ is an isomorphism.

We have an exact sequence of constructible sheaves

$$\mathcal{H}^j\Xi'|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}} \to \mathcal{H}^j\Xi|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}} \to \mathcal{H}^{j-k}(\Xi^k)|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}} \to \mathcal{H}^{j+1}\Xi'|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}}.$$

Hence it is enough to show that $\mathcal{H}^{j'}\Xi'|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}} = 0$ if $j' \ge j$. Assume that $\mathcal{H}^{j'}\Xi'|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}} \ne 0$ for some $j' \ge j$. Since $\mathcal{H}^{j'}\Xi'|_{\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}}$ is a local system (see (a)), we deduce that $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}}$ is contained in $\mathrm{supp}(\mathcal{H}^{j'}\Xi')$. We have $(\Xi'[k-1])^{\tilde{j}} = 0$ for any $\tilde{j} > 0$ hence $\dim\mathrm{supp}(\mathcal{H}^{j''}\Xi'[k-1]) \le -j''$ for any j''. Taking $j'' = j' - k + 1$, we deduce that

$$\dim\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{y',y''}} \le \dim\mathrm{supp}(\mathcal{H}^{j'}\Xi') \le -j' + k - 1 \le -j + k - 1$$

hence

$$|\mathbf{w}_{y',y''}| + \nu + (r+1)\rho \le -j + k - 1.$$

We have $|\mathbf{w}_{y',y''}| + \nu + (r+1)\rho = -j + k$ hence $-j + k \le -j + k - 1$, contradiction. This proves (b).

4.11. For $(y', y'') \in W^2$ we set

$$\begin{aligned}
\boldsymbol{\omega}_{y',y''} &= (\dot{w}_1, \dot{w}_2, \dots, \dot{w}_f, \dot{y}', \dot{w}_{f+2}, \dot{y}''^{-1}, \dot{w}_{f+3}, \dots, \dot{w}_r) \in W^r,\\
\boldsymbol{\lambda}_{y',y''} &= (\lambda_1, \lambda_2, \dots, \lambda_f, \mathbf{e}^{-s}(\lambda_{f+2}), \lambda_{f+2}, y''(\lambda_{f+2}), \lambda_{f+4}, \dots, \lambda_r) \in \mathfrak{s}_n^r,\\
K_{y',y''} &= M^{\boldsymbol{\omega}_{y',y''},\emptyset}_{\boldsymbol{\lambda}_{y',y''}}\langle|\mathbf{w}_{y',y''}| + \nu + (r+1)\rho\rangle \in \mathcal{M}_m(\tilde{\mathcal{B}}^{r+1}),\\
\tilde{K}_{y',y''} &= M^{\boldsymbol{\omega}_{y',y''},[1,r]}_{\boldsymbol{\lambda}_{y',y''}}\langle|\mathbf{w}_{y',y''}| + \nu + (r+1)\rho\rangle \in \mathcal{M}_m(\tilde{\mathcal{B}}^{r+1}).
\end{aligned}$$

Note that when $y' = \mathbf{e}^{-s}(y), y'' = y$, $\mathbf{w}_{y',y''}, \boldsymbol{\omega}_{y',y''}$, $\boldsymbol{\lambda}_{y',y''}$ and $\tilde{K}_{y',y''}$ become $\mathbf{w}_y, \boldsymbol{\omega}_y, \boldsymbol{\lambda}_y$ (see 4.6) and $\tilde{K}_y$ (see 4.8). We show that we have canonically

(a) $$gr_0(\Xi^k(k/2)) = \oplus_{y\in W}\tilde{K}_y.$$

Since $gr_0(\Xi^k(k/2))$ is a semisimple perverse sheaf of pure weight zero, it is a direct sum of simple perverse sheaves, necessarily of the form described in 4.9(a). Thus we have canonically

$$gr_0(\Xi^k(k/2)) = \oplus_{(y',y'')\in W^2} V_{y',y''} \otimes \tilde{K}_{y',y''}$$

where $V_{y',y''}$ are mixed $\bar{\mathbf{Q}}_l$-vector spaces of pure weight 0. By [1, 5.1.14], Ξ is mixed of weight ≤ 0 hence $\Xi^k(k/2)$ is mixed of weight ≤ 0. Hence we have an exact sequence in $\mathcal{M}_m(\tilde{\mathcal{B}}^{r+1})$:

(a) $$0 \to \mathcal{W}^{-1}(\Xi^k(k/2)) \to \Xi^k(k/2) \to gr_0(\Xi^k(k/2)) \to 0$$

that is,

$$0 \to \mathcal{W}^{-1}(\Xi^k(k/2)) \to \Xi^k(k/2) \to \oplus_{(y',y'')\in W^2} V_{y',y''} \otimes \tilde{K}_{y',y''} \to 0.$$

(Here $\mathcal{W}^{-1}(?)$ denotes the part of weight ≤ -1 of a mixed perverse sheaf.) Hence for any $(\tilde{y}', \tilde{y}'') \in W^2$ we have an exact sequence of (mixed) cohomology sheaves restricted to $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$ (where $h = 2\nu - |\tilde{y}'| - |\tilde{y}''| - k$):

(b) $\mathcal{H}^h(\mathcal{W}^{-1}(\Xi^k(k/2))) \xrightarrow{\alpha} \mathcal{H}^h(\Xi^k(k/2)) \to \oplus_{(y',y'')\in W^2} V_{y',y''} \otimes \mathcal{H}^h(\tilde{K}_{y',y''})$
$\to \mathcal{H}^{h+1}(\mathcal{W}^{-1}(\Xi^k(k/2)))$.

Moreover, by 4.10(b), we have an equality of local systems on $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$:

$$\mathcal{H}^h(\Xi^k(k/2)) = \mathcal{H}^{h+k}(\Xi(k/2)) = \mathcal{H}^{2\nu-|y'|-|y''|}(\Xi(k/2))$$

and this is $R_y(k/2 + |y| - \nu)$ if $\tilde{y}' = \mathbf{e}^{-s}(y), \tilde{y}'' = y$ (see 4.7(a)) and is 0 if $\tilde{y}' \ne \mathbf{e}^{-s}(\tilde{y}'')$ (see 4.4(a)) hence is pure of weight $-k - |\tilde{y}'| - |\tilde{y}''| + \nu = h$. On the other hand, $\mathcal{H}^h(\mathcal{W}^{-1}(\Xi^k(k/2)))$ is mixed of weight $\le h - 1$; it follows that α in (b) must be zero.

Assume that $\mathcal{H}^h(\tilde{K}_{y',y''})$ is not identically zero on $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$. Then, by 4.10(a), $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$ is contained in $\mathrm{supp}\mathcal{H}^h(\tilde{K}_{y',y''})$ which has dimension $\le -h$ (resp. $< -h$ if $(y', y'') \ne (\tilde{y}', \tilde{y}'')$); hence $-h = \dim \tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$ is $\le -h$ (resp. $< -h$); we see that we must have $(y', y'') = (\tilde{y}', \tilde{y}'')$ and we have $\mathcal{H}^h(\tilde{K}_{y',y''}) = \mathcal{H}^h(K_{y',y''})$ on $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$.

Assume that $\mathcal{H}^{h+1}(\mathcal{W}^{-1}(\Xi^k(k/2)))$ is not identically 0 on $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$. Then, by 4.10(a), $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$ is contained in $\mathrm{supp}\mathcal{H}^{h+1}(\mathcal{W}^{-1}(\Xi^k(k/2)))$ which has dimension $\le -h - 1$; hence $-h = \dim \tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}} \le -h - 1$, a contradiction. We see that (b) becomes an isomorphism of local systems on $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$:

$$0 = V_{\tilde{y}',\tilde{y}''} \otimes K_{\tilde{y}',\tilde{y}''} \text{ if } \mathbf{e}^s(\tilde{y}') \ne \tilde{y}'',$$

$$R_{\tilde{y}''}(-h/2) \xrightarrow{\sim} V_{\tilde{y}',\tilde{y}''} \otimes \mathcal{H}^h(K_{\tilde{y}',\tilde{y}''}) \text{ if } \mathbf{e}^s(\tilde{y}') = \tilde{y}''.$$

When $\mathbf{e}^s(\tilde{y}') = \tilde{y}'$ we have $\mathcal{H}^h(K_{\tilde{y}',\tilde{y}''}) = R_{\tilde{y}''}(-h/2)$ as local systems on $\tilde{\mathcal{O}}^{\emptyset}_{\mathbf{w}_{\tilde{y}',\tilde{y}''}}$. It follows that $V_{\tilde{y}',\tilde{y}''}$ is $\bar{\mathbf{Q}}_l$ if $\mathbf{e}^s(\tilde{y}') = \tilde{y}''$ and is 0 if $\mathbf{e}^s(\tilde{y}') \ne \tilde{y}''$. This proves (a).

4.12. Let $h \in [1,r]$. Let ${}_h\mathcal{D}^{\preceq}\tilde{\mathcal{B}}^{r+1}$ (resp. ${}_h\mathcal{D}^{\prec}\tilde{\mathcal{B}}^{r+1}$) be the subcategory of $\mathcal{D}\tilde{\mathcal{B}}^{r+1}$ consisting of objects K such that for any $j \in \mathbf{Z}$, any composition factor of K^j is of the form $M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},[1,r]}\langle|\mathbf{w}|+\nu+(r+1)\rho\rangle$ for some $\mathbf{w}=(w_1,\dots,w_r)\in W^r$, $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\dots,\lambda_r)\in\mathfrak{s}_n^r$ such that $w_h\cdot\lambda_h \preceq \mathbf{c}$ (resp. $w_h\cdot\lambda_h\prec\mathbf{c}$). (Here $\boldsymbol{\omega}=(\dot{w}_1,\dot{w}_2,\dots,\dot{w}_r)$.)

Let ${}_h\mathcal{M}^{\preceq}\tilde{\mathcal{B}}^{r+1}$ be the subcategory of ${}_h\mathcal{D}^{\preceq}\tilde{\mathcal{B}}^{r+1}$ consisting of perverse sheaves. Let ${}_h\mathcal{M}^{\prec}\tilde{\mathcal{B}}^{r+1}$ be the subcategory of ${}_h\mathcal{D}^{\prec}\tilde{\mathcal{B}}^{r+1}$ consisting of perverse sheaves.

If $K\in\mathcal{M}_m(\tilde{\mathcal{B}}^{r+1})$ is pure of weight 0 and is also in ${}_h\mathcal{D}^{\preceq}\tilde{\mathcal{B}}^{r+1}$, we denote by $\underline{K}$ the sum of all simple subobjects of K (without mixed structure) which are not in ${}_h\mathcal{D}^{\prec}\tilde{\mathcal{B}}^{r+1}$.

4.13. Let $Z_s \xrightarrow{\eta} \mathcal{Y} \xrightarrow{\vartheta} \tilde{\mathcal{B}}^4$ be as in 4.4 with $r=3, f=0$. We define $\mathfrak{b}:\mathcal{D}(Z_s)\to\mathcal{D}(\tilde{\mathcal{B}}^2)$ and $\mathfrak{b}:\mathcal{D}_m(Z_s)\to\mathcal{D}_m(\tilde{\mathcal{B}}^2)$ by

$$\mathfrak{b}(L)=p_{03!}\vartheta_!\eta^*L.$$

We show:

(a) *If* $L\in\mathcal{D}^{\preceq}(Z_s)$ *then* $\mathfrak{b}(L)\in\mathcal{D}^{\preceq}\tilde{\mathcal{B}}^2$.

(b) *If* $L\in\mathcal{D}^{\prec}(Z_s)$ *then* $\mathfrak{b}(L)\in\mathcal{D}^{\prec}\tilde{\mathcal{B}}^2$.

(c) *If* $L\in\mathcal{M}^{\preceq}(Z_s)$ *and* $h>5\rho+2\nu+2a$ *then* $(\mathfrak{b}(L))^h\in\mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$.

We can assume that $L=\mathbb{L}_{\lambda,s}^{\dot{z}}$ where $z\cdot\lambda\in I_n^s$, $z\cdot\lambda\preceq\mathbf{c}$. Applying 4.5(a) with $P=\eta^*\mathcal{L}_{\lambda,s}^{\dot{z}\sharp}$ we see that

$$\mathfrak{b}(\mathcal{L}_{\lambda,s}^{\dot{z}\sharp}) \approx \{L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}\langle-|z|-2\nu\rangle\,;y\in W\},$$

hence

$$\mathfrak{b}(\mathbb{L}_{\lambda,s}^{\dot{z}\sharp}) \approx \{L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}\langle-\nu+\rho\rangle\,;y\in W\}.$$

To prove (a) it is enough to show that for any $y\in W$ we have

$$L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}\in\mathcal{D}^{\preceq}\tilde{\mathcal{B}}^2.$$

When $z{\cdot}\lambda\in\mathbf{c}$ this follows from [21, 2.10(a)]. When $z{\cdot}\lambda\prec\mathbf{c}$ this again follows from [21, 2.10(a)], applied to the two-sided cell containing $z\cdot\lambda$ instead of $\mathbf{c}$.

The same argument proves (b). To prove (c) we can assume that $z \cdot \lambda \in \mathbf{c}$; it is enough to prove that for any $y \in W$ we have

$$(L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}\langle -\nu+\rho\rangle)^h \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$$

if $h > 5\rho + 2\nu + 2a$ or that

$$(L_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}})^j \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$$

if $j > 6\rho + \nu + 2a$. This follows from [21, 2.20(a)]. This completes the proof of (a), (b), (c).

We define $\underline{\mathfrak{b}} : \mathcal{C}_0^{\mathbf{c}}(Z_s) \to \mathcal{C}_0^{\mathbf{c}}(\tilde{\mathcal{B}}^2)$ by

$$\underline{\mathfrak{b}}(L) = \underline{gr_{5\rho+2\nu+2a}((\mathfrak{b}(L))^{5\rho+2\nu+2a})}((5\rho+2\nu+2a)/2).$$

We show:

(d) *Let* $z \cdot \lambda \in \mathbf{c}^s$. *If* $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$, *then*

$$\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = \oplus_{y \in W; y\cdot\lambda\in\mathbf{c}} \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \underline{\circ} \mathbf{L}_{\lambda}^{\dot{z}} \underline{\circ} \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}.$$

If $\mathbf{e}^s(\mathbf{c}) \neq \mathbf{c}$, *then* $\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = 0$.

We shall apply the method of [19, 1.12] with $\Phi : \mathcal{D}_m(Y_1) \to \mathcal{D}_m(Y_2)$ replaced by $p_{03!} : \mathcal{D}_m(\tilde{\mathcal{B}}^4) \to \mathcal{D}_m(\tilde{\mathcal{B}}^2)$ and with $\mathcal{D}^{\preceq}(Y_1)$, $\mathcal{D}^{\preceq}(Y_2)$ replaced by ${}_2\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^2)$, ${}_2\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$, see 4.12. We shall take $\mathbf{X}$ in *loc.cit.* equal to $\vartheta_!\eta^*\mathbb{L}_{\lambda,s}^{\dot{z}}$. The conditions of *loc.cit.* are satisfied: those concerning $\mathbf{X}$ are satisfied with $c' = 2\nu+3\rho$. (For $h > |z|+3\nu+4\rho$ we have $\Xi^h = 0$ that is $(\mathbf{X}[-|z|-\nu-\rho])^h = 0$, with Ξ as in 4.8(c). Hence if $j > 2\nu+3\rho$ we have $\mathbf{X}^j = 0$.) The conditions concerning $p_{03!}$ are satisfied with $c = 2\rho+2a$. (This follows from [21, 2.20(a)]) Since $\mathfrak{b}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = p_{03!}\mathbf{X}$ and $c + c' = 5\rho + 2\nu + 2a$, we see that

$$\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}}) = \underline{gr_{2\rho+2a}(p_{03!}((\underline{gr_{2\nu+3\rho}((\vartheta_!\eta^*\mathbb{L}_{\lambda,s}^{\dot{z}})^{2\nu+3\rho})}((2\nu+3\rho)/2)))^{2\rho+2a})}(\rho+a).$$

Using 4.11(a), we see that (with Ξ as in 4.11(a) and $k = |z| + 3\nu + 4\rho$) we have

$$\begin{aligned} &\underline{gr_{2\nu+3\rho}((\vartheta_!\eta^*\mathbb{L}_{\lambda,s}^{\dot{z}})^{2\nu+3\rho})}((2\nu+3\rho)/2)\\ = \ &\underline{gr_{2\nu+3\rho}((\Xi\langle|z|+\nu+\rho\rangle)^{2\nu+3\rho})}((2\nu+3\rho)/2)\end{aligned}$$

$$= \underline{gr_0(\Xi^k(k/2)} = \oplus_{y\in W} M^{\mathbf{e}^{-s}(\dot y),\dot z,\dot y^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}\langle 2|y|+|z|+\nu+4\rho\rangle .$$

Hence

$$\underline{\mathfrak{b}}(\mathbb{L}^{\dot z}_{\lambda,s}) = \underline{gr_{2\rho+2a}(\oplus_{y\in W}(p_{03!}M^{\mathbf{e}^{-s}(\dot y),\dot z,\dot y^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}\langle 2|y|+|z|+\nu+4\rho\rangle)^{2\rho+2a}})(\rho+a)$$
$$= \underline{gr_{2\rho+2a}(\oplus_{y\in W}(L^{\mathbf{e}^{-s}(\dot y),\dot z,\dot y^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)})^{6\rho+\nu+2a}((\nu+4\rho)/2))}(\rho+a).$$

Using [21, 2.26(a)], we see that in the last direct sum, the contribution of $y\in W$ is 0 unless $y\cdot\lambda\in\mathbf{c}$ and $\mathbf{e}^{-s}(y)\cdot\mathbf{e}^{-s}(\lambda)\in\mathbf{c}$. We see that the last direct sum is zero unless $\mathbf{e}^s(\mathbf{c})=\mathbf{c}$. If $\mathbf{e}^s(\mathbf{c})=\mathbf{c}$, for the terms corresponding to y such that $y\cdot\lambda\in\mathbf{c}$, we may apply [21, 2.24(a)]. Now (d) follows.

4.14. We set $\mathbf{Z_c}=\{s'\in\mathbf{Z};\mathbf{e}^{s'}(\mathbf{c})=\mathbf{c}\}$. This is a subgroup of $\mathbf{Z}$. In the remainder of this section we assume that $s\in\mathbf{Z_c}$.

Let $Z_s \xleftarrow{{}^!\eta} {}^!\mathcal{Y}$ be as in 4.4 with $r=3, f=0$. Let ${}^!\tilde{\mathcal{B}}^4$ be the space of orbits of the free $\mathbf{T}^2$-action on $\tilde{\mathcal{B}}^4$ given by

$$(t_1,t_2):(x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U})\mapsto(x_0\mathbf{U},x_1t_1\mathbf{U},x_2t_2\mathbf{U},x_3\mathbf{U});$$

let ${}^!\vartheta:{}^!\mathcal{Y}\to{}^!\tilde{\mathcal{B}}^4$ be the map induced by ϑ. We define $\mathfrak{b}':\mathcal{D}(Z_s)\to\mathcal{D}(\tilde{\mathcal{B}}^2)$ and $\mathfrak{b}':\mathcal{D}_m(Z_s)\to\mathcal{D}_m(\tilde{\mathcal{B}}^2)$ by

$$\mathfrak{b}'(L)=p_{03!}{}^!\vartheta_!{}^!\eta^*L.$$

(The map ${}^!\tilde{\mathcal{B}}^4\to\tilde{\mathcal{B}}^2$ induced by $p_{03}:\tilde{\mathcal{B}}^4\to\tilde{\mathcal{B}}^2$ is denoted again by p_{03}.) Let $\tau:\mathcal{Y}\to{}^!\mathcal{Y}$ be as in 4.4 (it is a principal $\mathbf{T}^2$-bundle). We have the following results.

(a) *If* $L\in\mathcal{D}^{\preceq}(Z_s)$, *then* $\mathfrak{b}'(L)\in\mathcal{D}^{\preceq}\tilde{\mathcal{B}}^2$.

(b) *If* $L\in\mathcal{D}^{\prec}(Z_s)$, *then* $\mathfrak{b}'(L)\in\mathcal{D}^{\prec}\tilde{\mathcal{B}}^2$.

(c) *If* $L\in\mathcal{M}^{\preceq}(Z_s)$ *and* $h>\rho+2\nu+2a$, *then* $(\mathfrak{b}'(L))^h\in\mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$.

We can assume that $L=\mathbb{L}^{\dot z}_{\lambda,s}$ where $z\cdot\lambda\in I^s_n$, $z\cdot\lambda\preceq\mathbf{c}$. A variant of the proof of 4.5(a) gives:

$$\mathfrak{b}'(\mathcal{L}^{\dot z\sharp}_{\lambda,s}) \approx \{{}'L^{\mathbf{e}^{-s}(\dot y),\dot z,\dot y^{-1},\{2\}}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}\langle -|z|-2\nu\rangle\,;y\in W\},$$

hence

$$\mathfrak{b}'(\mathbb{L}^{\dot{z}\sharp}_{\lambda,s}) \Leftrightarrow \{{}'L^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}\langle -\nu+\rho\rangle\,;y\in W\}.$$

To prove (a) it is enough to show that for any $y \in W$ we have

$${}'L^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)} \in \mathcal{D}^{\preceq}\tilde{\mathcal{B}}^2.$$

When $z{\cdot}\lambda \in \mathbf{c}$ this follows from [21, 2.10(c)]. When $z{\cdot}\lambda \prec \mathbf{c}$ this again follows from [21, 2.10(c)], applied to the two-sided cell containing $z\cdot\lambda$ instead of $\mathbf{c}$. The same argument proves (b). To prove (c) we can assume that $z\cdot\lambda \in \mathbf{c}$; it is enough to prove that for any $y \in W$ we have

$$\Big({}'L^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)}\langle -\nu+\rho\rangle\Big)^h \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$$

if $h > \rho + 2\nu + 2a$ or that $({}'L^{\mathbf{e}^{-s}(\dot{y}),\dot{z},\dot{y}^{-1},\{2\}}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda)})^j \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$ if $j > 2\rho+\nu+2a$. This follows from [21, 2.20(c)]. This completes the proof of (a), (b), (c).

We define $\underline{\mathfrak{b}'} : \mathcal{C}^{\mathbf{c}}_0(Z_s) \to \mathcal{C}^{\mathbf{c}}_0(\tilde{\mathcal{B}}^2)$ by

$$\underline{\mathfrak{b}'}(L) = \underline{gr_{\rho+2\nu+2a}((\mathfrak{b}'(L))^{\rho+2\nu+2a})}((\rho+2\nu+2a)/2).$$

In the remainder of this subsection we fix $z\cdot\lambda \in \mathbf{c}^s$ and we set $L = \mathbb{L}^{\dot{z}}_{\lambda,s}$. We show:

(d) *We have canonically* $\underline{\mathfrak{b}'}(L) = \underline{\mathfrak{b}}(L)$.

The method of proof is similar to that of [21, 2.22(a)]. It is based on the fact that

$$\mathfrak{b}(L) = \mathfrak{b}'(L) \otimes \mathfrak{L}^{\otimes 2}$$

which follows from the definitions. We define $\mathcal{R}_{i,j}$ for $i \in [0, 2\rho+1]$ and $\mathcal{P}_{i,j}$ for $i \in [0, 2\rho]$ as in [21, 2.17], but replacing $L^J, {}'L^J, r, \delta$ by $\mathfrak{b}(L), \mathfrak{b}'(L), 3, 2\rho$. In particular, we have

$$\mathcal{P}_{i,j} = \mathcal{X}_{4\rho-i}(i-2\rho) \otimes (\mathfrak{b}'(L))^{-4\rho+i+j} \text{ for } i \in [0, 2\rho]$$

where $\mathcal{X}_{4\rho-i}$ is a free abelian group of rank $\binom{2\rho}{i}$ and $\mathcal{X}_{4\rho} = \mathbf{Z}$. We have for any j an exact sequence analogous to [21, 2.17(a)]:

(e) $\quad \dots \to \mathcal{P}_{i,j-1} \to \mathcal{R}_{i+1,j} \to \mathcal{R}_{i,j} \to \mathcal{P}_{i,j} \to \mathcal{R}_{i+1,j+1} \to \mathcal{R}_{i,j+1} \to \dots,$

and we have

$$\mathcal{R}_{0,j} = (\mathfrak{b}(L))^j, \quad \mathcal{P}_{0,j} = (\mathfrak{b}'(L))^{j-4\rho}(-2\rho).$$

We show:

(f) *If* $i \in [0, 2\rho+1]$ *then* $\mathcal{R}_{i,j} \in \mathcal{M}^{\preceq}\tilde{\mathcal{B}}^2$.

