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THE SECOND COEFFICIENT OF THE ASYMPTOTIC EXPANSION OF THE WEIGHTED BERGMAN KERNEL FOR $(0, q)$ FORMS ON $\mathbb{C}^n$

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Abstract

Let $\phi \in C^\infty(\mathbb{C}^n)$ be a given real valued function. We assume that $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) on $\mathbb{C}^n$. When $q = n_-$, it is well-known that the Bergman kernel for $(0, q)$ forms with respect to the k-th weight $e^{-2k\phi}$, $k > 0$, admits a full asymptotic expansion in k. In this paper, we compute the trace of the second coefficient of the asymptotic expansion on the diagonal.

1. Introduction and Statement of the Main Result

Let L be a holomorphic line bundle over a Hermitian manifold (M, Θ), where Θ is a smooth positive $(1, 1)$-form on M, and let L^k be the k-th tensor power of L. Let $\Box_k^{(q)}$ be the Gaffney extension of the Kodaira Laplacian acting on $(0, q)$ forms with values in L^k. The Bergman kernel is the distribution kernel of the orthogonal projection onto $\operatorname{Ker}\Box_k^{(q)}$ in the L^2 space. We assume that the curvature of L is non-degenerate of constant signature (n_-, n_+) on M and let $q = n_-$. When M is compact, Catlin [2] and Zelditch [18] established the asymptotic expansion of the diagonal of the Bergman kernel

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for $q = n_- = 0$ and Berman-Sjöstrand [1], Ma-Marinescu [13] established the asymptotic expansion of the Bergman kernel for $q = n_- \geq 0$. When M is complete and L is uniformly positive on M with $\sqrt{-1}R^{K^*_M}$ and $\partial\Theta$ bounded below, where $R^{K^*_M}$ is the curvature of the bundle of $(n,0)$ forms, Ma-Marinescu [15] obtained the asymptotic expansion of the Bergman kernel for $q = n_- = 0$. More generally, if M is any complex manifold and $\Box^{(q)}_k$ has $O(k^{-n_0})$ small spectral gap on an open set $D \Subset M$ (see Definition 1.5 in [9], for the precise meaning of $O(k^{-n_0})$ small spectral gap), then it is known by a recent result (see Theorem 1.6 in [9]) that the Bergman kernel admits a full asymptotic expansion in k on D. The coefficients of these expansions turned out to be deeply related to various problem in complex geometry (see e.g. [3], [4], [5]).

The first four coefficients of the expansion of the Bergman kernel for $q = n_- = 0$ on the diagonal were computed by Lu [12]. The method of Lu is to construct appropriate peak sections as in [17], using Hörmander's L^2-method. Ma-Marinescu [16] calculated the first three coefficients of the expansion of the kernel of Berezin-Toeplitz quantization on the diagonal by using kernel calculations on $\mathbb{C}^n$. The author [8] gave a new method to calculated the first three coefficients of the expansion of the kernel of Berezin-Toeplitz quantization on the diagonal by using microlocal analysis. All these results are concern $q = n_- = 0$.

In this paper, we give for the first time a formula of the second coefficient of the expansion of the Bergman kernel for $q = n_- > 0$ on $\mathbb{C}^n$. We calculate the trace of the second coefficient of the expansion of the Bergman kernel for $q = n_- > 0$ when $M = \mathbb{C}^n$ and L is the trivial line bundle $\mathbb{C}$ endowed with the metric $|1|^2 = e^{-2\phi}$, where $\phi \in C^\infty(\mathbb{C}^n)$ is a given real valued function with $\partial\overline{\partial}\phi$ is non-degenerate of constant signature (n_-, n_+) on $\mathbb{C}^n$. There are two ingredients of our approach: the phase function version of the asymptotic expansion of the Bergman kernel and the method of stationary phase. Even through the calculation is quite complicate, the arguments in this paper are simple.

After the paper was completed Wen Lu [11] informed the author that he also obtained the formula for the coefficient b_1 by using the method of Ma-Marinescu [14]. Moreover, Wen Lu [11] obtained the formula for b_1 on general compact complex manifolds.

1.1. Notations

Let Ω be a C^∞ paracompact manifold equipped with a smooth density of integration. We let $T(\Omega)$ and $T^*(\Omega)$ denote the tangent bundle of Ω and the cotangent bundle of Ω respectively. The complexified tangent bundle of Ω and the complexified cotangent bundle of Ω will be denoted by $\mathbb{C}T(\Omega)$ and $\mathbb{C}T^*(\Omega)$ respectively. We write $<\ ,\ >$ to denote the pointwise duality between $T(\Omega)$ and $T^*(\Omega)$. We extend $<\ ,\ >$ bilinearly to $\mathbb{C}T(\Omega)\times\mathbb{C}T^*(\Omega)$. Let E be a C^∞ vector bundle over Ω. The fiber of E at $x\in\Omega$ will be denoted by E_x. Let $Y\subset\subset\Omega$ be an open set. From now on, the spaces of smooth sections of E over Y and distribution sections of E over Y will be denoted by $C^\infty(Y;\,E)$ and $\mathscr{D}'(Y;\,E)$ respectively. Let $\mathscr{E}'(Y;\,E)$ be the subspace of $\mathscr{D}'(Y;\,E)$ whose elements have compact support in Y. Put $C^\infty_0(Y;\,E)=C^\infty(Y;\,E)\bigcap\mathscr{E}'(Y;\,E)$. We let $L^2(Y;\,E)$ denote the L^2 space of sections of E over Y.

We shall denote the real coordinates by x_j, $j=1,\ldots,2n$, and the complex coordinates by $z=(z_1,\ldots,z_n)$, $z_j=x_{2j-1}+ix_{2j}$, $j=1,\ldots,n$. Let $\Lambda^{1,0}T(\mathbb{C}^n)$ and $\Lambda^{0,1}T(\mathbb{C}^n)$ denote the holomorphic tangent bundle and the anti-holomorphic tangent bundle of $\mathbb{C}^n$ respectively. We take the Hermitian metric $(\ |\ )$ on $\mathbb{C}T(\mathbb{C}^n)$ such that $(\frac{\partial}{\partial z_j}\mid\frac{\partial}{\partial z_k})=(\frac{\partial}{\partial\overline{z}_j}\mid\frac{\partial}{\partial\overline{z}_k})=\delta_{j,k}$, $j,k=1,\ldots,n$, $\Lambda^{1,0}T(\mathbb{C}^n)\perp\Lambda^{0,1}T(\mathbb{C}^n)$, where $\frac{\partial}{\partial z_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}-i\frac{\partial}{\partial x_{2j}})$, $\frac{\partial}{\partial\overline{z}_j}=\frac{1}{2}(\frac{\partial}{\partial x_{2j-1}}+i\frac{\partial}{\partial x_{2j}})$, $j=1,\ldots,n$, and $\delta_{j,k}=1$ if $j=k$, $\delta_{j,k}=0$ if $j\neq k$. For $p,q\geq 0$, $p,q\in\mathbb{Z}$, let $\Lambda^{p,q}T^*(\mathbb{C}^n)$ be the bundle of (p,q) forms of $\mathbb{C}^n$. We say that a multiindex $J=(j_1,\ldots,j_q)\in\{1,\ldots,n-1\}^q$ has length q and write $|J|=q$. We say that J is strictly increasing if $1\leqslant j_1<j_2<\cdots<j_q\leqslant n-1$. For multiindices $J=(j_1,\ldots,j_q)$, $K=(k_1,\ldots,k_p)$, we define $dz_K\wedge d\overline{z}_J:=dz_{k_1}\wedge\cdots\wedge dz_{k_p}\wedge d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}$. We take the Hermitian metric $(\ |\ )$ on $\Lambda^{p,q}T^*(\mathbb{C}^n)$ so that $\{dz_K\wedge d\overline{z}_J:|K|=p,\,|J|=q,\,K,\,J$ are strictly increasing$\}$ is an orthonormal frame for $\Lambda^{p,q}T^*(\mathbb{C}^n)$. Let $T\in\mathscr{L}(\Lambda^{p,q}T^*(\mathbb{C}^n),\Lambda^{p,q}T^*(\mathbb{C}^n))$. Then the trace of T is given by

$$\operatorname{Tr}T={\sum_{|K|=p,|J|=q}}'(T(dz_K\wedge d\overline{z}_J)\mid dz_K\wedge d\overline{z}_J),\tag{1.1}$$

where $\sum'$ means that the summation is performed only over strictly increasing multiindices. Thus $\operatorname{Tr} T = 0$ if

$$(T(dz_K \wedge d\overline{z}_J) \mid dz_K \wedge d\overline{z}_J) = 0 \tag{1.2}$$

for all strictly increasing multiinindices K, J, $|K| = p$, $|J| = q$.

If $w \in \Lambda^{0,1}T^*_z(\mathbb{C}^n)$, let $w^{\wedge,*} : \Lambda^{0,q+1}T^*_z(\mathbb{C}^n) \to \Lambda^{0,q}T^*_z(\mathbb{C}^n)$ be the adjoint of left exterior multiplication $w^\wedge : \Lambda^{0,q}T^*_z(\mathbb{C}^n) \to \Lambda^{0,q+1}T^*_z(\mathbb{C}^n)$. That is,

$$(w^\wedge u \mid v) = (u \mid w^{\wedge,*} v), \tag{1.3}$$

for all $u \in \Lambda^{0,q}T^*_z(\mathbb{C}^n)$, $v \in \Lambda^{0,q+1}T^*_z(\mathbb{C}^n)$. Notice that $w^{\wedge,*}$ depends anti-linearly on w.

Let E, F be C^∞ vector bundles over a smooth manifold M. We say that a k-dependent function $f(x, y, k) \in C^\infty(M \times M; \mathscr{L}(E_y, F_x))$ is negligible if for every compact set $K \subset M \times M$ and for all $N > 0$ and multiindice α, β, there is a constant $c_{N,\alpha,\beta,K} > 0$ independent of k such that for k sufficiently large, $\left|\partial^\alpha_x \partial^\beta_y f(x, y, k)\right| \leq c_{\alpha,\beta,K} k^{-N}$, $(x, y) \in K$. Let $b(x, y, k) \in C^\infty(M \times M; \mathscr{L}(E_y, F_x))$ be a k-dependent smooth function. We write

$$b \sim \sum_0^\infty b_j(x, y) k^{-N_0-j}$$

in $C^\infty(M \times M; \mathscr{L}(E_y, F_x))$, $b_j(x, y) \in C^\infty(M \times M; \mathscr{L}(E_y, F_x))$, $j = 0, 1, \ldots$, if for all $M_0 \in \mathbb{N}$, every compact set $K \subset M \times M$ and for all multiindice α, β, there is a constant $c_{M_0,\alpha,\beta,K} > 0$ independent of k such that for k sufficiently large,

$$\left|\partial^\alpha_x \partial^\beta_y (b - \sum_0^{M_0} b_j(x, y) k^{-N_0-j})\right| \leq c_{M_0,\alpha,\beta,K} k^{-N_0-M_0-1},$$

$(x, y) \in K$.

1.2. The asymptotic expansion of the Bergman kernel

Let $\phi(z) \in C^\infty(\mathbb{C}^n; \mathbb{R})$. In this work we assume that $\left(\frac{\partial^2 \phi}{\partial \overline{z}_j \partial z_k}\right)^n_{j,k=1}$ is non-degenerate of constant signatute (n_-, n_+). That is, the number of negative eigenvalues of $\left(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}\right)_{j,k=1}^{n}$ is n_- and $n_- + n_+ = n$.

We take $dm = 2^n dx_1 dx_2 \cdots dx_{2n}$ as the volume form on $\mathbb{C}^n$. Let $(\ |\ )$ be the inner product on $C_0^\infty(\mathbb{C}^n;\, \Lambda^{0,q}T^*(\mathbb{C}^n))$ defined by

$$(f \mid g) = \int_{\mathbb{C}^n} (f(z) \mid g(z))(dm), \quad f, g \in C_0^\infty(\mathbb{C}^n;\, \Lambda^{0,q}T^*(\mathbb{C}^n)). \tag{1.4}$$

For $k > 0$, let $(\ |\ )_k$ be the inner product on $C_0^\infty(\mathbb{C}^n;\, \Lambda^{0,q}T^*(\mathbb{C}^n))$ defined by

$$(f \mid g)_k = \int_{\mathbb{C}^n} (f(z) \mid g(z)) e^{-2k\phi(z)}(dm),\ \ f, g \in C_0^\infty(\mathbb{C}^n;\, \Lambda^{0,q}T^*(\mathbb{C}^n)). \tag{1.5}$$

Let $L^2_q(\mathbb{C}^n)$ and $L^2_{q,k}(\mathbb{C}^n)$ be the completions of $C_0^\infty(\mathbb{C}^n;\, \Lambda^{0,q}T^*(\mathbb{C}^n))$ with respect to $(\ |\ )$ and $(\ |\ )_k$ respectively. We extend the L^2 inner products $(\ |\ )$ and $(\ |\ )_k$ to $L^2_q(\mathbb{C}^n)$ and $L^2_{q,k}(\mathbb{C}^n)$ respectively.

Let $\overline{\partial} : C^\infty(\mathbb{C}^n;\, \Lambda^{0,q}T^*(\mathbb{C}^n)) \to C^\infty(\mathbb{C}^n;\, \Lambda^{0,q+1}T^*(\mathbb{C}^n))$ be the part of the exterior differential operator which maps forms of type $(0, q)$ to forms of type $(0, q+1)$. We extend $\overline{\partial}$ to $L^2_{q,k}(\mathbb{C}^n)$ by

$$\overline{\partial}_k : \mathrm{Dom}\,\overline{\partial}_k \subset L^2_{q,k}(\mathbb{C}^n) \to L^2_{q+1,k}(\mathbb{C}^n)\,, \tag{1.6}$$

where $\mathrm{Dom}\,\overline{\partial}_k := \{u \in L^2_{q,k}(\mathbb{C}^n);\, \overline{\partial}u \in L^2_{q+1,k}(\mathbb{C}^n)\}$, where $\overline{\partial}u$ is defined in the sense of distributions. We write

$$\overline{\partial}^*_k : \mathrm{Dom}\,\overline{\partial}^*_k \subset L^2_{q+1,k}(\mathbb{C}^n) \to L^2_{q,k}(\mathbb{C}^n) \tag{1.7}$$

to denote the Hilbert space adjoint of $\overline{\partial}_k$ in the L^2 space with respect to $(\ |\ )_k$. Let $\Box^{(q)}_k$ denote the Gaffney extension of the Kodaira Laplacian given by

$$\Box^{(q)}_k = \overline{\partial}_k\overline{\partial}^*_k + \overline{\partial}^*_k\overline{\partial}_k : \mathrm{Dom}\,\Box^{(q)}_k \subset L^2_{q,k}(\mathbb{C}^n) \to L^2_{q,k}(\mathbb{C}^n), \tag{1.8}$$

where

$$\mathrm{Dom}\,\Box^{(q)}_k = \Big\{s \in L^2_{q,k}(\mathbb{C}^n);\, s \in \mathrm{Dom}\,\overline{\partial}_k \cap \mathrm{Dom}\,\overline{\partial}^*_k,\ \overline{\partial}_k u \in \mathrm{Dom}\,\overline{\partial}^*_k,\ \overline{\partial}^*_k u \in \mathrm{Dom}\,\overline{\partial}_k\Big\}.$$

By a result of Gaffney [14, Prop. 3.1.2], $\Box^{(q)}_k$ is a positive self-adjoint operator.

Let

$$\Pi_k^{(q)} : L^2_{q,k}(\mathbb{C}^n) \to \operatorname{Ker} \Box_k^{(q)} \tag{1.9}$$

be the Bergman projection, i.e. the orthogonal projection onto $\operatorname{Ker} \Box_k^{(q)}$ with respect to $(\ |\ )_k$ and let $\Pi_k^{(q)}(z,w)$ be the distribution kernel of $\Pi_k^{(q)}$ with respect to the volume form dm. Since $\Box_k^{(q)}$ is elliptic, it is not difficult to see that

$$\Pi_k^{(q)}(z,w) \in C^\infty\big(\mathbb{C}^n \times \mathbb{C}^n; \mathscr{L}(\Lambda^{0,q}T^*(\mathbb{C}^n), \Lambda^{0,q}T^*(\mathbb{C}^n))\big).$$

We write

$$\Pi_k^{(q)} u(z) = \int_{\mathbb{C}^n} \Pi_k^{(q)}(z,w) u(w) dm(w),$$

$u \in C_0^\infty(\mathbb{C}^n; \Lambda^{0,q}T^*(\mathbb{C}^n))$.

