

EXACT WEAK LAWS AND ONE SIDED STRONG LAWS

ANDRÉ ADLER

Department of Applied Mathematics, Illinois Institute of Technology, Chicago, Illinois, 60616, USA.
E-mail: adler@iit.edu

Abstract

This paper explores Exact Weak Laws and their almost sure counterparts. First we show that the weighted sums of our infinite mean random variables can be balanced by a carefully selected sequence, i.e., that the ratio converges to one in probability. Then it is shown that the upper limit is almost surely infinity, while the lower limit is one.

1. Introduction

An exact limit theorem is where the numerator and denominator in some sense approaches one. That means that the sum of the random variables in the numerator are balanced appropriately by the constants in the denominator. The sum in the numerator can be considered the winnings of some game while the denominator would be the cumulative entrance fee after those n games have been played. The house and the gambler would only play this game if it was fair in some sense.

In this paper we extend the work done in [2] and [5]. Here we generalize the distribution used in both papers to $P\{X_j > x\} = \frac{\log^\alpha(x+j)}{x+j}$. We use the classic slowly varying function logarithm and since we want our random variables to have infinite expectations, we set $\alpha \geq -1$. Besides establishing the Weak Laws (Section 2) we also obtain the precise almost sure bounds (Section 3). As suggested in [2] and done in [5], it would be nice to extend this work for a more general sequence than just j . We would like to do both, we would like to obtain limit theorems when $P\{X_j > x\} = \frac{L(x+c_j)}{x+c_j}$

Received September 11, 2016.

AMS Subject Classification: 60F05, 60F15.

Key words and phrases: Almost sure convergence, weak law of large numbers, exact laws, one sided strong laws, slow variation.

for any sequence c_j and any slowly varying function $L(x)$. But that is very complicated, especially determining which random variables do not have a finite mean. So we will stay with $c_j = j$, but look at various powers of logarithm.

Since the random variables have $1 - F_{X_j}(x) = \frac{\log^\alpha(x+j)}{x+j}$ where $x > 0$ and $\alpha > -1$, we set $P\{X_j = 0\} = 1 - \frac{\log^\alpha(j)}{j}$ and by differentiating $F_{X_j}(x)$ we obtain

$$f_{X_j}(x) = \frac{\log^\alpha(x+j) - \alpha \log^{\alpha-1}(x+j)}{(x+j)^2}.$$

In terms of notation we use $\lg x = \log x$ with the base e , but whenever we have $\log 1$ in a denominator we will just set that equal to one, so we won't be dividing by zero. Hence $\lg x = \ln x$, except $\lg 1 = 1$. We set $\lg_2 x = \lg(\lg x)$, $\lg_3 x = \lg(\lg_2 x)$ and $\lg_4 x = \lg(\lg_3 x)$, so the i in $\lg_i x$ is not the base, it is the iteration of the logarithm. Many times in the proofs we will pull the logarithm out from an integral, since it is a slowly varying function, see [4], pages 279-284. Finally, note that the constant C will be used as a generic bound that is not necessarily the same in each appearance.

2. Exact Weak Laws

We first look at $\alpha > -1$, then we explore what happens when α equals minus one.

Theorem 1. *If $P\{X_j > x\} = \frac{\lg^\alpha(x+j)}{x+j}$ where $\alpha > -1$, then*

$$\frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg^{\alpha+1}(n) \lg_2(n)} \xrightarrow{P} 1.$$

Proof. We will establish all three parts of the Degenerate Convergence Criterion, see [3], page 356. Let $a_j = 1/j$ and $b_n = \lg^{\alpha+1}(n) \lg_2(n)$. Even though all the terms are quite similar, we will see that all of them go to zero except for one. And that one is only one of four in our truncated mean

$$\sum_{j=1}^n \frac{a_j}{b_n} EX_j I(a_j X_j \leq b_n)$$

$$\begin{aligned}
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_0^{b_n j} x \left[\frac{\lg^\alpha(x+j) - \alpha \lg^{\alpha-1}(x+j)}{(x+j)^2} \right] dx \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} (u-j) \left[\frac{\lg^\alpha(u) - \alpha \lg^{\alpha-1}(u)}{u^2} \right] du \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \left[\frac{\lg^\alpha(u)}{u} - \frac{j \lg^\alpha(u)}{u^2} - \frac{\alpha \lg^{\alpha-1}(u)}{u} + \frac{\alpha j \lg^{\alpha-1}(u)}{u^2} \right] du \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \frac{\lg^\alpha(u) du}{u} - \frac{1}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)j} \frac{\lg^\alpha(u) du}{u^2} \\
&\quad - \frac{\alpha}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \frac{\lg^{\alpha-1}(u) du}{u} + \frac{\alpha}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)j} \frac{\lg^{\alpha-1}(u) du}{u^2} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Letting $w = \lg u$, the first term becomes

$$\begin{aligned}
I_1 &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \frac{\lg^\alpha(u) du}{u} \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_{\lg(j)}^{\lg[(b_n+1)j]} w^\alpha dw \\
&= \frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{1}{j} [\lg^{\alpha+1}((b_n+1)j) - \lg^{\alpha+1}(j)] \\
&= \frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left[\left(\frac{\lg(b_n+1) + \lg(j)}{\lg(j)} \right)^{\alpha+1} - 1 \right] \\
&= \frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left[\left(1 + \frac{\lg(b_n+1)}{\lg(j)} \right)^{\alpha+1} - 1 \right].
\end{aligned}$$