(g) *If* $i \in [0, 2\rho+1]$, $j > 6\rho - i + \nu + 2a$ *then* $\mathcal{R}_{i,j} \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$.

We prove (f), (g) by descending induction on i as in [21, 2.21]. If $i = 2\rho+1$ then, since $\mathcal{R}_{2\rho+1,j} = 0$, there is nothing to prove. Now assume that $i \in [0, 2\rho]$. Assume that $\lambda' \cdot w$ is such that $\mathbf{L}_{\lambda'}^{\dot{w}}$ is a composition factor of $\mathcal{R}_{i,j}$ (without the mixed structure). We must show that $w \cdot \lambda' \preceq \mathbf{c}$ and that, if $j > 6\rho - i + \nu + 2a$, then $w \cdot \lambda' \prec \mathbf{c}$. Using (e), we see that $\mathbf{L}_{\lambda'}^{\dot{w}}$ is a composition factor of $\mathcal{R}_{i+1,j}$ or of $\mathcal{P}_{i,j}$. In the first case, using the induction hypothesis we see that $w \cdot \lambda' \preceq \mathbf{c}$ and that, if $j > 6\rho - i + \nu + 2a$ (so that $j > 6\rho - i - 1 + \nu + 2a$), then $w \cdot \lambda' \prec \mathbf{c}$. In the second case, $\mathbf{L}_{\lambda'}^{\dot{w}}$ is a composition factor of $(\mathfrak{b}'(L))^{-4\rho+i+j}$. Using (a),(c), we see that $w \cdot \lambda' \preceq \mathbf{c}$ and that, if $j > 6\rho - i + \nu + 2a$ (so that $-4\rho + i + j > \nu + 2\rho + 2a$), then $w \cdot \lambda' \prec \mathbf{c}$. This proves (f),(g).

We show:

(h) *Assume that* $i \in [0, 2\rho+1]$. *Then* $\mathcal{R}_{i,j}$ *is mixed of weight* $\leq j - i$.

We argue as in [21, 2.22] by descending induction on i. If $i = 2\rho+1$ there is nothing to prove. Assume now that $i \leq 2\rho$. By Deligne's theorem, $\mathfrak{b}'(L)$ is mixed of weight ≤ 0; hence $(\mathfrak{b}'(L))^{-4\rho+i+j}$ is mixed of weight $\leq -4\rho+i+j$ and $\mathcal{X}_{4\rho-i}(i-2\rho)\otimes(\mathfrak{b}'(L))^{-4\rho+i+j}$ is mixed of weight $\leq -4\rho+i+j-2(i-2\rho) = j-i$. In other words, $\mathcal{P}_{i,j}$ is mixed of weight $\leq j-i$. Thus in the exact sequence $\mathcal{R}_{i+1,j} \to \mathcal{R}_{i,j} \to \mathcal{P}_{i,j}$ coming from (e) in which $\mathcal{R}_{i+1,j}$ is mixed of weight $\leq j-i-1 < j-i$ (by the induction hypothesis) and $\mathcal{P}_{i,j}$ is mixed of weight $\leq j-i$, we must have that $\mathcal{R}_{i,j}$ is mixed of weight $\leq j-i$. This proves (h).

We now prove (d). From (e) we deduce an exact sequence

$$gr_j(\mathcal{R}_{1,j}) \to gr_j(\mathcal{R}_{0,j}) \to gr_j(\mathcal{P}_{0,j}) \to gr_j(\mathcal{R}_{1,j+1}).$$

By (h) we have $gr_j(\mathcal{R}_{1,j}) = 0$. We have $gr_j(\mathcal{R}_{0,j}) = gr_j(\mathfrak{b}(L)^j)$, $gr_j(\mathcal{P}_{0,j}) = gr_j((\mathfrak{b}'(L))^{-4\rho+j}(-2\rho))$. Moreover, by (g) we have $\mathcal{R}_{1,j+1} \in \mathcal{D}^{\prec}\tilde{\mathcal{B}}^2$ since

$j+1 > 6\rho-1+\nu+2a$. It follows that $gr_j(\mathcal{R}_{1,j+1}) \in \mathcal{D}^{\prec}\tilde{\mathcal{B}}^2$. Thus the exact sequence above induces an isomorphism as in (d).

Let $p'_{ij} : \tilde{\mathcal{B}}^3 \to \tilde{\mathcal{B}}^2$ be the projection to the ij-coordinate, where ij is 12, 23 or 13. Let

$$R = \mathbf{T}\backslash\{(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x'_3\mathbf{U}, \gamma) \in \tilde{\mathcal{B}}^4 \times G_s; \gamma \in x_2\mathbf{U}\tau^s x_1^{-1}\}$$

where $\mathbf{T}$ acts freely by

$$t : (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x'_3\mathbf{U}, \gamma) \mapsto (x_0\mathbf{U}, x_1\mathbf{e}^{-s}(t)\mathbf{U}, x_2 t\mathbf{U}, x'_3\mathbf{U}, \gamma).$$

We have cartesian diagrams

$$\begin{array}{ccc} R & \xrightarrow{d_1} & '\mathcal{Y} \times \tilde{\mathcal{B}}^2 \\ {\scriptstyle c_1}\downarrow & & {\scriptstyle s_1}\downarrow \\ \tilde{\mathcal{B}}^3 & \xrightarrow{p'} & \tilde{\mathcal{B}}^2 \times \tilde{\mathcal{B}}^2 \end{array}$$

$$\begin{array}{ccc} R & \xrightarrow{d_2} & \tilde{\mathcal{B}}^2 \times {}'\mathcal{Y} \\ {\scriptstyle c_2}\downarrow & & {\scriptstyle s_2}\downarrow \\ \tilde{\mathcal{B}}^3 & \xrightarrow{p'} & \tilde{\mathcal{B}}^2 \times \tilde{\mathcal{B}}^2 \end{array}$$

where

$$\begin{aligned} d_1(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) &= ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, \gamma x_0\tau^{-s}\mathbf{U}, \gamma), \\ &\qquad (\gamma x_0\tau^{-s}\mathbf{U}, x_3\mathbf{U})), \\ d_2(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) &= ((x_0\mathbf{U}, \gamma^{-1}x_3\tau^s\mathbf{U}), \\ &\qquad (\gamma^{-1}x_3\tau^s\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma)), \\ c_1(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) &= (x_0\mathbf{U}, \gamma x_0\tau^{-s}\mathbf{U}, x_3\mathbf{U}), \\ c_2(x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, \gamma) &= (x_0\mathbf{U}, \gamma^{-1}x_3\tau^s\mathbf{U}, x_3\mathbf{U}), \\ p' = (p'_{12}, p'_{23}), \qquad s_1 &= p_{03}{}'\vartheta \times 1, \quad s_2 = 1 \times p_{03}{}'\vartheta. \end{aligned}$$

It follows that $p'^*s_{1!} = c_{1!}d_1^*$, $p'^*s_{2!} = c_{2!}d_2^*$. Now let $L \in \mathcal{D}(Z_s)$, $L' \in \mathcal{D}(\tilde{\mathcal{B}}^2)$, $\tilde{L}' \in \mathcal{D}(\tilde{\mathcal{B}}^2)$, We have $\eta^*L \boxtimes L' \in \mathcal{D}('\mathcal{Y} \times \tilde{\mathcal{B}}^2$, $\tilde{L}' \boxtimes \eta^*L \in \mathcal{D}(\tilde{\mathcal{B}}^2 \times {}'\mathcal{Y})$. We have

$$p'_{12}{}^*\mathfrak{b}'(L)\otimes p'_{23}{}^*L' = p'^*s_{1!}(\eta^*L\boxtimes L') = c_{1!}d_1^*(\eta^*L\boxtimes L') = c_{1!}(e_1^*L\boxtimes e'_1{}^*L'),$$

$$p'_{12}{}^*\tilde{L}'\otimes p'_{23}{}^*\mathfrak{b}'(L) = p'^*s_{2!}(\tilde{L}'\boxtimes\eta^*L) = c_{2!}d_2^*(\tilde{L}'\boxtimes\eta^*L) = c_{2!}(e'_2{}^*\tilde{L}'\boxtimes e_1{}^*L),$$

where

$$\begin{aligned} e_1 &: R\to Z_s \text{ is } (x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U},\gamma)\mapsto \epsilon_s(x_1\mathbf{U},x_2\mathbf{U}),\\ e'_1 &: R\to\tilde{\mathcal{B}}^2 \text{ is } (x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U},\gamma)\mapsto(\gamma x_0\tau^{-s}\mathbf{U},x_3\mathbf{U}),\\ e'_2 &: R\to\tilde{\mathcal{B}}^2 \text{ is } (x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U},\gamma)\mapsto(x_0\mathbf{U},\gamma^{-1}x_3\tau^s\mathbf{U}).\end{aligned}$$

Applying $p'_{13!}$ we see that

$$\mathfrak{b}'(L)\circ L' = \tilde{c}_!(e_1^*L\boxtimes e'_1{}^*L'), \tilde{L}'\circ\mathfrak{b}'(L) = \tilde{c}_!(e'_2{}^*L\boxtimes e_1{}^*L),$$

where $\tilde{c}: R\to\tilde{\mathcal{B}}^2$ is $(x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U},\gamma)\mapsto(x_0\mathbf{U},x_3\mathbf{U})$.

We define $\mathbf{e}:\tilde{\mathcal{B}}^2\to\tilde{\mathcal{B}}^2$ by $\mathbf{e}(x\mathbf{U},y\mathbf{U}) = (\mathbf{e}(x)\mathbf{U},\mathbf{e}(y)\mathbf{U})$. We show:

(i) *If in addition $L'\in\mathcal{M}(\tilde{\mathcal{B}}^2)$ is G-equivariant, then we have canonically*

$$\mathfrak{b}'(L)\circ L' = (\mathbf{e}^{s*}L')\circ\mathfrak{b}'(L).$$

We take $\tilde{L}' = \mathbf{e}^{s*}L'$. It is enough to show that $\tilde{c}_!(e_1^*L\boxtimes e'_1{}^*L') = \tilde{c}_!(e'_2{}^*\tilde{L}'\boxtimes e_1^*L)$. Hence it is enough to show that we have canonically $e'_1{}^*L' = e'_2{}^*\tilde{L}'$ that is, $e'_1{}^*L' = e''^*_2L'$ where $e''_2 = \mathbf{e}^s e'_2: R\to\tilde{\mathcal{B}}^2$. We identify $\tilde{G}_s$ with G by $\gamma\mapsto g$ where $\gamma = g\tau^s$. Then $e'_1: R\to\tilde{\mathcal{B}}^2$ is $(x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U},\gamma)\mapsto(g\mathbf{e}^s(x_0)\mathbf{U},x_3\mathbf{U})$, $e''_2: R\to\tilde{\mathcal{B}}^2$ is $(x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U},\gamma)\mapsto(\mathbf{e}^s(x_0)\mathbf{U}, g^{-1}x_3\mathbf{U})$. The equality $e'_1{}^*L' = e''^*_2L'$ follows from the G-equivariance of L'. This proves (i).

We show:

(j) *If $L\in\mathcal{C}_0^{\mathbf{c}}Z_s$, $L'\in\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$, then we have canonically $\underline{\mathfrak{b}}(L)\underline{\circ}L' = (\mathbf{e}^{s*}L')\underline{\circ}\underline{\mathfrak{b}}(L)$.*

By (d), it is enough to prove that $\underline{\mathfrak{b}}'(L)\underline{\circ}L' = (\mathbf{e}^{s*}L')\underline{\circ}\underline{\mathfrak{b}}'(L)$. Using (i) together with (a), (b), (c) and results in [21, 2.23], we see that both sides are equal to

$$\underline{gr_{\rho+\nu+3a}(\tilde{c}_!(e_1^*L\otimes e'_1{}^*L'))^{\rho+\nu+3a}}((\rho+\nu+3a)/2)$$

$$= \underline{gr_{\rho+\nu+3a}\tilde{c}_!(e_1^* L \otimes e_2''^* L'))^{\rho+\nu+3a}}((\rho+\nu+3a)/2).$$

4.15. Let

$$\mathfrak{Z}_s = \{(z_0\mathbf{U}, z_1\mathbf{U}, z_2\mathbf{U}, z_3\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^4 \times \tilde{G}_s; \gamma \in z_2\mathbf{B}\tau^s z_1^{-1}\}.$$

Define $\tilde{\vartheta} : \mathfrak{Z}_s \to \tilde{\mathcal{B}}^4$ by $((z_0\mathbf{U}, z_1\mathbf{U}, z_2\mathbf{U}, z_3\mathbf{U}), \gamma) \mapsto (z_0\mathbf{U}, z_1\mathbf{U}, z_2\mathbf{U}, z_3\mathbf{U})$. Let

$$'\mathcal{Y} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^5 \times \tilde{G}_s; \gamma \in x_3\mathbf{U}\tau^s x_0^{-1}, \gamma \in x_2\mathbf{B}\tau^s x_1^{-1}\},$$
$$''\mathcal{Y} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in \tilde{\mathcal{B}}^5 \times \tilde{G}_s; \gamma \in x_4\mathbf{U}\tau^s x_1^{-1}, \gamma \in x_3\mathbf{B}\tau^s x_2^{-1}\}.$$

Define $'\vartheta : {'\mathcal{Y}} \to \tilde{\mathcal{B}}^5$, $''\vartheta : {''\mathcal{Y}} \to \tilde{\mathcal{B}}^5$ by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}).$$

We have isomorphisms $'\mathfrak{c} : {'\mathcal{Y}} \xrightarrow{\sim} \mathfrak{Z}_s$, $''\mathfrak{c} : {''\mathcal{Y}} \xrightarrow{\sim} \mathfrak{Z}_s$ given by

$$'\mathfrak{c} : ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_4\mathbf{U}), \gamma),$$
$$''\mathfrak{c} : ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \mapsto ((x_0\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma).$$

Define $'d : \tilde{\mathcal{B}}^5 \to \tilde{\mathcal{B}}^4$, $''d : \tilde{\mathcal{B}}^5 \to \tilde{\mathcal{B}}^4$ by

$$'d : (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}) \mapsto (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_4\mathbf{U}),$$
$$''d : (x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}) \mapsto (x_0\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}).$$

We fix w, u in W and λ, λ' in $\mathfrak{s}_n$. We assume that $w \cdot \lambda \in I_n^s$. The smooth subvarieties

$$'\mathcal{U} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in {'\mathcal{Y}}; x_1^{-1}x_2 \in G_w, x_3^{-1}x_4 \in G_{\mathbf{e}^s(u)}\},$$
$$\mathcal{U} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \in \mathfrak{Z}_s; x_1^{-1}x_2 \in G_w, x_0^{-1}g^{-1}x_3 \in G_u\},$$
$$''\mathcal{U} = \{((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) \in {''\mathcal{Y}}; x_2^{-1}x_3 \in G_w, x_0^{-1}x_1 \in G_u\},$$

of $'\mathcal{Y}, \mathfrak{Z}_s, ''\mathcal{Y}$ correspond to each other under the isomorphisms $'\mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} ''\mathcal{Y}$. Moreover, the maps $'\sigma : '\mathcal{U} \to Z_s$, $\sigma : \mathcal{U} \to Z_s$, $''\sigma : ''\mathcal{U} \to Z_s$ given by

$$\begin{aligned} ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) &\mapsto \epsilon_s(x_1\mathbf{U}, x_2\mathbf{U}), \\ ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) &\mapsto \epsilon_s(x_1\mathbf{U}, x_2\mathbf{U}), \\ ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) &\mapsto \epsilon_s(x_2\mathbf{U}, x_3\mathbf{U}), \end{aligned}$$

correspond to each other under the isomorphisms $'\mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} ''\mathcal{Y}$.

Also, the maps $'\tilde{\sigma} : '\mathcal{U} \to \tilde{\mathcal{O}}_{\mathbf{e}^s(u)}$, $\tilde{\sigma} : \mathcal{U} \to \tilde{\mathcal{O}}_{\mathbf{e}^s(u)}$, given by

$$\begin{aligned} ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) &\mapsto (x_3\mathbf{U}, x_4\mathbf{U}), \\ ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) &\mapsto (\gamma x_0 \tau^{-s}\mathbf{U}, x_3\mathbf{U}) \end{aligned}$$

correspond to each other under the isomorphism $'\mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s$ and the maps $\tilde{\sigma}_1 : \mathcal{U} \to \tilde{\mathcal{O}}_u$, $''\tilde{\sigma} : ''\mathcal{U} \to \tilde{\mathcal{O}}_u$ given by

$$\begin{aligned} ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) &\mapsto (x_0\mathbf{U}, \gamma^{-1} x_3 \tau^s \mathbf{U}), \\ ((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}, x_4\mathbf{U}), \gamma) &\mapsto (x_0\mathbf{U}, x_1\mathbf{U}), \end{aligned}$$

correspond to each other under the isomorphism $\mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} ''\mathcal{Y}$. It follows that the local systems $'\sigma^* \mathcal{L}^{\dot{w}}_{\lambda,s}$, $\sigma^* \mathcal{L}^{\dot{w}}_{\lambda,s}$, $''\sigma^* \mathcal{L}^{\dot{w}}_{\lambda,s}$ correspond to each other under the isomorphisms $'\mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} ''\mathcal{Y}$; the local systems $'\tilde{\sigma}^* L^{\mathbf{e}^s \dot{u})}_{\mathbf{e}^s(\lambda')}$, $\tilde{\sigma}^* L^{\mathbf{e}^s(\dot{u})}_{\mathbf{e}^s(\lambda')}$ correspond to each other under the isomorphism $'\mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s$; the local systems $\tilde{\sigma}_1^* L^{\dot{u}}_{\lambda'}$, $''\tilde{\sigma}^* L^{\dot{u}}_{\lambda'}$ correspond to each other under the isomorphism $\mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} ''\mathcal{Y}$. Moreover, by the G-equivariance of $L^{\dot{u}}_{\lambda'}$, we have as in the proof of 4.14(i): $\tilde{\sigma}^* L^{\mathbf{e}^s(\dot{u})}_{\mathbf{e}^s(\lambda')} = \tilde{\sigma}_1^*(L^{\dot{u}}_{\lambda'})$.

Let $'K, K, ''K$ be the intersection cohomology complex of the closure of $'\mathcal{U}, \mathcal{U}, ''\mathcal{U}$ respectively with coefficients in the local system

$$'\sigma^* \mathcal{L}^{\dot{w}}_{\lambda,s} \otimes {'\tilde{\sigma}^*} L^{\mathbf{e}^s(\dot{u})}_{\mathbf{e}^s(\lambda')}, \sigma^* \mathcal{L}^{\dot{w}}_{\lambda,s} \otimes \tilde{\sigma}^* L^{\mathbf{e}^s(\dot{u})}_{\mathbf{e}^s(\lambda')} = \sigma^* \mathcal{L}^{\dot{w}}_{\lambda,s} \otimes \tilde{\sigma}_1^*(L^{\dot{u}}_{\lambda'}), ''\sigma^* \mathcal{L}^{\dot{w}}_{\lambda,s} \otimes {''\tilde{\sigma}^*} L^{\dot{u}}_{\lambda'},$$

on $'\mathcal{U}, \mathcal{U}, ''\mathcal{U}$ (respectively), extended by 0 on the complement of this closure in $'\mathcal{Y}, \mathfrak{Z}_s, ''\mathcal{Y}$. We see that $'K, K, ''K$ correspond to each other under the isomorphisms $'\mathcal{Y} \xrightarrow{'\mathfrak{c}} \mathfrak{Z}_s \xleftarrow{''\mathfrak{c}} ''\mathcal{Y}$. Hence we have $'\mathfrak{c}_!('K) = K = ''\mathfrak{c}_!(''K)$. Using this and the commutative diagram

$$\begin{array}{ccccc} '\mathcal{Y} & \xrightarrow{'\mathfrak{c}} & \mathfrak{Z}_s & \xleftarrow{''\mathfrak{c}} & ''\mathcal{Y} \\ {\scriptstyle '\vartheta}\downarrow & & {\scriptstyle \tilde{\vartheta}}\downarrow & & {\scriptstyle ''\vartheta}\downarrow \\ \tilde{\mathcal{B}}^5 & \xrightarrow{'d} & \tilde{\mathcal{B}}^4 & \xleftarrow{''d} & \tilde{\mathcal{B}}^5 \end{array}$$

we see that

(a) $$'d_!{}'\vartheta_!('K) = ''d_!{}''\vartheta_!(''K).$$

(Both sides are equal to $\tilde{\vartheta}_! K$.)

4.16. In this subsection we study the functor $'d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \to \mathcal{D}_m(\tilde{\mathcal{B}}^4)$. Let $\mathbf{w} = (w_1, w_2, w_3, w_4)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathfrak{s}_n^4$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ (with $\omega_i \in \kappa_0^{-1}(w_i)$). Assume that $w_4 \cdot \lambda_4 \preceq \mathbf{c}$. Let $K = M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},[1,4]} \langle |\mathbf{w}| + 5\rho + \nu \rangle \in \mathcal{D}_m(\tilde{\mathcal{B}}^5)$. As in [21, 3.16], properties (a), (b), (c), (d) hold:

(a) *If $h > a + \rho$ then $('d_! K)^h \in {}'\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$. Moreover,*

$$\underline{gr_{a+\rho}(('dK)^{a+\rho})}((a+\rho)/2) = \oplus_{y' \in W; y'^{-1} \cdot \lambda_4 \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{\lambda_4}^{\dot{y}'^{-1}}, \mathbf{L}_{\lambda_3}^{\omega_3} \underline{\circ} \mathbf{L}_{\lambda_4}^{\omega_4}) \otimes M_{\lambda_1,\lambda_2,\lambda_4}^{\omega_1,\omega_2,\dot{y}'^{-1},[1,3]} \left\langle |w_1| + |w_2| + |y'| + 4\rho + \nu \right\rangle .$$

(b) *If $K \in {}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^5)$ then $'d_!(K) \in {}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$.*

(c) *If $K \in {}_4\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^5)$ then $'d_!(K) \in {}_4\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^4)$.*

(d) *If $K \in {}_4\mathcal{M}^{\preceq}(\tilde{\mathcal{B}}^5)$ and $h > a + \rho$ then $('d_!(K))^h \in {}_4\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$.*

4.17. In this subsection we study the functor $''d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \to \mathcal{D}_m(\tilde{\mathcal{B}}^4)$. Let $\mathbf{w} = (w_1, w_2, w_3, w_4)$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in \mathfrak{s}_n^4$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ (with $\omega_i \in \kappa_0^{-1}(w_i)$). Assume that $w_1 \cdot \lambda_1 \preceq \mathbf{c}$. Let $K = M_{\boldsymbol{\lambda}}^{\boldsymbol{\omega},[1,4]} \langle |\mathbf{w}| + 5\rho + \nu \rangle \in \mathcal{D}_m(\tilde{\mathcal{B}}^5)$. As in [21, 3.17], properties (a), (b), (c), (d) hold:

(a) *If $h > a + \rho$ then $(''d_! K)^h \in {}'\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$. Moreover,*

$$\underline{gr_{a+\rho}((''d_! K)^{a+\rho})}((a+\rho)/2) = \oplus_{y' \in W; y' \cdot \lambda_2 \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{\lambda_2}^{\dot{y}'}, \mathbf{L}_{\lambda_1}^{\omega_1} \underline{\circ} \mathbf{L}_{\lambda_2}^{\omega_2}) \otimes M_{\lambda_2,\lambda_3,\lambda_4}^{\dot{y}',\omega_3,\omega_4,[1,3]} \left\langle |w_3| + |w_4| + |y'| + 4\rho + \nu \right\rangle .$$

(b) *If $K \in {}_1\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^5)$ then $''d_!(K) \in {}_1\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$.*

(c) *If* $K \in {}_1\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^5)$ *then* $''d_!(K) \in {}_1\mathcal{D}^{\prec}(\tilde{\mathcal{B}}^4)$.

(d) *If* $K \in {}_1\mathcal{M}^{\preceq}(\tilde{\mathcal{B}}^5)$ *and* $h > a+\rho$ *then* $(''d_!(K))^h \in {}_1\mathcal{M}^{\prec}(\tilde{\mathcal{B}}^4)$.