It is well-known that $\Box_k^{(q)}$ has $O(k^{-n_0})$ small spectral gap on every open set $D \Subset M$ (see Definition 1.5 in [9], for the precise meaning of $O(k^{-n_0})$ small spectral gap). From this observation and Theorem 4.12, Theorem 4.14 in [9], we deduce the following

Theorem 1.1. *Let $D \Subset \mathbb{C}^n$ be an open set and let $0 \leq q \leq n$. If $q \neq n_-$, then $e^{-k\phi(z)}\Pi_k^{(q)}(z,w)e^{k\phi(w)}$ is negligible on $D \times D$. If $q = n_-$, then*

$$\Pi_k^{(q)} u(z) = \int_{\mathbb{C}^n} e^{ik\psi(z,w)} e^{k(\phi(z)-\phi(w))} b(z,w,k) u(w) dm(w) + Ru, \tag{1.10}$$

for $z \in D$, $u \in C_0^\infty(D, \Lambda^{0,q}T^*(\mathbb{C}^n))$, where

$$b \sim \sum_0^\infty b_j(z,w) k^{n-j}$$

in $C^\infty(D \times D; \mathscr{L}(\Lambda^{0,q}T^*_w(\mathbb{C}^n), \Lambda^{0,q}T^*_z(\mathbb{C}^n)))$,

$$b_j \in C^\infty(D \times D; \mathscr{L}(\Lambda^{0,q}T^*_w(\mathbb{C}^n), \Lambda^{0,q}T^*_z(\mathbb{C}^n))),$$

$j = 0, 1, \ldots,$

$$Ru = \int_{\mathbb{C}^n} e^{k(\phi(z)-\phi(w))} r(z,w,k) u(w) dm(w),$$

$r(z,w,k)$ is negligible and

$$\begin{aligned}&\psi \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n), \quad \psi(z,z)=0, \quad \psi(z,w) = -\overline{\psi}(w,z),\\ &\operatorname{Im}\psi(z,w) \geq c\,|z-w|^2,\ c>0.\end{aligned} \tag{1.11}$$

For $z=w$, we have

$$\frac{\partial\psi}{\partial\overline{z}} = i\frac{\partial\phi}{\partial\overline{z}}, \quad \frac{\partial\psi}{\partial z} = -i\frac{\partial\phi}{\partial z}, \quad \frac{\partial\psi}{\partial\overline{w}} = -i\frac{\partial\phi}{\partial\overline{z}}, \quad \frac{\partial\psi}{\partial w} = i\frac{\partial\phi}{\partial z}. \tag{1.12}$$

Moreover,

$$\sum_{j=1}^{n}\Big(i\frac{\partial\psi(z,w)}{\partial\overline{z}_j} + \frac{\partial\phi(z)}{\partial\overline{z}_j}\Big)\Big(-i\frac{\partial\psi(z,w)}{\partial z_j} + \frac{\partial\phi(z)}{\partial z_j}\Big) \tag{1.13}$$

vanishes to infinity order on $z=w$. Furthermore, the Taylor expansion of the phase $\psi(z,w)$ is uniquely determined at each point of $z=w$.

In particular,

$$\Pi_k^{(q)}(z,z) \sim k^n b_0(z,z) + k^{n-1}b_1(z,z) + k^{n-2}b_2(z,z) + \cdots \quad \text{locally uniformly on } \mathbb{C}^n. \tag{1.14}$$

The leading term $b_0(z,z)$ is essentially well-known (see formula (1.24) in Ma-Marinescu [13]).

Theorem 1.2. *Let $q=n_-$. For $p\in\mathbb{C}^n$, we assume that $\lambda_j(p)$, $j=1,\ldots,n$, are the eigenvalues of $\Big(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}(p)\Big)_{j,k=1}^n$ with respect to $(\,|\,)$ and that $\lambda_j(p)<0$ if $1\leq j\leq n_-$. Let $U_1(p),\ldots,U_n(p)$ be an orthonormal basis of $\Lambda^{1,0}T_p(\mathbb{C}^n)$ such that $< \partial\overline{\partial}\phi(p), U_s(p)\wedge\overline{U}_t(p) >= \delta_{s,t}\lambda_s(p)$, $s,t=1,\ldots,n$. Let $\overline{U}_j^*(p)$, $j=1,\ldots,n$, denote the orthonormal basis of $\Lambda^{0,1}T_p^*(\mathbb{C}^n)$, which is dual to $\overline{U}_j(p)$, $j=1,\ldots,n$. Then,*

$$b_0(p,p) = |\lambda_1(p)|\,|\lambda_2(p)|\cdots|\lambda_n(p)|\,\pi^{-n}\prod_{j=1}^{j=n_-}\overline{U}_j^*(p)^\wedge\overline{U}_j^*(p)^{\wedge,*}. \tag{1.15}$$

1.3. The main result

In order to state our result precisely, we have to introduce some notations and definitions. Let

$$F : \Lambda^{1,0}T(\mathbb{C}^n) \to \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p,q}T^*(\mathbb{C}^n)$$

be a linear operaor, where $p, q \in \mathbb{Z}$, $p, q \geq 0$. We write $F = (F_{j,k})_{j,k=1}^n$, $F_{j,k} \in \Lambda^{p,q}T^*(\mathbb{C}^n)$, $1 \leq j, k \leq n$,

$$F\frac{\partial}{\partial z_k} = \sum_{j=1}^n \frac{\partial}{\partial z_j} \otimes F_{j,k}, \tag{1.16}$$

$k = 1, \ldots, n$. We have

$$(FU \mid V) = \sum_{j,k=1}^n u_k \overline{v}_j F_{j,k} \in \Lambda^{p,q}T^*(\mathbb{C}^n), \tag{1.17}$$

where $U = \sum_{k=1}^n u_k \frac{\partial}{\partial z_k}, V = \sum_{j=1}^n v_j \frac{\partial}{\partial z_j} \in \Lambda^{1,0}T(\mathbb{C}^n)$.

Let $T : \Lambda^{1,0}T(\mathbb{C}^n)) \to \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{r,t}T^*(\mathbb{C}^n)$ be another linear operator, where $r, t \in \mathbb{Z}$, $r, s \geq 0$. We write $T = (T_{j,k})_{j,k=1}^n$, $T_{j,k} \in \Lambda^{r,t}T^*(\mathbb{C}^n)$, $j, k = 1, \ldots, n$, as in (1.17). Then

$$TF : \Lambda^{1,0}T(\mathbb{C}^n) \to \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p+r,q+t}T^*(\mathbb{C}^n)$$

is the linear operator defined by $TF\frac{\partial}{\partial z_k} = \sum_{s,j=1}^n \frac{\partial}{\partial z_s} \otimes (T_{s,j} \wedge F_{j,k})$, $k = 1, \ldots, n$.

We assume that F is smooth. That is,

$$F : C^\infty(\mathbb{C}^n;\, \Lambda^{1,0}T(\mathbb{C}^n)) \to C^\infty(\mathbb{C}^n;\, \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p,q}T^*(\mathbb{C}^n)).$$

Let

$$\overline{\partial}F : C^\infty(\mathbb{C}^n;\, \Lambda^{1,0}T(\mathbb{C}^n)) \to C^\infty(\mathbb{C}^n;\, \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p,q+1}T^*(\mathbb{C}^n)) \tag{1.18}$$

be the smooth linear operator defined by $\overline{\partial}F\frac{\partial}{\partial z_k} = \sum_{j=1}^n \frac{\partial}{\partial z_j} \otimes (\overline{\partial}F_{j,k})$, $k = 1, \ldots, n$. Similarly,

$$\partial F : C^\infty(\mathbb{C}^n;\, \Lambda^{1,0}T(\mathbb{C}^n)) \to C^\infty(\mathbb{C}^n;\, \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p+1,q}T^*(\mathbb{C}^n)) \tag{1.19}$$

is the smooth linear operator defined by $\partial F\frac{\partial}{\partial z_k}=\sum_{j=1}^n\frac{\partial}{\partial z_j}\otimes(\partial F_{j,k})$, $k=1,\ldots,n$.

Let

$$M_\phi: C^\infty(\mathbb{C}^n;\Lambda^{1,0}T(\mathbb{C}^n))\to C^\infty(\mathbb{C}^n;\,\Lambda^{1,0}T(\mathbb{C}^n)) \tag{1.20}$$

be the smooth linear map defined by

$$(M_\phi U\mid V)=<\partial\overline{\partial}\phi, U\wedge\overline{V}>, \tag{1.21}$$

$U,V\in C^\infty(\mathbb{C}^n;\,\Lambda^{1,0}T(\mathbb{C}^n))$. Note that $M_\phi=\left(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}\right)_{j,k=1}^n$ in the sense of (1.16). We write $M_\phi^{-1}: C^\infty(\mathbb{C}^n;\Lambda^{1,0}T(\mathbb{C}^n))\to C^\infty(\mathbb{C}^n;\,\Lambda^{1,0}T(\mathbb{C}^n))$ to denote the inverse of M_ϕ.

We recall that we work with the assumption that M_ϕ is non-degenerate of constant signature (n_-,n_+). For $z\in\mathbb{C}^n$, we can diagonalize $M_\phi(z)$, i.e. we can find an orthonormal basis $\{U_j\}_{j=1}^n$ of $\Lambda^{1,0}T(\mathbb{C}^n)$ such that $M_\phi(z)U_j(z)=\lambda_j(z)U_j(z)$, $j=1,\ldots,n$, $\lambda_j(z)\in\mathbb{R}$, $j=1,\ldots,n$. From now on, we assume that

$$\begin{aligned}&M_\phi(z)U_j(z)=\lambda_j(z)U_j(z),\quad U_j(z)\in\Lambda^{1,0}T_z(\mathbb{C}^n),\ (U_j\mid U_j)=1,\quad j=1,\ldots,n,\\&\lambda_j(z)<0,\ j=1,\ldots,n_-,\ \lambda_j(z)>0,\ j=n_-+1,\ldots,n.\end{aligned} \tag{1.22}$$

Let W_+ be the subbundle of $\Lambda^{1,0}T(\mathbb{C}^n)$ spanned by $\{U_{n_-+1},\ldots,U_n\}$ and let W_- be the subbundle of $\Lambda^{1,0}T(\mathbb{C}^n)$ spanned by $\{U_1,\ldots,U_{n_-}\}$. We take the Hermitian metric $(\mid)_{|\phi|}$ on $\Lambda^{1,0}T(\mathbb{C}^n)$ such that $W_+\perp W_-$, $(U\mid V)_{|\phi|}=(M_\phi U\mid V)$ if $U,V\in W_+$, $(U\mid V)_{|\phi|}=-(M_\phi U\mid V)$ if $U,V\in W_-$. The Hermitian metric $(\mid)_{|\phi|}$ on $\mathbb{C}T(\mathbb{C}^n)$ induces a Hermitian metric on $\Lambda^{p,q}T^*(\mathbb{C}^n)$ also denoted by $(\mid)_{|\phi|}$. We take the Hermitian metric $(\mid)_{|\phi|}$ on $\Lambda^{1,0}T(\mathbb{C}^n)$ $\mathbb{C}T(\mathbb{C}^n)$.

The two form $\partial\overline{\partial}\phi$ induces a connection D_ϕ on the bundle $\Lambda^{1,0}T(\mathbb{C}^n)$:

$$\begin{aligned}&D_\phi=d+\theta: C^\infty(\mathbb{C}^n;\,\Lambda^{1,0}(\mathbb{C}^n))\to C^\infty(\mathbb{C}^n;\,\mathbb{C}T^*(\mathbb{C}^n)\otimes\Lambda^{1,0}T(\mathbb{C}^n)),\\&D_\phi(\xi)=\sum_{j=1}^n(d\xi_j)\otimes\frac{\partial}{\partial z_j}+\sum_{1\leq j,k\leq n}\theta_{j,k}\xi_k\otimes\frac{\partial}{\partial z_j},\end{aligned} \tag{1.23}$$

where $\xi=\sum_{j=1}^n\xi_j\frac{\partial}{\partial z_j}$ and $\theta=h^{-1}\partial h=(\theta_{j,k})_{j,k=1}^n$, $h=\left(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}\right)_{j,k=1}^n$. We call θ the connection matrix for $\partial\overline{\partial}\phi$. The curvature of the connection D_ϕ is given by

$$\begin{aligned}&\Theta_\phi = \overline{\partial}\theta = \big(\overline{\partial}\theta_{j,k}\big)_{j,k=1}^n = (\Theta_{j,k})_{j,k=1}^n\,,\\&\Theta_\phi : C^\infty(\mathbb{C}^n;\, \Lambda^{1,0}T(\mathbb{C}^n)) \to C^\infty(\mathbb{C}^n;\, \Lambda^{1,1}T^*(\mathbb{C}^n)\otimes\Lambda^{1,0}T(\mathbb{C}^n)),\\&\xi = \sum_{j=1}^n \xi_j\frac{\partial}{\partial z_j} \to \sum_{j,k=1}^n \Theta_{j,k}\xi_k\otimes\frac{\partial}{\partial z_j}.\end{aligned}\tag{1.24}$$

For $j = 1,\ldots,n$, define

$$\begin{aligned}&\delta_j(k) = 0 \ \text{ if } \{j,k\}\subset\{1,\ldots,q\} \text{ or } \{j,k\}\subset\{q+1,\ldots,n\}\\&\delta_j(k) = 1 \ \text{ otherwise.}\end{aligned}\tag{1.25}$$

Define

$$\begin{aligned}&Q : \Lambda^{1,0}T(\mathbb{C}^n) \to \Lambda^{1,0}T^*(\mathbb{C}^n)\otimes\Lambda^{1,0}T(\mathbb{C}^n),\\&< (QU_j \mid U_k), U_s >=\Big(\frac{|\lambda_j|\,\delta_k(j)+|\lambda_s|\,\delta_k(s)}{|\lambda_k|+|\lambda_j|\,\delta_k(j)+|\lambda_s|\,\delta_k(s)} - \delta_k(j)\delta_k(s)\times\\&\frac{|\lambda_j|^2\,|\lambda_s|^2}{(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}\big(\frac{1}{|\lambda_j|+|\lambda_k|}+\frac{1}{|\lambda_k|+|\lambda_s|}\big)^2\Big) < (\partial M_\phi U_j \mid U_k), U_s > .\end{aligned}\tag{1.26}$$

It is not difficult to see that the definition (1.26) is independent of the choices of eigenvectors $U_1,\ldots,U_n$.

Define

$$R = \Theta_\phi - (\overline{\partial}M_\phi^{-1})Q : \Lambda^{1,0}T(\mathbb{C}^n) \to \Lambda^{1,1}T^*(\mathbb{C}^n)\otimes\Lambda^{1,0}T(\mathbb{C}^n). \tag{1.27}$$

Put $e_j = \frac{1}{\sqrt{|\lambda_j|}}U_j$, $j = 1,\ldots,n$, where U_j, $j = 1,\ldots,n$, are as in (1.22). The main result of this work is the following

Theorem 1.3. *Under the assumptions and notations above, let $q = n_-$. For $p\in\mathbb{C}^n$ and for b_1 in (1.14), we have*

$$\begin{aligned}\operatorname{Tr} b_1(p,p) \ = \ & 2^n\,|\lambda_1(p)|\,|\lambda_2(p)|\cdots|\lambda_n(p)|\,\pi^{-n}\\&\times\Big(\sum_{1\le k\le q, q+1\le j\le n, 1\le s\le n} a_{j,k,s}(p) < (\partial M_\phi U_j \mid U_k), U_s >^2 (p)\end{aligned}$$

$$+\frac{1}{4}\sum_{j,k=1}^{n}(1+\delta_j(k)\frac{|\lambda_j|-|\lambda_k|}{|\lambda_j|+|\lambda_k|}) < (Re_j \mid e_k)_{|\phi|}, \overline{e}_j \wedge e_k > (p)$$
$$-\sum_{j,k=1}^{n}\delta_j(k)\frac{|\lambda_j|}{|\lambda_j|+|\lambda_k|}\mathrm{Re}\,((Qe_j \mid e_j) \mid (\partial M_\phi e_k \mid e_k))_{|\phi|}(p)$$
$$+\frac{1}{2}\Big|\sum_{j=1}^{n}(Qe_j \mid e_j)\Big|^2_{|\phi|}(p)\Big), \tag{1.28}$$

where for $j,k,s=1,\ldots,n$,

$$a_{j,k,s}(p)=\frac{\delta_k(j)\delta_k(s)\,|\lambda_s(p)|}{2(|\lambda_j(p)|+|\lambda_k(p)|)^2(|\lambda_j(p)|+|\lambda_k(p)|+|\lambda_s(p)|)^2}, \tag{1.29}$$

Remark 1.4. It is straight forward to see that the right side of (1.28) is real (see (5.20)) and is independent of the choices of eigenvectors $U_1,\ldots,U_n$.