The Taylor Series for $(1+x)^{\alpha+1}$ is

$$(1+x)^{\alpha+1} = 1 + (\alpha+1)x + \frac{(\alpha+1)\alpha x^2}{2} + \frac{(\alpha+1)\alpha(\alpha-1)x^3}{6} + \dots$$

which is the usual Binomial Theorem if $\alpha+1$ is an integer, meaning it is a finite sum. Otherwise, it becomes an alternating series and the error is less than the ensuing term, which is just as good, since $x = \lg(b_n+1)/\lg(j)$

is small as long as j exceeds some value say J_0 . Thus the main term is essentially

$$\begin{aligned} & \frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left[\frac{(\alpha+1) \lg(b_n+1)}{\lg(j)} \right] \\ & \sim \frac{\lg(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \\ & \sim \left(\frac{\lg(b_n)}{b_n} \right) \left(\frac{\lg^{\alpha+1}(n)}{\alpha+1} \right) \\ & \sim \left(\frac{(\alpha+1) \lg_2(n)}{\lg^{\alpha+1}(n) \lg_2(n)} \right) \left(\frac{\lg^{\alpha+1}(n)}{\alpha+1} \right) = 1. \end{aligned}$$

As for any of the higher powers in this expansion, say $p \geq 2$, they are bounded above by

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left(\frac{\lg(b_n+1)}{\lg(j)} \right)^p \leq \frac{C \lg^p(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-p+1}(j)}{j}.$$

And by examining the the three cases we get, if $\alpha - p + 1 < -1$

$$\frac{C \lg^p(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-p+1}(j)}{j} \leq \frac{C \lg^p(b_n)}{b_n} \rightarrow 0$$

and if $\alpha - p + 1 = -1$, it is bounded above by

$$\begin{aligned} \frac{C \lg^p(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-p+1}(j)}{j} & \leq \frac{C \lg^p(b_n) \lg_2(n)}{b_n} \leq \frac{C \lg_2^{p+1}(n)}{\lg^{\alpha+1}(n) \lg_2(n)} \\ & = \frac{C \lg_2^p(n)}{\lg^{p-1}(n)} \rightarrow 0 \end{aligned}$$

and finally if $\alpha - p + 1 > -1$, then this term is bounded above by

$$\begin{aligned} & \frac{C \lg^p(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-p+1}(j)}{j} \\ & \leq \frac{C \lg^p(b_n) \lg^{\alpha-p+2}(n)}{b_n} \leq \frac{C \lg_2^p(n) \lg^{\alpha-p+2}(n)}{\lg^{\alpha+1}(n) \lg_2(n)} = \frac{C \lg_2^{p-1}(n)}{\lg^{p-1}(n)} \rightarrow 0. \end{aligned}$$

This shows how delicate the selection of b_n is. Now we need to examine the

other three terms in our truncated mean. Those and the other two terms in the Degenerate Convergence Theorem all go to zero. The second term in the truncated mean, by using slow variation, is bounded above by

$$\begin{aligned} |I_2| &= \left| \frac{-1}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)^j} \frac{\lg^\alpha(u) du}{u^2} \right| \leq \frac{1}{b_n} \sum_{j=1}^n \int_j^\infty \frac{\lg^\alpha(u) du}{u^2} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \leq \frac{C \lg^{\alpha+1}(n)}{b_n} = \frac{C}{\lg_2(n)} \rightarrow 0. \end{aligned}$$

The third term is similar to the first, but with one less power of logarithm, it is bounded above by

$$\begin{aligned} |I_3| &= \left| \frac{-\alpha}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)^j} \frac{\lg^{\alpha-1}(u) du}{u} \right| \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)^j} \frac{\lg^{\alpha-1}(u) du}{u} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} \int_{\lg(j)}^{\lg[(b_n+1)^j]} w^{\alpha-1} dw \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} [\lg^\alpha((b_n+1)j) - \lg^\alpha(j)] \\ &= \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left[\left(\frac{\lg(b_n+1) + \lg(j)}{\lg(j)} \right)^\alpha - 1 \right] \\ &= \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left[\left(1 + \frac{\lg(b_n+1)}{\lg(j)} \right)^\alpha - 1 \right]. \end{aligned}$$

Using the Binomial expansion once again, we have for the first term

$$\begin{aligned} \frac{C}{b_n} \sum_{j=1}^n \left(\frac{\lg^\alpha(j)}{j} \right) \left(\frac{\alpha \lg(b_n+1)}{\lg(j)} \right) &\leq \frac{C \lg(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-1}(j)}{j} \\ &\leq \frac{C \lg(b_n) \lg^\alpha(n)}{b_n} \leq \frac{C}{\lg n} \rightarrow 0. \end{aligned}$$

Likewise, the ensuing powers in the expansion make those terms go to zero

as well. Finally, the fourth term in our mean is

$$\begin{aligned} I_4 &= \frac{\alpha}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)j} \frac{\lg^{\alpha-1}(u) du}{u^2} \leq \frac{C}{b_n} \sum_{j=1}^n \int_j^\infty \frac{\lg^{\alpha-1}(u) du}{u^2} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-1}(j)}{j} \leq \frac{C \lg^\alpha(n)}{b_n} = \frac{C}{\lg(n) \lg_2(n)} \rightarrow 0. \end{aligned}$$

Note that, we just assumed that $\alpha > 0$, but if $-1 < \alpha \leq 0$, these terms also vanishes as n goes to infinity.