4.18. Let $w \cdot \lambda \in I_n^s$, $u \cdot \lambda' \in \mathbf{c}$. We shall apply the method of [19, 1.12] with $\Phi : \mathcal{D}_m(Y_1) \to \mathcal{D}_m(Y_2)$ replaced by $'d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \to \mathcal{D}_m(\tilde{\mathcal{B}}^4)$ and with $\mathcal{D}^{\preceq}(Y_1)$, $\mathcal{D}^{\preceq}(Y_2)$ replaced by ${}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^5)$, ${}_4\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$, see 4.15. We shall take $\mathbf{X}$ in *loc.cit.* equal to $\Xi = '\vartheta_!('K)$ as in 4.15, $(w_2, w_4) = (w, \mathbf{e}^s(u))$, $(\lambda_2, \lambda_4) = (\lambda, \mathbf{e}^s(\lambda'))$. The conditions of *loc.cit.* are satisfied: those concerning $\mathbf{X}$ are satisfied with $c' = k = |w| + |u| + 3\nu + 5\rho$ (see 4.8(c)); those concerning Φ are satisfied with $c = a+\rho$ (see 4.16). We see that

$$\begin{aligned}&\underline{gr_{a+\rho+k}(('d_!{'\vartheta_!}('K))^{a+\rho+k})}((a+\rho+k)/2)\\ &= \underline{gr_{a+\rho}(('d_!\underline{gr_k(('\vartheta_!('K))^k)}(k/2))^{a+\rho})}((a+\rho)/2).\end{aligned}$$

Using 4.11(a), we have:

$$\begin{aligned}gr_k('\vartheta_!('K))^k)(k/2) &= \oplus_{y\in W} M^{\mathbf{e}^{-s}(\dot{y}),\dot{w},\dot{y}^{-1},\mathbf{e}^s(\dot{u}),[1,4]}_{\mathbf{e}^{-s}(\lambda),\lambda,y(\lambda),\mathbf{e}^s(\lambda')}\,\langle 2|y|+|w|+|u|+5\rho+\nu\rangle\\ &= \underline{gr_k('\vartheta_!('K))^k}(k/2).\end{aligned}$$

Hence, using 4.16(a), we have

$$\begin{aligned}&\underline{gr_{a+\rho}(('d_!\underline{gr_k(('\vartheta_!('K))^k)}(k/2))^{a+\rho})}((a+\rho)/2)\\ &= \oplus_{y\in W}\oplus_{y'\in W; y'^{-1}\cdot\mathbf{e}^s(\lambda')\in\mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}'^{-1}}_{\mathbf{e}^s(\lambda')}, \mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}\underline{\circ}\mathbf{L}^{\mathbf{e}^s(\dot{u})}_{\mathbf{e}^s(\lambda')})\\ &\quad\otimes M^{\mathbf{e}^{-s}(\dot{y}),\dot{w},\dot{y}'^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,\mathbf{e}^s(\lambda')}\,\big\langle |y|+|w|+|y'|+4\rho+\nu\big\rangle.\end{aligned}$$

Since $y'^{-1}\cdot\mathbf{e}^s(\lambda') \in \mathbf{c}$, $\mathbf{e}^s(u)\cdot\mathbf{e}^s(\lambda') \in \mathbf{c}$ (recall that $\mathbf{e}^s\mathbf{c} = \mathbf{c}$), for $y \in W$ we have

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}'^{-1}}_{\mathbf{e}^s(\lambda')}, \mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}\underline{\circ}\mathbf{L}^{\mathbf{e}^s(\dot{u})}_{\mathbf{e}^s(\lambda')}) = 0$$

unless $\mathbf{e}^s(\lambda') = y'(\lambda)$ (see [21, 4.6(b)]) and $y^{-1}\cdot y(\lambda) \in \mathbf{c}$ (see [21, 2.26(a)]) or equivalently, $y\cdot\lambda \in \mathbf{c}$. Thus we have

(a)
$$\begin{aligned}&\underline{gr_{a+\rho+k}(('d_!{'\vartheta_!}('K))^{a+\rho+k})}((a+\rho+k)/2)\\ &= \oplus_{y\in W; y\cdot\lambda\in\mathbf{c}}\oplus_{y'\in W; y'^{-1}\cdot y'(\lambda)\in\mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}'^{-1}}_{y'(\lambda)}, \mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}\underline{\circ}\mathbf{L}^{\mathbf{e}^s(\dot{u})}_{\mathbf{e}^s(\lambda')})\\ &\quad\otimes M^{\mathbf{e}^{-s}(\dot{y}),\dot{w},\dot{y}'^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,y'(\lambda)}\,\big\langle |y|+|w|+|y'|+4\rho+\nu\big\rangle.\end{aligned}$$

4.19. In the setup of 4.18 we shall apply the method of [19, 1.12] with $\Phi : \mathcal{D}_m(Y_1) \to \mathcal{D}_m(Y_2)$ replaced by $''d_! : \mathcal{D}_m(\tilde{\mathcal{B}}^5) \to \mathcal{D}_m(\tilde{\mathcal{B}}^4)$ and with $\mathcal{D}^{\preceq}(Y_1)$, $\mathcal{D}^{\preceq}(Y_2)$ replaced by ${}_1\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^5)$, ${}_1\mathcal{D}^{\preceq}(\tilde{\mathcal{B}}^4)$, see 4.15. We shall take $\mathbf{X}$ in *loc.cit.* equal to $\Xi = ''\vartheta_!(''K)$ as in 4.15, $(w_1, w_3) = (u, w)$, $(\lambda_1, \lambda_3) = (\lambda', \lambda)$. The conditions of *loc.cit.* are satisfied: those concerning $\mathbf{X}$ are satisfied with $c' = k = |w| + |u| + 3\nu + 5\rho$ (see 4.8(c)); those concerning Φ are satisfied with $c = a + \rho$ (see 4.17). We see that

$$\begin{aligned}&\underline{gr_{a+\rho+k}((''d_!''\vartheta_!(''K))^{a+\rho+k}})((a+\rho+k)/2)\\ &= \underline{gr_{a+\rho}((''d_!\underline{gr_k((''\vartheta_!(''K))^k})(k/2))^{a+\rho}})((a+\rho)/2).\end{aligned}$$

Using 4.11(a), we have:

$$\begin{aligned}gr_k(''\vartheta_!(''K))^k)(k/2) &= \oplus_{y' \in W} M^{\dot{u}, \mathbf{e}^{-s}(\dot{y}'), \dot{w}, \dot{y}'^{-1}, [1,4]}_{\lambda', \mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)} \langle 2|y'| + |w| + |u| + 5\rho + \nu \rangle\\ &= \underline{gr_k(''\vartheta_!(''K))^k}(k/2).\end{aligned}$$

Hence, using 4.17(a), we have

$$\begin{aligned}&\underline{gr_{a+\rho}((''d_!\underline{gr_k((''\vartheta_!(''K))^k})(k/2))^{a+\rho}})((a+\rho)/2)\\ &= \oplus_{y' \in W} \oplus_{y_1 \in W; y_1 \cdot \mathbf{e}^{-s}(\lambda) \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}_1}_{\mathbf{e}^{-s}(\lambda)}, \mathbf{L}^{\dot{u}}_{\lambda'} \underline{\circ} \mathbf{L}^{\mathbf{e}^{-s}(\dot{y}')}_{\mathbf{e}^{-s}(\lambda)})\\ &\quad \otimes M^{\dot{y}_1, \dot{w}, \dot{y}'^{-1}, [1,3]}_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)} \langle |y_1| + |w| + |y'| + 4\rho + \nu \rangle.\end{aligned}$$

Since $u \cdot \lambda' \in \mathbf{c}$, for $y' \in W$ we have

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}_1}_{(\mathbf{e}^{-s}(\lambda)}, \mathbf{L}^{\dot{u}}_{\lambda'} \underline{\circ} \mathbf{L}^{\mathbf{e}^{-s}(\dot{y}')}_{\mathbf{e}^{-s}(\lambda)}) = 0$$

unless $\mathbf{e}^s(\lambda') = y'(\lambda)$ (see [21, 4.6(b)]) and $y'(\lambda) = \mathbf{e}^s(\lambda')$ (see [21, 2.26(a)]). Thus we have

$$\begin{aligned}&\underline{gr_{a+\rho+k}((''d_!''\vartheta_!(''K))^{a+\rho+k}})((a+\rho+k)/2)\\ &= \oplus_{y' \in W; y' \cdot \lambda \in \mathbf{c}} \oplus_{y_1 \in W; y_1 \cdot \mathbf{e}^{-s}(\lambda) \in \mathbf{c}} \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}_1}_{\mathbf{e}^{-s}(\lambda)}, \mathbf{L}^{\dot{u}}_{\lambda'} \underline{\circ} \mathbf{L}^{\mathbf{e}^{-s}(\dot{y}')}_{\mathbf{e}^{-s}(\lambda)})\\ &\quad \otimes M^{\dot{y}_1, \dot{w}, \dot{y}'^{-1}, [1,3]}_{\mathbf{e}^{-s}(\lambda), \lambda, y'(\lambda)} \langle |y_1| + |w| + |y'| + 4\rho + \nu \rangle.\end{aligned}$$

Setting $y_1 = \mathbf{e}^{-s} y$ and using that $\mathbf{e}^{-s} y \cdot \mathbf{e}^{-s}(\lambda) \in \mathbf{c}$ if and only if $y \cdot \lambda \in \mathbf{c}$, we can rewrite this as follows:

$$\text{(a)}\qquad \underline{gr_{a+\rho+k}(({}''d_!{}''\vartheta_!({}''K))^{a+\rho+k})}((a+\rho+k)/2)$$
$$= \oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\mathbf{e}^{-s}\dot{y}}_{\mathbf{e}^{-s}(\lambda)},\mathbf{L}^{\dot{u}}_{\lambda'}\underline{\circ}\mathbf{L}^{\mathbf{e}^{-s}(\dot{y}')}_{\mathbf{e}^{-s}(\lambda)})$$
$$\otimes M^{\mathbf{e}^{-s}\dot{y},\dot{w},\dot{y}'^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,y'(\lambda)}\langle|y|+|w|+|y'|+4\rho+\nu\rangle.$$

4.20. Let $y_1\cdot\lambda_1\in\mathbf{c}$, $y_2\cdot\lambda_2\in\mathbf{c}$, $y_3\cdot\lambda_3\in\mathbf{c}$. From [21, 3.20] we see that:

(a) *we have canonically*

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}_2^{-1}}_{y_2(\lambda_2)},\mathbf{L}^{\dot{y}_1^{-1}}_{y_1(\lambda_1)}\underline{\circ}\mathbf{L}^{\dot{y}_3}_{\lambda_3})=\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}_1}_{\lambda_1},\mathbf{L}^{\dot{y}_3}_{\lambda_3}\underline{\circ}\mathbf{L}^{\dot{y}_2}_{\lambda_2}).$$

In the setup of 4.18, we apply 4.18(a), 4.19(a) to $w\cdot\lambda$, $u\cdot\lambda'$ and we use the equality

$$\underline{gr_{a+\rho+k}(({}'d_!{}'\vartheta_!({}'K))^{a+\rho+k})}((a+\rho+k)/2)$$
$$=\underline{gr_{a+\rho+k}(({}''d_!{}''\vartheta_!({}''K))^{a+\rho+k})}((a+\rho+k)/2)$$

which comes from ${}'d_!{}'\vartheta_!({}'K)={}''d_!{}''\vartheta_!({}''K)$, see 4.15(a); we obtain

$$\text{(b)}\qquad \oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}'^{-1}}_{y'(\lambda)},\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}\underline{\circ}\mathbf{L}^{\mathbf{e}^{s}(\dot{u})}_{\mathbf{e}^{s}(\lambda')})$$
$$\otimes M^{\mathbf{e}^{-s}(\dot{y}),\dot{w},\dot{y}'^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,y'(\lambda)}\langle|y|+|w|+|y'|+4\rho+\nu\rangle$$
$$= \oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\mathbf{e}^{-s}\dot{y}}_{\mathbf{e}^{-s}(\lambda)},\mathbf{L}^{\dot{u}}_{\lambda'}\underline{\circ}\mathbf{L}^{\mathbf{e}^{-s}(\dot{y}')}_{\mathbf{e}^{-s}(\lambda)})$$
$$\otimes M^{\mathbf{e}^{-s}\dot{y},\dot{w},\dot{y}'^{-1},[1,3]}_{\mathbf{e}^{-s}(\lambda),\lambda,y'(\lambda)}\langle|y|+|w|+|y'|+4\rho+\nu\rangle.$$

4.21. We assume that $w\cdot\lambda, u\cdot\lambda'$ in 4.18 satisfy in addition $w\cdot\lambda\in\mathbf{c}$. We apply $p_{03!}$ and $\langle N\rangle$ for some N to the two sides of 4.20(b). (Recall that $p_{03}:\tilde{\mathcal{B}}^4\to\tilde{\mathcal{B}}^2$.) We obtain

$$\oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{y}'^{-1}}_{y'(\lambda)},\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}\underline{\circ}\mathbf{L}^{\mathbf{e}^{s}(\dot{u})}_{\mathbf{e}^{s}(\lambda')})\otimes\mathbf{L}^{\mathbf{e}^{-s}(\dot{y})}_{\mathbf{e}^{-s}(\lambda)}\circ\mathbf{L}^{\dot{w}}_{\lambda}\circ\mathbf{L}^{\dot{y}'^{-1}}_{y'(\lambda)}$$
$$= \oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\mathbf{e}^{-s}\dot{y}}_{\mathbf{e}^{-s}(\lambda)},\mathbf{L}^{\dot{u}}_{\lambda'}\underline{\circ}\mathbf{L}^{\mathbf{e}^{-s}(\dot{y}')}_{\mathbf{e}^{-s}(\lambda)})$$
$$\otimes\mathbf{L}^{\mathbf{e}^{-s}\dot{y}}_{\mathbf{e}^{-s}(\lambda)}\circ\mathbf{L}^{\dot{w}}_{\lambda}\circ\mathbf{L}^{\dot{y}'^{-1}}_{y'(\lambda)}.$$

Applying $\underline{()^{\{2(a-\nu)\}}}$ to both sides and using [21, 2.24(a)] we obtain

$$\oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}},\mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}\underline{\circ}\mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})})\otimes\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})}\underline{\circ}\mathbf{L}_{\lambda}^{\dot{w}}\underline{\circ}\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}$$
$$=\oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}},\mathbf{L}_{\lambda'}^{\dot{u}}\underline{\circ}\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')})$$
$$\otimes\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}\dot{y}}\underline{\circ}\mathbf{L}_{\lambda}^{\dot{w}}\underline{\circ}\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}},$$

or equivalently

$$\oplus_{y\in W;y\cdot\lambda\in\mathbf{c}}\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})}\underline{\circ}\mathbf{L}_{\lambda}^{\dot{w}}\underline{\circ}\mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}\underline{\circ}\mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}$$
$$=\oplus_{y'\in W;y'\cdot\lambda\in\mathbf{c}}\mathbf{L}_{\lambda'}^{\dot{u}}\underline{\circ}\mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y}')}\underline{\circ}\mathbf{L}_{\lambda}^{\dot{w}}\underline{\circ}\mathbf{L}_{y'(\lambda)}^{\dot{y}'^{-1}}.$$

Using 4.13(d), this can be rewritten as follows:

(a) $$\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{w}})\underline{\circ}\mathbf{L}_{\mathbf{e}^s(\lambda')}^{\mathbf{e}^s(\dot{u})}=\mathbf{L}_{\lambda'}^{\dot{u}}\underline{\circ}\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{w}}).$$

Another identification of the two sides in (a) is given by 4.14(j) with $L=\mathbb{L}_{\lambda,s}^{\dot{w}}$, $L'=\mathbf{L}_{\lambda'}^{\dot{u}}$ (note that $\underline{\mathfrak{b}}(L)=\underline{\mathfrak{b}}'(L)$ by 4.14(d)). In fact, the arguments in 4.13-4.20 and in this subsection show that

(b) *these two identifications of the two sides of (a) coincide.*

4.22. Let $s',s''\in\mathbf{Z}$. Let

$$V=\{(B_0,B_1,B_2,\gamma U_{B_0},\gamma' U_{B_1});$$
$$(B_0,B_1,B_2)\in\mathcal{B}^3,\gamma\in\tilde{G}_{s'},\gamma'\in\tilde{G}_{s''},\gamma B_0\gamma^{-1}=B_1,\gamma' B_1\gamma'^{-1}=B_2\}.$$

Define $p_{01}:V\to Z_{s'}$, $p_{12}:V\to Z_{s''}$, $p_{02}:V\to Z_{s'+s''}$ by

$$p_{01}:(B_0,B_1,B_2,\gamma U_{B_0},\gamma' U_{B_1})\mapsto(B_0,B_1,\gamma U_{B_0}),$$
$$p_{12}:(B_0,B_1,B_2,gU_{B_0},\gamma' U_{B_1})\mapsto(B_1,B_2,\gamma' U_{B_1}),$$
$$p_{02}:(B_0,B_1,B_2,\gamma U_{B_0},\gamma' U_{B_1})\mapsto(B_0,B_2,\gamma'\gamma U_{B_0}).$$

For $L\in\mathcal{D}(Z_{s'}),L'\in\mathcal{D}(Z_{s''})$ we set

$$L\bullet L'=p_{02!}(p_{01}^*L\otimes p_{12}^*L')\in\mathcal{D}(Z_{s'+s''}).$$

This operation defines a monoidal structure on $\sqcup_{s'\in\mathbf{Z}}\mathcal{D}(Z_{s'})$. Hence if ${}^1L\in\mathcal{D}(Z_{s_1}),{}^2L\in\mathcal{D}(Z_{s_2}),\dots,{}^rL\in\mathcal{D}(Z_{s_r})$, then ${}^1L\bullet{}^2L\bullet\dots\bullet{}^rL\in\mathcal{D}(Z_{s_1+\dots+s_r})$

is well defined. Note that, if $L \in \mathcal{D}_m(Z_{s'}), L'_m \in \mathcal{D}(Z_{s''})$ then we have naturally $L \bullet L' \in \mathcal{D}_m(Z_{s'+s''})$. We show:

(a) *For $L \in \mathcal{D}(Z_{s'}), L' \in \mathcal{D}(Z_{s''})$ we have canonically $\epsilon^*_{s'+s''}(L \bullet L') = \epsilon^*_{s'}(L) \circ \epsilon^*_{s''}(L')$.*

Let

$$Y = \{(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}); x\mathbf{U} \in \tilde{\mathcal{B}}, y\mathbf{U} \in \tilde{\mathcal{B}}; \gamma \in \tilde{G}_{s'}\}.$$

Define $j : Y \to \tilde{\mathcal{B}}^2$, $j_1 : Y \to Z_{s'}$, $j_2 : Y \to Z_{s''}$ by

$$\begin{aligned} j(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}) &= (x\mathbf{U}, y\mathbf{U}), \\ j_1(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}) &= (x\mathbf{B}x^{-1}, \gamma x\mathbf{B}x^{-1}\gamma^{-1}, \gamma U_{x\mathbf{B}x^{-1}}), \\ j_2(x\mathbf{U}, y\mathbf{U}, \gamma U_{x\mathbf{B}x^{-1}}) &= (\gamma x\mathbf{B}x^{-1}\gamma^{-1}, y\mathbf{B}y^{-1}, y\mathbf{U}\tau^{s'+s''}x^{-1}\gamma^{-1}). \end{aligned}$$

From the definitions we have

$$\epsilon^*_{s'+s''}(L \bullet L') = j_!(j_1^*(L) \otimes j_2^*(L')) = \epsilon^*_{s'}(L) \circ \epsilon^*_{s''}(L')$$

and (a) follows.

4.23. Let $s' \in \mathbf{Z_c}$. Let $L \in \mathcal{D}^{\spadesuit} Z_s, L' \in \mathcal{D}^{\spadesuit} Z_{s'}$. We show:

(a) *If $L \in \mathcal{D}^{\preceq} Z_s$ or $L' \in \mathcal{D}^{\preceq} Z_{s'}$ then $L \bullet L' \in \mathcal{D}^{\preceq} Z_{s+s'}$. If $L \in \mathcal{D}^{\prec} Z_s$ or $L' \in \mathcal{D}^{\prec} Z_{s'}$ then $L \bullet L' \in \mathcal{D}^{\prec} Z_{s+s'}$.*

For the first assertion of (a) we can assume that $L = \mathbb{L}^{\dot{w}}_{\lambda,s}$, $L' = \mathbb{L}^{\dot{w}'}_{\lambda',s'}$ with $w \cdot \lambda \in I^s_n, w' \cdot \lambda' \in I^{s'}_n$ and either $w \cdot \lambda \preceq \mathbf{c}$ or $w' \cdot \lambda' \preceq \mathbf{c}$. Assume that $w_1 \cdot \lambda_1 \in I^{s+s'}_n$ and $\mathbb{L}^{\dot{w}_1}_{\lambda_1,s+s'}$ is a composition factor of $(L \bullet L')^j$. Then $\mathbf{L}^{\dot{w}_1}_{\lambda_1} = \tilde{\epsilon}_{s+s'}\mathbb{L}^{\dot{w}_1}_{\lambda_1,s+s'}$ is a composition factor of

$$\begin{aligned} \epsilon^*_{s+s'}(L \bullet L')^j \langle\rho\rangle &= (\epsilon^*_{s+s'}(L \bullet L'))^{j+\rho}(\rho/2) = (\epsilon^*_s L \circ \epsilon^*_{s'} L')^{j+\rho}(\rho/2) \\ &= (\epsilon^*_s L\langle\rho\rangle \circ \epsilon^*_{s'} L'\langle\rho\rangle)^{j-\rho}(-\rho/2) = (\mathbf{L}^{\dot{w}}_{\lambda} \circ \mathbf{L}^{\dot{w}'}_{\lambda'})^{j-\rho}(\rho/2). \end{aligned}$$

From [21, 2.23(b)] we see that $w_1 \cdot \lambda_1 \preceq \mathbf{c}$. This proves the first assertion of (a). The second assertion of (a) can be reduced to the first assertion.

We show:

(b) *Assume that $L \in \mathcal{M}^{\spadesuit} Z_s, L' \in \mathcal{M}^{\spadesuit} Z_{s'}$ and that either $L \in \mathcal{D}^{\preceq} Z_s$ or $L' \in \mathcal{D}^{\preceq} Z_{s'}$. If $j > a + \rho - \nu$ then $(L \bullet L')^j \in \mathcal{M}^{\prec} Z_{s+s'}$.*

We can assume that $L = \mathbb{L}_{\lambda,s}^{\dot{w}}$, $L' = \mathbb{L}_{\lambda',s'}^{\dot{w}'}$ with $w \cdot \lambda \in I_n^s, w' \cdot \lambda' \in I_n^{s'}$ and either $w \cdot \lambda \in \mathbf{c}$ or $w' \cdot \lambda' \in \mathbf{c}$. Assume that $w_1 \cdot \lambda_1 \in I_n^{s+s'}$ and that $\mathbb{L}_{\lambda_1,s+s'}^{\dot{w}_1}$ is a composition factor of $(L \bullet L')^j$. Then as in the proof of (a), $\mathbf{L}_{\lambda_1}^{\dot{w}_1}$ is a composition factor of

$$\tilde{e}_{s+s'}(L \bullet L')^j = (\mathbf{L}_\lambda^{\dot{w}} \circ \mathbf{L}_{\lambda'}^{\dot{w}'})^{j-\rho}(-\rho/2).$$

Since $j - \rho > a - \nu$ we see from [21, 2.23(a)] that $w_1 \cdot \lambda_1 \prec \mathbf{c}$. This proves (b).