2. The Taylor Expansion of $\psi(z,w)$ at $z=w$

From now on, we assume that $q=n_-$. The goal of this work is to compute $\mathrm{Tr}\, b_1(p,p)$, for $p\in\mathbb{C}^n$. We may assume that $p=0$ and by taking unitary transformation, we can assume that near 0,

$$\phi(z)=a_0+\sum_{j=1}^{n}(a_jz_j+\overline{a}_j\overline{z}_j)+\sum_{j=1}^{n}\lambda_j\,|z_j|^2+O(|z|^3),$$

where $\lambda_1,\ldots,\lambda_n$ are eigenvalues of M_ϕ, $a_0\in\mathbb{R}$, $a_j\in\mathbb{C}$, $j=1,\ldots,n$. Put $\widetilde{\phi}=\phi-a_0-\sum_{j=1}^{n}(a_jz_j+\overline{a}_j\overline{z}_j)$ and let $\widetilde{\Pi}_k^{(q)}$ be the Bergman projection with respect to $e^{-2k\widetilde{\phi}}$ as in (1.9). It is easy to see that

$$\Pi_k^{(q)}=e^{ka_0+k\sum_{j=1}^{n}a_jz_j}\circ\widetilde{\Pi}_k^{(q)}\circ e^{-ka_0-k\sum_{j=1}^{n}a_jz_j}.$$

Thus, the coefficients of the asymptotic expansion of the kernel of $\Pi_k^{(q)}$ on the diagonal are the same as the coefficients of the asymptotic expansion of the kernel of $\widetilde{\Pi}_k^{(q)}$.

From the discussion above, we may assume that

$$\phi(z)=\lambda_1\,|z_1|^2+\lambda_2\,|z_2|^2+\cdots+\lambda_n\,|z_n|^2+O(|z|^3) \tag{2.1}$$

near $z=0$. Suppose that $\lambda_j<0$, $j=1,\ldots,q$, and $\lambda_j>0$, $j=q+1,\ldots,n$. In this section, we are going to compute the Taylor expansion of $\psi(z,w)$ at $z=w=0$. We introduce some notations. For $\alpha=(\alpha_1,\ldots,\alpha_n)$, $\alpha_j\in\mathbb{N}\bigcup 0$, $j=1,\ldots,n$, we put $\alpha'=(\alpha_1,\ldots,\alpha_q)$, $\alpha''=(\alpha_{q+1},\ldots,\alpha_n)$. We write $<\lambda',\alpha'>:=\sum_{j=1}^{q}\lambda_j\alpha_j$, $<\lambda'',\alpha''>:=\sum_{j=q+1}^{n}\lambda_j\alpha_j$, $<|\lambda'|,\alpha'>:=\sum_{j=1}^{q}|\lambda_j|\,\alpha_j$ and $<|\lambda''|,\alpha''>:=\sum_{j=q+1}^{n}|\lambda_j|\,\alpha_j$. The main goal of this section is to prove the following

Theorem 2.1. *Under the assumptions and notations before, we have*

$$
\begin{aligned}
\psi(z,0) &= i\sum_{j=1}^{n}|\lambda_j|\,|z_j|^2 \\
&\quad +i\sum_{|\alpha|+|\beta|=3,(\alpha'',\beta')\neq 0}\frac{<\lambda'',\alpha''>+<\lambda',\beta'>}{<|\lambda''|,\alpha''>+<|\lambda'|,\beta'>}\frac{\partial^3\phi}{\partial\overline{z}^\alpha\partial z^\beta}(0)\frac{\overline{z}^\alpha z^\beta}{\alpha!\beta!} \\
&\quad +\frac{i}{2}\sum_{q+1\leq j,k\leq n,1\leq s\leq q}\frac{1}{|\lambda_j|+|\lambda_k|+|\lambda_s|}\Big(-|\lambda_j|-|\lambda_k|-|\lambda_s| \\
&\quad +\frac{2\,|\lambda_j|\,|\lambda_k|}{|\lambda_j|+|\lambda_s|}+\frac{2\,|\lambda_j|\,|\lambda_k|}{|\lambda_k|+|\lambda_s|}\Big)\frac{\partial^3\phi}{\partial z_j\partial z_k\partial\overline{z}_s}(0)z_jz_k\overline{z}_s \\
&\quad +\frac{i}{2}\sum_{q+1\leq j\leq n,1\leq t,s\leq q}\frac{1}{|\lambda_j|+|\lambda_t|+|\lambda_s|}\Big(|\lambda_j|+|\lambda_t|+|\lambda_s| \\
&\quad -\frac{2\,|\lambda_t|\,|\lambda_s|}{|\lambda_j|+|\lambda_t|}-\frac{2\,|\lambda_t|\,|\lambda_s|}{|\lambda_j|+|\lambda_s|}\Big)\frac{\partial^3\phi}{\partial z_j\partial\overline{z}_t\partial\overline{z}_s}(0)z_j\overline{z}_t\overline{z}_s \\
&\quad -\frac{i}{3}\sum_{q+1\leq j,k,s\leq n}\frac{\partial^3\phi}{\partial z_j\partial z_k\partial z_s}(0)z_jz_kz_s \\
&\quad +\frac{i}{3}\sum_{1\leq j,k,s\leq q}\frac{\partial^3\phi}{\partial\overline{z}_j\partial\overline{z}_k\partial\overline{z}_s}(0)\overline{z}_j\overline{z}_k\overline{z}_s+O(|z|^4),
\end{aligned}
\tag{2.2}
$$

in some neighborhood of 0. *Moreover, we have*

$$
\begin{aligned}
&\frac{\partial^4\psi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0,0) \\
&=\sum_{t=1}^{q}\Big(\frac{2i\,|\lambda_j|}{\big(|\lambda_j|+|\lambda_k|\big)\big(|\lambda_t|+|\lambda_j|\big)}\left|\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_j\partial z_k}(0)\right|^2 \\
&\quad +\frac{2i\,|\lambda_t|\,|\lambda_k|\,|\lambda_j|}{\big(|\lambda_j|+|\lambda_k|\big)\big(|\lambda_t|+|\lambda_k|+|\lambda_j|\big)^2}\Big(\frac{1}{|\lambda_j|+|\lambda_t|}+\frac{1}{|\lambda_j|+|\lambda_k|}\Big)\left|\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_j\partial\overline{z}_k}(0)\right|^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2i\,|\lambda_j|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_k|)}\frac{\partial^3\phi}{\partial z_t\partial\overline{z}_j\partial z_j}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial\overline{z}_k\partial z_k}(0)\Big)\\
&+\sum_{t=q+1}^{n}\Big(\frac{2i\,|\lambda_k|}{\big(|\lambda_j|+|\lambda_k|\big)\big(|\lambda_t|+|\lambda_k|\big)}\Big|\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_j\partial z_k}(0)\Big|^2\\
&+\frac{2i\,|\lambda_t|\,|\lambda_k|\,|\lambda_j|}{\big(|\lambda_j|+|\lambda_k|\big)\big(|\lambda_t|+|\lambda_k|+|\lambda_j|\big)^2}\big(\frac{1}{|\lambda_j|+|\lambda_k|}+\frac{1}{|\lambda_t|+|\lambda_k|}\big)\Big|\frac{\partial^3\phi}{\partial\overline{z}_t\partial\overline{z}_j\partial z_k}(0)\Big|^2\\
&+\frac{2i\,|\lambda_k|}{(|\lambda_k|+|\lambda_t|)(|\lambda_j|+|\lambda_k|)}\frac{\partial^3\phi}{\partial z_t\partial\overline{z}_j\partial z_j}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial\overline{z}_k\partial z_k}(0)\Big)\\
&+i\frac{\lambda_j+\lambda_k}{|\lambda_j|+|\lambda_k|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0),
\end{aligned}
\tag{2.3}
$$

where $q+1\le j\le n$, $1\le k\le q$,

$$
\begin{aligned}
\frac{\partial^4\psi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0,0) \;=\; &\sum_{t=1}^{q}\Big(\frac{2i}{\big(|\lambda_t|+|\lambda_j|+|\lambda_k|\big)^2}\big(|\lambda_t|+|\lambda_j|+|\lambda_k|\\
&-\frac{|\lambda_j|\,|\lambda_k|}{|\lambda_j|+|\lambda_t|}-\frac{|\lambda_j|\,|\lambda_k|}{|\lambda_k|+|\lambda_t|}\big)\Big|\frac{\partial^3\phi}{\partial z_t\partial\overline{z}_j\partial\overline{z}_k}(0)\Big|^2\Big)\\
&+i\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0),
\end{aligned}
\tag{2.4}
$$

where $q+1\le j,k\le n$, *and*

$$
\begin{aligned}
\frac{\partial^4\psi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0,0) \;=\; &\sum_{t=q+1}^{n}\Big(\frac{2i}{\big(|\lambda_t|+|\lambda_j|+|\lambda_k|\big)^2}\big(|\lambda_t|+|\lambda_j|+|\lambda_k|\\
&-\frac{|\lambda_j|\,|\lambda_k|}{|\lambda_k|+|\lambda_t|}-\frac{|\lambda_j|\,|\lambda_k|}{|\lambda_j|+|\lambda_t|}\big)\Big|\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_j\partial z_k}(0)\Big|^2\Big)\\
&-i\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0),
\end{aligned}
\tag{2.5}
$$

where $1\le j,k\le q$.

To prove Theorem 2.1, we first need the following (see [7], for a proof).

Proposition 2.2. *Under the assumptions above, we have*

$$
\psi(z,w)=i\sum_{j=1}^{n}|\lambda_j|\,|z_j-w_j|^2+i\sum_{j=1}^{n}\lambda_j(\overline{z}_jw_j-\overline{w}_jz_j)+O(|(z,w)|^3) \tag{2.6}
$$

near $z = w = 0$.

From (2.1) and (2.6), we may write

$$\begin{aligned}\psi(z,w) \;&=\; i\sum_{j=1}^{n}|\lambda_j|\,|z_j-w_j|^2+i\sum_{j=1}^{n}\lambda_j(\overline{z}_jw_j-\overline{w}_jz_j)\\ &\quad+\psi_3(z,w)+\psi_4(z,w)+\cdots\,,\end{aligned}\tag{2.7}$$

where $\psi_j(z,w)$ is a homogeneous polynomial of degree j in (z,w), $j = 3,4,\ldots$, and

$$\phi(z)=\sum_{j=1}^{n}\lambda_j\,|z_j|^2+\phi_3(z)+\phi_4(z)+\cdots\,,\tag{2.8}$$

where $\phi_j(z)$ is a homogeneous polynomial of degree j in z, $j=3,4,\ldots$. Now, using (2.7) and (2.8) in (1.13) , we get

$$\begin{aligned}&\sum_{j=1}^{q}\Big((2\lambda_j(z_j-w_j)+i\frac{\partial\psi_3}{\partial\overline{z}_j}+\frac{\partial\phi_3}{\partial\overline{z}_j}+i\frac{\partial\psi_4}{\partial\overline{z}_j}+\frac{\partial\phi_4}{\partial\overline{z}_j})\\ &\quad\times(-i\frac{\partial\psi_3}{\partial z_j}+\frac{\partial\phi_3}{\partial z_j}-i\frac{\partial\psi_4}{\partial z_j}+\frac{\partial\phi_4}{\partial z_j})\Big)\\ &\quad+\sum_{j=q+1}^{n}\Big((i\frac{\partial\psi_3}{\partial\overline{z}_j}+\frac{\partial\phi_3}{\partial\overline{z}_j}+i\frac{\partial\psi_4}{\partial\overline{z}_j}+\frac{\partial\phi_4}{\partial\overline{z}_j})\\ &\quad\times(2\lambda_j(\overline{z}_j-\overline{w}_j)-i\frac{\partial\psi_3}{\partial z_j}+\frac{\partial\phi_3}{\partial z_j}-i\frac{\partial\psi_4}{\partial z_j}+\frac{\partial\phi_4}{\partial z_j})\Big)=O(|(z,w)|^5).\end{aligned}\tag{2.9}$$

We regroup the terms in (2.9) according to the order. Then, the order three and order four terms are the following

$$\begin{aligned}&T\psi_3(z,w)-\sum_{j=1}^{q}2i\,|\lambda_j|\,w_j\frac{\partial\psi_3(z,w)}{\partial z_j}-\sum_{j=q+1}^{n}2i\,|\lambda_j|\,\overline{w}_j\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}\\ &\quad=\;-\sum_{j=1}^{q}2\lambda_j(z_j-w_j)\frac{\partial\phi_3(z)}{\partial z_j}-\sum_{j=q+1}^{n}2\lambda_j(\overline{z}_j-\overline{w}_j)\frac{\partial\phi_3(z)}{\partial\overline{z}_j},\end{aligned}\tag{2.10}$$

$$T\psi_4(z,w)-\sum_{j=1}^{q}2i\,|\lambda_j|\,w_j\frac{\partial\psi_4(z,w)}{\partial z_j}-\sum_{j=q+1}^{n}2i\,|\lambda_j|\,\overline{w}_j\frac{\partial\psi_4(z,w)}{\partial\overline{z}_j}$$

$$= -\sum_{j=1}^{q} 2\lambda_j (z_j - w_j)\frac{\partial\phi_4(z)}{\partial z_j} - \sum_{j=q+1}^{n} 2\lambda_j(\overline{z}_j - \overline{w}_j)\frac{\partial\phi_4(z)}{\partial\overline{z}_j}$$
$$-\sum_{j=1}^{n}\Big(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j} + \frac{\partial\phi_3(z)}{\partial\overline{z}_j}\Big)\Big(-i\frac{\partial\psi_3(z,w)}{\partial z_j} + \frac{\partial\phi_3(z)}{\partial z_j}\Big), \quad (2.11)$$

where

$$T = \sum_{j=1}^{q} 2i\,|\lambda_j|\,z_j\frac{\partial}{\partial z_j} + \sum_{j=q+1}^{n} 2i\,|\lambda_j|\,\overline{z}_j\frac{\partial}{\partial\overline{z}_j}. \quad (2.12)$$

Let

$$\psi_3 = \psi_3^0 + \psi_3^1 + \psi_3^2 + \psi_3^3, \quad (2.13)$$

where ψ_3^j is a homogeneous polynomial of degree j in w, $j = 0, 1, 2, 3$. Now, we write (2.10) according to the degree of homogenity in w, we get

$$T\psi_3^0 = -\sum_{j=1}^{q} 2\lambda_j z_j\frac{\partial\phi_3}{\partial z_j} - \sum_{j=q+1}^{n} 2\lambda_j\overline{z}_j\frac{\partial\phi_3}{\partial\overline{z}_j}, \quad (2.14)$$
$$T\psi_3^1 = \sum_{j=1}^{q} 2\lambda_j w_j(-i\frac{\partial\psi_3^0}{\partial z_j} + \frac{\partial\phi_3}{\partial z_j}) + \sum_{j=q+1}^{n} 2\lambda_j\overline{w}_j(i\frac{\partial\psi_3^0}{\partial\overline{z}_j} + \frac{\partial\phi_3}{\partial\overline{z}_j}), \quad (2.15)$$
$$T\psi_3^2 = \sum_{j=1}^{q} 2i\,|\lambda_j|\,w_j\frac{\partial\psi_3^1}{\partial z_j} + \sum_{j=q+1}^{n} 2i\,|\lambda_j|\,\overline{w}_j\frac{\partial\psi_3^1}{\partial\overline{z}_j}. \quad (2.16)$$