Next we look at the second moment term in the Degenerate Convergence Theorem

$$\begin{aligned} &\frac{1}{b_n^2} \sum_{j=1}^n a_j^2 EX_j^2 I(a_j X_j \leq b_n) \\ &= \frac{1}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_0^{b_n j} x^2 \left[\frac{\lg^\alpha(x+j) - \alpha \lg^{\alpha-1}(x+j)}{(x+j)^2} \right] dx \\ &\leq \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_0^{b_n j} \frac{x^2 \lg^\alpha(x+j) dx}{(x+j)^2} \\ &\leq \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_0^{b_n j} \lg^\alpha(x+j) dx \\ &= \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_j^{(b_n+1)j} \lg^\alpha(u) du. \end{aligned}$$

If $\alpha = 0$, which was already established in [2], then this term is

$$\frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} [(b_n+1)j - j] = \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} \leq \frac{C}{\lg_2(n)} \rightarrow 0.$$

And if $-1 < \alpha < 0$, then this term is

$$\frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_j^{(b_n+1)j} \lg^\alpha(u) du \leq \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \leq \frac{C \lg^{\alpha+1}(n)}{b_n} = \frac{C}{\lg_2(n)} \rightarrow 0.$$

Finally, if $\alpha > 0$, then this term is

$$\begin{aligned}
 \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_j^{(b_n+1)j} \lg^\alpha(u) du &\leq \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \lg^\alpha((b_n+1)j) \int_j^{(b_n+1)j} du \\
 &= \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \lg^\alpha((b_n+1)j) [(b_n+1)j - j] \\
 &= \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \lg^\alpha((b_n+1)j) [b_n j] \\
 &= \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} \lg^\alpha((b_n+1)j) \\
 &= \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} \left(\lg(b_n+1) + \lg j \right)^\alpha \\
 &= \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left(\frac{\lg(b_n+1)}{\lg(j)} + 1 \right)^\alpha.
 \end{aligned}$$

Once again using the expansion $(1+x)^\alpha = 1 + \alpha x + \alpha(\alpha-1)x^2/2 \dots$ which is either finite or eventually an alternating series, we see that this limit is zero. The first term is

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} (1) \leq \frac{C \lg^{\alpha+1}(n)}{b_n} = \frac{C}{\lg_2(n)} \rightarrow 0.$$

The next term is αx , where $x = \lg(b_n+1)/\lg(j)$, which is small for most values of j . We could split the sum into two parts, where one sum is $\sum_{j=1}^{J_0}$ and the second sum is $\sum_{j=J_0+1}^n$ where $x < 1$, when j exceeds J_0 . The first sum would go to zero as n went to infinity since $b_n \rightarrow \infty$. Thus the second term is in our series expansion of $(1+x)^\alpha$ is

$$\begin{aligned}
 \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left(\frac{\alpha \lg(b_n+1)}{\lg(j)} \right) &\leq \frac{C \lg(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-1}(j)}{j} \\
 &\leq \frac{C \lg_2(n) \lg^\alpha(n)}{\lg^{\alpha+1}(n) \lg_2(n)} = \frac{C}{\lg n} \rightarrow 0.
 \end{aligned}$$

Now the rest of the binomial series has powers x^p for $p \geq 2$. Either this series terminates or eventually alternates. In either case for the term x^p , we

have

$$\begin{aligned} \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left(\frac{\lg(b_n + 1)}{\lg(j)} \right)^p &\leq \frac{C \lg^p(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-p}(j)}{j} \\ &\leq \frac{C \lg_2^p(n) \lg^{\alpha-p+1}(n)}{\lg^{\alpha+1}(n) \lg_2(n)} = \frac{C \lg_2^{p-1}(n)}{\lg^p(n)} \rightarrow 0 \end{aligned}$$

where we are assuming that $\alpha - p + 1 > 0$, but we easily show that all of these terms go to zero in every setting. Thus the variance term does vanish as n goes to infinity.

The last limit in the Degenerate Convergence Theorem are the tail probabilities, but that limit is zero just like the last one, since for all $\epsilon > 0$

$$\begin{aligned} \sum_{j=1}^n P\{X_j > \epsilon b_n j\} &= \sum_{j=1}^n \frac{(\lg(\epsilon b_n j + j))^\alpha}{\epsilon b_n j + j} \\ &= \frac{1}{\epsilon b_n + 1} \sum_{j=1}^n \frac{(\lg(\epsilon b_n + 1) + \lg(j))^\alpha}{j} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{(\lg(\epsilon b_n + 1) + \lg(j))^\alpha}{j}. \end{aligned}$$

If $-1 < \alpha \leq 0$, then this is bounded above by

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \leq \frac{C \lg^{\alpha+1}(n)}{b_n} = \frac{C}{\lg_2(n)} \rightarrow 0.$$

However, if $\alpha > 0$, this term becomes

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left(\frac{\lg(\epsilon b_n + 1)}{\lg(j)} + 1 \right)^\alpha.$$

Again using $(1 + x)^\alpha = 1 + \alpha x + \alpha(\alpha - 1)x^2/2 \cdots$, which is either a finite sum or an alternating series, we have for the first term

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} (1) \leq \frac{C \lg^{\alpha+1}(n)}{b_n} = \frac{C}{\lg_2(n)} \rightarrow 0.$$

Next, observing the αx term, which is

$$\begin{aligned} \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left(\frac{\alpha \lg(\epsilon b_n + 1)}{\lg(j)} \right) &\leq \frac{C \lg(\epsilon b_n + 1)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-1}(j)}{j} \\ &\leq \frac{C \lg_2(n) \lg^\alpha(n)}{\lg^{\alpha+1}(n) \lg_2(n)} = \frac{C}{\lg n} \rightarrow 0. \end{aligned}$$