4.24. Let $s' \in \mathbf{Z_c}$. For $L \in \mathcal{C}_0^{\mathbf{c}} Z_s, L' \in \mathcal{C}_0^{\mathbf{c}} Z_{s'}$ we set

$$L \underline{\bullet} L' = \underline{(L \bullet L')^{\{a+\rho-\nu\}}} \in \mathcal{C}_0^{\mathbf{c}} Z_{s+s'}.$$

Using 4.23(a),(b) we see as in [21, 2.24] that for $L \in \mathcal{C}_0^{\mathbf{c}} Z_s, L' \in \mathcal{C}_0^{\mathbf{c}} Z_{s'}, L'' \in \mathcal{C}_0^{\mathbf{c}} Z_{s''}$ we have

$$L \underline{\bullet} (L' \underline{\bullet} L'') = (L \underline{\bullet} L') \underline{\bullet} L'' = \underline{(L \bullet L' \bullet L'')^{\{2a+2\rho-2\nu\}}}.$$

We see that $L, L' \mapsto L \underline{\bullet} L'$ defines a monoidal structure on $\sqcup_{s' \in \mathbf{Z_c}} \mathcal{C}_0^{\mathbf{c}} Z_{s'}$. Hence if ${}^1L \in \mathcal{C}_0^{\mathbf{c}} Z_{s_1}, {}^2L \in \mathcal{C}_0^{\mathbf{c}} Z_{s_2}, \dots, {}^rL \in \mathcal{C}_0^{\mathbf{c}} Z_{s_r}$, then ${}^1L \underline{\bullet} {}^2L \underline{\bullet} \dots \underline{\bullet} {}^rL \in \mathcal{C}_0^{\mathbf{c}} Z_{s_1+\dots+s_r}$ is well defined; we have

(a) $${}^1L \underline{\bullet} {}^2L \underline{\bullet} \dots \underline{\bullet} {}^rL = \underline{({}^1L \bullet {}^2L \bullet \dots \bullet {}^rL)^{\{(r-1)(a+\rho-\nu)\}}}.$$

For $L \in \mathcal{C}_0^{\mathbf{c}} Z_s, L' \in \mathcal{C}_0^{\mathbf{c}} Z_{s'}$ we have $\tilde{\epsilon}_s L, \tilde{\epsilon}_{s'} L' \in \mathcal{C}_0^{\mathbf{c}} \tilde{\mathcal{B}}^2$. We show:

(b) $$\tilde{\epsilon}_{s+s'}(L \underline{\bullet} L') = (\tilde{\epsilon}_s L) \underline{\circ} (\tilde{\epsilon}_{s'} L').$$

It is enough to show that

$$\begin{aligned}&\epsilon_{s+s'}^*(gr_0((L \bullet L')^{a+\rho-\nu})((a+\rho-\nu)/2))[\rho](\rho/2)\\ &\quad = gr_0((\epsilon_s^* L[\rho](\rho/2) \circ \epsilon_{s'}^* L'[\rho](\rho/2))^{a-\nu})((a-\nu)/2))).\end{aligned}$$

The left hand side is equal to

$$gr_0(\epsilon_{s+s'}^*((L \bullet L')^{a+\rho-\nu})((a+\rho-\nu)/2))[\rho](\rho/2))$$

hence it is enough to show:

$$\epsilon_{s+s'}^*((L\bullet L')^{a+\rho-\nu})((a+\rho-\nu)/2))[\rho](\rho/2)$$
$$= (\epsilon_s^* L[\rho](\rho/2)\circ\epsilon_{s'}^* L'[\rho](\rho/2))^{a-\nu}((a-\nu)/2))$$

that is,

$$\epsilon_{s+s'}^*((L\bullet L')^{a+\rho-\nu})[\rho] = (\epsilon_s^* L[\rho]\circ\epsilon_{s'}^* L'[\rho])^{a-\nu},$$

or, after using 4.3(b):

$$(\epsilon_{s+s'}^*(L\bullet L'))^{a+2\rho-\nu} = (\epsilon_s^* L\circ\epsilon_{s'}^* L')^{a+2\rho-\nu}.$$

It remains to use that $\epsilon_{s+s'}^*(L\bullet L') = \epsilon_s^* L\circ\epsilon_{s'}^* L'$, see 4.22(a).

4.25. In the setup of 4.14 let

$${}^\diamond\mathcal{Y}=\mathbf{T}^2\backslash\{((x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U}),\gamma)\in\tilde{\mathcal{B}}^4\times\tilde{G}_s;\gamma\in x_3\mathbf{U}\tau^s x_0^{-1},\gamma\in x_2\mathbf{U}\tau^s x_1^{-1}\}$$

where $\mathbf{T}^2$ acts freely by

$$(t_1,t_2):((x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U}),\gamma)\mapsto((x_0t_1\mathbf{U},x_1t_2\mathbf{U},x_2t_2\mathbf{U},x_3t_1\mathbf{U}),\gamma).$$

We define ${}^\diamond\eta : {}^\diamond\mathcal{Y}\to Z_s$ by

$$((x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U}),\gamma)\mapsto\epsilon_s(x_1\mathbf{U},x_2\mathbf{U}).$$

We define $d : {}^\diamond\mathcal{Y}\to Z_s$ by

$$((x_0\mathbf{U},x_1\mathbf{U},x_2\mathbf{U},x_3\mathbf{U}),\gamma)\mapsto\epsilon_s(x_0\mathbf{U},x_3\mathbf{U}).$$

We define $\mathfrak{b}'' : \mathcal{D}(Z_s)\to\mathcal{D}(Z_s)$ and $\mathfrak{b}'' : \mathcal{D}_m(Z_s)\to\mathcal{D}_m(Z_s)$ by

$$\mathfrak{b}''(L) = d_!({}^\diamond\eta)^*L.$$

From the definitions it is clear that

(a) $$\mathfrak{b}'(L) = \epsilon_s^*\mathfrak{b}''(L).$$

Using (a) we see that 4.14(a),(b),(c) imply the following statements.

(b) *If* $L\in\mathcal{D}^{\preceq}(Z_s)$, *then* $\mathfrak{b}''(L)\in\mathcal{D}^{\preceq}Z_s$. *If* $L\in\mathcal{D}^{\prec}(Z_s)$ *then* $\mathfrak{b}''(L)\in\mathcal{D}^{\prec}Z_s$.

(c) *If* $L\in\mathcal{M}^{\preceq}(Z_s)$ *and* $h>2\nu+2a$ *then* $(\mathfrak{b}''(L))^h\in\mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$.

We define $\underline{\mathfrak{b}''}:\mathcal{C}_0^{\mathbf{c}}(Z_s)\to\mathcal{C}_0^{\mathbf{c}}(Z_s)$ by

$$\underline{\mathfrak{b}''}(L)=\underline{gr_{2\nu+2a}((\mathfrak{b}''(L))^{2\nu+2a})}(\nu+a).$$

Using results in 4.3 we see that, if $L\in\mathcal{C}_0^{\mathbf{c}}Z_s$, then

(d) $\underline{\mathfrak{b}'}(L)=\tilde{\epsilon}_s(\underline{\mathfrak{b}''}(L))$.

5. The monoidal category $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$

5.1. *In this section,* $\mathbf{c},a,\mathfrak{o},n,\Psi$ *are as in 3.1(a).*

Define $\delta:\tilde{\mathcal{B}}\to\tilde{\mathcal{B}}^2$ by $x\mathbf{U}\mapsto(x\mathbf{U},x\mathbf{U})$. For $w\cdot\lambda\in\mathbf{c}$ we set

$$\beta_{w\cdot\lambda}=\mathcal{H}^{-a+|w|}(\delta^*(L_\lambda^{\dot{w}\sharp}))((-a+|w|)/2).$$

By [21, 4.1] we have

(a) $\dim\beta_{w\cdot\lambda}=1$ if $w\cdot\lambda\in\mathbf{D_c}$, $\dim\beta_{w\cdot\lambda}=0$ if $w\cdot\lambda\notin\mathbf{D_c}$.

We set

$$\mathbf{1}'=\oplus_{d\cdot\lambda\in\mathbf{D_c}}\beta^*_{d\cdot\lambda}\otimes\mathbf{L}_\lambda^{\dot{d}}\in\mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2.$$

Here $\beta^*_{d\cdot\lambda}$ is the vector space dual to $\beta_{d\cdot\lambda}$.

5.2. For $L\in\mathcal{D}_m(\tilde{\mathcal{B}}^2)$ we set $L^\dagger=\tilde{\mathfrak{h}}^*L$ where $\tilde{\mathfrak{h}}:\tilde{\mathcal{B}}^2\to\tilde{\mathcal{B}}^2$ is as in 3.1. By [21, 4.4(b)], we have:

(a) *If* $L\in\mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2$ *then* $\mathfrak{D}(L^\dagger)\in\mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2$. *If* $L\in\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ *then* $\mathfrak{D}(L^\dagger)\in\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$.

5.3. The bifunctor $\mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2\times\mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2\to\mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2$, $L,L'\mapsto L\underline{\circ}L'$ in 3.10 gives rise to a bifunctor $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2\times\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2\to\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ denoted again by $L,L'\mapsto L\underline{\circ}L'$ as follows. Let $L\in\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$, $L'\in\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$; by replacing if necessary Ψ by a power, we choose mixed structures of pure weight 0 on L,L', we define $L\underline{\circ}L'$ as in 3.10 in terms of these mixed structures and we then disregard the mixed structure on $L\underline{\circ}L'$. The resulting object of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ is denoted again by $L\underline{\circ}L'$; it is independent of the choice of Ψ which defines the mixed structures.

Similarly for s,s' in $\mathbf{Z_c}$, the bifunctor $\mathcal{C}_0^{\mathbf{c}}Z_s\times\mathcal{C}_0^{\mathbf{c}}Z_{s'}\to\mathcal{C}_0^{\mathbf{c}}Z_{s+s'}$, $L,L'\mapsto L\underline{\bullet}L'$ in 4.24 gives rise to a bifunctor $\mathcal{C}^{\mathbf{c}}Z_s\times\mathcal{C}^{\mathbf{c}}Z_{s'}\to\mathcal{C}^{\mathbf{c}}Z_{s+s'}$ denoted again by $L, L' \mapsto L\underline{\bullet}L'$. Moreover, $\underline{\mathfrak{b}} : \mathcal{C}_0^{\mathbf{c}} Z_s \to \mathcal{C}_0^{\mathbf{c}} \tilde{\mathcal{B}}^2$ in 4.13 can be also viewed as a functor $\underline{\mathfrak{b}} : \mathcal{C}^{\mathbf{c}} Z_s \to \mathcal{C}^{\mathbf{c}} \tilde{\mathcal{B}}^2$.

The operation $L\underline{\bullet}L'$ (resp. $L\underline{\circ}L'$) makes $\sqcup_{s\in\mathbf{Z_c}}\mathcal{C}^{\mathbf{c}} Z_s$ (resp. $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$) into a monoidal abelian category (see 4.24, 3.10). By [21, 4.5(a)], we have:

(a) *For L, L' in $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ we have canonically*

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{1}', L\underline{\circ}L') = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathfrak{D}(L'^{\dagger}), L).$$

5.4. We set

(a) $$\mathbf{1} = \oplus_{d\cdot\lambda\in\mathbf{D_c}} \beta_{d\cdot\lambda} \otimes \mathbf{L}_{\lambda}^{\dot{d}^{-1}} \in \mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2.$$

Here $\beta_{d\cdot\lambda}$ is as in 5.1. By [21, 4.7(g)],

(a) $\mathbf{1} = \mathbf{1}'$ *is a unit object of the monoidal category* $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$.

By [21, 4.8], this monoidal category has a natural rigid structure.

5.5. *In the remainder of this section we fix $s \in \mathbf{Z_c}$.*

In this case, $(\mathbf{e}^s)^*$ defines an equivalence of categories $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2 \to \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$; this follows from 3.11(a).

By analogy with [20, 6.2] and slightly extending a definition in [22, 3.1], we define an $\mathbf{e}^s$*-half-braiding* for an object $\mathcal{L} \in \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$, as a collection $e_{\mathcal{L}} = \{e_{\mathcal{L}}(L); L \in \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2\}$ where $e_{\mathcal{L}}(L)$ is an isomorphism $(\mathbf{e}^s)^*(L)\underline{\circ}\mathcal{L} \xrightarrow{\sim} \mathcal{L}\underline{\circ}L$ such that $e_{\mathcal{L}}(\mathbf{1}) = Id_{\mathcal{L}}$ and such that (i), (ii) below hold:

(i) If $L \xrightarrow{t} L'$ is a morphism in $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ then the diagram

$$\begin{array}{ccc} (\mathbf{e}^s)^*(L)\underline{\circ}\mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L}\underline{\circ}L \\ {\scriptstyle (\mathbf{e}^s)^*(t)\underline{\bullet}1}\downarrow & & \downarrow{\scriptstyle 1\underline{\bullet}t} \\ (\mathbf{e}^s)^*(L')\underline{\circ}\mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L')} & \mathcal{L}\underline{\circ}L' \end{array}$$

is commutative.

(ii) If $L, L' \in \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ then $e_{\mathcal{L}}(L\underline{\circ}L') : (\mathbf{e}^s)^*(L\underline{\circ}L')\underline{\circ}\mathcal{L} \to \mathcal{L}\underline{\circ}(L\underline{\circ}L')$ is equal to the composition

$$(\mathbf{e}^s)^*(L)\underline{\circ}(\mathbf{e}^s)^*(L')\underline{\circ}\mathcal{L} \xrightarrow{1\underline{\circ}e_{\mathcal{L}}(L')} (\mathbf{e}^s)^*(L)\underline{\circ}\mathcal{L}\underline{\circ}L' \xrightarrow{e_{\mathcal{L}}(L)\underline{\circ}1} \mathcal{L}\underline{\circ}L\underline{\circ}L'.$$

(When $s = 0$ this reduces to the definition of a half-braiding for $\mathcal{L}$ given in [22, 3.1].)

Let $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$ be the category whose objects are the pairs $(\mathcal{L}, e_{\mathcal{L}})$ where $\mathcal{L}$ is an object of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ and $e_{\mathcal{L}}$ is an $\mathbf{e}^s$-half-braiding for $\mathcal{L}$. For $(\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'})$ in $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$ we define $\mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}((\mathcal{L}, e_{\mathcal{L}}), (\mathcal{L}', e_{\mathcal{L}'}))$ to be the vector space consisting of all $t \in \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathcal{L}, \mathcal{L}')$ such that for any $L \in \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ the diagram

$$\begin{array}{ccc}
(\mathbf{e}^s)^*(L)\underline{\circ}\mathcal{L} & \xrightarrow{e_{\mathcal{L}}(L)} & \mathcal{L}\underline{\circ}L \\
{\scriptstyle 1\underline{\circ}t}\downarrow & & {\scriptstyle t\underline{\circ}1}\downarrow \\
(\mathbf{e}^s)^*(L)\underline{\circ}\mathcal{L}' & \xrightarrow{e_{\mathcal{L}}(L')} & \mathcal{L}'\underline{\circ}L
\end{array}$$

is commutative. We say that $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$ is the $\mathbf{e}^s$-*centre* of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$. By a variation of a result of [22], [4] (which concerns the usual centre), the additive category $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$ is semisimple, with finitely many isomorphism classes of simple objects. By a variation of a general result on semisimple rigid monoidal categories in [4, Proposition 5.4], for any $L \in \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ one can define directly an $\mathbf{e}^s$-half-braiding on the object

$$\mathcal{I}_s(L) = \oplus_{y\cdot\lambda\in\mathbf{c}}(\mathbf{e}^s)^*(\mathbf{L}^{\dot{y}}_{\lambda})\underline{\circ}L\underline{\circ}\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)} = \oplus_{y\cdot\lambda\in\mathbf{c}}\mathbf{L}^{\mathbf{e}^{-s}(\lambda)}_{\mathbf{e}^{-s}(y)}\underline{\circ}L\underline{\circ}\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}$$

of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ such that, denoting by $\overline{\mathcal{I}_s(L)}$ the corresponding object of $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$, we have canonically

(a) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(L, L') = \mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}(\overline{\mathcal{I}_s(L)}, L')$$

for any $L' \in \mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$. (We use that for $y\cdot\lambda \in \mathbf{c}$, the dual of the simple object $\mathbf{L}^{\dot{y}}_{\lambda}$ is $\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}$, see [21, 4.4(c)]; we also use 3.11(a).) The $\mathbf{e}^s$-half-braiding on $\mathcal{I}_s(L)$ can be described as follows: for any $X \in \mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ we have canonically

$$\begin{aligned}
&(\mathbf{e}^s)^*(X)\underline{\circ}\mathcal{I}_s(L) \\
&= \oplus_{y\cdot\lambda\in\mathbf{c}}(\mathbf{e}^s)^*(X)\underline{\circ}(\mathbf{e}^s)^*(\mathbf{L}^{\dot{y}}_{\lambda})\underline{\circ}L\underline{\circ}\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)} \\
&= \oplus_{y\cdot\lambda\in\mathbf{c},z\cdot\lambda'\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}((\mathbf{e}^s)^*(\mathbf{L}^{\dot{z}}_{\lambda'}), (\mathbf{e}^s)^*(X\underline{\circ}\mathbf{L}^{\dot{y}}_{\lambda})) \otimes (\mathbf{e}^s)^*(\mathbf{L}^{\dot{z}}_{\lambda'})\underline{\circ}L\underline{\circ}\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)} \\
&= \oplus_{y\cdot\lambda\in\mathbf{c},z\cdot\lambda'\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{z}}_{\lambda'}, X\underline{\circ}\mathbf{L}^{\dot{y}}_{\lambda}) \otimes (\mathbf{e}^s)^*(\mathbf{L}^{\dot{z}}_{\lambda'})\underline{\circ}L\underline{\circ}\mathbf{L}^{\dot{y}^{-1}}_{y(\lambda)}
\end{aligned}$$

$$= \oplus_{y\cdot\lambda\in\mathbf{c},z\cdot\lambda'\in\mathbf{c}}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}},\mathbf{L}_{z(\lambda')}^{\dot{z}^{-1}}\otimes X)\otimes(\mathbf{e}^s)^*(\mathbf{L}_{\lambda'}^{\dot{z}})\underline{\circ}L\underline{\circ}\mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}$$
$$= \oplus_{z\cdot\lambda'\in\mathbf{c}}(\mathbf{e}^s)^*(\mathbf{L}_{\lambda'}^{\dot{z}})\underline{\circ}L\underline{\circ}\mathbf{L}_{z(\lambda')}^{\dot{z}^{-1}}\otimes X=\mathcal{I}_s(L)\underline{\circ}X.$$

(The fourth equality uses 4.20(a); we have also used 3.11(a).) We show:

(b) *If $z\cdot\lambda\in\mathbf{c}$ and $\mathcal{I}_s(\mathbf{L}_\lambda^{\dot{z}})\neq 0$ then $z\cdot\lambda\in\mathbf{c}^s$.*

For some $y\cdot\lambda'\in\mathbf{c}$ we have $\mathbf{L}_{\mathbf{e}^{-s}(\lambda')}^{\mathbf{e}^{-s}(\dot{y})}\underline{\circ}\mathbf{L}_\lambda^{\dot{z}}\neq 0$ (hence $\mathbf{e}^{-s}(\lambda')=z(l)$) and $\mathbf{L}_\lambda^{\dot{z}}\underline{\circ}\mathbf{L}_{y(\lambda')}^{\dot{y}^{-1}}\neq 0$ (hence $\lambda=\lambda'$). It follows that $z(\lambda)=\mathbf{e}^{-s}(\lambda)$ and (b) is proved.

5.6. By 4.13(d), for $z\cdot\lambda\in\mathbf{c}^s$ we have canonically

(a) $$\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}})=\mathcal{I}_s(\mathbf{L}_\lambda^{\dot{z}})$$

as objects of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$. Here $\underline{\mathfrak{b}}:\mathcal{C}^{\mathbf{c}}Z_s\to\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ is as in 5.3. Now $\mathcal{I}_s(\mathbf{L}_\lambda^{\dot{z}})$ has a natural $\mathbf{e}^s$-half-braiding (by 5.5) and $\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}})$ has a natural $\mathbf{e}^s$-half-braiding (by 4.14(j)). By 4.21(b),

(b) *these two $\mathbf{e}^s$-half-braidings are compatible with the identification (a).*

In view of (a), (b) we can reformulate 5.5(a) as follows.

Theorem 5.7. *For any $z\cdot\lambda\in\mathbf{c}^s$, $L'\in\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$, we have canonically*

(a) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}_\lambda^{\dot{z}},L')=\mathrm{Hom}_{\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}}(\overline{\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}})},L')$$

where $\overline{\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}})}$ is $\underline{\mathfrak{b}}(\mathbb{L}_{\lambda,s}^{\dot{z}})$ viewed as an object of $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$ with the $\mathbf{e}^s$-half-braiding given by 4.14(j).

5.8. We set

$$\mathbf{1}_0'=\oplus_{d\cdot\lambda\in\mathbf{D_c}}\beta_{d\cdot\lambda}^*\otimes\mathbb{L}_{\lambda,0}^{\dot{d}}\in\mathcal{C}^{\mathbf{c}}Z_0.$$

From the definitions we have $\tilde{\epsilon}_0\mathbf{1}_0'=\mathbf{1}'$. Since $\mathbf{1}'=\mathbf{1}$, we have also $\tilde{\epsilon}_0\mathbf{1}_0'=\mathbf{1}$. We show:

(a) *For $L\in\mathcal{C}^{\mathbf{c}}Z_{-s},L'\in\mathcal{C}^{\mathbf{c}}Z_s$ we have*

$$\mathrm{Hom}_{\mathcal{M}(Z_0)}(\mathbf{1}_0',L\underline{\bullet}L')=\mathrm{Hom}_{\mathcal{M}(Z_{-s})}(\mathfrak{D}(L'^{\dagger}),L).$$

We can assume that $L=\mathbb{L}^{\dot w}_{\lambda,-s}$, $L'=\mathbb{L}^{\dot w'}_{\lambda',s}$ where $w\cdot\lambda\in\mathbf{c}^{-s}$, $w'\cdot\lambda'\in\mathbf{c}^{s}$. Using the fully faithfulness of $\tilde\epsilon_0:\mathcal{M}(Z_0)\to\mathcal{M}\tilde{\mathcal{B}}^2$, $\tilde\epsilon_{-s}:\mathcal{M}(Z_{-s})\to\mathcal{M}\tilde{\mathcal{B}}^2$, and the equality $\tilde\epsilon_0\mathbf{1}'_0=\mathbf{1}$, we see that it is enough to prove that

$$\mathrm{Hom}_{\mathcal{M}(\tilde{\mathcal{B}}^2)}(\mathbf{1},\tilde\epsilon_0(L\underline{\bullet}L'))=\mathrm{Hom}_{\mathcal{M}\tilde{\mathcal{B}}^2}(\tilde\epsilon_{-s}(\mathfrak{D}(L'^{\dagger})),\tilde\epsilon_{-s}(L)).$$

From 4.3 we have $\tilde\epsilon_{-s}(L)=\mathbf{L}^{\dot w}_{\lambda}$, $\tilde\epsilon_{s}(L')=\mathbf{L}^{\dot w'}_{\lambda'}$, $\tilde\epsilon_{-s}(\mathbb{L}^{\dot w'^{-1}}_{w'(\lambda'),-s})=\mathbf{L}^{\dot w'^{-1}}_{w'(\lambda')}$.

From 4.3(e) we have

$$\tilde\epsilon_{-s}(\mathfrak{D}(L'^{\dagger}))=\tilde\epsilon_{-s}(\mathfrak{D}(\mathbb{L}^{\dot w'^{-1}}_{w'(\lambda'^{-1}),-s}))=\tilde\epsilon_{-s}(\mathbb{L}^{\dot w'^{-1}}_{w'(\lambda'),-s})=\mathbf{L}^{\dot w'^{-1}}_{w'(\lambda')}.$$

(We have use that $\mathfrak{D}(\mathbb{L}^{\dot w'^{-1}}_{w'(\lambda'^{-1}),-s})=\mathbb{L}^{\dot w'^{-1}}_{w'(\lambda'),-s}$ which follows from [21, 4.4(a)]) Using 4.24(b), we have

$$\tilde\epsilon_0(L\underline{\bullet}L')=(\tilde\epsilon_{-s}L)\underline{\circ}(\tilde\epsilon_sL')=\mathbf{L}^{\dot w}_{\lambda}\underline{\circ}\mathbf{L}^{\dot w'}_{\lambda'}.$$

Hence it is enough to prove

$$\mathrm{Hom}_{\mathcal{M}\tilde{\mathcal{B}}^2}(\mathbf{1},\mathbf{L}^{\dot w}_{\lambda}\underline{\circ}\mathbf{L}^{\dot w'}_{\lambda'})=\mathrm{Hom}_{\mathcal{M}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot w'^{-1}}_{w'(\lambda')},\mathbf{L}^{\dot w}_{\lambda}).$$

This follows from [21, 4.5(a)].

6. Truncated induction, truncated restriction, truncated convolution

6.1. *In this section we fix $s\in\mathbf{Z}$.*

Let $\dot Z_s=\{(B,B',\gamma)\in\mathcal{B}\times\mathcal{B}\times\tilde G_s;\gamma B\gamma^{-1}=B'\}$. We have a diagram

(a) $$Z_s\xleftarrow{f}\dot Z_s\xrightarrow{\pi}\tilde G_s$$

where $f(B,B',\gamma)=(B,B',\gamma U_B)$, $\pi(B,B',\gamma)=\gamma$. Note that G acts on Z_s by $g:(B,B',\gamma U_B)\mapsto(gBg^{-1},gB'g^{-1},g\gamma g^{-1}U_{gBg^{-1}})$, on $\dot Z_s$ by $g:(B,B',\gamma)\mapsto(gBg^{-1},gB'g^{-1},g\gamma g^{-1})$, on $\tilde G_s$ by $g:\gamma\mapsto g\gamma g^{-1}$; moreover, f and π are compatible with these G-actions. We define $\chi:\mathcal{D}(Z_s)\to\mathcal{D}(\tilde G_s)$ by

$$\chi(L)=\pi_!f^*L.$$

For any $w \cdot \lambda \in I$ we define $\mathfrak{R}^{\dot{w}}_{\lambda,s} \in \mathcal{D}(\tilde{G}_s)$, $R^{\dot{w}}_{\lambda,s} \in \mathcal{D}(\tilde{G}_s)$ by

$$\mathfrak{R}^{\dot{w}}_{\lambda,s} = \chi(\mathcal{L}^{\dot{w}}_{\lambda,s}), R^{\dot{w}}_{\lambda,s} = \chi(\mathcal{L}^{\dot{w}\sharp}_{\lambda,s}), \text{ if } w \cdot \lambda \in I^s,$$

$$\mathfrak{R}^{\dot{w}}_{\lambda} = 0, R^{\dot{w}}_{\lambda} = 0 \text{ if } w \cdot \lambda \notin I^s.$$

Assume now that $s \neq 0$ and that we are in case A. In this case, the conjugation G-action on $\tilde{G}_s$ is transitive, see 2.1, and the stabilizer of τ^s for this G-action is the finite group $G^{\mathbf{e}^s} = \{g \in G; \mathbf{e}^s(g) = g\}$.