We need

Lemma 2.3. *We use the same notations as before. Let*

$$g = \sum_{\alpha,\beta,\gamma,\delta} g_{\alpha,\beta,\gamma,\delta}\overline{z}^\alpha z^\beta\overline{w}^\gamma w^\delta, h = \sum_{\alpha,\beta,\gamma,\delta} h_{\alpha,\beta,\gamma,\delta}\overline{z}^\alpha z^\beta\overline{w}^\gamma w^\delta,$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}\bigcup 0$, $j = 1, \ldots, n$, *and similar for* β, γ, δ. *If* $Tg = h$, *then*

$$h_{\alpha,\beta,\gamma,\delta} = 0 \quad \text{if} \quad (\alpha'', \beta') = 0, \quad (2.17)$$
$$g_{\alpha,\beta,\gamma,\delta} = \frac{1}{2i}\frac{1}{<|\lambda''|, \alpha''> + <|\lambda'|, \beta'>}h_{\alpha,\beta,\gamma,\delta} \quad \text{if} \quad (\alpha'', \beta') \neq 0. \quad (2.18)$$

Proof. From the definition of T (see (2.12)), we can compute

$$T(g_{\alpha,\beta,\gamma,\delta}\overline{z}^\alpha z^\beta \overline{w}^\gamma w^\delta) = 0$$

if $(\alpha'', \beta') = 0$, and

$$T(g_{\alpha,\beta,\gamma,\delta}\overline{z}^\alpha z^\beta \overline{w}^\gamma w^\delta) = 2i(< \left|\lambda''\right|, \alpha'' > + < \left|\lambda'\right|, \beta' >)\overline{z}^\alpha z^\beta \overline{w}^\gamma w^\delta g_{\alpha,\beta,\gamma,\delta}$$

if $(\alpha'', \beta') \neq 0$. From this, the lemma follows. □

From Lemma 2.3, (2.14), (2.15), (2.16) and very complicated computation, we can determine ψ_3^0, ψ_3^1 and ψ_3^2 modulo terms $r(z,w) = O(|(z,w)|^3)$ with $\frac{\partial^{|\alpha|+|\beta|}r}{\partial\overline{z}^\alpha\partial z^\beta} = 0$ if $(\alpha'', \beta') \neq 0$. Moreover, we have the following(see Section 2 in [10] for the details)

Proposition 2.4. *Under the assumptions and notations before, we have*

$$\begin{aligned}
\psi(z,w) &= i\sum_{j=1}^{n}|\lambda_j|\,|z_j - w_j|^2 + i\sum_{j=1}^{n}\lambda_j(\overline{z}_j w_j - \overline{w}_j z_j)\\
&+i\sum_{|\alpha|+|\beta|=3,(\alpha'',\beta')\neq 0}\frac{< \lambda'', \alpha'' > + < \lambda', \beta' >}{< |\lambda''|, \alpha'' > + < |\lambda'|, \beta' >}\frac{\partial^3\phi}{\partial\overline{z}^\alpha\partial z^\beta}(0)\frac{\overline{z}^\alpha z^\beta}{\alpha!\beta!}\\
&+i\sum_{j=1}^{q}\sum_{|\alpha|+|\beta|=2,\alpha''\neq 0}\Big(\frac{\overline{z}^\alpha z^\beta w_j}{\alpha!\beta!}\frac{\partial^3\phi}{\partial\overline{z}^\alpha\partial z^\beta\partial z_j}(0)\\
&\times\frac{2\,|\lambda_j| < |\lambda''|, \alpha'' >}{(< |\lambda''|, \alpha'' > + < |\lambda'|, \beta' >)(< |\lambda''|, \alpha'' > + < |\lambda'|, \beta' > + |\lambda_j|)}\Big)\\
&-i\sum_{j=q+1}^{n}\sum_{|\alpha|+|\beta|=2,\beta'\neq 0}\Big(\frac{\overline{z}^\alpha z^\beta \overline{w}_j}{\alpha!\beta!}\frac{\partial^3\phi}{\partial\overline{z}^\alpha\partial z^\beta\partial\overline{z}_j}(0)\\
&\times\frac{2\,|\lambda_j| < |\lambda'|, \beta' >}{(< |\lambda''|, \alpha'' > + < |\lambda'|, \beta' >)(< |\lambda''|, \alpha'' > + < |\lambda'|, \beta' > + |\lambda_j|)}\Big)\\
&+i\sum_{q+1\leq k\leq n,1\leq j,s\leq q}\Big(\overline{z}_k w_j w_s\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_j\partial z_s}(0)\frac{|\lambda_j|\,|\lambda_s|}{|\lambda_k| + |\lambda_j| + |\lambda_s|}\\
&\times\big(\frac{1}{|\lambda_j| + |\lambda_k|} + \frac{1}{|\lambda_k| + |\lambda_s|}\big)\Big)\\
&-i\sum_{1\leq k\leq q,q+1\leq j,s\leq n}\Big(z_k\overline{w}_j\overline{w}_s\frac{\partial^3\phi}{\partial z_k\partial\overline{z}_j\partial\overline{z}_s}(0)\frac{|\lambda_j|\,|\lambda_s|}{|\lambda_k| + |\lambda_j| + |\lambda_s|}
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\frac{1}{|\lambda_j|+|\lambda_k|}+\frac{1}{|\lambda_k|+|\lambda_s|}\Big)\Big)\\
&+i\sum_{q+1\leq s,k\leq n,1\leq j\leq q}\overline{z}_s\overline{w}_k w_j\frac{2\,|\lambda_j|\,|\lambda_k|}{(|\lambda_s|+|\lambda_j|)(|\lambda_s|+|\lambda_j|+|\lambda_k|)}\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial\overline{z}_k}(0)\\
&-i\sum_{q+1\leq k\leq n,1\leq j,s\leq q}z_s\overline{w}_k w_j\frac{2\,|\lambda_j|\,|\lambda_k|}{(|\lambda_s|+|\lambda_k|)(|\lambda_s|+|\lambda_j|+|\lambda_k|)}\frac{\partial^3\phi}{\partial z_s\partial z_j\partial\overline{z}_k}(0)\\
&+R(z,w)+O(|(z,w)|^4), \qquad (2.19)
\end{aligned}
$$

where $R(z,w)=O(|(z,w)|^3)$ *and* $\frac{\partial^{|\alpha|+|\beta|}R}{\partial\overline{z}^\alpha\partial z^\beta}=0$ *if* $(\alpha'',\beta')\neq 0$.

Now, we are ready to compute $\frac{\partial^3\psi}{\partial\overline{z}^\alpha\partial z^\beta}(0,0)$, where $|\alpha|+|\beta|=3$ and $(\alpha'',\beta')=0$. We compute $\frac{\partial^3\psi}{\partial\overline{z}_{s_0}\partial z_{j_0}\partial z_{k_0}}(0,0)$, $1\leq s_0\leq q$, $q+1\leq j_0,k_0\leq n$. From $\psi(z,w)=-\overline{\psi}(w,z)$, we have

$$
\frac{\partial^3\psi(z,w)}{\partial\overline{z}_{s_0}\partial z_{j_0}\partial z_{k_0}}=-\overline{\frac{\partial^3\psi(z,w)}{\partial w_{s_0}\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}}}. \qquad (2.20)
$$

To compute $\frac{\partial^3\psi}{\partial\overline{z}_{s_0}\partial z_{j_0}\partial z_{k_0}}(0,0)$, it is equivalent to compute $\frac{\partial^3\psi}{\partial w_{s_0}\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}}(0,0)$. From (1.12), we know that

$$
\frac{\partial\psi(z,z)}{\partial\overline{w}_{j_0}}=-i\frac{\partial\phi(z)}{\partial\overline{z}_{j_0}}. \qquad (2.21)
$$

Differentiate (2.21) with respect to $\overline{z}_{k_0}$, we get

$$
\frac{\partial^2\psi(z,z)}{\partial\overline{w}_{j_0}\partial\overline{z}_{k_0}}+\frac{\partial^2\psi(z,z)}{\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}}=-i\frac{\partial^2\phi(z)}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}}. \qquad (2.22)
$$

Again, differentiate (2.22) with respect to z_{s_0}, we get

$$
\begin{aligned}
&\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{z}_{k_0}z_{s_0}}(0,0)+\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{z}_{k_0}\partial w_{s_0}}(0,0)\\
&+\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}\partial z_{s_0}}(0,0)+\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}\partial w_{s_0}}(0,0)=-i\frac{\partial^3\phi}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}\partial z_{s_0}}(0). \qquad (2.23)
\end{aligned}
$$

From (2.23), we have

$$
\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}\partial w_{s_0}}(0,0) \;=\; -i\frac{\partial^3\phi}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}\partial z_{s_0}}(0)-\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{z}_{k_0}z_{s_0}}(0,0)
$$

$$-\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{z}_{k_0}\partial w_{s_0}}(0,0)-\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}\partial z_{s_0}}(0,0). \quad (2.24)$$

In view of (2.19), we see that

$$\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{z}_{k_0}z_{s_0}}(0,0)=\frac{-2i\,|\lambda_{j_0}|\,|\lambda_{s_0}|}{\big(|\lambda_{j_0}|+|\lambda_{k_0}|+|\lambda_{s_0}|\big)\big(|\lambda_{k_0}|+|\lambda_{s_0}|\big)}\frac{\partial^3\phi}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}\partial z_{s_0}}(0)$$
$$\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{z}_{k_0}\partial w_{s_0}}(0,0)=\frac{2i\,|\lambda_{j_0}|\,|\lambda_{s_0}|}{\big(|\lambda_{j_0}|+|\lambda_{k_0}|+|\lambda_{s_0}|\big)\big(|\lambda_{k_0}|+|\lambda_{s_0}|\big)}\frac{\partial^3\phi}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}\partial z_{s_0}}(0)$$
$$\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}\partial z_{s_0}}(0,0)=\frac{-2i\,|\lambda_{j_0}|\,|\lambda_{k_0}|}{|\lambda_{k_0}|+|\lambda_{s_0}|+|\lambda_{j_0}|}\times\Big(\frac{1}{|\lambda_{j_0}|+|\lambda_{s_0}|}+\frac{1}{|\lambda_{k_0}|+|\lambda_{s_0}|}\Big)\frac{\partial^3\phi}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}\partial z_{s_0}}(0).$$

Combining this with (2.24), we have

$$\begin{aligned}
&\frac{\partial^3\psi}{\partial\overline{w}_{j_0}\partial\overline{w}_{k_0}\partial w_{s_0}}(0,0)\\
&=\Big(-i-\frac{-2i\,|\lambda_{j_0}|\,|\lambda_{s_0}|}{\big(|\lambda_{j_0}|+|\lambda_{k_0}|+|\lambda_{s_0}|\big)\big(|\lambda_{k_0}|+|\lambda_{s_0}|\big)}\\
&\quad-\frac{2i\,|\lambda_{j_0}|\,|\lambda_{s_0}|}{\big(|\lambda_{j_0}|+|\lambda_{k_0}|+|\lambda_{s_0}|\big)\big(|\lambda_{k_0}|+|\lambda_{s_0}|\big)}\\
&\quad-\frac{-2i\,|\lambda_{j_0}|\,|\lambda_{k_0}|}{|\lambda_{k_0}|+|\lambda_{s_0}|+|\lambda_{j_0}|}\Big(\frac{1}{|\lambda_{j_0}|+|\lambda_{s_0}|}+\frac{1}{|\lambda_{k_0}|+|\lambda_{s_0}|}\Big)\Big)\frac{\partial^3\phi}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}\partial z_{s_0}}(0)\\
&=\frac{i}{|\lambda_{j_0}|+|\lambda_{k_0}|+|\lambda_{s_0}|}\Big(-|\lambda_{j_0}|-|\lambda_{k_0}|-|\lambda_{s_0}|\\
&\quad+\frac{2\,|\lambda_{j_0}|\,|\lambda_{k_0}|}{|\lambda_{j_0}|+|\lambda_{s_0}|}+\frac{2\,|\lambda_{j_0}|\,|\lambda_{k_0}|}{|\lambda_{k_0}|+|\lambda_{s_0}|}\Big)\frac{\partial^3\phi}{\partial\overline{z}_{k_0}\partial\overline{z}_{j_0}\partial z_{s_0}}(0).
\end{aligned}\quad (2.25)$$

From (2.20), we obtain

$$\begin{aligned}
\frac{\partial^3\psi_3}{\partial z_{j_0}\partial z_{k_0}\partial\overline{z}_{s_0}}&=\frac{i}{|\lambda_{j_0}|+|\lambda_{k_0}|+|\lambda_{s_0}|}\Big(-|\lambda_{j_0}|-|\lambda_{k_0}|-|\lambda_{s_0}|\\
&\quad+\frac{2\,|\lambda_{j_0}|\,|\lambda_{k_0}|}{|\lambda_{j_0}|+|\lambda_{s_0}|}+\frac{2\,|\lambda_{j_0}|\,|\lambda_{k_0}|}{|\lambda_{k_0}|+|\lambda_{s_0}|}\Big)\frac{\partial^3\phi}{\partial z_{j_0}\partial z_{k_0}\partial\overline{z}_{s_0}}(0).
\end{aligned}\quad (2.26)$$

We can repeat the method above several times to determine all the terms $\frac{\partial^3\psi_3}{\partial\overline{z}^\alpha\partial z^\beta}$, $(\alpha'',\beta')=0$. The computation is straight forward. We omit the process. We state our result

Proposition 2.5. *Under the assumptions and notations before, we have*

$$\begin{aligned}
\psi(z,0) \;=\; & i\sum_{j=1}^{n}|\lambda_j|\,|z_j|^2 \\
& +i\sum_{|\alpha|+|\beta|=3,(\alpha'',\beta')\neq 0}\frac{<\lambda'',\alpha''>+<\lambda',\beta'>}{<|\lambda''|,\alpha''>+<|\lambda'|,\beta'>}\frac{\partial^3\phi}{\partial\overline{z}^\alpha\partial z^\beta}(0)\frac{\overline{z}^\alpha z^\beta}{\alpha!\beta!} \\
& +\frac{i}{2}\sum_{q+1\le j,k\le n,1\le s\le q}\frac{1}{|\lambda_j|+|\lambda_k|+|\lambda_s|}\Big(-|\lambda_j|-|\lambda_k|-|\lambda_s| \\
& +\frac{2\,|\lambda_j|\,|\lambda_k|}{|\lambda_j|+|\lambda_s|}+\frac{2\,|\lambda_j|\,|\lambda_k|}{|\lambda_k|+|\lambda_s|}\Big)\frac{\partial^3\phi}{\partial z_j\partial z_k\partial\overline{z}_s}(0)z_jz_k\overline{z}_s \\
& +\frac{i}{2}\sum_{q+1\le j\le n,1\le k,s\le q}\frac{1}{|\lambda_j|+|\lambda_k|+|\lambda_s|}\Big(|\lambda_j|+|\lambda_k|+|\lambda_s| \\
& -\frac{2\,|\lambda_k|\,|\lambda_s|}{|\lambda_j|+|\lambda_k|}-\frac{2\,|\lambda_k|\,|\lambda_s|}{|\lambda_j|+|\lambda_s|}\Big)\frac{\partial^3\phi}{\partial z_j\partial\overline{z}_k\partial\overline{z}_s}(0)z_j\overline{z}_k\overline{z}_s \\
& -\frac{i}{3}\sum_{q+1\le j,k,s\le n}\frac{\partial^3\phi}{\partial z_j\partial z_k\partial z_s}(0)z_jz_kz_s \\
& +\frac{i}{3}\sum_{1\le j,k,s\le q}\frac{\partial^3\phi}{\partial\overline{z}_j\partial\overline{z}_k\partial\overline{z}_s}(0)\overline{z}_j\overline{z}_k\overline{z}_s+O(|z|^4), \qquad (2.27)
\end{aligned}$$

in some neighborhood of 0.