The value of $x = \lg(\epsilon b_n + 1)/\lg(j)$, is small for most values of j in our series expansion of $(1+x)^\alpha$. We can make this rigorous by splitting the sum into two parts, where one sum is $\sum_{j=1}^{J_0}$ and the second sum is $\sum_{j=J_0+1}^n$ where $x < 1$, when j exceeds J_0 , but that isn't necessary. This series either terminates, when α is an integer or it eventually alternates, so looking at the x^p term with p an integer greater than one, we have

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j)}{j} \left(\frac{\lg(\epsilon b_n + 1)}{\lg(j)} \right)^p \leq \frac{C \lg^p(b_n)}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-p}(j)}{j}.$$

If $\alpha - p < -1$, this is bounded above by

$$\frac{C \lg^p(b_n)}{b_n} \leq \frac{C \lg_2^p(n)}{\lg^{\alpha+1}(n) \lg_2(n)} = \frac{C \lg_2^{p-1}(n)}{\lg^{\alpha+1}(n)} \rightarrow 0.$$

If $\alpha - p = -1$, this is bounded above by

$$\frac{C \lg_2^{p+1}(n)}{\lg^{\alpha+1}(n) \lg_2(n)} = \frac{C \lg_2^p(n)}{\lg^p(n)} \rightarrow 0.$$

Finally, if $\alpha - p$ exceeds minus one, an upper bound is

$$\frac{C \lg_2^p(n) \lg^{\alpha-p+1}(n)}{\lg^{\alpha+1}(n) \lg_2(n)} = \frac{C \lg_2^{p-1}(n)}{\lg^p(n)} \rightarrow 0$$

completing this proof. \square

We set the parameter α larger than minus one, so that these random variables have infinite expectation. If the expectation was finite, then the classic Laws of Large Numbers would apply. Next, we examine what happens when $\alpha = -1$. In keeping with the spirit of non-integrable random variables, we now assume $P\{X_j > x\} = \frac{1}{(x+j)\lg(x+j)}$. Thus $P\{X_j = 0\} = 1 - \frac{1}{j\lg(j)}$

and by differentiating $F_{X_j}(x) = 1 - P\{X_j > x\}$ we obtain

$$f_{X_j}(x) = \frac{\lg(x+j) + 1}{(x+j)^2 \lg^2(x+j)}.$$

Theorem 2. *If $P\{X_j > x\} = \frac{1}{(x+j)\lg(x+j)}$, then*

$$\frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg_2(n) \lg_3(n)} \xrightarrow{P} 1.$$

Proof. We start once again by examining the truncated mean, which shows that our selection of $b_n = \lg_2(n) \lg_3(n)$ is the correct choice

$$\begin{aligned} & \sum_{j=1}^n \frac{a_j}{b_n} EX_j I(a_j X_j \leq b_n) \\ &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_0^{b_n j} x \left[\frac{\lg(x+j) + 1}{(x+j)^2 \lg^2(x+j)} \right] dx \\ &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} (u-j) \left[\frac{\lg(u) + 1}{u^2 \lg^2(u)} \right] du \\ &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \frac{du}{u \lg(u)} - \frac{1}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)j} \frac{du}{u^2 \lg(u)} \\ &\quad + \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \frac{du}{u \lg^2(u)} - \frac{1}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)j} \frac{du}{u^2 \lg^2(u)} \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Letting $w = \lg(u)$, the first term becomes

$$\begin{aligned} I_1 &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \frac{du}{u \lg u} = \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_{\lg j}^{\lg[(b_n+1)j]} \frac{dw}{w} \\ &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} [\lg_2((b_n+1)j) - \lg_2(j)] = \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \lg \left[\frac{\lg(b_n+1) + \lg(j)}{\lg(j)} \right] \\ &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \lg \left[\frac{\lg(b_n+1)}{\lg(j)} + 1 \right]. \end{aligned}$$

We use the series expansion of $\lg(x+1) = x - x^2/2 + x^3/3 + \dots$ with $x = \lg(b_n+1)/\lg(j)$, which is small if j exceeds some value, say J_0 . This is

an alternating series and the only significant term is the x term. Plugging

in x into I_1 we have

$$\frac{1}{b_n} \sum_{j=1}^n \frac{\lg(b_n + 1)}{j \lg(j)} \sim \frac{\lg(b_n)}{b_n} \sum_{j=1}^n \frac{1}{j \lg(j)} \sim \frac{\lg_3(n) \lg_2(n)}{b_n} = 1.$$

Any other term, x^p , with $p \geq 2$ is

$$\frac{1}{b_n} \sum_{j=1}^n \frac{\lg^p(b_n + 1)}{j \lg^p(j)} \leq \frac{C \lg^p(b_n)}{b_n} \leq \frac{\lg_3^{p-1}(n)}{\lg_2(n)} \rightarrow 0$$

since $\lg(b_n) = \lg_3(n) + \lg_4(n) \sim \lg_3(n)$. That is the only nonzero limit, since

$$\begin{aligned} |I_2| &= \left| \frac{-1}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)j} \frac{du}{u^2 \lg(u)} \right| \leq \frac{1}{b_n} \sum_{j=1}^n \int_j^\infty \frac{du}{u^2 \lg(u)} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg(j)} \leq \frac{C \lg_2(n)}{b_n} = \frac{C}{\lg_3(n)} \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} I_3 &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{(b_n+1)j} \frac{du}{u \lg^2(u)} \\ &\leq \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^\infty \frac{du}{u \lg^2(u)} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg(j)} \leq \frac{C \lg_2(n)}{b_n} = \frac{C}{\lg_3(n)} \rightarrow 0 \end{aligned}$$

and finally

$$\begin{aligned} |I_4| &= \left| \frac{-1}{b_n} \sum_{j=1}^n \int_j^{(b_n+1)j} \frac{du}{u^2 \lg^2(u)} \right| \leq \frac{1}{b_n} \sum_{j=1}^n \int_j^\infty \frac{du}{u^2 \lg^2(u)} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg^2(j)} \leq \frac{C}{b_n} \rightarrow 0. \end{aligned}$$