With the notation of 4.1, for $w \in W$ we have isomorphisms

$$X^w_s \xrightarrow{\sim} \pi^{-1}(\tau^s) \cap f^{-1}(Z^w_s), \bar{X}^w_s \xrightarrow{\sim} \pi^{-1}(\tau^s) \cap f^{-1}(\bar{Z}^w_s)$$

given by $B \mapsto (B, \mathbf{e}^s(B), \tau^s)$. Using this, and the transitivity of the G-action on $\tilde{G}_s$, we see that for $w \cdot \lambda \in I^s$ and for $j \in \mathbf{Z}$, $(\mathfrak{R}^{\dot{w}}_{\lambda,s})^j[-\Delta]$ (resp. $(R^{\dot{w}}_{\lambda,s})^j[-\Delta]$) is the G-equivariant local system on $\tilde{G}_s$ whose stalk at τ^s is $H^{j-\Delta}_c(X^z_s, \mathcal{F}^{\dot{w}}_{\lambda,s})[\Delta]$ (resp. $IH^{j-\Delta}(\bar{X}^z_s, \mathcal{F}^{\dot{w}}_{\lambda,s})[\Delta]$) with the $G^{\mathbf{e}^s}$-action considered in 4.1.

We return to the general case. We say that a simple perverse sheaf A on $\tilde{G}_s$ is a *character sheaf* if the following equivalent conditions are satisfied:

(i) there exists $w \cdot \lambda \in I$ such that $(A : \oplus_j (\mathfrak{R}^{\dot{w}}_{\lambda,s})^j) \neq 0$;

(ii) there exists $w \cdot \lambda \in I$ such that $(A : (R^{\dot{w}}_{\lambda,s})^j) \neq 0$.

In case A with $s \neq 0$, if A satisfies either (i) or (ii), then it must be G-equivariant, hence $A[-D]$ must be a G-equivariant local system whose stalk at τ^s viewed as a $G^{\mathbf{e}^s}$-module is irreducible, so that in this case the equivalence of (i),(ii) follows from the equivalence of (i), (ii) in 4.1. In case A with $s = 0$ the equivalence of (i),(ii) follows from [11, 12.7]; a similar proof applies in case B (see also [14, 28.13]).

A character sheaf A determines a W-orbit $\mathfrak{o}$ on $\mathfrak{s}_\infty$: the set of $\lambda \in \mathfrak{s}_\infty$ such that $(A : \oplus_j (\mathfrak{R}^{\dot{w}}_{\lambda,s})^j) \neq 0$ for some $w \in W$ (or equivalently $(A : \oplus_j (R^{\dot{w}}_{\lambda,s})^j) \neq 0$ for some $w \in W$); we have necessarily $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$. In case A with $s \neq 0$ this follows from 4.1. In case A with $s = 0$ this follows from [11, 11.2(a), 12.7]; a similar proof applies in case B.

We now fix $\mathfrak{o} \in W\backslash\mathfrak{s}_\infty$ such that $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$. We say that A is an $\mathfrak{o}$-*character sheaf* if the W-orbit on $\mathfrak{s}_\infty$ determined by A is $\mathfrak{o}$. Let $CS_{\mathfrak{o},s}$ be a set of representatives for the isomorphism classes of $\mathfrak{o}$-character sheaves on $\tilde{G}_s$. In case A with $s \neq 0$ we have a natural bijection $CS_{\mathfrak{o},s} \leftrightarrow \mathrm{Irr}_{\mathfrak{o}}(G^{\mathbf{e}^s})$ (notation of 4.1); to $A \in CS_{\mathfrak{o},s}$ corresponds the stalk of the G-equivariant local system $A[-\Delta]$ at τ^s, viewed as an irreducible $G^{\mathbf{e}^s}$-module.

Let $\mathfrak{o} \in W\backslash\mathfrak{s}_\infty$ be such that $\mathbf{e}^s(\mathfrak{o}) = \mathfrak{o}$. With notation in 2.4 we have the following result.

(b) *There exists a pairing* $CS_{\mathfrak{o},s} \times \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}}) \to \bar{\mathbf{Q}}_l$, $(A, E) \mapsto b_{A,E}$ *such that for any* $A \in CS_{\mathfrak{o},s}$, *any* $z \cdot \lambda \in I$ *with* $\lambda \in \mathfrak{o}$ *and any* $j \in \mathbf{Z}$ *we have*

$$(A : (R^{\dot{z}}_{\lambda,s})^j) = (-1)^{j+\Delta}(j - \Delta - |z|; \sum_{E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})} b_{A,E}\mathrm{tr}(\mathbf{e}_s c_{z\cdot\lambda}, E^v)).$$

Assume first that $z \cdot \lambda \in I^s$. In case A with $s \neq 0$, (b) follows from 4.1(b). In case A with $s = 0$, (b) is a reformulation of [11, 14.11], see [21, 5.1]. In case B, (b) can be deduced from [15, 34.19] and the quasi-rationality result [16, 39.8]. (In *loc.cit.* there is the assumption that the adjoint group of G is simple, which was made to simplify the arguments.)

Next we assume tha $z \cdot \lambda \in I - I^s$. Then the left hand side of (a) is zero; hence it is enough to show that $\mathrm{tr}(\mathbf{e}_s c_{z\cdot\lambda}, E^v) = 0$ for any $E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})$. We have a direct sum decomposition $E^v = \oplus_{\lambda' \in \mathfrak{s}_\infty} 1_{\lambda'} E^v$. It is enough to show that for $\lambda' \in \mathfrak{s}_\infty$ we have $\mathbf{e}_s c_{z\cdot\lambda}(1_{\lambda'}E^v) \subset 1_{\lambda''}E^v$ where $\lambda'' \in \mathfrak{s}_\infty$, $\lambda'' \neq \lambda'$. We can assume that $\lambda' = \lambda$. We have

$$\mathbf{e}_s c_{z\cdot\lambda}(1_\lambda E^v) \subset \mathbf{e}_s(1_{z(\lambda)}E^v) = 1_{\mathbf{e}^s(z(\lambda))}E^v.$$

It is enough to show that $\mathbf{e}^s(z(\lambda)) \neq \lambda$ that is, $z(\lambda) \neq \mathbf{e}^{-s}(\lambda)$; this follows from $z \cdot \lambda \notin I^s$.

Given $A \in CS_{\mathfrak{o},s}$, there is a unique two-sided cell $\mathbf{c}_A$ of I such that $b_{A,E} = 0$ whenever $E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})$ satisfies $\mathbf{c}_E \neq \mathbf{c}_A$. In case A with $s \neq 0$ this follows from results in [6], under the assumption that the centre of G is connected; but the argument in [6] extends to the general case. In case A with $s = 0$ this follows from [11, 16.7]. In case B this follows from [17, §41]. We have necessarily $\mathbf{c}_A \subset I_{\mathfrak{o}}$. As in [17, 41.8], [18, 44.18], we see that:

(c) *We have* $(A:\oplus_j(R^{\dot z}_{\lambda,s})^j)\ne 0$ *for some* $z\cdot\lambda\in\mathbf{c}_A$*; conversely, if* $z\cdot\lambda\in I$ *is such that* $(A:\oplus_j(R^{\dot z}_{\lambda,s})^j)\ne 0$*, then* $\mathbf{c}_A\preceq z\cdot\lambda$.

Let a_A be the value of the a-function on $\mathbf{c}_A$. If $z\cdot\lambda\in I^s$, $E\in\mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})$ satisfy $\mathrm{tr}(\mathbf{e}_s c_{z\cdot\lambda},E^v)\ne 0$ then $\mathbf{c}_E\preceq z\cdot\lambda$; if in addition we have $z\cdot\lambda\in\mathbf{c}_E$ then from the definitions we have

$$\mathrm{tr}(\mathbf{e}_s c_{z\cdot\lambda},E^v)=\sum_{h\ge 0}c_{z\cdot\lambda,E,h,s}v^{a_E-h}$$

where $c_{z\cdot\lambda,E,h,s}\in\bar{\mathbf{Q}}_l$ is zero for large h, $c_{z\cdot\lambda,E,0,s}=\mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda},E^\infty)$ and a_E is as in 1.13. Hence from (b) we see that for $A\in CS_{\mathfrak{o},s}$ and $z\cdot\lambda\in I_{\mathfrak{o}}$, $j\in\mathbf{Z}$, the following holds:

(d) *We have* $(A:(R^{\dot z}_{\lambda,s})^j)=0$ *unless* $\mathbf{c}_A\preceq z\cdot\lambda$*; if* $z\cdot\lambda\in\mathbf{c}_A$*, then*

$$(A:(R^{\dot z}_{\lambda,s})^j)=(-1)^{j+\Delta}(j-\Delta-|z|;\sum_{E\in\mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}});\mathbf{c}_E=\mathbf{c}_A;h\ge 0}b_{A,E}c_{z\cdot\lambda,E,h,s}v^{a_A-h})$$

which is 0 *unless* $j-\Delta-|z|\le a_A$.

In the remainder of this section let $\mathbf{c},a,n,\Psi$ *be as in 3.1(a). We assume that* $w\cdot\lambda\in\mathbf{c}\implies\lambda\in\mathfrak{o}$.

Note that χ can be also viewed as a functor $\chi:\mathcal{D}_m(Z_s)\to\mathcal{D}_m(\tilde G_s)$.

Let $\mathcal{M}^{\preceq}\tilde G_s$ (resp. $\mathcal{M}^{\prec}\tilde G_s$) be the category of perverse sheaves on $\tilde G_s$ whose composition factors are all of the form $A\in CS_{\mathfrak{o},s}$ with $\mathbf{c}_A\preceq\mathbf{c}$ (resp. $\mathbf{c}_A\prec\mathbf{c}$). Let $\mathcal{D}^{\preceq}\tilde G_s$ (resp. $\mathcal{D}^{\prec}\tilde G_s$) be the subcategory of $\mathcal{D}(\tilde G_s)$ whose objects are complexes K such that K^j is in $\mathcal{M}^{\preceq}\tilde G_s$ (resp. $\mathcal{M}^{\prec}\tilde G_s$) for any j. Let $\mathcal{D}^{\preceq}_m\tilde G_s$ (resp. $\mathcal{D}^{\prec}_m\tilde G_s$) be the subcategory of $\mathcal{D}_m(\tilde G_s)$ whose objects are also in $\mathcal{D}^{\preceq}\tilde G_s$ (resp. $\mathcal{D}^{\prec}\tilde G_s$).

Let $z\cdot\lambda\in I_{\mathfrak{o}}$. From (d) we deduce:

(e) *If* $z\cdot\lambda\preceq\mathbf{c}$*, then* $(R^{\dot z}_{\lambda,s})^j\in\mathcal{M}^{\preceq}\tilde G_s$ *for all* $j\in\mathbf{Z}$.

(f) *If* $z\cdot\lambda\in\mathbf{c}$ *and* $j>a+\Delta+|z|$ *then* $(R^{\dot z}_{\lambda,s})^j\in\mathcal{M}^{\prec}\tilde G_s$.

(g) *If* $z\cdot\lambda\prec\mathbf{c}$ *then* $(R^{\dot z}_{\lambda,s})^j\in\mathcal{M}^{\prec}\tilde G_s$ *for all* $j\in\mathbf{Z}$.

6.2. Let $CS_{\mathbf{c},s}=\{A\in CS_{\mathfrak{o},s};\mathbf{c}_A=\mathbf{c}\}$. For any $z\cdot\lambda\in I$ we set

$$n_z=a(z)+\Delta+|z|.$$

Let $A \in CS_{\mathbf{c},s}$ and let $z \cdot \lambda \in \mathbf{c}$. We have

(a) $$(A : (R^{\dot{z}}_{\lambda,s})^{n_z}) = (-1)^{a+|z|} \sum_{E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})} b_{A,E}\mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty).$$

Indeed, from 6.1(b) we have

$$(A : (R^{\dot{z}}_{\lambda,s})^{n_z}) = (-1)^{a+|z|} \sum_{E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})} b_{A,E}(a; \mathrm{tr}(\mathbf{e}_s c_{z\cdot\lambda}, E^v))$$

and it remains to use that $(a; \mathrm{tr}(\mathbf{e}_s c_{z\cdot\lambda}, E^v)) = \mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty)$. We show:

(b) *For any* $A \in CS_{\mathbf{c},s}$ *there exists* $E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})$ *such that* $b_{A,E} \neq 0$ *hence* $\mathbf{c}_E = \mathbf{c}$.

Assume that this is not so. Then, using 6.1(b), for any $z \cdot \lambda \in I_{\mathfrak{o}}$ we have $(A : \oplus_j (R^{\dot{z}}_{\lambda,s})^j) = 0$. This contradicts the assumption that $A \in CS_{\mathfrak{o},s}$. We show:

(c) *For any* $A \in CS_{\mathbf{c},s}$ *there exists* $z \cdot \lambda \in \mathbf{c}$ *such that* $(A : (R^{\dot{z}}_{\lambda,s})^{n_z}) \neq 0$.

Assume that this is not so. Then, using (a), we see that

$$\sum_{E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}}); \mathbf{c}_E = \mathbf{c}} b_{A,E}\mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty) = 0$$

for any $z \cdot \lambda \in \mathbf{c}$. If $z \cdot \lambda \in I_{\mathfrak{o}} - \mathbf{c}$ then the last sum is automatically zero since $t_{z\cdot\lambda}$ acts as 0 on E^∞ for each E in the sum. Thus we have

$$\sum_{E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}}); \mathbf{c}_E = \mathbf{c}} b_{A,E}\mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty) = 0$$

for any $z \cdot \lambda \in I_{\mathfrak{o}}$. In the last sum the condition $\mathbf{c}_E = \mathbf{c}$ is automatically satisfied if $b_{A,E} \neq 0$. Thus we have

$$\sum_{E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})} b_{A,E}\mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty) = 0$$

for any $z \cdot \lambda \in I_{\mathfrak{o}}$. By a general argument (see for example [15, 34.14(e)]), the linear functions $t_{z\cdot\lambda} \mapsto \mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty)$, $\mathbf{J}_{\mathfrak{o}} \to \bar{\mathbf{Q}}_l$ (for various E as in the last sum) are linearly independent. It follows that $b_{A,E} = 0$ for each E as in the last sum. This contradicts (b).

We show:

(d) *Let* $z \cdot \lambda \in \mathbf{c}$ *be such that* $(R^{\dot{z}}_{\lambda,s})^{n_z} \neq 0$. *Then* $z \cdot \lambda \underset{\text{left}}{\sim} ee^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda))$ *and* $z \cdot \lambda \underset{\text{left}}{\sim} ee^s(z^{-1}) \cdot \lambda$.

Using (a) we see that there exists $E \in \mathrm{Irr}_s(\mathbf{H}^1_{\mathfrak{o}})$ such that $\mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty) \neq 0$. We have $E^\infty = \oplus_{d\cdot\lambda_1 \in \mathbf{D}\cap\mathfrak{o}} t_{d\cdot\lambda_1} E^\infty$. We define $d \cdot \lambda_1 \in \mathbf{D} \cap \mathfrak{o}$ by the condition that $z \cdot \lambda \underset{\text{left}}{\sim} d \cdot \lambda_1$. We define $d' \cdot \lambda'_1 \in \mathbf{D} \cap \mathfrak{o}$ by the condition that $z^{-1} \cdot z(\lambda) \underset{\text{left}}{\sim} d' \cdot \lambda'_1$. Now $t_{z\cdot\lambda} : E^\infty \to E^\infty$ maps the summand $t_{d\cdot\lambda_1} E^\infty$ into the summand $t_{d'\cdot\lambda'_1} E^\infty$ and all other summands to zero. Moreover, $\mathbf{e}_s$ maps $t_{d'\cdot\lambda'_1} E^\infty$ into $t_{\mathbf{e}^s(d')\cdot\mathbf{e}^s(\lambda'_1)} E^\infty$. Hence $\mathbf{e}_s t_{z\cdot\lambda} : E^\infty \to E^\infty$ maps the summand $t_{d\cdot\lambda_1} E^\infty$ into the summand $t_{\mathbf{e}^s(d')\cdot\mathbf{e}^s(\lambda'_1)} E^\infty$ and all other summands to zero. Since $\mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^\infty) \neq 0$ it follows that $t_{d\cdot\lambda_1} E^\infty = t_{\mathbf{e}^s(d')\cdot\mathbf{e}^s(\lambda'_1)} E^\infty \neq 0$. Since $\mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1) \in \mathbf{D} \cap \mathfrak{o}$, it follows that $d \cdot \lambda_1 = \mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1)$. Since $\mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda)) \underset{\text{left}}{\sim} \mathbf{e}^s(d') \cdot \mathbf{e}^s(\lambda'_1)$, we see that $z \cdot \lambda \underset{\text{left}}{\sim} \mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda))$. To complete the proof, it remains to note that $\mathbf{e}^s(z(\lambda)) = \lambda$ that is $z \cdot \lambda \in I^s$. This follows from the fact that $(R^{\dot{z}}_{\lambda,s})^{n_z} \neq 0$.

We show:

(e) *If* $CS_{\mathbf{c},s} \neq \emptyset$ *then* $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$.

Using (c) and the hypothesis we see that there exists $z \cdot \lambda \in \mathbf{c}$ such that $(R^{\dot{z}}_{\lambda,s})^{n_z} \neq 0$. Using (d), we see that $\mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda)) \in \mathbf{c}$. Since $z^{-1} \cdot z(\lambda) \in \mathbf{c}$ (see Q10 in 1.9) we have also $\mathbf{e}^s(z^{-1}) \cdot \mathbf{e}^s(z(\lambda)) \in \mathbf{e}^s(\mathbf{c})$. Thus, $\mathbf{c} \cap \mathbf{e}^s(\mathbf{c}) \neq \emptyset$. It follows that $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$.

6.3. *Until the end of* 6.7 *we assume that* $s \in \mathbf{Z}_{\mathbf{c}}$.

We show:

(a) *If* $L \in \mathcal{D}^{\preceq} Z_s$ *then* $\chi(L) \in \mathcal{D}^{\preceq} \tilde{G}_s$. *If* $L \in \mathcal{D}^{\prec} Z_s$ *then* $\chi(L) \in \mathcal{D}^{\prec} \tilde{G}_s$.

(b) *If* $L \in \mathcal{M}^{\preceq} Z_s$ *and* $j > a + \nu$ *then* $(\chi(L))^j \in \mathcal{M}^{\prec} \tilde{G}_s$.

It is enough to prove (a),(b) assuming in addition that $L = \mathbb{L}^{\dot{z}}_{\lambda,z}$ where $z \cdot \lambda \in I^s$, $z \cdot \lambda \preceq \mathbf{c}$. Then (a) follows from 6.1(e), (g). In the setup of (b) we have

$$(\chi(\mathbb{L}^{\dot{z}}_{\lambda,s}))^j = (R^{\dot{z}}_{\lambda})^{j+|z|+\nu+\rho}((|z| + \nu + \rho)/2)$$

and this is in $\mathcal{M}^{\prec} G$ since $j + |z| + \nu + \rho > a + \Delta + |z|$, see 6.1(f).

6.4. Let $\mathcal{C}^{\spadesuit}\tilde{G}_s$ be the subcategory of $\mathcal{M}(\tilde{G}_s)$ consisting of semisimple objects. Let $\mathcal{C}_0^{\spadesuit}\tilde{G}_s$ be the subcategory of $\mathcal{M}_m(\tilde{G}_s)$ consisting of objects of pure of weight zero. Let $\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ be the subcategory of $\mathcal{M}(\tilde{G}_s)$ consisting of objects which are direct sums of objects in $CS_{\mathbf{c},s}$. Let $\mathcal{C}_0^{\mathbf{c}}\tilde{G}_s$ be the subcategory of $\mathcal{C}_0^{\spadesuit}\tilde{G}_s$ consisting of those K such that, as an object of $\mathcal{C}^{\spadesuit}\tilde{G}_s$, K belongs to $\mathcal{C}^{\mathbf{c}}\tilde{G}_s$. For $K \in \mathcal{C}_0^{\spadesuit}\tilde{G}_s$ let $\underline{K}$ be the largest subobject of K such that as an object of $\mathcal{C}^{\spadesuit}\tilde{G}_s$, we have $\underline{K} \in \mathcal{C}^{\mathbf{c}}\tilde{G}_s$.

6.5. For $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$ we set

$$\underline{\chi}(L) = \underline{(\chi(L))^{a+\nu}}((a+\nu)/2) = \underline{(\chi(L))^{\{a+\nu\}}} \in \mathcal{C}_0^{\mathbf{c}}\tilde{G}_s.$$

(The last equality uses that π in 6.1 is proper hence it preserves purity.) The functor $\underline{\chi} : \mathcal{C}_0^{\mathbf{c}} Z_s \to \mathcal{C}_0^{\mathbf{c}}\tilde{G}_s$ is called *truncated induction*. For $z \cdot \lambda \in \mathbf{c}^s$ we have

(a) $$\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}) = \underline{(R^{\dot{z}}_{\lambda,s})^{n_z}}(n_z/2).$$

Indeed,

$$\begin{aligned}
\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}) &= \underline{(\chi(\mathbb{L}^{\dot{z}}_{\lambda,s}))^{a+\nu}}((a+\nu)/2) = \underline{(\chi(\mathcal{L}^{\dot{z}\sharp}_{\lambda,s}\langle |z|+\nu+\rho\rangle))^{a+\nu}}((a+\nu)/2) \\
&= \underline{(\chi(\mathcal{L}^{\dot{z}\sharp}_{\lambda,s}))^{|z|+a+\Delta}}((|z|+a+\Delta)/2) = \underline{(\chi(\mathcal{L}^{\dot{z}\sharp}_{\lambda,s}))^{n_z}}(n_z/2) \\
&= \underline{(R^{\dot{z}}_{\lambda,s})^{n_z}}(n_z/2).
\end{aligned}$$

Using (a) and 6.2(d) we see that:

(d) *If* $z \cdot \lambda \in \mathbf{c}^s$ *is such that* $\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}) \neq 0$ *then* $z \cdot \lambda \underset{\text{left}}{\sim} ee^s(z^{-1}) \cdot \lambda$.

6.6. For $z \cdot \lambda, z' \cdot \lambda'$ in $\mathbf{c}^s$ we show:

(a) $$\dim \operatorname{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}), \underline{\chi}(\mathbb{L}^{\dot{z}'}_{\lambda',s})) = \sum_{u \cdot \lambda_1 \in \mathbf{c}} \mathbf{t}(t_{u^{-1}\cdot u(\lambda_1)} t_{z\cdot\lambda} t_{\mathbf{e}^s(u)\cdot \mathbf{e}^s(\lambda_1)} t_{z'^{-1}\cdot z'(\lambda')})$$

where $\mathbf{t} : \mathbf{H}^\infty \to \mathbf{Z}$ is as in 1.9.

Let $()^{\spadesuit} : \bar{\mathbf{Q}}_l \to \bar{\mathbf{Q}}_l$ be a field automorphism which maps any root of 1 in $\bar{\mathbf{Q}}_l$ to its inverse. The field automorphism $\bar{\mathbf{Q}}_l(v) \to \bar{\mathbf{Q}}_l(v)$ which maps v to v and $x \in \bar{\mathbf{Q}}_l$ to $x^{\spadesuit}$ is denoted again by $^{\spadesuit}$.