Now, to complete the proof of Theorem 2.1, we only need to compute the terms $\frac{\partial^4\psi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0,0)$, $1\le j,k\le n$. Take $w=0$ in (2.11), we obtain

$$\begin{aligned}
T\psi_4(z,0) \;=\; & -\sum_{j=1}^{n}\big(i\frac{\partial\psi_3}{\partial\overline{z}_j}(z,0)+\frac{\partial\phi_3}{\partial\overline{z}_j}(z)\big)\big(-i\frac{\partial\psi_3}{\partial z_j}(z,0)+\frac{\partial\phi_3}{\partial z_j}(z)\big) \\
& -\sum_{j=1}^{q}2\lambda_jz_j\frac{\partial\phi_4}{\partial z_j}(z)-\sum_{j=q+1}^{n}2\lambda_j\overline{z}_j\frac{\partial\phi_4}{\partial\overline{z}_j}(z). \qquad (2.28)
\end{aligned}$$

From (2.28) and by some direct but very complicated computation, we can determine the terms $\frac{\partial^4\psi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0,0)$, $1\le j,k\le n$ and hence finish the proof of Theorem 2.1. We refer the reader to Section 2 in [10] for the details.

3. The Transport Equations for $\Box_k^{(q)}$

3.1. The transport equations

In this section, we will write down the transport equations for $\Box_k^{(q)}$ and we will solve the first transport equation at $z = w$ in some sence. The main reference for this section is [1]. We first derive representations for $\overline{\partial}$, $\overline{\partial}_k^*$ in spaces without exponential weights, by using the following unitary identifications:

$$\begin{cases} L_q^2(\mathbb{C}^n) \leftrightarrow L_{q,k}^2(\mathbb{C}^n) \\ \qquad u \leftrightarrow \hat{u} = e^{k\phi}u. \end{cases} \tag{3.1}$$

Using (3.1), we get

$$\overline{\partial}\hat{u} = e^{k\phi}\overline{\partial}_s u, \tag{3.2}$$

where

$$\overline{\partial}_s = \sum_{j=1}^{n}\Big(d\overline{z}_j^{\wedge} \circ (\frac{\partial}{\partial \overline{z}_j} + k\frac{\partial \phi}{\partial \overline{z}_j})\Big). \tag{3.3}$$

Now the formal adjoint of $\overline{\partial}_s$ for the inner product $(\ |\ )$ given by (1.4) is

$$\overline{\partial}_s^* = \sum_{j=1}^{n}\Big(d\overline{z}_j^{\wedge,*} \circ (-\frac{\partial}{\partial z_j} + k\frac{\partial \phi}{\partial z_j})\Big), \tag{3.4}$$

where in view of the unitarity of the relation (3.1),

$$\overline{\partial}^*\hat{u} = e^{k\phi}\overline{\partial}_s^* u. \tag{3.5}$$

We can identify the Kodaira Laplacian with

$$\hat{\Box}_k^{(q)} = \overline{\partial}_s\overline{\partial}_s^* + \overline{\partial}_s^*\overline{\partial}_s. \tag{3.6}$$

Put

$$\hat{\Pi}_k^{(q)} : L_q^2(\mathbb{C}^n) \to \operatorname{Ker}\hat{\Box}_k^{(q)} \tag{3.7}$$

be the orthogonal projection with respect to $(\ |\ )$ and let $\hat{\Pi}_k^{(q)}(z,w)$ be the distribution kernel of $\hat{\Pi}_k^{(q)}$. From (3.2) and (3.5), we have

$$\hat{\Pi}_k^{(q)}(z,w) = e^{-k\phi(z)}\Pi_k^{(q)}(z,w)e^{k\phi(w)}. \tag{3.8}$$

In view of Theorem 1.1, we see that

$$\hat{\Pi}_k^{(q)}(z,w)=e^{ik\psi(z,w)}b(z,w,k)+S(z,w,k), \tag{3.9}$$

where $b(z.w,k)\sim\sum_o^\infty b_j(z,w)k^{n-j}$ in
$C^\infty(\mathbb{C}^n\times\mathbb{C}^n;\mathscr{L}(\Lambda^{0,q}T^*_w(\mathbb{C}^n),\Lambda^{0,q}T^*_z(\mathbb{C}^n)))$ and $S(z,w,k)$ is negligible.

From (3.3) and (3.9), we have

$$e^{-ik\psi}\overline{\partial}_s\hat{\Pi}_k^{(q)}(z,w)=\sum_{j=1}^n k(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_j^\wedge b+\sum_{j=1}^n d\overline{z}_j^\wedge\frac{\partial b}{\partial\overline{z}_j}. \tag{3.10}$$

From this and (3.4), we can compute

$$\begin{aligned}e^{-ik\psi}\overline{\partial}_s^*\overline{\partial}_s\hat{\Pi}_k^{(q)}(z,w)&=k^2\sum_{j,t=1}^n(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^\wedge b\\ -k\sum_{j,t=1}^n\frac{\partial}{\partial z_t}(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^\wedge b&-k\sum_{j,t=1}^n(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^\wedge\frac{\partial b}{\partial z_t}\\ +k\sum_{j,t=1}^n(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})d\overline{z}_t^{\wedge,*}d\overline{z}_j^\wedge\frac{\partial b}{\partial\overline{z}_j}&-\sum_{j,t=1}^n d\overline{z}_t^{\wedge,*}d\overline{z}_j^\wedge\frac{\partial^2 b}{\partial z_t\partial\overline{z}_j}.\end{aligned} \tag{3.11}$$

Similarly, we have

$$\begin{aligned}e^{-ik\psi}\overline{\partial}_s\overline{\partial}_s^*\hat{\Pi}_k^{(q)}(z,w)&=k^2\sum_{j,t=1}^n(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})d\overline{z}_j^\wedge d\overline{z}_t^{\wedge,*}b\\ +k\sum_{j,t=1}^n\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})d\overline{z}_j^\wedge d\overline{z}_t^{\wedge,*}b&+k\sum_{j,t=1}^n(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})d\overline{z}_j^\wedge d\overline{z}_t^{\wedge,*}\frac{\partial b}{\partial\overline{z}_j}\\ -k\sum_{j,t=1}^n(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_j^\wedge d\overline{z}_t^{\wedge,*}\frac{\partial b}{\partial z_t}&-\sum_{j,t=1}^n d\overline{z}_j^\wedge d\overline{z}_t^{\wedge,*}\frac{\partial^2 b}{\partial z_t\partial\overline{z}_j}.\end{aligned} \tag{3.12}$$

Combining (3.11), (3.12) with $d\overline{z}_j^\wedge\circ d\overline{z}_k^{\wedge,*}+d\overline{z}_k^{\wedge,*}\circ d\overline{z}_j^\wedge=\delta_{j,k}$, we get

$$\begin{aligned}e^{-ik\psi}\hat{\Box}_k^{(q)}\hat{\Pi}_k^{(q)}(z,w)&=k^2\sum_{j=1}^n(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})(-i\frac{\partial\psi}{\partial z_j}+\frac{\partial\phi}{\partial z_j})b\\ +k\sum_{j,t=1}^n\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})d\overline{z}_j^\wedge d\overline{z}_t^{\wedge,*}b&-k\sum_{j,t=1}^n\frac{\partial}{\partial z_t}(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^\wedge b\end{aligned}$$

$$+k\sum_{j=1}^{n}(-i\frac{\partial\psi}{\partial z_j}+\frac{\partial\phi}{\partial z_j})\frac{\partial b}{\partial\overline{z}_j}-k\sum_{j=1}^{n}(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})\frac{\partial b}{\partial z_j}-\sum_{j=1}^{n}\frac{\partial^2 b}{\partial z_j\partial\overline{z}_j}. \quad (3.13)$$

We regroup (3.13) according to the degree of k and notice that the leading term in (3.13) vanishes to infinite order on $z=w$ and $\hat{\Box}_k^{(q)}\hat{\Pi}_k^{(q)}=0$, we obtain the following

Proposition 3.1. *We have*

$$\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}b_0-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0$$
$$+\sum_{j=1}^{n}(-i\frac{\partial\psi}{\partial z_j}+\frac{\partial\phi}{\partial z_j})\frac{\partial b_0}{\partial\overline{z}_j}-\sum_{j=1}^{n}(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})\frac{\partial b_0}{\partial z_j} \quad (3.14)$$

vanishes to infinite order on $z=w$ and

$$\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi}{\partial z_t}+\frac{\partial\phi}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}b_1-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_1$$
$$+\sum_{j=1}^{n}(-i\frac{\partial\psi}{\partial z_j}+\frac{\partial\phi}{\partial z_j})\frac{\partial b_1}{\partial\overline{z}_j}-\sum_{j=1}^{n}(i\frac{\partial\psi}{\partial\overline{z}_j}+\frac{\partial\phi}{\partial\overline{z}_j})\frac{\partial b_1}{\partial z_j}-\sum_{j=1}^{n}\frac{\partial^2 b_0}{\partial z_j\partial\overline{z}_j} \quad (3.15)$$

vanishes to infinite order on $z=w$.

3.2. The first order of the Taylor expansion of $b_0(z,w)$ at $z=w$

Now, as in section 2, we assume that $\phi(z)=\sum_{j=1}^{n}\lambda_j\left|z_j\right|^2+O(\left|z\right|^3)$ near 0 and that $\lambda_j<0$, $j=1,\ldots,q$ and $\lambda_j>0$, $j=q+1,\ldots,n$. We work in some neighborhood of $(0,0)$. Put

$$b_0(z,w)=b_0^0(z,w)+b_0^1(z,w)+b_0^2(z,w)+\cdots, \quad (3.16)$$

where b_0^j is a homogeneous polynomialof degree j in (z,w). We recall that

$$b_0^0=\pi^{-n}\left|\lambda_1\right|\left|\lambda_2\right|\cdots\left|\lambda_n\right|\prod_{j=1}^{q}d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}.$$

(See Theorem 1.2.) For $1\leq s\leq n$, put $\chi_1(s)=1$ if $1\leq s\leq q$ and $\chi_1(s)=0$ if $q+1\leq s\leq n$ and put $\chi_2(s)=1-\chi_1(s)$. We recall that for any operator

$T \in \mathscr{L}(\Lambda^{p,q}T^*(\mathbb{C}^n), \Lambda^{p,q}T^*(\mathbb{C}^n))$, the trace of T is given by (1.1). The first goal of this section is to prove the following

Theorem 3.2. *Under the notations above, we have*

$$
\begin{aligned}
&b_0^1(z,0) = \\
&\sum_{q+1\le j\le n, 1\le k\le q, 1\le s\le n} \frac{1}{|\lambda_j|+|\lambda_k|+|\lambda_s|\,\chi_1(s)} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0) z_s d\overline{z}_k^{\wedge,*} d\overline{z}_j^{\wedge} b_0^0 \\
&+ \sum_{q+1\le j\le n, 1\le k\le q, 1\le s\le n} \frac{1}{|\lambda_j|+|\lambda_k|+|\lambda_s|\,\chi_2(s)} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial \overline{z}_s}(0) \overline{z}_s d\overline{z}_k^{\wedge,*} d\overline{z}_j^{\wedge} b_0^0 \\
&+ \sum_{q+1\le j\le n, 1\le k\le q, q+1\le s\le n} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_k|)(|\lambda_j|+|\lambda_k|+|\lambda_s|)} \frac{\partial^3\phi}{\partial z_j\partial \overline{z}_k\partial z_s}(0) z_s b_0^0 d\overline{z}_j^{\wedge,*} d\overline{z}_k^{\wedge} \\
&+ \sum_{q+1\le j\le n, 1\le k\le q, 1\le s\le q} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_k|)(|\lambda_j|+|\lambda_k|+|\lambda_s|)} \frac{\partial^3\phi}{\partial z_j\partial \overline{z}_k\partial \overline{z}_s}(0) \overline{z}_s b_0^0 d\overline{z}_j^{\wedge,*} d\overline{z}_k^{\wedge} \\
&- \sum_{q+1\le s\le n, 1\le j\le q} \frac{1}{|\lambda_j|+|\lambda_s|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial \overline{z}_s}(0) \overline{z}_s b_0^0 - \sum_{1\le s,j\le q} \frac{1}{|\lambda_j|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial \overline{z}_s}(0) \overline{z}_s b_0^0 \\
&+ \sum_{1\le s\le q, q+1\le j\le n} \frac{|\lambda_s|}{|\lambda_j|\,(|\lambda_j|+|\lambda_s|)} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial \overline{z}_s}(0) \overline{z}_s b_0^0 \\
&+ \sum_{1\le s\le q, q+1\le j\le n} \frac{1}{|\lambda_j|+|\lambda_s|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0) z_s b_0^0 \\
&- \sum_{q+1\le s\le n, 1\le j\le q} \frac{|\lambda_s|}{|\lambda_j|\,(|\lambda_j|+|\lambda_s|)} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0) z_s b_0^0 \\
&+ \sum_{q+1\le s,j\le n} \frac{1}{|\lambda_j|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0) z_s b_0^0.
\end{aligned} \tag{3.17}
$$

As in section 2, we write

$$
\begin{aligned}
\psi(z,w) = i\sum_{j=1}^n |\lambda_j|\,|z_j-w_j|^2 + i\sum_{j=1}^n \lambda_j(\overline{z}_j w_j - \overline{w}_j z_j) \\
+ \psi_3(z,w) + \psi_4(z,w) + \cdots ,
\end{aligned} \tag{3.18}
$$

and

$$
\phi(z) = \sum_{j=1}^n \lambda_j\,|z_j|^2 + \phi_3(z) + \phi_4(z) + \cdots , \tag{3.19}
$$

where $\psi_j(z,w)$ is a homogeneous polynomial of degree j in (z,w), $j=3,4,\ldots$, $\phi_j(z)$ is a homogeneous polynomial of degree j in z, $j=3,4,\ldots$. Now, using (3.16), (3.18) and (3.19) in (3.14), we get

$$\begin{aligned}
&\Big(\sum_{j=q+1}^{n}2\left|\lambda_j\right|d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}+\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_t}+\frac{\partial\phi_3(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}\\
&+\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_4(z,w)}{\partial z_t}+\frac{\partial\phi_4(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}\Big)(b_0^0+b_0^1(z,w)+b_0^2(z,w))\\
&+\Big(\sum_{j=1}^{q}2\left|\lambda_j\right|d\overline{z}_j^{\wedge,*}d\overline{z}_j^{\wedge}-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\\
&-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_4(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_4(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)(b_0^0+b_0^1(z,w)+b_0^2(z,w))\\
&+\sum_{j=q+1}^{n}2\left|\lambda_j\right|(\overline{z}_j-\overline{w}_j)(\frac{\partial b_0^1(z,w)}{\partial\overline{z}_j}+\frac{\partial b_0^2(z,w)}{\partial\overline{z}_j})+\sum_{j=1}^{n}(-i\frac{\partial\psi_3}{\partial z_j}+\frac{\partial\phi_3}{\partial z_j})\frac{\partial b_0^1(z,w)}{\partial\overline{z}_j}\\
&+\sum_{j=1}^{q}2\left|\lambda_j\right|(z_j-w_j)(\frac{\partial b_0^1(z,w)}{\partial z_j}+\frac{\partial b_0^2(z,w)}{\partial z_j})-\sum_{j=1}^{n}(i\frac{\partial\psi_3}{\partial\overline{z}_j}+\frac{\partial\phi_3}{\partial\overline{z}_j})\frac{\partial b_0^1(z,w)}{\partial z_j}\\
&=O(\left|(z,w)\right|^3).
\end{aligned}\tag{3.20}$$

It is straight forward to see that the order 1 and 2 terms in (3.20) are the following

$$\begin{aligned}
Lb_0^1(z,w) \;=\; &-\Big(\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_t}+\frac{\partial\phi_3(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}\\
&-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_0^0\\
&+\sum_{j=q+1}^{n}2\left|\lambda_j\right|\overline{w}_j\frac{\partial b_0^1(z,w)}{\partial\overline{z}_j}+\sum_{j=1}^{q}2\left|\lambda_j\right|w_j\frac{\partial b_0^1(z,w)}{\partial z_j}
\end{aligned}\tag{3.21}$$

and

$$\begin{aligned}
Lb_0^2(z,w) \;=\; &-\Big(\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_4(z,w)}{\partial z_t}+\frac{\partial\phi_4(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}\\
&-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_4(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_4(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_0^0
\end{aligned}$$

$$
\begin{aligned}
&-\Big(\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_t}+\frac{\partial\phi_3(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}\\
&-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_0^1(z,w)\\
&-\sum_{j=1}^{n}(-i\frac{\partial\psi_3(z,w)}{\partial z_j}+\frac{\partial\phi_3(z)}{\partial z_j})\frac{\partial b_0^1(z,w)}{\partial\overline{z}_j}\\
&+\sum_{j=1}^{n}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})\frac{\partial b_0^1(z,w)}{\partial z_j}\\
&+\sum_{j=q+1}^{n}2\left|\lambda_j\right|\overline{w}_j\frac{\partial b_0^2(z,w)}{\partial\overline{z}_j}+\sum_{j=1}^{q}2\left|\lambda_j\right|w_j\frac{\partial b_0^2(z,w)}{\partial z_j},
\end{aligned}
\tag{3.22}
$$

where

$$
\begin{aligned}
L \;=\; &\sum_{j=q+1}^{n}2\left|\lambda_j\right|d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}+\sum_{j=1}^{q}2\left|\lambda_j\right|d\overline{z}_j^{\wedge,*}d\overline{z}_j^{\wedge}+\sum_{j=q+1}^{n}2\left|\lambda_j\right|\overline{z}_j\frac{\partial}{\partial\overline{z}_j}\\
&+\sum_{j=1}^{q}2\left|\lambda_j\right|z_j\frac{\partial}{\partial z_j}.
\end{aligned}
\tag{3.23}
$$