Next, we look at the second moment term in the Degenerate Convergence

Theorem

$$\frac{1}{b_n^2} \sum_{j=1}^n a_j^2 EX_j^2 I(a_j X_j \leq b_n) = \frac{1}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_0^{b_n j} x^2 \left[\frac{\lg(x+j) + 1}{(x+j)^2 \lg^2(x+j)} \right] dx$$

$$\begin{aligned}
&= \frac{1}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_j^{(b_n+1)j} \frac{(u-j)^2 [\lg(u)+1] du}{u^2 \lg^2(u)} \\
&\leq \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2} \int_j^{(b_n+1)j} \frac{du}{\lg(u)} \\
&\leq \frac{C}{b_n^2} \sum_{j=1}^n \frac{1}{j^2 \lg(j)} \int_j^{(b_n+1)j} du = \frac{C}{b_n^2} \sum_{j=1}^n \frac{b_n j}{j^2 \lg(j)} \\
&= \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg(j)} \leq \frac{C \lg_2(n)}{b_n} = \frac{C}{\lg_3(n)} \rightarrow 0.
\end{aligned}$$

As for the tail probabilities, we have for all $\epsilon > 0$

$$\begin{aligned}
\sum_{j=1}^n P\{X_j > \epsilon b_n j\} &= \sum_{j=1}^n \frac{1}{(\epsilon b_n j + j) \lg(\epsilon b_n j + j)} \\
&= \sum_{j=1}^n \frac{1}{(\epsilon b_n + 1) j \lg[(\epsilon b_n + 1) j]} \\
&= \frac{1}{\epsilon b_n + 1} \sum_{j=1}^n \frac{1}{j [\lg(\epsilon b_n + 1) + \lg(j)]} \\
&\leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg(j)} \\
&\leq \frac{C \lg_2(n)}{b_n} = \frac{C}{\lg_3(n)} \rightarrow 0.
\end{aligned}$$

3. Almost Sure Results

The Weak Laws established in the last section will be used to obtain one sided Strong Laws. The easy part was done in both [2] and [5]. By using the technique from [1] we can get an exact lower bound. But the problem is in selecting the right sequence d_n . Note that the in both proofs we need one series to converge and the other series to diverge where both series differ by just a power of $\lg_2 n$ and $\lg_3 n$, respectively.

Theorem 3. *If $P\{X_j > x\} = \frac{\lg^\alpha(x+j)}{x+j}$ where $\alpha > -1$, then*

$$\liminf_{n \rightarrow \infty} \frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg^{\alpha+1}(n) \lg_2(n)} = 1 \quad \text{almost surely}$$

and

$$\limsup_{n \rightarrow \infty} \frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg^{\alpha+1}(n) \lg_2(n)} = \infty \quad \text{almost surely.}$$

Proof. Let $a_j = 1/j$, $b_n = \lg^{\alpha+1}(n) \lg_2(n)$, $c_n = b_n/a_n = n \lg^{\alpha+1}(n) \lg_2(n)$ and $d_n = n \lg^{\alpha+1}(n)$. The difference between c_n and d_n is subtle, but that is exactly what helps us achieve our lower limit. The upper limit occurs since for all $M > 0$

$$\begin{aligned} \sum_{n=1}^{\infty} P\{a_n X_n / b_n > M\} &= \sum_{n=1}^{\infty} P\{X_n > M c_n\} \\ &= \sum_{n=1}^{\infty} \frac{\lg^{\alpha}(M c_n + n)}{M c_n + n} \\ &\geq \sum_{n=1}^{\infty} \frac{\lg^{\alpha}(n)}{M n \lg^{\alpha+1}(n) \lg_2(n) + n} \\ &\geq C \sum_{n=1}^{\infty} \frac{\lg^{\alpha}(n)}{n \lg^{\alpha+1}(n) \lg_2(n)} \\ &= C \sum_{n=1}^{\infty} \frac{1}{n \lg(n) \lg_2(n)} = \infty \end{aligned}$$

when $\alpha \geq 0$. Similarly if $-1 < \alpha < 0$, then

$$\begin{aligned} \sum_{n=1}^{\infty} P\{a_n X_n / b_n > M\} &= \sum_{n=1}^{\infty} P\{X_n > M c_n\} \\ &= \sum_{n=1}^{\infty} \frac{\lg^{\alpha}(M c_n + n)}{M c_n + n} \\ &\geq C \sum_{n=1}^{\infty} \frac{\lg^{\alpha}(c_n)}{c_n} \\ &\geq C \sum_{n=1}^{\infty} \frac{\lg^{\alpha}(n)}{n \lg^{\alpha+1}(n) \lg_2(n)} \\ &= C \sum_{n=1}^{\infty} \frac{1}{n \lg(n) \lg_2(n)} = \infty. \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{a_n X_n}{b_n} = \infty \quad \text{almost surely}$$

hence

$$\limsup_{n \rightarrow \infty} \frac{\sum_{j=1}^n a_j X_j}{b_n} \geq \limsup_{n \rightarrow \infty} \frac{a_n X_n}{b_n} = \infty \quad \text{almost surely}$$

which completes the easy half of the proof.