Let N_1 (resp. N_2) be the left (resp. right) hand side of (a). Using 6.5(a) and the definitions we see that

(b) $$N_1 = \sum_{A \in CS_{\mathbf{c},s}} (A : (R_{\lambda,s}^{\dot{z}})^{n_z})(A : (R_{\lambda',s}^{\dot{z}'})^{n_{z'}}).$$

Using 6.2(a) and the analogous identity for $(A : (R_{\lambda',s}^{\dot{z}'})^{n_{z'}})$ in which the field automorphism $()^{\spadesuit} : \bar{\mathbf{Q}}_l \to \bar{\mathbf{Q}}_l$ is applied to both sides (the left hand side is fixed by $()^{\spadesuit}$), we deduce that

$$N_1 = (-1)^{|z|+|z'|} \sum_{E,E' \in \mathrm{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)} \sum_{A \in CS_{\mathbf{c},s}} b_{A,E} b_{A,E'}^{\spadesuit} \mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^{\infty}) \mathrm{tr}(\mathbf{e}_s t_{z'\cdot\lambda'}, E'^{\infty})^{\spadesuit}.$$

In the last sum we replace $\sum_{A \in CS_{\mathbf{c},s}} b_{A,E} b_{A,E'}^{\spadesuit}$ by 1 if $E' = E$ and by 0 if $E' \neq E$. (In case A with $s \neq 0$ we use [6, 3.9(i)] which assumes that the centre of G is connected, but a similar proof applies without assumption on the centre. In case A with $s = 0$ and in case B we use [15, 35.18(g)].)

We see that

$$N_1 = (-1)^{|z|+|z'|} \sum_{E \in \mathrm{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)} \mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^{\infty}) \mathrm{tr}(\mathbf{e}_s t_{z'\cdot\lambda'}, E^{\infty})^{\spadesuit}.$$

We now use the equality (for $E \in \mathrm{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)$):

$$\mathrm{tr}(\mathbf{e}_s t_{z'\cdot\lambda'}, E^{\infty})^{\spadesuit} = \mathrm{tr}(t_{z'^{-1}\cdot z'(\lambda')} \mathbf{e}_s^{-1}, E^{\infty})$$

which can be deduced from [15, 34.17]. We see that

$$N_1 = (-1)^{|z|+|z'|} \sum_{E \in \mathrm{Irr}_s(\mathbf{H}_{\mathfrak{o}}^1)} \mathrm{tr}(\mathbf{e}_s t_{z\cdot\lambda}, E^{\infty}) \mathrm{tr}(t_{z'^{-1}\cdot z'(\lambda')} \mathbf{e}_s^{-1}, E^{\infty}).$$

This is equal to $(-1)^{|z|+|z'|}$ times the trace of the linear map $\xi \mapsto t_{z\cdot\lambda} \mathbf{e}^s(\xi) t_{z'^{-1}\cdot z'(\lambda')}$ from $\mathbf{J}_{\mathfrak{o}}$ to $\mathbf{J}_{\mathfrak{o}}$; hence it is equal to

$$(-1)^{|z|+|z'|} \sum_{u\cdot\lambda_1 \in \mathfrak{o}} \mathbf{t}(t_{u^{-1}\cdot u(\lambda_1)} t_{z\cdot\lambda} t_{\mathbf{e}^s(u)\cdot\mathbf{e}^s(\lambda_1)} t_{z'^{-1}\cdot z'(\lambda')}) = (-1)^{|z|+|z'|} N_2.$$

(In the last sum, the terms with $u \cdot \lambda_1 \in \mathfrak{o} - \mathbf{c}$ contribute 0.) Thus, $N_1 = (-1)^{|z|+|z'|} N_2$. Since N_1 and N_2 are natural numbers it follows that $N_1 = N_2$. This proves (a).

The proof above shows also that $\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}), \underline{\chi}(\mathbb{L}^{\dot{z}'}_{\lambda',s})) = 0$ whenever $(-1)^{|z|+|z'|} = -1$.

Replacing in (a) $u \cdot \lambda_1$ by $\mathbf{e}^{-s}(y) \cdot \mathbf{e}^{-s}\lambda_1$ (recall that $\mathbf{e}^s : \mathbf{c} \to \mathbf{c}$ is a bijection) we can rewrite (a) as follows:

$$\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}), \underline{\chi}(\mathbb{L}^{\dot{z}'}_{\lambda',s})) = \sum_{y\cdot\lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\epsilon^{-s}(y^{-1})\cdot \mathbf{e}^{-s}(y(\lambda_1))} t_{z\cdot\lambda} t_{y\cdot\lambda_1} t_{z'^{-1}\cdot z'(\lambda')}).$$

Since N_1 (in the form (b)) is symmetric in $z \cdot \lambda, z' \cdot \lambda'$, we have also

$$\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}), \underline{\chi}(\mathbb{L}^{\dot{z}'}_{\lambda',s})) = \sum_{y\cdot\lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\epsilon^{-s}(y^{-1})\cdot \mathbf{e}^{-s}(y(\lambda_1))} t_{z'\cdot\lambda'} t_{y\cdot\lambda_1} t_{z^{-1}\cdot z(\lambda)}).$$

Replacing $y \cdot \lambda_1$ by $y^{-1} \cdot y(\lambda_1)$ (recall that $y \cdot \lambda_1 \mapsto y^{-1} \cdot y(\lambda_1)$ is an involution $\mathbf{c} \to \mathbf{c}$) we can rewrite this as follows:

(c)
$$\begin{aligned}&\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}), \underline{\chi}(\mathbb{L}^{\dot{z}'}_{\lambda',s}))\\ &= \sum_{y\cdot\lambda_1 \in \mathbf{c}} \mathbf{t}(t_{\epsilon^{-s}(y)\cdot \mathbf{e}^{-s}(\lambda_1)} t_{z'\cdot\lambda'} t_{y^{-1}\cdot y(\lambda_1)} t_{z^{-1}\cdot z(\lambda)}).\end{aligned}$$

We show:

(d) *There exist $z \cdot \lambda \in \mathbf{c}^s$ such that $\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}) \neq 0$.*

Let $k = u \cdot \lambda_1 \in \mathbf{c}$. Then $\mathbf{e}^s(k) \in \mathbf{c}$, $k^! \in \mathbf{c}$ hence by 1.15(d) we have $t_{k^!} t_j t_{\mathbf{e}^s(k)} \neq 0$ for some $j \in I$. From 2.5(a) we deduce that $j \in \mathbf{c}^s$. We can find $j' = z' \cdot \lambda' \in \mathbf{c}$ such that $t_{j'}$ appears with nonzero coefficient in $t_{k^!} t_j t_{\mathbf{e}^s(k)}$. It follows that $\mathbf{t}(t_{k^!} t_j t_{\mathbf{e}^s(k)} t_{j'^!}) \neq 0$. Since $\mathbf{t}(\xi\xi') = \mathbf{t}(\xi'\xi)$ for $\xi, \xi' \in \mathbf{H}^\infty$ we deduce that $\mathbf{t}(t_{\mathbf{e}^s(k)} t_{j'^!} t_{k^!} t_j) \neq 0$. In particular we have $t_{\mathbf{e}^s(k)} t_{j'^!} t_{k^!} \neq 0$. Applying the antiautomorphism $t_u \mapsto t_{u^!}$ of $\mathbf{H}^\infty$ we deduce $t_k t_{j'} t_{\mathbf{e}^s(k^!)} \neq 0$. Using again 2.5(a) we deduce that $j' \in \mathbf{c}^s$. If $i \in \mathbf{c}$, $j \in I$ satisfy $t_{i^!} t_j t_{\mathbf{e}^s(i)} \neq 0$ then $j \in \mathbf{c}^s$. Since $\mathbf{t}(t_{h^!} t_j t_{\mathbf{e}^s(h)} t_{j'^!}) \in \mathbf{N}$ for any $h \in \mathbf{c}$ and $\mathbf{t}(t_{k^!} t_j t_{\mathbf{e}^s(k)} t_{j'^!}) \neq 0$, we see that $\sum_{h\in\mathbf{c}} \mathbf{t}(t_{h^!} t_j t_{\mathbf{e}^s(h)} t_{j'^!}) \in \mathbf{N}_{>0}$. Using this and (a), we see that

$$\dim \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}), \underline{\chi}(\mathbb{L}^{\dot{z}'}_{\lambda',s})) \in \mathbf{N}_{>0}.$$

This proves (d).

The following converses to 6.2(e) is an immediate consequence of (d):

(e) *We have $CS_{\mathbf{c},s} \neq \emptyset$.*

6.7. Let $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$. We show that $\mathfrak{D}(L) \in \mathcal{C}_0^{\widetilde{\mathbf{c}}} Z_s$. (Here $\widetilde{\mathbf{c}}$ is as in 1.14.) It is enough to note that for $w \cdot \lambda \in \mathbf{c}^s$ and $\omega \in \kappa_0^{-1}(w)$ we have

(a) (a) $\mathfrak{D}(\mathbb{L}_{\lambda,s}^{\omega}) = \mathbb{L}_{\lambda^{-1},s}^{\omega}$.

We show:

(b) *For $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$ we have canonically $\underline{\chi}(\mathfrak{D}(L)) = \mathfrak{D}(\underline{\chi}(L))$ where the first $\underline{\chi}$ is relative to $\widetilde{\mathbf{c}}$ instead of* $\mathbf{c}$.

Let $\pi, f, \dot{Z}_s$ be as in 6.1. By the relative hard Lefschetz theorem [1, 5.4.10] applied to the projective morphism π and to $f^* L \langle \nu \rangle$ (a perverse sheaf of pure weight 0 on $\dot{Z}_s$) we have canonically for any $j \in \mathbf{Z}$:

(c)
$$(\pi_! f^* L \langle \nu \rangle)^{-j} = (\pi_! f^* L \langle \nu \rangle)^j (j).$$

We have used the fact that f is smooth with fibres of dimension ν. This also shows that

(d)
$$\mathfrak{D}(\chi(\mathfrak{D}(L))) = \chi(L) \langle 2\nu \rangle .$$

Using (d) we have

$$\begin{aligned} \mathfrak{D}(\underline{\chi}(\mathfrak{D}(L))) &= \mathfrak{D}((\chi(\mathfrak{D}(L)))^{a+\nu}((a+\nu)/2))) \\ &= (\mathfrak{D}(\chi(\mathfrak{D}(L))))^{-a-\nu}((-a-\nu)/2) \\ &= (\chi(L) \langle 2\nu \rangle)^{-a-\nu}((-a-\nu)/2) = (\chi(L) \langle \nu \rangle)^{-a}(-a/2). \end{aligned}$$

Hence using (c) we have

$$\mathfrak{D}(\underline{\chi}(\mathfrak{D}(L))) = (\chi(L) \langle \nu \rangle)^{a}(a/2) = (\chi(L))^{a+\nu}((a+\nu)/2) = \underline{\chi}(L).$$

This proves (b).

6.8. We define $\zeta : \mathcal{D}(\tilde{G}_s) \to \mathcal{D}(Z_s)$ and $\zeta : \mathcal{D}_m(\tilde{G}_s) \to \mathcal{D}_m(Z_s)$ by $\zeta(K) = f_! \pi^* K$ where $Z_s \xleftarrow{f} \dot{Z}_s \xleftarrow{\pi} \tilde{G}_s$ is as in 6.1(a). We show:

(a) *For any $L \in \mathcal{D}(Z_s)$ or $L \in \mathcal{D}_m(Z_s)$ we have $\mathfrak{b}''(L) = \zeta(\chi(L))$.*

We have $\zeta(\chi(L)) = f_! \pi^* \pi_! f^*(L)$. We have

$$\dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s = \{((B_0, B_1, B_2, B_3), \gamma) \in \mathcal{B}^4 \times \tilde{G}_s; \gamma B_0 \gamma^{-1} = B_3, \tilde{g} B_1 \tilde{g}^{-1} = B_2\}.$$

We have a cartesian diagram

$$\begin{array}{ccc} \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s & \xrightarrow{\tilde{\pi}_1} & \dot{Z}_s \\ {\scriptstyle \tilde{\pi}_2}\downarrow & & {\scriptstyle \pi}\downarrow \\ \dot{Z}_s & \xrightarrow{\pi} & \tilde{G}_s \end{array}$$

where $\tilde{\pi}_1((B_0, B_1, B_2, B_3), \gamma) = (B_0, B_3, \gamma)$, $\tilde{\pi}_2((B_0, B_1, B_2, B_3), \gamma) = (B_1, B_2, \gamma)$. It follows that $\pi^* \pi_! = \tilde{\pi}_{1!} \tilde{\pi}_2^*$. Thus,

$$\zeta(\chi(L)) = f_! \tilde{\pi}_{1!} \tilde{\pi}_2^* f^*(L) = (f \tilde{\pi}_1)_! (f \tilde{\pi}_2)^*(L).$$

Define $\pi'_1 : \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s \to Z_s$, $\pi'_2 : \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s \to Z_s$ by

$$\begin{aligned} \pi'_1((B_0, B_1, B_2, B_3), \gamma) &= (B_0, B_3, \gamma U_{B_0}), \\ \pi'_2((B_0, B_1, B_2, B_3), \gamma) &= (B_1, B_2, \gamma U_{B_1}). \end{aligned}$$

Then $\pi'_1 = f \tilde{\pi}_1$, $\pi'_2 = f \tilde{\pi}_2$ and $\zeta(\chi(L)) = \pi'_{1!} \pi'^*_2(L)$. Let ${}^\diamond\mathcal{Y}$ be as in 4.14. We have an isomorphism ${}^\diamond\mathcal{Y} \to \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s$ induced by

$$((x_0\mathbf{U}, x_1\mathbf{U}, x_2\mathbf{U}, x_3\mathbf{U}), \gamma) \mapsto ((x_0\mathbf{B}x_0^{-1}, x_1\mathbf{B}x_1^{-1}, x_2\mathbf{B}x_2^{-1}, x_3\mathbf{B}x_3^{-1}), \gamma).$$

We use this to identify ${}^\diamond\mathcal{Y} = \dot{Z}_s \times_{\tilde{G}_s} \dot{Z}_s$. Then π'_1, π'_2 become $d, {}^\diamond\eta$ of 4.25. We see that (a) holds.

6.9. *In the remainder of this section we assume that $s \in \mathbf{Z_c}$.*

Let $z \cdot \lambda \in \mathfrak{o}$. We set $\Sigma = \epsilon_s^* \zeta(R_{\lambda,s}^{\dot{z}}) \langle 2\nu + |z| \rangle \in \mathcal{D}(\tilde{\mathcal{B}}^2)$. Let $j \in \mathbf{Z}$. We show:

(a) *If $z \cdot \lambda \preceq \mathbf{c}$, then $\Sigma^j \in \mathcal{M}^{\preceq} \tilde{\mathcal{B}}^2$.*

(b) *If $z \cdot \lambda \prec \mathbf{c}$, then $\Sigma^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$.*

(c) *If $z \cdot \lambda \in \mathbf{c}$ and $j > \nu + 2\rho + 2a$, then $\Sigma^j \in \mathcal{M}^{\prec} \tilde{\mathcal{B}}^2$.*

If $z \cdot \lambda \notin I^s$, then $\Sigma = 0$ and there is nothing to prove. Now assume that

$z \cdot \lambda \in I^s$. Using 4.9(a), we have

$$\Sigma = \epsilon_s^* \zeta(\chi(\mathcal{L}_{\lambda,s}^{\dot{z}\sharp})) \langle 2\nu + |z| \rangle = \mathfrak{b}'(\mathcal{L}_{\lambda,s}^{\dot{z}\sharp}) \langle 2\nu + |z| \rangle = \mathfrak{b}'(\mathbb{L}_{\lambda,s}^{\dot{z}}) \langle \nu - \rho \rangle .$$

Now (a),(b) follow from 4.14(a),(b); (c) follows from 4.14(c). (If $j > \nu + 2\rho + 2a$, then $j + \nu - r > 2\nu + \rho + 2a$.)

6.10. We show:

(a) *If $K \in \mathcal{D}^{\preceq}\tilde{G}_s$, then $\zeta(K) \in \mathcal{D}^{\preceq} Z_s$.*

(b) *If $K \in \mathcal{D}^{\prec}\tilde{G}_s$, then $\zeta(K) \in \mathcal{D}^{\prec} Z_s$.*

(c) *If $K \in \mathcal{D}^{\preceq}\tilde{G}_s$ and $j > \nu + a$, then $(\zeta(K))^j \in \mathcal{M}^{\prec} Z_s$.*

We can assume in addition that $K = A \in CS_{\mathbf{c}',s}$ for a two-sided cell $\mathbf{c}'$ such that $\mathbf{c}' \preceq \mathbf{c}$. Assume first that $\mathbf{c}' = \mathbf{c}$. By 6.2(c) we can find $z \cdot \lambda \in \mathbf{c}$ such that $(A : (R_{\lambda,s}^{\dot{z}})^{n_z}) \neq 0$. Then $A[-n_z]$ (without mixed structure) is a direct summand of the semisimple complex $R_{\lambda,s}^{\dot{z}}$. Hence $\epsilon_s^* \zeta(A)[-n_z]$ is a direct summand of $\epsilon_s^* \zeta(R_{\lambda,s}^{\dot{z}})$ and $\epsilon_s^* \zeta(A)[-n_z + 2\nu + |z|]$ is a direct summand of Σ (in 6.9), that is, $\epsilon_s^* \zeta(A)[-a - \rho]$ is a direct summand of Σ. By 6.9, if $j \in \mathbf{Z}$ (resp. $j > \nu + 2\rho + 2a$) then $\Sigma^j \in \mathcal{M}^{\preceq}\tilde{\mathcal{B}}^2$ (resp. $\Sigma^j \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$) hence $(\epsilon_s^* \zeta(A)[-a - \rho])^j \in \mathcal{M}^{\preceq}\tilde{\mathcal{B}}^2$ (resp. $(\epsilon_s^* \zeta(A)[-a - \rho])^j \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$), that is, $(\epsilon_s^* \zeta(A))^{j-a-\rho} \in \mathcal{M}^{\preceq}\tilde{\mathcal{B}}^2$ (resp. $(\epsilon_s^* \zeta(A))^{j-a-\rho} \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$). We see that if $j' \in \mathbf{Z}$ (resp. $j' > \nu + \rho + a$) then $(\epsilon_s^* \zeta(A))^{j'} \in \mathcal{M}^{\preceq}\tilde{\mathcal{B}}^2$ (resp. $(\epsilon_s^* \zeta(A))^{j'} \in \mathcal{M}^{\prec}\tilde{\mathcal{B}}^2$), so that $(\zeta(A))^{j'-\rho} \in \mathcal{M}^{\preceq} Z_s$ (resp. $(\zeta(A))^{j'-\rho} \in \mathcal{M}^{\prec} Z_s$); here we use 4.3(a). We see that if $j \in \mathbf{Z}$ (resp. $j > \nu + a$, so that $j + \rho > \nu + \rho + a$), then $(\zeta(A))^j \in \mathcal{M}^{\preceq} Z_s$ (resp. $(\zeta(A))^j \in \mathcal{M}^{\prec} Z_s$). Thus the desired results hold when $\mathbf{c}' = \mathbf{c}$.

Assume now that $\mathbf{c}' \prec \mathbf{c}$. Applying the above argument with $\mathbf{c}$ replaced by $\mathbf{c}'$, we see that the desired results hold.

6.11. For $K \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_s$ we set

$$\underline{\zeta}(K) = \underline{(\zeta(K))^{\{\nu+a\}}} \in \mathcal{C}_0^{\mathbf{c}} Z_s.$$

We say that $\underline{\zeta}(K)$ is the *truncated restriction* of K.

6.12. Let $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$. We show:

(a) *We have canonically $\underline{\zeta}(\underline{\chi}(L)) = \underline{\mathfrak{b}''}(L)$.*

We shall apply the method of [19, 1.12] with $\Phi : \mathcal{D}_m(Y_1) \to \mathcal{D}_m(Y_2)$ replaced by $\zeta : \mathcal{D}_m(\tilde{G}_s) \to \mathcal{D}_m(Z_s)$ and with $\mathcal{D}^{\preceq}(Y_1)$, $\mathcal{D}^{\preceq}(Y_2)$ replaced by $\mathcal{D}^{\preceq}\tilde{G}_s$, $\mathcal{D}^{\preceq}Z_s$. We shall take $\mathbf{X}$ in *loc.cit.* equal to $\chi(L)$. The conditions of *loc.cit.* are satisfied: those concerning $\mathbf{X}$ are satisfied with $c' = a + \nu$, see 6.3. The conditions concerning ζ are satisfied with $c = a + \nu$, see 6.10. We see that

(b) $$(\zeta(\chi(L)))^j = 0 \text{ if } j > 2a + 2\nu$$

and

(c) $$\underline{gr_{2a+2\nu}((\zeta(\chi(L)))^{2a+2\nu}})(a + \nu) = \underline{\zeta}(\underline{\chi}(L)).$$

Since $\zeta(\chi(L)) = \mathfrak{b}''(L)$, we see that the left hand side of (c) equals $\underline{\mathfrak{b}''}(L)$. Thus (a) is proved.

Combining (a) with 4.25(d) and 4.14(d) we see that

(b) *we have canonically* $\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(L)) = \underline{\mathfrak{b}}(L)$.

6.13. Let $K \in \mathcal{D}(\tilde{G}_s)$ and let $L \in \mathcal{D}^{\spadesuit}\tilde{\mathcal{B}}^2$. Let $\tilde{L} = (\mathbf{e}^s)^*L$. In (a) below the assumption $s \in \mathbf{Z_c}$ is not used:

(a) *there is a canonical isomorphism* $\tilde{L} \circ \epsilon_s^*\zeta(K) \xrightarrow{\sim} \epsilon_s^*\zeta(K) \circ L$.

Let $Y = \tilde{\mathcal{B}}^2 \times \tilde{G}_s$. Define $j : Y \to \tilde{G}_s$ by $j(x_0\mathbf{U}, x_1\mathbf{U}, \gamma) = \gamma$. Define $j_1 : Y \to \tilde{\mathcal{B}}^2$ by $j_1(x_0\mathbf{U}, x_1\mathbf{U}, \gamma) = (x_0\mathbf{U}, \gamma^{-1}x_1\tau^s\mathbf{U})$. Define $j_2 : Y \to \tilde{\mathcal{B}}^2$ by $j_2(x_0\mathbf{U}, x_1\mathbf{U}, \gamma) = (\gamma x_0\tau^{-s}\mathbf{U}, x_1\mathbf{U})$. From the definitions we have $\tilde{L}\circ\epsilon_s^*\zeta(K) = j_{2!}(j_1^*(\tilde{L})\otimes j^*(K))$, $\epsilon_s^*\zeta(K)\circ L = j_{2!}(j_2^*(L)\otimes j^*(K))$. It remains to prove that $j_1^*(\tilde{L}) = j_2^*L$ that is, $j_1'^*L = j_2^*L$ where $j_1' = \mathbf{e}^s j_1 : Y \to \tilde{\mathcal{B}}^2$ is given by $j_1'(x_0\mathbf{U}, x_1\mathbf{U}, \gamma) = (\tau^s x_0\tau^{-s}\mathbf{U}, \tau^s\gamma^{-1}x_1\mathbf{U})$. The equality $j_1'^*L = j_2^*L$ follows from the G-equivariance of L. This proves (a).

Now let $K \in \mathcal{C}_0^{\mathbf{c}}\tilde{G}_s$ and let $L \in \mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2$. Since $\mathbf{e}^s(\mathbf{c}) = \mathbf{c}$, we have $(\mathbf{e}^s)^*L \in \mathcal{C}_0^{\mathbf{c}}\tilde{\mathcal{B}}^2$, see 3.11(a). We show that

(b) *there is a canonical isomorphism* $(\mathbf{e}^s)^*(L)\underline{\circ}\tilde{\epsilon}_s\underline{\zeta}(K) \xrightarrow{\sim} (\tilde{\epsilon}_s\underline{\zeta}(K))\underline{\circ}L$.

We apply the method of [19, 1.12] with $\Phi : \mathcal{D}_m^{\preceq}\tilde{\mathcal{B}}^2 \to \mathcal{D}_m^{\preceq}\tilde{\mathcal{B}}^2$, $L' \mapsto L' \circ L$, $\mathbf{X} = \tilde{\epsilon}_s\zeta(K)$ and with $(c, c') = (a - \nu, \nu + a)$, see [21, 2.23(a)], 6.10(c). We deduce that we have canonically

(c) $$\underline{(\underline{(\tilde{\epsilon}_s\zeta(K))^{\{a+\nu\}}} \circ L)^{\{a-\nu\}}} = \underline{(\tilde{\epsilon}_s\zeta(K) \circ L)^{\{2a\}}}.$$

We apply the method of [19, 1.12] with $\Phi : \mathcal{D}^{\preceq}_{m}\tilde{\mathcal{B}}^2 \to \mathcal{D}^{\preceq}_{m}\tilde{\mathcal{B}}^2$, $L' \mapsto (\mathbf{e}^s)^* L \circ L'$, $\mathbf{X} = \tilde{\epsilon}_s\zeta(K)$ and with $(c, c') = (a-\nu, \nu+a)$, see [21, 2.23(a)], 6.10(c). We deduce that we have canonically

(d) $$\underline{(((\mathbf{e}^s)^* L \circ \underline{\underline{(\tilde{\epsilon}\zeta(K))^{\{a+\nu\}}}})^{\{a-\nu\}}} = \underline{((\mathbf{e}^s)^* L \circ \tilde{\epsilon}\zeta(K))^{\{2a\}}}.$$

We now combine (c), (d) with (a); we obtain (b).