We rewrite first term of the right side of (3.21):

$$
\begin{aligned}
&-\Big(\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_t}+\frac{\partial\phi_3(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}\\
&\quad-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_0^0\\
&=-\sum_{j=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_j}+\frac{\partial\phi_3(z)}{\partial z_j})d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}b_0^0\\
&\quad+\sum_{j=1}^{n}\frac{\partial}{\partial z_j}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})d\overline{z}_j^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0\\
&\quad+\sum_{j,t=1,j\neq t}^{n}i\frac{\partial^2\psi_3(z,w)}{\partial\overline{z}_j\partial z_t}(d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}+d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge})b_0^0\\
&\quad-\sum_{j,t=1.j\neq t}^{n}\frac{\partial^2\phi_3(z)}{\partial\overline{z}_j\partial z_t}(d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}-d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge})b_0^0.
\end{aligned}
\tag{3.24}
$$

Note that $d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}+d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}=\delta_{j,t}$. From the form of b_0^0 (see Theorem 1.2), we can check that $d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}b_0^0=0$ if $q+1\leq j\leq n$, $d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}b_0^0=b_0^0$ if $1\leq j\leq q$, $d\overline{z}_j^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0=0$ if $1\leq j\leq q$, $d\overline{z}_j^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0=b_0^0$ if $q+1\leq j\leq n$ and when $j\neq t$, $d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0\neq 0$ if and only if $q+1\leq j\leq n$ and $1\leq t\leq q$. From this observaton, (3.24) becomes

$$\begin{aligned}
&-\Big(\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_t}+\frac{\partial\phi_3(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}\\
&\quad-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_0^0\\
&=-\sum_{j=1}^{q}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_j}+\frac{\partial\phi_3(z)}{\partial z_j})b_0^0+\sum_{j=q+1}^{n}\frac{\partial}{\partial z_j}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})b_0^0\\
&\quad+2\sum_{q+1\leq j\leq n,1\leq t\leq q}^{n}\frac{\partial^2\phi_3(z)}{\partial\overline{z}_j\partial z_t}d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0. \qquad (3.25)
\end{aligned}$$

From (2.19), it is straight forward to see that

$$\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,w)}{\partial z_j}+\frac{\partial\phi_3(z)}{\partial z_j})=\sum_{q+1\leq s\leq n}\frac{2\left|\lambda_s\right|}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)(\overline{z}_s-\overline{w}_s) \qquad (3.26)$$

where $1\leq j\leq q$ and

$$\frac{\partial}{\partial z_j}(i\frac{\partial\psi_3(z,w)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})=\sum_{1\leq s\leq q}\frac{2\left|\lambda_s\right|}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)(z_s-w_s), \quad (3.27)$$

where $q+1\leq j\leq n$. From (3.25), (3.26) and (3.27), (3.21) becomes

$$\begin{aligned}
Lb_0^1(z,w) \;=\; &-\sum_{q+1\leq s\leq n,1\leq j\leq q}\frac{2\left|\lambda_s\right|}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)(\overline{z}_s-\overline{w}_s)b_0^0\\
&+\sum_{1\leq s\leq q,q+1\leq j\leq n}\frac{2\left|\lambda_s\right|}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)(z_s-w_s)b_0^0\\
&+2\sum_{q+1\leq j\leq n,1\leq t\leq q,1\leq s\leq n}^{n}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0)z_s d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0
\end{aligned}$$

$$+2\sum_{q+1\le j\le n,1\le t\le q,1\le s\le n}^{n}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{z}_s d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0$$

$$+\sum_{j=q+1}^{n}2\left|\lambda_j\right|\overline{w}_j\frac{\partial b_0^1(z,w)}{\partial\overline{z}_j}+\sum_{j=1}^{q}2\left|\lambda_j\right|w_j\frac{\partial b_0^1(z,w)}{\partial z_j}. \quad (3.28)$$

Now, we write (3.28) according to the degree of homogenity in w, we get

$$\begin{aligned}Lb_0^1(z,0) &= -\sum_{q+1\le s\le n,1\le j\le q}\frac{2\left|\lambda_s\right|}{\left|\lambda_j\right|+\left|\lambda_s\right|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)\overline{z}_s b_0^0\\&\quad+\sum_{1\le s\le q,q+1\le j\le n}\frac{2\left|\lambda_s\right|}{\left|\lambda_j\right|+\left|\lambda_s\right|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)z_s b_0^0\\&\quad+2\sum_{q+1\le j\le n,1\le t\le q,1\le s\le n}^{n}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0)z_s d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0\\&\quad+2\sum_{q+1\le j\le n,1\le t\le q,1\le s\le n}^{n}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{z}_s d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 \qquad (3.29)\end{aligned}$$

and

$$\begin{aligned}Lb_0^1(0,w) &= \sum_{q+1\le s\le n,1\le j\le q}\frac{2\left|\lambda_s\right|}{\left|\lambda_j\right|+\left|\lambda_s\right|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)\overline{w}_s b_0^0\\&\quad-\sum_{1\le s\le q,q+1\le j\le n}\frac{2\left|\lambda_s\right|}{\left|\lambda_j\right|+\left|\lambda_s\right|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)w_s b_0^0\\&\quad+\sum_{j=q+1}^{n}2\left|\lambda_j\right|\overline{w}_j\frac{\partial b_0^1(z,0)}{\partial\overline{z}_j}+\sum_{j=1}^{q}2\left|\lambda_j\right|w_j\frac{\partial b_0^1(z,0)}{\partial z_j}. \qquad (3.30)\end{aligned}$$

We pause and introduce some notations. For multi-index J, we write $|J|=q$, $J\nearrow$, if $J=(j_1,\ldots,j_q)$, $1\le j_1<j_2<\cdots<j_q\le n$. Set $d\overline{z}^J=d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}$, $J=(j_1,\ldots,j_q)$. Then, $d\overline{z}^J$, $|J|=q$, $J\nearrow$, is an orthonormal basis of $\Lambda^{0,q}T_0^*(\mathbb{C}^n)$, where 0 is the origin in $\mathbb{C}^n$. Let $M_{d\overline{z}^J,d\overline{z}^K}$, $|J|=|K|=q$, $J,K\nearrow$, be the $\mathbb{C}$-linear operator:

$$\begin{aligned}M_{d\overline{z}^J,d\overline{z}^K}:\Lambda^{0,q}T_0^*(\mathbb{C}^n)&\to\Lambda^{0,q}T_0^*(\mathbb{C}^n)\\d\overline{z}^J&\to d\overline{z}^K\\d\overline{z}^I&\to 0\ \text{ if } I\ne J. \qquad (3.31)\end{aligned}$$

It is clear that $M_{d\overline{z}^J,d\overline{z}^K}$, $|J|=|K|=q$, $J,K\nearrow$, is a basis of the vector space $\mathscr{L}(\Lambda^{0,q}T_0^*(\mathbb{C}^n),\Lambda^{0,q}T_0^*(\mathbb{C}^n))$. For $m\in\mathbb{N}\bigcup\{0\}$, put

$$\begin{aligned}&P^m(\mathscr{L}(\Lambda^{0,q}T_0^*(\mathbb{C}^n),\Lambda^{0,q}T_0^*(\mathbb{C}^n)))\\&=\left\{\sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=m}A_{\alpha,\beta}\overline{z}^\alpha z^\beta\overline{w}^\gamma w^\delta;\ A_{\alpha,\beta}\in\mathscr{L}(\Lambda^{0,q}T_0^*(\mathbb{C}^n),\Lambda^{0,q}T_0^*(\mathbb{C}^n))\right\}.\end{aligned}$$

For multi-index J, $|J|=q$, we define

$$F(J)=2\sum_{j\in J,q+1\leq j\leq n}|\lambda_j|+2\sum_{j\notin J,1\leq j\leq q}|\lambda_j|\,. \tag{3.32}$$

Put $I_0=(1,\ldots,q)$. Note that $F(J)\neq 0$ if and only if $J\neq I_0$. We have the following

Lemma 3.3. *We use the same notations as in the discussion before Theorem* 2.1 *and before. If we consider* L *as the operator (We recall that* L *is given by* (3.23).*)*

$$L:P^m(\mathscr{L}(\Lambda^{0,q}T_0^*(\mathbb{C}^n),\Lambda^{0,q}T_0^*(\mathbb{C}^n)))\rightarrow P^m(\mathscr{L}(\Lambda^{0,q}T_0^*(\mathbb{C}^n),\Lambda^{0,q}T_0^*(\mathbb{C}^n))).$$

Then,

$$\begin{aligned}&\operatorname{Ker}L \\&=\left\{\sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=m,(\alpha'',\beta')=0,|J|=q,J\nearrow}c^{\alpha,\beta,\gamma,\delta}_{J,I_0}M_{d\overline{z}J,d\overline{z}^{I_0}}\overline{z}^\alpha z^\beta\overline{w}^\gamma w^\delta;c^{\alpha,\beta,\gamma,\delta}_{J,I_0}\in\mathbb{C}\right\}.\end{aligned}\tag{3.33}$$

Moreover, for $A\in P^m(\mathscr{L}(\Lambda^{0,q}T_0^*(\mathbb{C}^n),\Lambda^{0,q}T_0^*(\mathbb{C}^n)))$, *we write*

$$A=\sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=m,|J|=|K|=q,J,K\nearrow}c^{\alpha,\beta,\gamma,\delta}_{J,K}M_{d\overline{z}^J,d\overline{z}^K}\overline{z}^\alpha z^\beta\overline{w}^\gamma w^\delta,$$

$c^{\alpha,\beta,\gamma,\delta}_{J,K}\in\mathbb{C}$. *If* $c^{\alpha,\beta,\gamma,\delta}_{J,I_0}=0$ *when* $(\alpha'',\beta')=0$. *Then, we have* $LB=A$, *where*

$$\begin{aligned}B\ =&\sum_{|\alpha|+|\beta|+|\gamma|+|\delta|=m,|J|=|K|=q,J,K\nearrow,K\neq I_0\ \textit{if}\ (\alpha'',\beta')=0}c^{\alpha,\beta,\gamma,\delta}_{J,K}M_{d\overline{z}^J,d\overline{z}^K}\\&\times\frac{1}{F(K)+2<|\lambda''|\,,\alpha''>+2<|\lambda'|\,,\beta'>}\overline{z}^\alpha z^\beta\overline{w}^\gamma w^\delta+u(z),\end{aligned}\tag{3.34}$$

where $u \in \operatorname{Ker} L$.

Proof. We recall that

$$L = \sum_{j=q+1}^{n} 2\left|\lambda_j\right| d\overline{z}_j^{\wedge} d\overline{z}_j^{\wedge,*} + \sum_{j=1}^{q} 2\left|\lambda_j\right| d\overline{z}_j^{\wedge,*} d\overline{z}_j^{\wedge} + \sum_{j=q+1}^{n} 2\left|\lambda_j\right| \overline{z}_j \frac{\partial}{\partial \overline{z}_j} + \sum_{j=1}^{q} 2\left|\lambda_j\right| z_j \frac{\partial}{\partial z_j}.$$

For $M_{d\overline{z}^J, d\overline{z}^K}$, $|J| = |K| = q$, $J, K \nearrow$, we have

$$\begin{aligned}
&\big(L(M_{d\overline{z}^J, d\overline{z}^K} \overline{z}^{\alpha} z^{\beta} \overline{w}^{\gamma} w^{\delta})\big) d\overline{z}^J \\
&= \Big(\sum_{j=q+1}^{n} 2\left|\lambda_j\right| d\overline{z}_j^{\wedge} d\overline{z}_j^{\wedge,*} + \sum_{j=1}^{q} 2\left|\lambda_j\right| d\overline{z}_j^{\wedge,*} d\overline{z}_j^{\wedge} \\
&\quad + \sum_{j=q+1}^{n} 2\left|\lambda_j\right| \overline{z}_j \frac{\partial}{\partial \overline{z}_j} + \sum_{j=1}^{q} 2\left|\lambda_j\right| z_j \frac{\partial}{\partial z_j} \Big) d\overline{z}^K \overline{z}^{\alpha} z^{\beta} \\
&= \Big(2 \sum_{j \in K, q+1 \le j \le n} \left|\lambda_j\right| + 2 \sum_{j \notin K, 1 \le j \le q} \left|\lambda_j\right| \\
&\quad + 2(< \left|\lambda''\right|, \alpha'' > + < \left|\lambda'\right|, \beta' >)\Big) d\overline{z}^K \overline{z}^{\alpha} z^{\beta} \overline{w}^{\gamma} w^{\delta} \\
&= \Big(F(K) + 2(< \left|\lambda''\right|, \alpha'' > + < \left|\lambda'\right|, \beta' >)\Big) d\overline{z}^K \overline{z}^{\alpha} z^{\beta} \overline{w}^{\gamma} w^{\delta}
\end{aligned}$$

and

$$\big(L(M_{d\overline{z}^J, d\overline{z}^K} \overline{z}^{\alpha} z^{\beta} \overline{w}^{\gamma} w^{\delta})\big) d\overline{z}^I = 0$$

if $I \neq J$. Thus,

$$\begin{aligned}
&L(M_{d\overline{z}^J, d\overline{z}^K}) \overline{z}^{\alpha} z^{\beta} \overline{w}^{\gamma} w^{\delta} \\
&= \Big(F(K) + 2(< \left|\lambda''\right|, \alpha'' > + < \left|\lambda'\right|, \beta' >)\Big) M_{d\overline{z}^J, d\overline{z}^K} \overline{z}^{\alpha} z^{\beta} \overline{w}^{\gamma} w^{\delta}. \quad (3.35)
\end{aligned}$$

From (3.35), the lemma follows. □

It is not difficult to see that $b_0^0 = \left|\lambda_1\right| \left|\lambda_2\right| \cdots \left|\lambda_n\right| \pi^{-n} M_{d\overline{z}^{I_0}, d\overline{z}^{I_0}}$ and

$$d\overline{z}_j^{\wedge} d\overline{z}_t^{\wedge,*} b_0^0 = \left|\lambda_1\right| \left|\lambda_2\right| \cdots \left|\lambda_n\right| \pi^{-n} M_{d\overline{z}^{I_0}, d\overline{z}_j^{\wedge} d\overline{z}_t^{\wedge,*} d\overline{z}^{I_0}}, \quad (3.36)$$

where $q+1 \le j \le n$, $1 \le t \le q$. Combining this with (3.29) and Lemma 3.3, we get the following

Proposition 3.4. *We have that*

$$\begin{aligned}b^1_0(z,0)=&\sum_{q+1\leq j\leq n,1\leq t\leq q,1\leq s\leq n}\frac{1}{|\lambda_j|+|\lambda_t|+|\lambda_s|\,\chi_1(s)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0)z_s d\overline{z}^{\wedge,*}_t d\overline{z}^{\wedge}_j b^0_0\\&+\sum_{q+1\leq j\leq n,1\leq t\leq q,1\leq s\leq n}\frac{1}{|\lambda_j|+|\lambda_t|+|\lambda_s|\,\chi_2(s)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{z}_s d\overline{z}^{\wedge,*}_t d\overline{z}^{\wedge}_j b^0_0\\&-\sum_{q+1\leq s\leq n,1\leq j\leq q}\frac{1}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)\overline{z}_s b^0_0\\&+\sum_{1\leq s\leq q,q+1\leq j\leq n}\frac{1}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)z_s b^0_0+u(z),\end{aligned}\tag{3.37}$$

where $u(z)\in\operatorname{Ker}L$.