From Theorem 1, we obtain a lower bound on the lower limit, i.e.,

$$\liminf_{n \rightarrow \infty} \frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg^{\alpha+1}(n) \lg_2(n)} \leq 1 \quad \text{almost surely.}$$

In order to obtain equality here, we use the sequence d_n and partition our sum as

$$\begin{aligned} \frac{1}{b_n} \sum_{j=1}^n a_j X_j &\geq \frac{1}{b_n} \sum_{j=1}^n a_j X_j I(0 < X_j \leq d_j) \\ &= \frac{1}{b_n} \sum_{j=1}^n a_j [X_j I(0 < X_j \leq d_j) - EX_j I(0 < X_j \leq d_j)] \\ &\quad + \frac{1}{b_n} \sum_{j=1}^n a_j EX_j I(0 < X_j \leq d_j). \end{aligned}$$

The first term converges to zero almost surely by the Khintchine-Kolmogorov Convergence Theorem, see [3], page 113 and the Kronecker Lemma since $b_n \uparrow \infty$ and

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_n^2 I(0 < X_n \leq d_n) \\ &= \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_0^{d_n} x^2 \left(\frac{\lg^\alpha(x+n) - \alpha \lg^{\alpha-1}(x+n)}{(x+n)^2} \right) dx \\ &\leq \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_n^{d_n+n} \frac{(u-n)^2 \lg^\alpha(u) du}{u^2} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_n^{d_n+n} \lg^\alpha(u) du \\ &\leq \sum_{n=1}^{\infty} \frac{\lg^\alpha(d_n+n)}{c_n^2} \int_n^{d_n+n} du \\ &\leq C \sum_{n=1}^{\infty} \frac{d_n \lg^\alpha(d_n)}{c_n^2} \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{n=1}^{\infty} \frac{n \lg^{2\alpha+1}(n)}{n^2 \lg^{2\alpha+2}(n) \lg_2^2(n)} \\
&= C \sum_{n=1}^{\infty} \frac{1}{n \lg(n) \lg_2^2(n)} < \infty
\end{aligned}$$

whenever $\alpha \geq 0$. If $-1 < \alpha < 0$, then

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_n^2 I(0 < X_n \leq d_n) \\
&= \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_0^{d_n} x^2 \left(\frac{\lg^\alpha(x+n) - \alpha \lg^{\alpha-1}(x+n)}{(x+n)^2} \right) dx \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_n^{d_n+n} \frac{(u-n)^2 \lg^\alpha(u) du}{u^2} \\
&\leq C \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_n^{d_n+n} \lg^\alpha(u) du \\
&\leq C \sum_{n=1}^{\infty} \frac{\lg^\alpha(n)}{c_n^2} \int_n^{d_n+n} du \\
&= C \sum_{n=1}^{\infty} \frac{d_n \lg^\alpha(n)}{c_n^2} \\
&= C \sum_{n=1}^{\infty} \frac{n \lg^{2\alpha+1}(n)}{n^2 \lg^{2\alpha+2}(n) \lg_2^2(n)} \\
&= C \sum_{n=1}^{\infty} \frac{1}{n \lg(n) \lg_2^2(n)} < \infty.
\end{aligned}$$

Next, we need to show that the other term converges to one as in the proof of Theorem 1. As with the expectation in Theorem 1, we have four terms.

The first term is the important one

$$\begin{aligned}
&\frac{1}{b_n} \sum_{j=1}^n a_j EX_j I(0 < X_j \leq d_j) \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_0^{d_j} x \left(\frac{\lg^\alpha(x+j) - \alpha \lg^{\alpha-1}(x+j)}{(x+j)^2} \right) dx \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{(u-j)(\lg^\alpha(u) - \alpha \lg^{\alpha-1}(u)) du}{u^2}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{\lg^\alpha(u) du}{u} - \frac{\alpha}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{\lg^{\alpha-1}(u) du}{u} \\
&\quad - \frac{1}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{\lg^\alpha(u) du}{u^2} + \frac{\alpha}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{\lg^{\alpha-1}(u) du}{u^2} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The first term is

$$\begin{aligned}
&\frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{\lg^\alpha(u) du}{u} = \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_{\lg(j)}^{\lg(d_j+j)} w^\alpha dw \\
&= \frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{1}{j} \left(\lg^{\alpha+1}(d_j+j) - \lg^{\alpha+1}(j) \right) \\
&= \frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{1}{j} \left([\lg(j) + \lg(\lg^{\alpha+1}(j) + 1)]^{\alpha+1} - \lg^{\alpha+1}(j) \right) \\
&= \frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left(\left[1 + \frac{\lg(\lg^{\alpha+1}(j) + 1)}{\lg(j)} \right]^{\alpha+1} - 1 \right).
\end{aligned}$$

The only term of significance in $(1+x)^{\alpha+1}$ is the $(\alpha+1)x$ term, where

$x = \lg(\lg^{\alpha+1}(j) + 1)/\lg(j)$, which is certainly smaller than one. Thus I_1

becomes

$$\begin{aligned}
&\frac{1}{(\alpha+1)b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left(\frac{(\alpha+1) \lg(\lg^{\alpha+1}(j) + 1)}{\lg(j)} \right) \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left(\frac{\lg(\lg^{\alpha+1}(j) + 1)}{\lg(j)} \right) \\
&\sim \frac{1}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left(\frac{(\alpha+1) \lg_2(j)}{\lg(j)} \right) \\
&= \frac{\alpha+1}{b_n} \sum_{j=1}^n \frac{\lg^\alpha(j) \lg_2(j)}{j} \\
&\sim \left(\frac{\alpha+1}{b_n} \right) \left(\frac{\lg^{\alpha+1}(n) \lg_2(n)}{\alpha+1} \right) = 1.
\end{aligned}$$