6.14. Let s', s'' be integers. Let $\mu : \tilde{G}_{s'} \times \tilde{G}_{s''} \to \tilde{G}_{s'+s''}$ be the multiplication map. For $K \in \mathcal{D}(\tilde{G}_{s'}), K' \in \mathcal{D}(\tilde{G}_{s''})$ (resp. $K \in \mathcal{D}_m(\tilde{G}_{s'}), K' \in \mathcal{D}_m(\tilde{G}_{s''})$) we set $K * K' = \mu_!(K \boxtimes K')$; this is in $\mathcal{D}(\tilde{G}_{s'+s''})$ (resp. in $\mathcal{D}_m(\tilde{G}_{s'+s''})$). For $K \in \mathcal{D}(\tilde{G}_{s_1}), K' \in \mathcal{D}(\tilde{G}_{s_2})$, $K'' \in \mathcal{D}(\tilde{G}_{s_3})$ we have canonically $(K * K') * K'' = K * (K' * K'')$ (and we denote this by $K * K' * K''$). For $K \in \mathcal{M}(\tilde{G}_{s'}), K' \in \mathcal{M}(\tilde{G}_{s''})$ we show:

(a) *If K' is G-equivariant then we have canonically $K * K' = ((\mathbf{e}^{-s'})^* K') * K'$. If K is G-equivariant then we have canonically $K * K' = K' * ((\mathbf{e}^{s''})^* K)$.*

The proof is immediate. It will be omitted. (Compare [19, 4.1].)

6.15. Let $s', s'' \in \mathbf{Z}$. We show:

(a) *For $K \in \mathcal{D}(\tilde{G}_{s'})$, $L \in \mathcal{D}(Z_{s''})$ we have canonically $K * \chi(L) = \chi(L \bullet \zeta(K))$.*

Let $Y = \tilde{G}_{s'} \times \tilde{G}_{s''} \times \mathcal{B}$. Define $c : Y \to \tilde{G}_{s'} \times Z_{s''}$ by

$$c(\gamma_1, \gamma_2, B) = (\gamma_1, (B, \gamma_2 B \gamma_2^{-1}, \gamma_2 U_B));$$

define $d : Y \to \tilde{G}_{s'+s''}$ by $d(\gamma_1, \gamma_2, B) = \gamma_1\gamma_2$. From the definitions we see that both $K * \chi(L)$, $\chi(L \bullet \zeta(K))$ can be identified with $d_! c^*(K \boxtimes L)$. This proves (a).

Now let $L \in \mathcal{D}(Z_{s'}), L' \in \mathcal{D}(Z_{s''})$. Replacing in (a) K, L by $\chi(L), L'$ and using 6.8(a), we obtain

(b) $$\chi(L) * \chi(L') = \chi(L' \bullet \mathfrak{b}''(L)).$$

6.16. Let $s' \in \mathbf{Z_c}$. Let $L \in \mathcal{D}^{\spadesuit}(Z_s)$, $L' \in \mathcal{D}^{\spadesuit}(Z_{s'})$, $j \in \mathbf{Z}$. We show:

(a) *If $L \in \mathcal{D}^{\preceq} Z_s$ or $L' \in \mathcal{D}^{\preceq} Z_{s'}$ then $L' \bullet \mathfrak{b}''(L) \in \mathcal{D}^{\preceq} Z_{s+s'}$.*

(b) *If* $L\in\mathcal{D}^{\prec}Z_s$ *or* $L'\in\mathcal{D}^{\prec}Z_{s'}$ *then* $L'\bullet\mathfrak{b}''(L)\in\mathcal{D}^{\prec}Z_{s+s'}$.

(c) *If* $L\in\mathcal{M}^{\preceq}Z_s$, $L'\in\mathcal{M}^{\spadesuit}Z_{s'}$ *and* $j>3a+\rho+\nu$ *then* $(L'\bullet\mathfrak{b}''(L))^j\in\mathcal{D}^{\prec}Z_{s+s'}$.

Now (a), (b) follow from 4.25(b) and 4.23(a). To prove (c) we may assume that $L=\mathbb{L}^{\dot w}_{\lambda,s}$, $L'=\mathbb{L}^{\dot w'}_{\lambda',s'}$ with $w\cdot\lambda\in I_n^s$, $w'\cdot\lambda'\in I_n^{s'}$ and $w\cdot\lambda\preceq\mathbf{c}$. We apply the method of [19, 1.12] with $\Phi:\mathcal{D}^{\preceq}Z_s\to\mathcal{D}^{\preceq}Z_{s+s'}$, $L_1\mapsto L'\bullet L_1$ and $\mathbf{X}=\mathfrak{b}''(L)$ and with $c'=2\nu+2a$ (see 4.25(c)), $c=a+\rho-\nu$ (see 4.23(b)). We have $c+c'=\nu+\rho+3a$ hence (c) holds.

6.17. Let $s'\in\mathbf{Z_c}$. Let $L\in\mathcal{D}^{\spadesuit}(Z_s)$, $L'\in\mathcal{D}^{\spadesuit}(Z_{s'})$, $j\in\mathbf{Z}$. We show:

(a) *If* $L\in\mathcal{D}^{\preceq}Z_s$ *or* $L'\in\mathcal{D}^{\preceq}Z_{s'}$ *then* $\chi(L'\bullet\mathfrak{b}''(L))\in\mathcal{D}^{\preceq}\tilde G_{s+s'}$.

(b) *If* $L\in\mathcal{D}^{\prec}Z_s$ *or* $L'\in\mathcal{D}^{\prec}Z_{s'}$ *then* $\chi(L'\bullet\mathfrak{b}''(L))\in\mathcal{D}^{\prec}\tilde G_{s+s'}$.

(c) *If* $L\in\mathcal{M}^{\preceq}Z_s$, $L'\in\mathcal{M}^{\spadesuit}Z_{s'}$ *and* $j>4a+2\nu+\rho$ *then* $(\chi(L'\bullet\mathfrak{b}''(L)))^j\in\mathcal{M}^{\prec}\tilde G_{s+s'}$.

(a), (b) follow from 6.3(a) using 6.16(a), (b). To prove (c) we can assume that $L=\mathbb{L}^{\dot w}_{\lambda,s}$, $L'=\mathbb{L}^{\dot w'}_{\lambda',s'}$ with $w\cdot\lambda\in I_n^s$, $w'\cdot\lambda'\in I_n^{s'}$ and $w\cdot\lambda\preceq\mathbf{c}$. We apply the method of [19, 1.12] with $\Phi:\mathcal{D}^{\preceq}Z_{s+s'}\to\mathcal{D}^{\preceq}\tilde G_{s+s'}$, $L_1\mapsto\chi(L_1)$, $\mathbf{X}=L'\bullet\mathfrak{b}''(L)$ and with $c'=\nu+\rho+3a$ (see 6.16(c)), $c=a+\nu$ (see 6.3(b)). We have $c+c'=2\nu+\rho+4a$ hence (c) holds.

6.18. Let $s'\in\mathbf{Z_c}$. Let $K\in\mathcal{D}^{\spadesuit}(\tilde G_s)$, $K'\in\mathcal{D}^{\spadesuit}(\tilde G_{s'})$. We show:

(a) *If* $K\in\mathcal{D}^{\preceq}\tilde G_s$ *or* $K'\in\mathcal{D}^{\preceq}\tilde G_{s'}$ *then* $K*K'\in\mathcal{D}^{\preceq}G_{s+s'}$.

(b) *If* $K\in\mathcal{D}^{\prec}\tilde G_s$ *or* $K'\in\mathcal{D}^{\prec}\tilde G_{s'}$ *then* $K*K'\in\mathcal{D}^{\prec}\tilde G_{s+s'}$.

(c) *If* $K\in\mathcal{D}^{\preceq}\tilde G_s$ *or* $K'\in\mathcal{D}^{\preceq}\tilde G_{s'}$ *and* $j>2a+\rho$ *then* $(K*K')^j\in\mathcal{D}^{\prec}\tilde G_{s+s'}$.

We can assume that $K=A\in CS_{\mathfrak{o},s}$, $K'=A'\in CS_{\mathfrak{o},s'}$. Let $A''\in\mathcal{M}(\tilde G_{s+s'})$ be a composition factor of $(A*A')^j$. By 6.2(c) we can find $w\cdot\lambda\in\mathbf{c}_A$, $w'\cdot\lambda'\in\mathbf{c}_{A'}$ such that $(A:(R^{\dot w}_{\lambda,s})^{n_w})\neq0$, $(A':(R^{\dot w'}_{\lambda',s'})^{n_{w'}})\neq0$. Then A is a direct summand of $R^{\dot w}_{\lambda,s}[n_w]$ and A' is a direct summand of $R^{\dot w'}_{\lambda',s'}[n_{w'}]$. Hence $A*A'$ is a direct summand of

$$R^{\dot w}_{\lambda,s}*R^{\dot w'}_{\lambda',s'}[a(w\cdot\lambda)+a(w'\cdot\lambda')+|w|+|w'|+2\Delta]$$

and $(A*A')^j$ is a direct summand of

$$(R^{\dot w}_{\lambda,s}*R^{\dot w'}_{\lambda',s'}[|w|+|w'|+2\nu+2\rho])^{j+a(w\cdot\lambda)+a(w'\cdot\lambda')+2\nu}$$

$$= (\chi(\mathbb{L}^{\dot w}_{\lambda,s}) * \chi(\mathbb{L}^{\dot w'}_{\lambda',s'}))^{j+a(w\cdot\lambda)+a(w'\cdot\lambda')+2\nu}.$$

Using 6.15(b) we see that $(A * A')^j$ is a direct summand of

(d) $$(\chi(\mathbb{L}^{\dot w'}_{\lambda',s'} \bullet \mathfrak{b}''(\mathbb{L}^{\dot w}_{\lambda,s}))^{j+a(w\cdot\lambda)+a(w'\cdot\lambda')+2\nu}.$$

Hence A'' is a composition factor of (d). Using 6.17(a) we see that $A'' \in CS_{\mathfrak{o},s+s'}$, that $\mathbf{c}_{A''} \preceq w\cdot\lambda$ and that $\mathbf{c}_{A''} \preceq w'\cdot\lambda'$. In the setup of (a) we have $w\cdot\lambda \preceq \mathbf{c}$ or $w'\cdot\lambda' \preceq \mathbf{c}$ hence $\mathbf{c}_{A''} \le \mathbf{c}$. Thus (a) holds. Similarly, (b) holds. In the setup of (c) we have $w\cdot\lambda \preceq \mathbf{c}$ and $w'\cdot\lambda' \preceq \mathbf{c}$. Hence $a(w\cdot\lambda) \ge a$, $a(w'\cdot\lambda') \ge a$. (See Q3 in 1.9.) Assume that $\mathbf{c}_{A''} = \mathbf{c}$. Since A'' is a composition factor of (d), we see from 6.17(c) that

$$j + a(w\cdot\lambda) + a(w'\cdot\lambda') + 2\nu \le 4a + 2\nu + \rho$$

hence $j + 2a + 2\nu \le 4a + 2\nu + \rho$ and $j \le 2a + \rho$. This proves (c).

6.19. Let $s' \in \mathbf{Z_c}$. For $K \in \mathcal{C}^{\mathbf{c}}_0\tilde G_s$, $K' \in \mathcal{C}^{\mathbf{c}}_0\tilde G_{s'}$, we set

$$K\underline{*}K' = \underline{(K * K')^{\{2a+\rho\}}} \in \mathcal{C}^{\mathbf{c}}_0\tilde G_{s+s'}.$$

We say that $K\underline{*}K'$ is the *truncated convolution* of K, K'. Note that 6.14(a) induces for $K, K' \in \mathcal{C}^{\mathbf{c}}_0 G$ a canonical isomorphism

(a) $$K\underline{*}K' = K'\underline{*}((\mathbf{e}^{s'})^* K).$$

Let $L \in \mathcal{C}^{\mathbf{c}}_0 Z_{s'}$, $K \in \mathcal{C}^{\mathbf{c}}_0\tilde G_s$. Using the method of [19, 1.2] several times, we see that

$$K\underline{*}\underline{\chi}(L) = \underline{gr_k((K * \chi(L))^k)}(k/2)$$

where $k = (a+\nu) + (2a+\rho) = 3a + \nu + \rho$ and

$$\underline{\chi}(L\underline{\bullet}\underline{\zeta}(K)) = \underline{gr_{k'}((\chi(L \bullet \zeta(K))^{k'})}(k'/2)$$

where $k' = (a+\nu) + (a+\nu) + (a+\rho-\nu) = 3a + \nu + \rho$. Using now 6.15(a) and the equality $k = k'$ we obtain

(b) $$K\underline{*}\underline{\chi}(L) = \underline{\chi}(L\underline{\bullet}\underline{\zeta}(K)).$$

Let $L \in \mathcal{C}_0^{\mathbf{c}} Z_s$, $L' \in \mathcal{C}_0^{\mathbf{c}} Z_{s'}$. Using the method of [19, 1.12] several times, we see that

$$\underline{\chi}(L)\underline{*}\underline{\chi}(L') = \underline{gr_k((\chi(L) * \chi(L'))^k)}(k/2)$$

where $k = (a+\nu) + (a+\nu) + (2a+\rho) = 4a + 2\nu + \rho$ and

$$\underline{\chi}(L'\underline{\bullet}\underline{\mathfrak{b}''}(L) = \underline{gr_{k'}((\chi(L' \bullet \mathfrak{b}''(L)))^{k'})}(k'/2)$$

where $k' = (2a+2\nu) + (a+\rho-\nu) + (a+\nu) = 4a + 2\nu + \rho$. Using now 6.15(b) and the equality $k = k'$ we obtain

(c) $$\underline{\chi}(L)\underline{*}\underline{\chi}(L') = \underline{\chi}(L'\underline{\bullet}(\underline{\mathfrak{b}''}(L))).$$

We show (assuming that $s_h \in \mathbf{Z_c}$ for $h = 1, 2, 3$):

(d) *For $K \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_{s_1}, K' \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_{s_2}, K'' \in \mathcal{C}_0^{\mathbf{c}} \tilde{G}_{s_3}$, there is a canonical isomorphism $(K\underline{*}K')\underline{*}K'' \xrightarrow{\sim} K\underline{*}(K'\underline{*}K'')$.*

Indeed, just as in [19, 4.7] we can identify, using the method of [19, 1.12], both $(K\underline{*}K')\underline{*}K''$ and $K\underline{*}(K'\underline{*}K'')$ with $\underline{(K * K' * K'')^{\{4a+2\rho\}}}$.

6.20. Let $s', s'' \in \mathbf{Z}$. For $K \in \mathcal{D}(\tilde{G}_{s'})$, $K' \in \mathcal{D}(\tilde{G}_{s''})$, we show:

(a) *We have canonically $\zeta(K * K') = \zeta(K') \bullet \zeta(K)$.*

Let

$$Y = \{(B, \gamma U_B, \gamma_1, \gamma_2); B \in \mathcal{B}, \gamma \in \tilde{G}_{s'+s''}, \gamma_1 \in \tilde{G}_{s'}, \gamma_2 \in \tilde{G}_{s''}; \gamma_1\gamma_2 \in \gamma U_B\}.$$

Define $j_1 : Y \to \tilde{G}_{s'}$, $j_2 : Y \to \tilde{G}_{s''}$ by $j_1(B, \gamma U_B, \gamma_1, \gamma_2) = \gamma_1$, $j_2(B, \gamma U_B, \gamma_1, \gamma_2) = \gamma_2$. Define $j : Y \to Z_{s'+s''}$ by $j(B, \gamma U_B, \gamma_1, \gamma_2) = (B, \gamma B \gamma^{-1}, \gamma U_B)$. From the definitions we have $\zeta(K * K') = j_!(j_1^*(K) \otimes j_2^*(K')) = \zeta(K') \bullet \zeta(K)$; (a) follows.

Let $s' \in \mathbf{Z_c}$. For $K \in \mathcal{D}_0^{\mathbf{c}}(G_s)$, $K' \in \mathcal{D}_0^{\mathbf{c}}(G_{s'})$, we show:

(b) *We have canonically $\underline{\zeta}(K\underline{*}K') = \underline{\zeta}(K')\underline{\bullet}\underline{\zeta}(K)$.*

Using the method of [19, 1.12] we see that

$$\underline{\zeta}(K\underline{*}K') = \underline{gr_k((\zeta(K * K'))^k)}(k/2)$$

where $k=(a+\nu)+(2a+\rho)=3a+\nu+\rho$ and that

$$\underline{\zeta}(K')\underline{\bullet}\underline{\zeta}(K)=\underline{gr_{k'}((\zeta(K)\bullet\zeta(K'))^{k'}})(k'/2)$$

where $k'=(a+\rho-\nu)+(a+\nu)+(a+\nu)=3a+\nu+\rho$. It remains to use (a) and the equality $k=k'$.

6.21. Let $s'\in\mathbf{Z}$. Define $h:\tilde{G}_{s'}\to\tilde{G}_{-s'}$ by $\gamma\mapsto\gamma^{-1}$. For $K\in\mathcal{D}(\tilde{G}_{-s'})$ we set $K^\dagger=h^*K\in\mathcal{D}(\tilde{G}_{s'})$. We show:

(a) *For $L\in\mathcal{D}(Z_{-s'})$ we have $(\chi(L))^\dagger=\chi(L^\dagger)$ with $L^\dagger$ as in 4.2.*

This follows from the definition of χ using the commutative diagram

$$\begin{array}{ccccc} Z_{s'} & \xleftarrow{f} & \dot{Z}_{s'} & \xrightarrow{\pi} & \tilde{G}_{s'} \\ \mathfrak{h}\downarrow & & \dot{\mathfrak{h}}\downarrow & & h\downarrow \\ Z_{-s'} & \xleftarrow{f} & \dot{Z}_{-s'} & \xrightarrow{\pi} & \tilde{G}_{-s'} \end{array}$$

where f,π are as in 6.1, $\mathfrak{h}$ is as in 4.2 and $\dot{\mathfrak{h}}:\dot{Z}_{s'}\to\dot{Z}_{-s'}$ is $(B,B',\gamma)\mapsto(B',B,\gamma^{-1})$.

From (a) and 4.3(e) we see that, if $w\cdot\lambda\in I_n^{-s}$, then

(b) $$(\chi(\mathbb{L}^{\dot{w}}_{\lambda,-s}))^\dagger=\chi(\mathbb{L}^{\dot{w}^{-1}}_{w(\lambda)^{-1},s}).$$

We deduce that

(c) *if $A\in CS_{\mathbf{c},-s}$, then $A^\dagger\in CS_{\tilde{\mathbf{c}},s}$.*

From (a), (c) we deduce:

(d) *For $L\in\mathcal{C}_0^{\mathbf{c}}Z_{-s}$ we have $(\underline{\chi}(L))^\dagger=\underline{\chi}(L^\dagger)$ where the second $\underline{\chi}$ is relative to $\tilde{\mathbf{c}},\mathfrak{o}^{-1}$ instead of $\mathbf{c},\mathfrak{o}$.*

7. Equivalence of $\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ with the $\mathbf{e}^s$-centre of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$

7.1. *In this section (except in 7.8) let $\mathbf{c},\mathfrak{o},a,n,\Psi$ be as in 3.1(a).*

In this subsection we assume that $s\in\mathbf{Z}_{\mathbf{c}}$. Let $u:\tilde{G}_{-s}\to\mathbf{p}$ be the obvious map; let $\phi:\mathbf{p}\to G$ be the map with image $\{1\}$. From [10, 7.4] we see that for K, K' in $\mathcal{M}_m\tilde{G}_{-s}$ we have canonically

$$(u_!(K \otimes K'))^0 = \mathrm{Hom}_{\mathcal{M}(\tilde{G}_{-s})}(\mathfrak{D}(K), K'), \quad (u_!(K \otimes K'))^j = 0 \text{ if } j > 0.$$

We deduce that if K, K' are also pure of weight 0 then $(u_!(K \otimes K'))^0$ is pure of weight 0 that is, $(u_!(K \otimes K'))^0 = gr_0(u_!(K \otimes K'))^0$. From the definitions we see that we have $u_!(K \otimes K') = \phi^*(K^\dagger * K')$ where $K^\dagger \in \mathcal{M}_m(\tilde{G}_s)$ is as in 6.21. Hence, for K' in $\mathcal{C}_0^{\mathbf{c}}\tilde{G}_{-s}$ and K in $\mathcal{C}_0^{\tilde{\mathbf{c}}}\tilde{G}_{-s}$ (so that $K^\dagger \in \mathcal{C}_0^{\mathbf{c}}\tilde{G}_s$, see 6.21(c)) we have

(a) $$\mathrm{Hom}_{\mathcal{M}(\tilde{G}_{-s})}(\mathfrak{D}(K), K') = (\phi^*(K^\dagger * K'))^0 = (\phi^*(K^\dagger * K'))^{\{0\}}.$$

Using [19, 8.2] with $\Phi : \mathcal{D}_m^{\preceq}\tilde{G}_0 \to \mathcal{D}_m\mathbf{p}$, $K_1 \mapsto \phi^* K_1$, $c = -2a - \rho$ (see [21, 6.8(a)]), K replaced by $K^\dagger * K' \in \mathcal{D}_m(\tilde{G}_0)$ and $c' = 2a + \rho$, we see that we have canonically

$$(\phi^*(K^\dagger \underline{*} K'))^{\{-2a-\rho\}} \subset (\phi^*(K^\dagger * K'))^{\{0\}}.$$

In particular, if $L \in \mathcal{C}_0^{\mathbf{c}} Z_{-s}$, $L' \in \mathcal{C}_0^{\mathbf{c}} Z_s$, then we have canonically

$$(\phi^*(\underline{\chi}(L') \underline{*} \underline{\chi}(L)))^{\{-2a-\rho\}} \subset (\phi^*(\underline{\chi}(L') * \underline{\chi}(L)))^{\{0\}}.$$

Using the equality

$$(\phi^*(\underline{\chi}(L') \underline{*} \underline{\chi}(L)))^{\{-2a-\rho\}} = \phi^*(\underline{\chi}(L \underline{\bullet} \underline{\zeta}(\underline{\chi}(L')))))^{-2a-\rho}$$

which comes from 6.19(b), we deduce that we have canonically

$$\phi^*(\underline{\chi}(L \underline{\bullet} \underline{\zeta}(\underline{\chi}(L')))))^{-2a-\rho} \subset (\phi^*(\underline{\chi}(L') * \underline{\chi}(L)))^{\{0\}},$$

or equivalently, using (a) with K, K' replaced by $\underline{\chi}(L')^\dagger$, $\underline{\chi}(L)$,

$$\begin{aligned}\phi^*(\underline{\chi}(L \underline{\bullet} \underline{\zeta}(\underline{\chi}(L'))))^{-2a-\rho} &\subset \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_{-s}}(\mathfrak{D}(\underline{\chi}(L')^\dagger), \underline{\chi}(L)) \\ &= \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L')).\end{aligned}$$

Using now [21, 6.9(d)] with L replaced by $L \underline{\bullet} \underline{\zeta}(\underline{\chi}(L')) \in \mathcal{C}_0^{\mathbf{c}} Z_0$, we have canonically

$$\phi^*(\underline{\chi}(L \underline{\bullet} \underline{\zeta}(\underline{\chi}(L'))))^{-2a-\rho} = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} Z_0}(\mathbf{1}_0', L \underline{\bullet} \underline{\zeta}(\underline{\chi}(L'))).$$

Thus we have canonically

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_0}(\mathbf{1}_0', L\underline{\bullet}\underline{\zeta}(\underline{\chi}(L'))) \subset \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L'))$$

or equivalently (using 5.8(a))

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_{-s}}(\mathfrak{D}(\underline{\zeta}(\underline{\chi}(L'))^\dagger), L) \subset \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L')).$$

Now we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_{-s}}(\mathfrak{D}(\underline{\zeta}(\underline{\chi}(L'))^\dagger), L) &= \mathrm{Hom}_{\mathcal{C}^{\tilde{\mathbf{c}}}Z_{-s}}(\mathfrak{D}(L), \underline{\zeta}(\underline{\chi}(L'))^\dagger) \\ &= \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_s}((\mathfrak{D}(L))^\dagger, \underline{\zeta}(\underline{\chi}(L'))),\end{aligned}$$

hence

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_s}((\mathfrak{D}(L))^\dagger, \underline{\zeta}(\underline{\chi}(L'))) \subset \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\mathfrak{D}(\underline{\chi}(L)^\dagger), \underline{\chi}(L')).$$