Now, we compute $b^1_0(0,w)$. From (3.37), we can compute the last two terms of the right side of (3.30):

$$\begin{aligned}&\sum_{s=q+1}^{n}2\,|\lambda_s|\,\overline{w}_s\frac{\partial b^1_0(z,0)}{\partial\overline{z}_s}+\sum_{s=1}^{q}2\,|\lambda_s|\,w_s\frac{\partial b^1_0(z,0)}{\partial z_s}\\&=\sum_{q+1\leq j\leq n,1\leq t\leq q,1\leq s\leq q}\frac{2\,|\lambda_s|}{|\lambda_j|+|\lambda_t|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0)w_s d\overline{z}^{\wedge,*}_t d\overline{z}^{\wedge}_j b^0_0\\&\quad+\sum_{q+1\leq j\leq n,1\leq t\leq q,q+1\leq s\leq n}\frac{2\,|\lambda_s|}{|\lambda_j|+|\lambda_t|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{w}_s d\overline{z}^{\wedge,*}_t d\overline{z}^{\wedge}_j b^0_0\\&\quad-\sum_{q+1\leq s\leq n,1\leq j\leq q}\frac{2\,|\lambda_s|}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)\overline{w}_s b^0_0\\&\quad+\sum_{1\leq s\leq q,q+1\leq j\leq n}\frac{2\,|\lambda_s|}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)w_s b^0_0.\end{aligned}\tag{3.38}$$

Combining this with (3.30), we obtain

$$\begin{aligned}Lb^1_0(0,w)=&\sum_{q+1\leq j\leq n,1\leq t\leq q,1\leq s\leq q}\frac{2\,|\lambda_s|}{|\lambda_j|+|\lambda_t|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0)w_s d\overline{z}^{\wedge,*}_t d\overline{z}^{\wedge}_j b^0_0\\+&\sum_{q+1\leq j\leq n,1\leq t\leq q,q+1\leq s\leq n}\frac{2\,|\lambda_s|}{|\lambda_j|+|\lambda_t|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{w}_s d\overline{z}^{\wedge,*}_t d\overline{z}^{\wedge}_j b^0_0.\end{aligned}\tag{3.39}$$

From this, (3.36) and Lemma 3.3, we get

Proposition 3.5. *We have that*

$$\begin{aligned} b_0^1(0,w) \;=\; & \sum_{q+1\leq j\leq n, 1\leq t\leq q, 1\leq s\leq q} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)} \\ & \qquad\qquad \times\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0)w_s d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 \\ & + \sum_{q+1\leq j\leq n, 1\leq t\leq q, q+1\leq s\leq n} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)} \\ & \qquad\qquad \times\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{w}_s d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0+v(w), \end{aligned} \tag{3.40}$$

where $v(w)\in\operatorname{Ker}L$.

In view of Proposition 3.4, we know that to prove Theorem 3.2, we only need to compute $u(z)$, where $u(z)$ is as in (3.37). Now, we compute $u(z)$. Note that $u(z)\in\operatorname{Ker}L$. From Lemma 3.3, we may write

$$\begin{aligned} u(z) \;=\; & \sum_{q+1\leq s\leq n, |J|=q, J\nearrow, J\neq I_0} c^s_{J,I_0}M_{d\overline{z}^J, d\overline{z}^{I_0}}z_s \\ & + \sum_{1\leq s\leq q, |J|=q, J\nearrow, J\neq I_0} c^s_{J,I_0}M_{d\overline{z}^J, d\overline{z}^{I_0}}\overline{z}_s \\ & + \sum_{q+1\leq s\leq n} z_s c^s M_{d\overline{z}^{I_0}, d\overline{z}^{I_0}} + \sum_{1\leq s\leq q}\overline{z}_s c^s M_{d\overline{z}^{I_0}, d\overline{z}^{I_0}}, \end{aligned} \tag{3.41}$$

where $c^s_{J,I_0}, c^s\in\mathbb{C}$, for all $s=1,\ldots,n$, $|J|=q$, $J\nearrow$, $J\neq I_0$. Let $u^*(z)$ be the adjoint of $u(z)$ with respect to $(\,|\,)$ in the space $\mathscr{L}(\Lambda^{0,q}T^*(\mathbb{C}^n),\Lambda^{0,q}T^*(\mathbb{C}^n))$. We can check that

$$\begin{aligned} u^*(z) \;=\; & \sum_{q+1\leq s\leq n, |J|=q, J\nearrow, J\neq I_0} \overline{c^s_{J,I_0}}M_{d\overline{z}^{I_0}, d\overline{z}^J}\overline{z}_s \\ & + \sum_{1\leq s\leq q, |J|=q, J\nearrow, J\neq I_0} \overline{c^s_{J,I_0}}M_{d\overline{z}^{I_0}, d\overline{z}^J}z_s \\ & + \sum_{1\leq s\leq q}\overline{c^s}z_s M_{d\overline{z}^{I_0}, d\overline{z}^{I_0}} + \sum_{q+1\leq s\leq n}\overline{c^s}\overline{z}_s M_{d\overline{z}^{I_0}, d\overline{z}^{I_0}}. \end{aligned} \tag{3.42}$$

We notice that the Bergman projection $\Pi^{(q)}_k$ is self-adjoint. From this observation, we deduce that

$$(b_0^1(w,0))^* = b_0^1(0,w), \tag{3.43}$$

where $(b_0^1(w,0))^*$ is the adjoint of $b_0^1(w,0)$ with respect to the inner product $(\ |\ )$ in the space $\mathscr{L}(\Lambda^{0,q}T^*(\mathbb{C}^n), \Lambda^{0,q}T^*(\mathbb{C}^n))$. From (3.37) and (3.42) and recall that $\operatorname{Ker} L$ is given by (3.33), we deduce that

$$\begin{aligned}(b_0^1(w,0))^* \ =& \sum_{q+1\leq s\leq n, |J|=q, J\nearrow, J\neq I_0} \overline{c^s_{J,I_0}} M_{d\overline{z}^{I_0}, d\overline{z}^J}\overline{w}_s \\ &+ \sum_{1\leq s\leq q, |J|=q, J\nearrow, J\neq I_0} \overline{c^s_{J,I_0}} M_{d\overline{z}^{I_0}, d\overline{z}^J} w_s + r(w),\end{aligned} \tag{3.44}$$

where $r(w) \in \operatorname{Ker} L$. From (3.40) and (3.36), we have

$$\begin{aligned}b_0^1(0,w) =& \left|\lambda_1\right|\cdots\left|\lambda_n\right|\pi^{-n} \\ &\times\Big(\sum_{q+1\leq j\leq n, 1\leq t\leq q, 1\leq s\leq q} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)} \\ &\qquad\times\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0) w_s M_{d\overline{z}^{I_0}, d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}d\overline{z}^{I_0}} \\ &+\sum_{q+1\leq j\leq n, 1\leq t\leq q, q+1\leq s\leq n} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)} \\ &\qquad\times\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{w}_s M_{d\overline{z}^{I_0}, d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}d\overline{z}^{I_0}}\Big) + v(w),\end{aligned} \tag{3.45}$$

where $v(w) \in \operatorname{Ker} L$. From (3.43), (3.44) and (3.45), we get

$$c^s_{J,I_0} = |\lambda_1|\cdots|\lambda_n|\,\pi^{-n}\times\frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)}\times\frac{\partial^3\phi}{\partial z_j\partial\overline{z}_t\partial\overline{z}_s}(0)$$

if $J = d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}d\overline{z}^{I_0}$, $q+1\leq j\leq n$, $1\leq t\leq q$, and $s = 1,\ldots,q$, (3.46)

$$c^s_{J,I_0} = |\lambda_1|\cdots|\lambda_n|\,\pi^{-n}\times\frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)}\times\frac{\partial^3\phi}{\partial z_j\partial\overline{z}_t\partial z_s}(0)$$

if $J = d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}d\overline{z}^{I_0}$, $q+1\leq j\leq n$, $1\leq t\leq q$, and $s = q+1,\ldots,n$, (3.47)

$$c^s_{J,I_0} = 0 \quad \text{otherwise}. \tag{3.48}$$

Combining above with (3.41) and (3.36), we obtain

$$
\begin{aligned}
b_0^1(z,0) = & \sum_{q+1\leq j\leq n, 1\leq t\leq q, 1\leq s\leq n} \frac{1}{|\lambda_j|+|\lambda_t|+|\lambda_s|\,\chi_1(s)} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0) z_s d\overline{z}_t^{\wedge,*} d\overline{z}_j^{\wedge} b_0^0 \\
& + \sum_{q+1\leq j\leq n, 1\leq t\leq q, 1\leq s\leq n} \frac{1}{|\lambda_j|+|\lambda_t|+|\lambda_s|\,\chi_2(s)} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0) \overline{z}_s d\overline{z}_t^{\wedge,*} d\overline{z}_j^{\wedge} b_0^0 \\
& - \sum_{q+1\leq s\leq n, 1\leq j\leq q} \frac{1}{|\lambda_j|+|\lambda_s|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0) \overline{z}_s b_0^0 \\
& + \sum_{1\leq s\leq q, q+1\leq j\leq n} \frac{1}{|\lambda_j|+|\lambda_s|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0) z_s b_0^0 \\
& + \sum_{q+1\leq j\leq n, 1\leq t\leq q, q+1\leq s\leq n} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)} \\
& \times \frac{\partial^3\phi}{\partial z_j\partial\overline{z}_t\partial z_s}(0) z_s b_0^0 d\overline{z}_j^{\wedge,*} d\overline{z}_t^{\wedge} \\
& + \sum_{q+1\leq j\leq n, 1\leq t\leq q, 1\leq s\leq q} \frac{|\lambda_s|}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_t|+|\lambda_s|)} \\
& \times \frac{\partial^3\phi}{\partial z_j\partial\overline{z}_t\partial\overline{z}_s}(0) \overline{z}_s b_0^0 d\overline{z}_j^{\wedge,*} d\overline{z}_t^{\wedge} \\
& + \sum_{q+1\leq s\leq n} z_s c^s M_{d\overline{z}^{I_0}, d\overline{z}^{I_0}} + \sum_{1\leq s\leq q} \overline{z}_s c^s M_{d\overline{z}^{I_0}, d\overline{z}^{I_0}}. \qquad (3.49)
\end{aligned}
$$

Now, to complete the proof of Theorem 3.2, we only need to know c^s, $s = 1, \ldots, n$. From (3.49), we know that

$$
\begin{aligned}
\operatorname{Tr} b_0^1(z,0) \; = \; & -|\lambda_1|\cdots|\lambda_n|\,\pi^{-n}\Big(\sum_{q+1\leq s\leq n, 1\leq j\leq q} \frac{1}{|\lambda_j|+|\lambda_s|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)\overline{z}_s \Big) \\
& + |\lambda_1|\cdots|\lambda_n|\,\pi^{-n}\Big(\sum_{1\leq s\leq q, q+1\leq j\leq n} \frac{1}{|\lambda_j|+|\lambda_s|} \frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0) z_s \Big) \\
& + \sum_{q+1\leq s\leq n} z_s c^s + \sum_{1\leq s\leq q} \overline{z}_s c^s. \qquad (3.50)
\end{aligned}
$$

From Theorem 1.2, we know that $\operatorname{Tr} b_0(z,z) = \pi^{-n}(-1)^q \det\left(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}(z)\right)_{j,k=1}^n$.

From this, we can compute

$$\begin{aligned}\frac{\partial}{\partial z_s}\operatorname{Tr} b_0(z,z)\Big|_{z=0} &= \pi^{-n}(-1)^q\frac{\partial}{\partial z_s}\det\left(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}(z)\right)^n_{j,k=1}\Big|_{z=0}\\ &= \pi^{-n}\,|\lambda_1|\cdots|\lambda_n|\Big(-\sum_{1\leq j\leq q}\frac{1}{|\lambda_j|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\\ &\quad+\sum_{q+1\leq j\leq n}\frac{1}{|\lambda_j|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\Big).\end{aligned}\tag{3.51}$$

From (3.43), we know that $\overline{\operatorname{Tr} b^1_0(z,0)} = \operatorname{Tr} b^1_0(0,z)$. Note that $b^1_0(z,z) = b^1_0(z,0)+b^1_0(0,z)$. From this, we see that

$$\begin{aligned}\frac{\partial}{\partial z_s}\operatorname{Tr} b_0(z,z)|_{z=0} &= \frac{\partial}{\partial z_s}\operatorname{Tr} b^1_0(z,0)+\frac{\partial}{\partial z_s}\operatorname{Tr} b^1_0(0,z)\\ &= \frac{\partial}{\partial z_s}\operatorname{Tr} b^1_0(z,0)+\overline{\frac{\partial}{\partial \overline{z}_s}\operatorname{Tr} b^1_0(z,0)}.\end{aligned}\tag{3.52}$$

Thus, if $q+1\leq s\leq n$, from (3.50) and (3.52), we can check that

$$\begin{aligned}&\frac{\partial}{\partial z_s}\operatorname{Tr} b_0(z,z)|_{z=0}\\ &= -|\lambda_1|\cdots|\lambda_n|\,\pi^{-n}\Big(\sum_{1\leq j\leq q}\frac{1}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\Big)+c^s.\end{aligned}\tag{3.53}$$

From (3.51) and (3.53), we can compute

$$\begin{aligned}c_s &= \pi^{-n}\,|\lambda_1|\cdots|\lambda_n|\Big(-\sum_{1\leq j\leq q}\frac{1}{|\lambda_j|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)+\sum_{q+1\leq j\leq n}\frac{1}{|\lambda_j|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\Big)\\ &\quad+\pi^{-n}\,|\lambda_1|\cdots|\lambda_n|\Big(\sum_{1\leq j\leq q}\frac{1}{|\lambda_j|+|\lambda_s|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\Big)\\ &= \pi^{-n}\,|\lambda_1|\cdots|\lambda_n|\Big(-\sum_{1\leq j\leq q}\frac{|\lambda_s|}{|\lambda_j|\,(|\lambda_j|+|\lambda_s|)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\\ &\quad+\sum_{q+1\leq j\leq n}\frac{1}{|\lambda_j|}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\Big),\end{aligned}\tag{3.54}$$

$q+1 \leq s \leq n$. Similarly, we can repeat the procedure above and get

$$\begin{aligned} c_s &= \pi^{-n} |\lambda_1| \cdots |\lambda_n| \Big(- \sum_{1 \leq j \leq q} \frac{1}{|\lambda_j|} \frac{\partial^3 \phi}{\partial \overline{z}_j \partial z_j \partial \overline{z}_s}(0) \\ &\quad + \sum_{q+1 \leq j \leq n} \frac{|\lambda_s|}{|\lambda_j| (|\lambda_j| + |\lambda_s|)} \frac{\partial^3 \phi}{\partial \overline{z}_j \partial z_j \partial \overline{z}_s}(0) \Big), \end{aligned} \tag{3.55}$$

if $1 \leq s \leq q$. Combining (3.55), (3.54) with (3.49), Theorem 3.2 follows.