As for any other powers of x , those vanish as n goes to infinity since they

are smaller than

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha+1}(j)}{j} \left(\frac{\lg(\lg^{\alpha+1}(j) + 1)}{\lg(j)} \right)^p \rightarrow 0$$

for all $p \geq 2$. As for I_2 we have as an upper bound

$$\left| \frac{-\alpha}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{\lg^{\alpha-1}(u) du}{u} \right| \leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} \int_{\lg(j)}^{\lg(d_j+j)} w^{\alpha-1} dw.$$

If $\alpha < 0$ then this is less than

$$\frac{C}{b_n} \sum_{j=1}^n \frac{1}{j} \int_{\lg(j)}^{\infty} w^{\alpha-1} dw \leq \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha}(j)}{j} \leq \frac{C \lg^{\alpha+1}(n)}{b_n} \leq \frac{C}{\lg_2(n)} \rightarrow 0.$$

If $\alpha = 0$ then this term doesn't even exist and if $\alpha > 0$, then this is bounded above by

$$\frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha}(d_j + j)}{j} \leq \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha}(j)}{j} \leq \frac{C \lg^{\alpha+1}(n)}{b_n} = \frac{C}{\lg_2(n)} \rightarrow 0.$$

As for I_3 we have

$$\begin{aligned} \left| \frac{-1}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{\lg^{\alpha}(u) du}{u^2} \right| &\leq \frac{1}{b_n} \sum_{j=1}^n \int_j^{\infty} \frac{\lg^{\alpha}(u) du}{u^2} \\ &\leq \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha}(j)}{j} \\ &\leq \frac{C \lg^{\alpha+1}(n)}{b_n} = \frac{C}{\lg_2(n)} \rightarrow 0. \end{aligned}$$

Finally, an upper bound for I_4 is

$$\left| \frac{\alpha}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{\lg^{\alpha-1}(u) du}{u^2} \right| \leq \frac{C}{b_n} \sum_{j=1}^n \int_j^{\infty} \frac{\lg^{\alpha-1}(u) du}{u^2} \leq \frac{C}{b_n} \sum_{j=1}^n \frac{\lg^{\alpha-1}(j)}{j} \rightarrow 0$$

for any α . Putting everything together we see that the lower limit is indeed one, with probability one. \square

We can also obtain a one sided strong law when α is minus one.

Theorem 4. *If $P\{X_j > x\} = \frac{1}{(x+j)\lg(x+j)}$, then*

$$\liminf_{n \rightarrow \infty} \frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg_2(n) \lg_3(n)} = 1 \quad \text{almost surely}$$

and

$$\limsup_{n \rightarrow \infty} \frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg_2(n) \lg_3(n)} = \infty \quad \text{almost surely.}$$

Proof. Let $a_j = 1/j$, $b_n = \lg_2(n) \lg_3(n)$, $c_n = b_n/a_n = n \lg_2(n) \lg_3(n)$ and $d_n = n \lg_2(n)$. Once again, the difference between c_n and d_n is subtle, but that is exactly what helps us achieve our lower limit. The upper limit occurs since for all $M > 0$

$$\begin{aligned} \sum_{n=1}^{\infty} P\{a_n X_n / b_n > M\} &= \sum_{n=1}^{\infty} P\{X_n > M c_n\} \\ &= \sum_{n=1}^{\infty} \frac{1}{(M c_n + n) \lg(M c_n + n)} \\ &= \sum_{n=1}^{\infty} \frac{1}{(M n \lg_2(n) \lg_3(n) + n) \lg(M n \lg_2(n) \lg_3(n) + n)} \\ &\geq C \sum_{n=1}^{\infty} \frac{1}{n \lg(n) \lg_2(n) \lg_3(n)} = \infty. \end{aligned}$$

Thus

$$\limsup_{n \rightarrow \infty} \frac{a_n X_n}{b_n} = \infty \quad \text{almost surely}$$

hence

$$\limsup_{n \rightarrow \infty} \frac{\sum_{j=1}^n a_j X_j}{b_n} \geq \limsup_{n \rightarrow \infty} \frac{a_n X_n}{b_n} = \infty \quad \text{almost surely}$$

which completes the easy half of the proof.

From Theorem 2, we obtain a lower bound on the lower limit, i.e.,

$$\liminf_{n \rightarrow \infty} \frac{\sum_{j=1}^n \frac{1}{j} X_j}{\lg_2(n) \lg_3(n)} \leq 1 \quad \text{almost surely.}$$

Using the same partition as in Theorem 3, but with a different d_n we have

$$\begin{aligned} \frac{1}{b_n} \sum_{j=1}^n a_j X_j &\geq \frac{1}{b_n} \sum_{j=1}^n a_j X_j I(0 < X_j \leq d_j) \\ &= \frac{1}{b_n} \sum_{j=1}^n a_j [X_j I(0 < X_j \leq d_j) - EX_j I(0 < X_j \leq d_j)] \\ &\quad + \frac{1}{b_n} \sum_{j=1}^n a_j EX_j I(0 < X_j \leq d_j). \end{aligned}$$

The first term converges to zero almost surely by the Khintchine-Kolmogorov Convergence Theorem, see [3], page 113 and the Kronecker Lemma since $b_n \uparrow \infty$ and