We set ${}^1L = \mathfrak{D}(L^\dagger) = (\mathfrak{D}(L))^\dagger \in \mathcal{C}_0^{\mathbf{c}}Z_s$ and note that

$$\mathfrak{D}(\underline{\chi}(L)^\dagger) = \mathfrak{D}(\underline{\chi}(L^\dagger)) = \underline{\chi}(\mathfrak{D}(L^\dagger)) = \underline{\chi}({}^1L),$$

see 6.21(d), 6.7(b). We obtain

(b) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_s}({}^1L, \underline{\zeta}(\underline{\chi}(L'))) \subset \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}({}^1L), \underline{\chi}(L'))$$

for any ${}^1L, L'$ in $\mathcal{C}_0^{\mathbf{c}}Z_s$. We show that (b) is an equality:

(c) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_s}({}^1L, \underline{\zeta}(\underline{\chi}(L'))) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}({}^1L), \underline{\chi}(L')).$$

Let N' (resp. N'') be the dimension of the left (resp. right) hand side of (b). It is enough to show that $N' = N''$. We can assume that ${}^1L = \mathbb{L}_{\lambda',s}^{\dot{z}'}$, $L' = \mathbb{L}_{\lambda,s}^{\dot{z}}$ where $z \cdot \lambda \in \mathbf{c}^s$, $z'' \cdot \lambda' \in \mathbf{c}^s$. By 6.12(a), N' is the multiplicity of ${}^1L$ in $\underline{\mathfrak{b}''}(L')$; by the fully faithfulness of $\tilde{\epsilon}_s$ this is the same as the multiplicity of $\tilde{\epsilon}_s{}^1L$ in $\tilde{\epsilon}_s\underline{\mathfrak{b}''}(L') = \underline{\mathfrak{b}'}(L') = \underline{\mathfrak{b}}(L')$ (the last two equalities use 4.25(d) and 4.14(d)). By 4.13(d), this is the same as the multiplicity of $\mathbf{L}_{\lambda'}^{\dot{z}'}$ in

$$\oplus_{y \in W; y \cdot \lambda \in \mathbf{c}} \mathbf{L}_{\mathbf{e}^{-s}(\lambda)}^{\mathbf{e}^{-s}(\dot{y})} \underline{\circ} \mathbf{L}_{\lambda}^{\dot{z}} \underline{\circ} \mathbf{L}_{y(\lambda)}^{\dot{y}^{-1}}.$$

Using now [21, 2.22(c)] we see that N' is the coefficient of $t_{z'\cdot\lambda'}$ in

$$\sum_{y\in W;y\cdot\lambda\in\mathbf{c}} t_{\mathbf{e}^{-s}(y)\cdot\mathbf{e}^{-s}(\lambda)}t_{z\cdot\lambda}t_{y^{-1}\cdot y(\lambda)} \in \mathbf{H}^\infty.$$

Hence if $\mathbf{t}:\mathbf{H}^\infty\to\mathbf{Z}$ is as in 1.9, then

$$N' = \sum_{y\in W;y\cdot\lambda\in\mathbf{c}} \mathbf{t}(t_{\mathbf{e}^{-s}(y)\cdot\mathbf{e}^{-s}(\lambda)}t_{z\cdot\lambda}t_{y^{-1}\cdot y(\lambda)}t_{z'^{-1}\cdot z'(\lambda')}).$$

This can be rewritten as

$$N' = \sum_{y\cdot\lambda_1\in\mathbf{c}} \mathbf{t}(t_{\mathbf{e}^{-s}(y)\cdot\mathbf{e}^{-s}(\lambda_1)}t_{z\cdot\lambda}t_{y^{-1}\cdot y(\lambda_1)}t_{z'^{-1}\cdot z'(\lambda')}).$$

(In the last sum, the terms corresponding to $y\cdot\lambda_1$ with $\lambda_1\neq\lambda$ are equal to zero.) By 6.6(c) (with $z\cdot\lambda, z'\cdot\lambda'$ interchanged) we have

$$N'' = \sum_{y\cdot\lambda_1\in\mathbf{c}} \mathbf{t}(t_{\epsilon^{-s}(y)\cdot\mathbf{e}^{-s}(\lambda_1)}t_{z\cdot\lambda}t_{y^{-1}\cdot y(\lambda_1)}t_{z'^{-1}\cdot z'(\lambda)}).$$

Thus, $N' = N''$. This completes the proof of (c).

7.2. Let $s,s'\in\mathbf{Z_c}$. We define a bifunctor $\mathcal{C}^{\mathbf{c}}\tilde{G}_s\times\mathcal{C}^{\mathbf{c}}\tilde{G}_{s'}\to\mathcal{C}^{\mathbf{c}}\tilde{G}_{s+s'}$ denoted by $K,K'\mapsto K\underline{*}K'$ as follows. By replacing if necessary Ψ in 7.1 by a power, we can assume that any $A\in CS_{\mathbf{c},s}$ and any $A\in CS_{\mathbf{c},s'}$ admits a mixed structure (defined in terms of Ψ) of pure weight zero. Let $K\in\mathcal{C}^{\mathbf{c}}\tilde{G}_s$, $K'\in\mathcal{C}^{\mathbf{c}}\tilde{G}_{s'}$; we choose mixed structures of pure weight 0 on K,K' with respect to Ψ (this is possible by our choice of Ψ). We define $K\underline{*}K'$ as in 6.19 in terms of these mixed structures and we then disregard the mixed structure on $K\underline{*}K'$. The resulting object of $\mathcal{C}^{\mathbf{c}}\tilde{G}_{s+s'}$ is denoted again by $K\underline{*}K'$; it is independent of the choice made.

In the same way the functor $\underline{\chi}:\mathcal{C}_0^{\mathbf{c}}Z_s\to\mathcal{C}_0^{\mathbf{c}}\tilde{G}_s$ gives rise to a functor $\mathcal{C}^{\mathbf{c}}Z_s\to\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ denoted again by $\underline{\chi}$; the functor $\underline{\zeta}:\mathcal{C}_0^{\mathbf{c}}\tilde{G}_s\to\mathcal{C}_0^{\mathbf{c}}Z_s$ gives rise to a functor $\mathcal{C}^{\mathbf{c}}\tilde{G}_s\to\mathcal{C}^{\mathbf{c}}Z_s$ denoted again by $\underline{\zeta}$.

The operation $K\underline{*}K'$ is again called truncated convolution. It has a canonical associativity isomorphism (deduced from that in 6.19(d)); this makes $\sqcup_{s\in\mathbf{Z_c}}\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ into a monoidal category.

From 6.20 we see that under $\underline{\zeta}:\sqcup_{s\in\mathbf{Z_c}}\mathcal{C}^{\mathbf{c}}\tilde{G}_s\to\sqcup_{s\in\mathbf{Z_c}}\mathcal{C}^{\mathbf{c}}Z_s$, the monoidal structure on $\sqcup_{s\in\mathbf{Z_c}}\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ is compatible with the opposite of the monoidal structure on $\sqcup_{s\in\mathbf{Z_c}}\mathcal{C}^{\mathbf{c}}Z_s$.

If $K\in\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ then the isomorphisms 6.13(b) provide an $\mathbf{e}^s$-half-braiding for $\tilde{\epsilon}_s\underline{\zeta}(K)\in\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ so that $\tilde{\epsilon}_s\underline{\zeta}(K)$ can be naturally viewed as an object of $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$ denoted by $\overline{\tilde{\epsilon}_s\underline{\zeta}(K)}$. (Note that 6.13(b) is stated in the mixed category but it implies the corresponding result in the unmixed category.) Then $K\mapsto\overline{\tilde{\epsilon}_s\underline{\zeta}(K)}$ is a functor $\mathcal{C}^{\mathbf{c}}\tilde{G}_s\to\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$.

Theorem 7.3. *Let $s\in\mathbf{Z_c}$. The functor $\mathcal{C}^{\mathbf{c}}\tilde{G}_s\to\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$, $K\mapsto\overline{\tilde{\epsilon}_s\underline{\zeta}(K)}$ is an equivalence of categories.*

From 6.12(a), 4.14(d), 4.25(d) we have canonically for any $z\cdot\lambda\in\mathbf{c}^s$:

(a) $$\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))=\underline{\mathfrak{b}}(\mathbb{L}^{\dot{z}}_{\lambda,s})$$

as objects of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$. From the definitions we see that the $\mathbf{e}^s$-half-braiding on the left hand side of (a) provided by 7.2 is the same as the $\mathbf{e}^s$-half-braiding on the right hand side of (a) provided by 4.14(j). Hence we have

(b) $$\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))}=\overline{\underline{\mathfrak{b}}(\mathbb{L}^{\dot{z}}_{\lambda,s})}$$

as objects of $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$. Using this and 5.7(a) with $L'=\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))$ (where $z\cdot\lambda, w\cdot\lambda'$ are in $\mathbf{c}^s$), we have

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{z}}_{\lambda},\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s})))=\mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}(\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))},\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))}).$$

Combining this with the equalities

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}),\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))&=\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_s}(\mathbb{L}^{\dot{z}}_{l,s},\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s})))\\&=\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2}(\mathbf{L}^{\dot{z}}_{l},\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))),\end{aligned}$$

of which the first comes from 6.10(c) and the second comes from the fully faithfulness of $\tilde{\epsilon}_s$, we obtain

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}),\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))=\mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}(\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))},\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))}).$$

In other words, setting

$$\mathbf{A}_{z\cdot\lambda,w\cdot\lambda'} = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}),\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s})),$$
$$\mathbf{A}'_{z\cdot\lambda,w\cdot\lambda'} = \mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}(\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))},\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))}),$$

we have

(c) $$\mathbf{A}_{z\cdot\lambda,w\cdot\lambda'} = \mathbf{A}'_{z\cdot\lambda,w\cdot\lambda'}.$$

Note that the identification (c) is induced by the functor $K \mapsto \overline{\tilde{\epsilon}_s\underline{\zeta}(K)}$. Let $\mathbf{A} = \oplus\mathbf{A}_{z\cdot\lambda,w\cdot\lambda'}$, $\mathbf{A}' = \oplus\mathbf{A}_{z\cdot\lambda,w\cdot\lambda'}$ (both direct sums are taken over all $z\cdot\lambda, w\cdot\lambda'$ in $\mathbf{c}^s$). Then from (c) we have $\mathbf{A} = \mathbf{A}'$. Note that this identification is compatible with the obvious algebra structures of $\mathbf{A}, \mathbf{A}'$.

For any $A \in CS_{\mathbf{c},s}$ we denote by $\mathbf{A}_A$ the set of all $f \in \mathbf{A}$ such that for any $z\cdot\lambda, w\cdot\lambda'$, the $(z\cdot\lambda, w\cdot\lambda')$-component of f maps the A-isotypic component of $\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s})$ to the A-isotypic component of $\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s})$ and any other isotypic component of $\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s})$ to 0. Thus, $\mathbf{A} = \oplus_{A\in CS_{\mathbf{c},s}}\mathbf{A}_A$ is the decomposition of $\mathbf{A}$ into a sum of simple algebras. (Each $\mathbf{A}_A$ is nonzero since, by 6.2(c) and 6.5(a), any A is a summand of some $\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s})$.)

Let $\mathfrak{S}$ be a set of representatives for the isomorphism classes of simple objects of $\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}$. For any $\sigma \in \mathfrak{S}$ we denote by $\mathbf{A}'_\sigma$ the set of all $f' \in \mathbf{A}'$ such that for any $z\cdot\lambda, w\cdot\lambda'$, the $(z\cdot\lambda, w\cdot\lambda')$-component of f' maps the σ-isotypic component of $\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))}$ to the σ-isotypic component of $\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{w}}_{\lambda',s}))}$ and all other isotypic components of $\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))}$ to zero. Then $\mathbf{A}' = \oplus_{\sigma\in\mathfrak{S}}\mathbf{A}'_\sigma$ is the decomposition of $\mathbf{A}'$ into a sum of simple algebras. (Each $\mathbf{A}'_\sigma$ is nonzero since any σ is a summand of some $\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))}$ with $z\cdot\lambda \in \mathbf{c}^s$. Indeed, we can find $z\cdot\lambda \in \mathbf{c}$ such that $\mathbf{L}^{\dot{z}}_\lambda$ is a direct summand of σ, viewed as an object of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$; then, by 5.5(a), σ is a summand of $\overline{\mathcal{I}_s(\mathbf{L}^{\dot{z}}_\lambda)}$. If in addition, $z\cdot\lambda \in \mathbf{c}^s$ then, by 5.6(a),(b), we have $\overline{\mathcal{I}_s(\mathbf{L}^{\dot{z}}_\lambda)} = \overline{\underline{\mathfrak{b}}(\mathbb{L}^{\dot{z}}_{\lambda,s})}$ hence σ is a summand of $\overline{\underline{\mathfrak{b}}(\mathbb{L}^{\dot{z}}_{\lambda,s})}$ hence, by (a), σ is a summand of $\overline{\tilde{\epsilon}_s\underline{\zeta}(\underline{\chi}(\mathbb{L}^{\dot{z}}_{\lambda,s}))}$, as required. If $z\cdot\lambda \notin \mathbf{c}^s$ then, by 5.5(b), we have $\mathcal{I}_s(\mathbf{L}^{\dot{z}}_\lambda) = 0$ which is a contradiction.) Since $\mathbf{A} = \mathbf{A}'$, from the uniqueness of decomposition of a semisimple algebra as a direct sum of simple algebras, we see that there is a unique bijection $CS_{\mathbf{c},s} \leftrightarrow \mathfrak{S}$, $A \leftrightarrow \sigma_A$ such that $\mathbf{A}_A = \mathbf{A}'_{\sigma_A}$ for any $A \in CS_{\mathbf{c},s}$. From the definitions we now see that for any $A \in CS_{\mathbf{c},s}$ we have $\overline{\tilde{\epsilon}_s\underline{\zeta}(K)} \cong \sigma_A$. Therefore, Theorem 7.3 holds.

Theorem 7.4. *We preserve the setup of Theorem 7.3. Let* $L \in \mathcal{C}^{\mathbf{c}} Z_s$, $K \in \mathcal{C}^{\mathbf{c}} \tilde{G}_s$. *We have canonically*

(a) $$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} Z_s}(L, \underline{\zeta}(K)) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(L), K).$$

We can assume that $L = \mathbb{L}^{\dot{z}}_{\lambda,s}$ where $z \cdot \lambda \in \mathbf{c}^s$. From 7.3 and its proof we see that

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(L), K) = \mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}(\overline{\tilde{\epsilon}_s \underline{\zeta}(\underline{\chi}(L))}, \overline{\tilde{\epsilon}_s \underline{\zeta}(K)}) = \mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}(\overline{\mathcal{I}_s(\mathbf{L}^{\dot{z}}_{\lambda})}, \overline{\tilde{\epsilon}_s \underline{\zeta}(K)}).$$

Using 5.5(a) we see that

$$\mathrm{Hom}_{\mathcal{Z}^{\mathbf{c}}_{\mathbf{e}^s}}(\overline{\mathcal{I}_s(\mathbf{L}^{\dot{z}}_{\lambda})}, \overline{\tilde{\epsilon}_s \underline{\zeta}(K)}) \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \mathcal{B}^2}(\mathbf{L}^{\dot{z}}_{\lambda}, \tilde{\epsilon}_s \underline{\zeta}(K)) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} Z_s}(L, \underline{\zeta}(K)).$$

This proves the theorem.

7.5. We preserve the setup of Theorem 7.3. We show that for $K \in \mathcal{C}^{\mathbf{c}} \tilde{G}_s$ we have canonically

(a) $$\mathfrak{D}(\underline{\zeta}(\mathfrak{D}(K)))) = \underline{\zeta}(K).$$

Here the first $\underline{\zeta}$ is relative to $\tilde{\mathbf{c}}$. It is enough to show that for any $L \in \mathcal{C}^{\mathbf{c}} Z_s$ we have canonically

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} Z_s}(L, \mathfrak{D}(\underline{\zeta}(\mathfrak{D}(K))))) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} Z_s}(L, \underline{\zeta}(K)).$$

Here the left side equals

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}^{\tilde{\mathbf{c}}} Z_s}(\underline{\zeta}(\mathfrak{D}(K)), \mathfrak{D}(L)) &= \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\mathfrak{D}(K), \underline{\chi}(\mathfrak{D}(L))) \\ &= \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\mathfrak{D}(K), \mathfrak{D}(\underline{\chi}(L))). \end{aligned}$$

(We have used 7.4(a) for $\tilde{\mathbf{c}}$ and 6.7(b).) The right hand side equals

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\underline{\chi}(L), K) = \mathrm{Hom}_{\mathcal{C}^{\mathbf{c}} \tilde{G}_s}(\mathfrak{D}(K), \mathfrak{D}(\underline{\chi}(L))).$$

(We have again used 7.4(a).) This proves (a).

Theorem 7.6. *Let $s \in \mathbf{Z_c}$. Let $K \in \mathcal{C}^{\mathbf{c}}\tilde{G}_s$. In $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ we have*

$$\tilde{\epsilon}_s \underline{\zeta}(K) \cong \oplus_{z\cdot\lambda \in \mathbf{c}^s; z\cdot\lambda \underset{\text{left}}{\sim} ee^s(z^{-1})\cdot\lambda} (\mathbf{L}_\lambda^{\dot{z}})^{\oplus N(z,\lambda)}$$

where $N(z,\lambda) \in \mathbf{N}$.

In $\mathcal{C}^{\mathbf{c}}Z_s$ we have

(a) $$\underline{\zeta}(K) \cong \oplus_{z\cdot\lambda \in \mathbf{c}^s} (\mathbb{L}_{\lambda,s}^{\dot{z}})^{\oplus N(z,\lambda)}$$

where $N(z,\lambda) \in \mathbf{N}$. If $N(z,\lambda) > 0$ then

$$\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}Z_s}(\mathbb{L}_{\lambda,s}^{\dot{z}}, \underline{\zeta}(K)) \neq 0$$

hence by 7.4 we have $\mathrm{Hom}_{\mathcal{C}^{\mathbf{c}}\tilde{G}_s}(\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}), K) \neq 0$ and in particular $\underline{\chi}(\mathbb{L}_{\lambda,s}^{\dot{z}}) \neq 0$. Using 6.5(d) we deduce that

(b) $z \cdot \lambda \underset{\text{left}}{\sim} e^s(z^{-1}) \cdot \lambda$.

Thus the direct sum in (a) can be restricted to $z\cdot\lambda$ satisfying (b). We now apply $\tilde{\epsilon}_s$ to both sides of (a) and use that $\tilde{\epsilon}_s \mathbb{L}_{\lambda,s}^{\dot{z}} = \mathbf{L}_\lambda^{\dot{z}}$. The theorem follows.

7.7. Let $s \in \mathbf{Z_c}$. From 7.3 and 7.6 we see that any object of $\mathcal{Z}_{\mathbf{e}^s}^{\mathbf{c}}$, when viewed as an object of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$, is a direct sum of objects of the form $\mathbf{L}_\lambda^{\dot{z}}$ with $z\cdot\lambda \in \mathbf{c}^s$ such that $z \cdot \lambda \underset{\text{left}}{\sim} e^s(z^{-1}) \cdot \lambda$.

In the remainder of this subsection we assume that $\tilde{G}$ is as in case A with G simple of type A_2 (resp. B_2 or G_2). In this case W is generated by σ_1, σ_2 in S with relation $(\sigma_1\sigma_2)^m = 1$ where $m = 3$ (resp. $m = 4$ or $m = 6$). We assume that $\mathbf{c}$ is the two-sided cell of I consisting of all $w \cdot 1$ where $w \in W$, $1 \le |w| \le m-1$. We shall write $\mathbf{L}^{iji\dots}$ instead of $\mathbf{L}_1^{\dot{\sigma}_i\dot{\sigma}_j\dot{\sigma}_i\dots}$ where $iji\dots$ is $121\dots$ or $212\dots$. The objects of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ of the form $\tilde{\epsilon}_s\underline{\zeta}(K)$ with K a simple object of $\mathcal{C}^{\mathbf{c}}\tilde{G}_s$ are (up to isomorphism) the following ones:

$\mathbf{L}^1 \oplus \mathbf{L}^2$ for type A_2;

$\mathbf{L}^1 \oplus \mathbf{L}^2, \mathbf{L}^1 \oplus \mathbf{L}^{212}, \mathbf{L}^2 \oplus \mathbf{L}^{121}, \mathbf{L}^{121} \oplus \mathbf{L}^{212}$ for type B_2;

$\mathbf{L}^1 \oplus \mathbf{L}^2, \mathbf{L}^1 \oplus \mathbf{L}^2 \oplus \mathbf{L}^{121} \oplus \mathbf{L}^{212}, \mathbf{L}^2 \oplus \mathbf{L}^{121} \oplus \mathbf{L}^{21212}$,
$\mathbf{L}^1 \oplus \mathbf{L}^{212} \oplus \mathbf{L}^{12121}, \mathbf{L}^{121} \oplus \mathbf{L}^{212} \oplus \mathbf{L}^{12121} \oplus \mathbf{L}^{21212}, \mathbf{L}^{121} \oplus \mathbf{L}^{212}$ for type G_2.

Note that in type G_2, $\mathbf{L}^{121} \oplus \mathbf{L}^{212}$ comes from two nonisomorphic objects K of $\mathcal{C}^{\mathbf{c}}\tilde{G}_s$.

7.8. In this subsection we assume that $\tilde{G}$ is as in case A with $G = SL_2(\mathbf{k})$ and $p \neq 2$. In this case we may identify $\mathbf{T} = \mathbf{k}^*$ and $W = \{1, \sigma\}$ with $\sigma(t) = t^{-1}$ for $t \in \mathbf{T}$. We take $\tau \in \tilde{G}_1$ such that $\mathbf{e} : G \to G$ in 2.3 satisfies $\mathbf{e}(t) = t^q$ for any $t \in T$. Then for $\lambda \in \mathfrak{s}_\infty \cong \mathbf{k}^*$ we have $\mathbf{e}(\lambda) = \lambda^{q^{-1}}$, $\sigma(\lambda) = \lambda^{-1}$. Let λ_0 be the unique element of $\mathfrak{s}_\infty$ such that $\lambda_0^2 = 1, \lambda_0 \neq 1$. In $\mathbf{H}$ we have $c_{1 \cdot \lambda} = T_1 1_\lambda$ for all λ, $c_{\sigma \cdot \lambda} = v^{-1} T_\sigma 1_\lambda$ if $\lambda \neq 1$, $c_{\sigma \cdot 1} = v^{-1} T_\sigma 1_1 + v^{-1} T_1 1_1$. It follows that the two-sided cells in $I = \{w \cdot \lambda; w \in W, \lambda \in \mathfrak{s}_\infty\}$ are the following subsets of I:

$\mathbf{c}_\lambda = \mathbf{c}_{\lambda^{-1}} = \{1 \cdot \lambda, 1 \cdot \lambda^{-1}, \sigma \cdot \lambda, \sigma \cdot \lambda^{-1}\}$ with $\lambda \in \mathfrak{s}_\infty; \lambda^2 \neq 1$;

$\mathbf{c}_{\lambda_0} = \{1 \cdot \lambda_0, \sigma \cdot \lambda_0\}$;

$\mathbf{c}'_1 = \{\sigma \cdot 1\}$;

$\mathbf{c}_1 = \{1 \cdot 1\}$.

Let $s \in \mathbf{Z}$. The two-sided cells of I which are stable under $\mathbf{e}^s$ are:

(i) $\mathbf{c}_\lambda = \mathbf{c}_{\lambda^{-1}}$ where $\lambda \in \mathfrak{s}_\infty$, $\lambda^2 \neq 1$, $\lambda^{q^{-s}} = \lambda$ (note that $\mathbf{e}^s$ acts as 1 on this two-sided cell);

(ii) $\mathbf{c}_\lambda = \mathbf{c}_{\lambda^{-1}}$ where $\lambda \in \mathfrak{s}_\infty$, $\lambda^2 \neq 1$, $\lambda^{q^{-s}} = \lambda^{-1}$ (note that $\mathbf{e}^s$ acts as a fixed point free involution on this two-sided cell and that we have necessarily $s \neq 0$);

(iii) $\mathbf{c}_{\lambda_0}$ (note that $\mathbf{e}^s$ acts as 1 on this two-sided cell);

(iv) $\mathbf{c}'_1$ (note that $\mathbf{e}^s$ acts as 1 on this two-sided cell);

(v) $\mathbf{c}_1$ (note that $\mathbf{e}^s$ acts as 1 on this two-sided cell).

For $\mathbf{c}$ in (i)-(v), the $\mathbf{e}^s$-centre of $\mathcal{C}^{\mathbf{c}}\tilde{\mathcal{B}}^2$ has exactly N simple objects (up to isomorphism) where $N = 1$ in the cases (i), (ii), (iv), (v) and $N = 4$ in the case (iii).

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