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{c_n^2} EX_n^2 I(0 < X_n \leq d_n) &= \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_0^{d_n} x^2 \left(\frac{\lg(x+n)+1}{(x+n)^2 \lg^2(x+n)} \right) dx \\ &= \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_n^{d_n+n} \frac{(u-n)^2 (\lg(u)+1) du}{u^2 \lg^2(u)} \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_n^{d_n+n} \frac{du}{\lg(u)} \\ &\leq C \sum_{n=1}^{\infty} \frac{1}{c_n^2} \int_n^{d_n+n} \frac{du}{\lg(n)} \\ &= C \sum_{n=1}^{\infty} \frac{d_n}{c_n^2 \lg(n)} \\ &= C \sum_{n=1}^{\infty} \frac{1}{n \lg(n) \lg_2(n) \lg_3^2(n)} < \infty. \end{aligned}$$

Note that this barely converges and the previous infinite sum diverges, by the slimmest of margins, a factor of $\lg_3(n)$. Next, we need to show that the other term converges to one

$$\begin{aligned} \frac{1}{b_n} \sum_{j=1}^n a_j EX_j I(0 < X_j \leq d_j) &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_0^{d_j} x \left(\frac{\lg(x+j)+1}{(x+j)^2 \lg^2(x+j)} \right) dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{(u-j)(\lg(u)+1)du}{u^2 \lg^2(u)} \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{du}{u \lg(u)} - \frac{1}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{du}{u^2 \lg(u)} \\
&\quad + \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{du}{u \lg^2(u)} - \frac{1}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{du}{u^2 \lg^2(u)} \\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

The first integral is

$$\begin{aligned}
\frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{du}{u \lg(u)} &= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_{\lg(j)}^{\lg(d_j+j)} \frac{dw}{w} \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} [\lg_2(d_j+j) - \lg_2(j)] \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \lg\left(\frac{\lg(d_j+j)}{\lg(j)}\right) \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \lg\left(\frac{\lg(j) + \lg(\lg_2(j)+1)}{\lg(j)}\right) \\
&= \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \lg\left(1 + \frac{\lg(\lg_2(j)+1)}{\lg(j)}\right).
\end{aligned}$$

The only term of significance in $\lg(1+x)$ is $x = \lg(\lg_2(j)+1)/\lg(j)$. Thus

I_1 becomes

$$\frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \left(\frac{\lg(\lg_2(j)+1)}{\lg(j)}\right) \sim \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \left(\frac{\lg_3(j)}{\lg(j)}\right) \sim \frac{\lg_2(n) \lg_3(n)}{b_n} = 1.$$

As for any other powers of x , those vanish as n goes to infinity since they

are smaller than

$$\frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \left(\frac{\lg(\lg_2(j)+1)}{\lg(j)}\right)^p \leq \frac{1}{b_n} \sum_{j=1}^n \frac{\lg_3^p(j)}{j \lg^p(j)} \leq \frac{C}{b_n} \rightarrow 0$$

since $p \geq 2$. As for I_2 we have

$$\begin{aligned} \frac{1}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{du}{u^2 \lg u} &\leq \frac{1}{b_n} \sum_{j=1}^n \int_j^\infty \frac{du}{u^2 \lg u} \leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg j} \\ &\leq \frac{C \lg_2(n)}{b_n} = \frac{C}{\lg_3(n)} \rightarrow 0. \end{aligned}$$

As for I_3 we have

$$\begin{aligned} \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^{d_j+j} \frac{du}{u \lg^2 u} &\leq \frac{1}{b_n} \sum_{j=1}^n \frac{1}{j} \int_j^\infty \frac{du}{u \lg^2 u} \leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg j} \\ &\leq \frac{C \lg_2 n}{b_n} = \frac{C}{\lg_3 n} \rightarrow 0. \end{aligned}$$

Finally for I_4 we have

$$\frac{1}{b_n} \sum_{j=1}^n \int_j^{d_j+j} \frac{du}{u^2 \lg^2 u} \leq \frac{1}{b_n} \sum_{j=1}^n \int_j^\infty \frac{du}{u^2 \lg^2 u} \leq \frac{C}{b_n} \sum_{j=1}^n \frac{1}{j \lg^2 j} \leq \frac{C}{b_n} \rightarrow 0.$$

Putting everything together we see that the lower limit is once again one, with probability one. \square

4. Discussion

It shouldn't be too difficult to obtain similar Weak Laws and one-sided Strong Laws when $P\{X_j > x\} = \frac{\lg_n^\alpha(x+j)}{(x+j) \prod_{i=1}^{n-1} \lg_i(x+j)}$ whenever $\alpha \geq -1$ for any positive integer n . But to extend this to any sequence m_j and any slowly varying function, $L(\cdot)$, where $P\{X_j > x\} = \frac{L(x+m_j)}{x+m_j}$ would be quite difficult.

References

1. A. Adler, Generalized one-sided laws of the iterated logarithm for random variables barely with or without finite mean, *J. Theoret. Probab.*, **3** (1990), 587-597.
2. A. Adler, An exact weak law of large numbers, *Bull. Inst. Math. Acad. Sinica*, **7** (2012), 417-422.
3. Y. S. Chow and H. Teicher, *Probability Theory: Independence, Interchangeability, Martingales*, 3rd ed., Springer-Verlag, New York, 1997.
4. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol 2, 2nd ed., John Wiley, New York, 1966.

5. T. Nakata, Weak law of large numbers for weighted independent random variables with infinite mean, *Statist. Probab. Letters* **109** (2016), 124-129.