3.3. The second order of the Taylor expansion of $b_0(z, w)$ at $z = w$ and the b_1 term

The second goal of this section is to prove the following

Theorem 3.6. *Put*

$$\operatorname{Tr} b_0^1(z, 0) = \pi^{-n} |\lambda_1| \cdots |\lambda_n| \Big(\sum_{s=1}^{n} (a_s z_s + b_s \overline{z}_s) \Big). \tag{3.56}$$

Then for $b_0^2(z, w)$ in (3.16)*, we have*

$$\begin{aligned} & b_0^2(z, 0) \\ &= \sum_{q+1 \leq j \leq n, 1 \leq k \leq q, 1 \leq s \leq n} \frac{1}{|\lambda_s| (|\lambda_j| + |\lambda_k| + |\lambda_s| \chi_1(s))} \left| \frac{\partial^3 \phi}{\partial \overline{z}_j \partial z_k \partial z_s}(0) \right|^2 |z_s|^2 b_0^0 \\ &\quad + \sum_{q+1 \leq j \leq n, 1 \leq k \leq q, 1 \leq s \leq n} \frac{1}{|\lambda_s| (|\lambda_j| + |\lambda_k| + |\lambda_s| \chi_2(s))} \left| \frac{\partial^3 \phi}{\partial \overline{z}_j \partial z_k \partial \overline{z}_s}(0) \right|^2 |z_s|^2 b_0^0 \\ &\quad + \sum_{q+1 \leq j \leq n, 1 \leq k \leq q, q+1 \leq s \leq n} \frac{|\lambda_s|}{(|\lambda_j| + |\lambda_k|)(|\lambda_j| + |\lambda_k| + |\lambda_s|)^2} \left| \frac{\partial^3 \phi}{\partial \overline{z}_j \partial z_k \partial \overline{z}_s}(0) \right|^2 \\ &\qquad\qquad \times |z_s|^2 \, d\overline{z}_k^{\wedge,*} d\overline{z}_j^{\wedge} b_0^0 d\overline{z}_j^{\wedge,*} d\overline{z}_k^{\wedge} \\ &\quad + \sum_{q+1 \leq j \leq n, 1 \leq k \leq q, 1 \leq s \leq q} \frac{|\lambda_s|}{(|\lambda_j| + |\lambda_k|)(|\lambda_j| + |\lambda_k| + |\lambda_s|)^2} \left| \frac{\partial^3 \phi}{\partial \overline{z}_j \partial z_k \partial z_s}(0) \right|^2 \\ &\qquad\qquad \times |z_s|^2 \, d\overline{z}_k^{\wedge,*} d\overline{z}_j^{\wedge} b_0^0 d\overline{z}_j^{\wedge,*} d\overline{z}_k^{\wedge} \\ &\quad - \sum_{q+1 \leq u \leq n, 1 \leq s \leq n} \frac{1}{|\lambda_u| + |\lambda_s| \chi_1(s)} b_s \frac{\partial^3 \phi}{\partial \overline{z}_u \partial z_u \partial z_s}(0) \, |z_u|^2 \, b_0^0 \end{aligned}$$

$$+\sum_{1\leq u\leq q,1\leq s\leq n}\frac{1}{|\lambda_u|+|\lambda_s|\,\chi_2(s)}a_s\frac{\partial^3\phi}{\partial\overline{z}_u\partial z_u\partial\overline{z}_s}(0)\,|z_u|^2\,b_0^0$$
$$-\sum_{1\leq j\leq q,q+1\leq s\leq n}\frac{1}{|\lambda_j|+|\lambda_s|}a_s\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}(0)\,|z_s|^2\,b_0^0$$
$$+\sum_{q+1\leq j\leq n,1\leq s\leq q}\frac{1}{|\lambda_j|+|\lambda_s|}b_s\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\,|z_s|^2\,b_0^0$$
$$-\sum_{1\leq j\leq q,1\leq k\leq n}\frac{\partial^4(-i\psi(z,0)+\phi)}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0)\frac{|z_k|^2}{2\,|\lambda_k|}b_0^0$$
$$+\sum_{q+1\leq j\leq n,1\leq k\leq n}\frac{\partial^4(i\psi(z,0)+\phi)}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0)\frac{|z_k|^2}{2\,|\lambda_k|}b_0^0+r(z)+h(z) \tag{3.57}$$

where $\operatorname{Tr}h=0$ *and* $\frac{\partial^2 r}{\partial\overline{z}_j\partial z_j}=0$, $j=1,\ldots,n$.

Furthermore, we have

$$b_1(0,0)=\sum_{q+1\leq j\leq n,1\leq k\leq q,1\leq s\leq q}\frac{|\lambda_s|}{2(|\lambda_j|+|\lambda_k|)^2(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}$$
$$\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}$$
$$+\sum_{q+1\leq j\leq n,1\leq k\leq q,q+1\leq s\leq n}\frac{|\lambda_s|}{2(|\lambda_j|+|\lambda_k|)^2(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}$$
$$\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial\overline{z}_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}+cb_0^0+R, \tag{3.58}$$

where $c\in\mathbb{C}$, $\operatorname{Tr}R=0$.

Take $w=0$ in (3.22), we get

$$Lb_0^2(z,0)=-\Big(\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_4(z,0)}{\partial z_t}+\frac{\partial\phi_4(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}$$
$$-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_4(z,0)}{\partial\overline{z}_j}+\frac{\partial\phi_4(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_0^0$$
$$-\Big(\sum_{j,t=1}^{n}\frac{\partial}{\partial\overline{z}_j}(-i\frac{\partial\psi_3(z,0)}{\partial z_t}+\frac{\partial\phi_3(z)}{\partial z_t})d\overline{z}_j^{\wedge}d\overline{z}_t^{\wedge,*}$$

$$
\begin{aligned}
&-\sum_{j,t=1}^{n}\frac{\partial}{\partial z_t}(i\frac{\partial\psi_3(z,0)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})d\overline{z}_t^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_0^1(z,0)\\
&-\sum_{j=1}^{n}(-i\frac{\partial\psi_3(z,0)}{\partial z_j}+\frac{\partial\phi_3(z)}{\partial z_j})\frac{\partial b_0^1(z,0)}{\partial\overline{z}_j}\\
&+\sum_{j=1}^{n}(i\frac{\partial\psi_3(z,0)}{\partial\overline{z}_j}+\frac{\partial\phi_3(z)}{\partial\overline{z}_j})\frac{\partial b_0^1(z,0)}{\partial z_j}. \qquad (3.59)
\end{aligned}
$$

From Lemma 3.3, (3.59) and some very complicated computation (see Section 3.3 in [10] for the details), we get (3.57).

Now, to complete the proof of Theorem 3.6, we only need to prove (3.58). From (3.15) and (2.2), we see that

$$
\Big(\sum_{j=1}^{q}2\left|\lambda_j\right|d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}+\sum_{j=q+1}^{n}2\left|\lambda_j\right|d\overline{z}_j^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_1(0,0)=\sum_{j=1}^{n}\frac{\partial^2 b_0}{\partial z_j\partial\overline{z}_j}(0,0). \qquad (3.60)
$$

From (3.57), we see that

$$
\begin{aligned}
\sum_{j=1}^{n}\frac{\partial^2 b_0}{\partial z_j\partial\overline{z}_j}(0,0) = &\sum_{q+1\leq j\leq n,1\leq k\leq q,1\leq s\leq q}\frac{|\lambda_s|}{(|\lambda_j|+|\lambda_k|)(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}\\
&\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}\\
&+\sum_{q+1\leq j\leq n,1\leq k\leq q,q+1\leq s\leq n}\frac{|\lambda_s|}{(|\lambda_j|+|\lambda_k|)(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}\\
&\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial\overline{z}_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}+\alpha b_0^0+f, \quad (3.61)
\end{aligned}
$$

where $\alpha\in\mathbb{C}$ and $\operatorname{Tr} f=0$. Since we can solve (3.60), we conclude that $\alpha=0$. Thus, (3.60) becomes:

$$
\begin{aligned}
&\Big(\sum_{j=1}^{q}2\left|\lambda_j\right|d\overline{z}_j^{\wedge}d\overline{z}_j^{\wedge,*}+\sum_{j=q+1}^{n}2\left|\lambda_j\right|d\overline{z}_j^{\wedge,*}d\overline{z}_j^{\wedge}\Big)b_1(0,0)\\
&=\sum_{q+1\leq j\leq n,1\leq k\leq q,1\leq s\leq q}\frac{|\lambda_s|}{(|\lambda_j|+|\lambda_k|)(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}
\end{aligned}
$$

$$\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}$$

$$+\sum_{q+1\le j\le n,1\le k\le q,q+1\le s\le n}\frac{|\lambda_s|}{(|\lambda_j|+|\lambda_k|)(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}$$

$$\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial\overline{z}_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}+f. \quad (3.62)$$

Again, from Lemma 3.3, we get (3.58). Theorem 3.6 follows.

4. The trace of the b_1 term

As before, in this section, we assume that $\phi(z)=\sum_{j=1}^n\lambda_j|z_j|^2+O(|z|^3)$ near 0 and that $\lambda_j<0$, $j=1,\ldots,q$, and $\lambda_j>0$, $j=q+1,\ldots,n$. We work in some neighborhood of $(0,0)$ and we shall use the same notaions as before. In view of (3.58), we know that

$$\begin{aligned}b_1(0,0) \;=\;& \sum_{q+1\le j\le n,1\le k\le q,1\le s\le q}\frac{|\lambda_s|}{2(|\lambda_j|+|\lambda_k|)^2(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}\\ &\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}\\ &+\sum_{q+1\le j\le n,1\le k\le q,q+1\le s\le n}\frac{|\lambda_s|}{2(|\lambda_j|+|\lambda_k|)^2(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}\\ &\times\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial\overline{z}_s}(0)\right|^2 d\overline{z}_k^{\wedge,*}d\overline{z}_j^{\wedge}b_0^0 d\overline{z}_j^{\wedge,*}d\overline{z}_k^{\wedge}\\ &+cb_0^0+R, \end{aligned} \quad (4.1)$$

where $c\in\mathbb{C}$, $\operatorname{Tr}R=0$. The main goal of this section is to determine the constant c. We notice that the projection $\hat{\Pi}_k^{(q)}$ has the following property:

$$\hat{\Pi}_k^{(q)}\circ\hat{\Pi}_k^{(q)}=\hat{\Pi}_k^{(q)}. \quad (4.2)$$

We recall that $\hat{\Pi}_k^{(q)}$ is given by (3.7). From (3.9), we have

$$\hat{\Pi}_k^{(q)}\circ\hat{\Pi}_k^{(q)}(u,w)=\int e^{ik\psi(z,u)}b(u,z,k)e^{ik\psi(z,w)}b(z,w,k)dm(z)+F(u,w,k), \quad (4.3)$$

where $F(u,w,k)$ is negligible. Take $u=w=0$ in (4.3) and from (4.2) and

(3.9), we get

$$\int e^{ik(\psi(0,z)+\psi(z,0))}b(0,z,k)b(z,0,k)dm(z) \sim k^n b_0(0,0) + k^{n-1}b_1(0,0) + \cdots . \tag{4.4}$$

We use $z_j = x_{2j-1} + ix_{2j}$, $j = 1, \dots, n$. We write ∂_{x_j} to denote the operator $\frac{\partial}{\partial x_j}$, $j = 1, \dots, 2n$. For multi-indix $\alpha = (\alpha_1, \dots, \alpha_{2n})$, $\alpha_j \in \mathbb{N}\bigcup\{0\}$, $j = 1, \dots, 2n$. We write $|\alpha| = N$ if $\sum_{j=1}^{2n} \alpha_j = N$ and we write ∂_x^α to denote the operator $\partial_{x_1}^{\alpha_1} \cdots \partial_{x_{2n}}^{\alpha_{2n}}$. We recall the stationary phase formula of Hörmander (see Theorem 7.7.5 in Hörmander [6])

Theorem 4.1. *Let $K \subset \mathbb{C}^n$ be a compact set, X an open neighborhood of K and N a positive integer. If $u \in C_0^\infty(K)$, $f \in C^\infty(X)$ and $\operatorname{Im} f \geq 0$ in X, $\operatorname{Im} f(x_0) = 0$, $f'(x_0) = 0$, $\det f''(x_0) \neq 0$, $f' \neq 0$ in $K \setminus \{x_0\}$ then*

$$\left| \int u(z)e^{ikf(z)}dm - 2^n e^{ikf(x_0)} \det \left(\frac{kf''(x_0)}{2\pi i} \right)^{-\frac{1}{2}} \sum_{j<N} k^{-j} L_j u \right| \leq Ck^{-N} \sum_{|\alpha| \leq 2N} \sup |\partial_x^\alpha u|, \quad k > 0, \tag{4.5}$$

where C is bounded when f stays in a bounded set in $C^\infty(X)$ and $\frac{|x-x_0|}{|f'(x)|}$ has a uniform bounded and

$$L_j u = \sum_{\nu-\mu=j} \sum_{2\nu \geq 3\mu} i^{-j} 2^{-\nu} < f''(x_0)^{-1}D, D >^\nu \frac{(g_{x_0}^\mu u)(x_0)}{\nu!\mu!}. \tag{4.6}$$

Here

$$g_{x_0}(x) = f(x) - f(x_0) - \frac{1}{2} < f''(x_0)(x-x_0), x - x_0 > \tag{4.7}$$

and $D = \begin{pmatrix} -i\partial_{x_1} \\ \vdots \\ -i\partial_{x_{2n}} \end{pmatrix}$.

Now, we apply (4.5) to the left side of (4.4). From (2.2), we know that

$$\psi(z,0) + \psi(0,z) = 2i \operatorname{Im}\psi(z,0) = 2i \sum_{j=1}^n |\lambda_j|\, |z_j|^2 + O(|z|^3). \tag{4.8}$$

Since $\operatorname{Im}(\psi(0,z)+\psi(z,0))>0$ when $z\neq 0$, we may assume that $b(0,z,k)$ has compact supports in some small neighborhood K of $0\in\mathbb{C}^n$. From (4.5), we have

$$\begin{aligned}&\int e^{ik(\psi(0,z)+\psi(z,0))}b(0,z,k)b(z,0,k)dm(z)\\&= |\lambda_1|^{-1}\cdots|\lambda_n|^{-1}\,\pi^n k^{-n}\Big(b(0,0,k)^2+k^{-1}L_1(b(0,z,k)b(z,0,k))\Big|_{z=0}\\&\quad +O(k^{2n-2})\Big).\end{aligned}\tag{4.9}$$

We can check that

$$b(0,0,k)^2=k^{2n}b_0(0,0)^2+k^{2n-1}(b_0(0,0)b_1(0,0)+b_1(0,0)b_0(0,0))+O(k^{2n-2}).\tag{4.10}$$

The computation of the term $L_1(b(0,z,k)b(z,0,k))|_{z=0}$ is straight forward. We omit the process. We state our result

Proposition 4.2. *Under the notations above, we have*

$$\begin{aligned}&k^{-2n}L_1(b(0,z,k)b(z,0,k))\Big|_{z=0}+O(k^{-1})\\&=\frac{1}{2}\sum_{j=1}^n\frac{1}{|\lambda_j|}\Big(\frac{\partial^2 b_0}{\partial z_j\partial\overline{z}_j}(z,0)\Big|_{z=0}b_0(0,0)+\frac{\partial^2 b_0}{\partial z_j\partial\overline{z}_j}(0,z)\Big|_{z=0}b_0(0,0)\\&\quad+(\frac{\partial b_0}{\partial\overline{z}_j})(0,z)\Big|_{z=0}\frac{\partial b_0}{\partial z_j}(z,0)\Big|_{z=0}+\frac{\partial b_0}{\partial z_j}(0,z)\Big|_{z=0}\frac{\partial b_0}{\partial\overline{z}_j}(z,0)\Big|_{z=0}\Big)\\&\quad-\frac{1}{4}\sum_{j,t=1}^n\frac{1}{|\lambda_j|\,|\lambda_t|}\frac{\partial^4\operatorname{Im}\psi(z,0)}{\partial\overline{z}_j\partial z_j\partial\overline{z}_t\partial z_t}\Big|_{z=0}b_0(0,0)^2\\&\quad-\frac{1}{2}\sum_{j,s=1}^n\frac{1}{|\lambda_j|\,|\lambda_s|}\frac{\partial^3\operatorname{Im}\psi(z,0)}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}\Big|_{z=0}\Big(\frac{\partial b_0}{\partial z_s}(0,z)\Big|_{z=0}+\frac{\partial b_0}{\partial z_s}(z,0)\Big|_{z=0}\Big)b_0(0,0)\\&\quad-\frac{1}{2}\sum_{j,s=1}^n\frac{1}{|\lambda_j|\,|\lambda_s|}\frac{\partial^3\operatorname{Im}\psi(z,0)}{\partial\overline{z}_j\partial z_j\partial z_s}\Big|_{z=0}\Big(\frac{\partial b_0}{\partial\overline{z}_s}(0,z)\Big|_{z=0}+\frac{\partial b_0}{\partial\overline{z}_s}(z,0)\Big|_{z=0}\Big)b_0(0,0)\\&\quad+\frac{1}{4}\sum_{j,t,s=1}^n\frac{1}{|\lambda_j|\,|\lambda_t|\,|\lambda_s|}\left|\frac{\partial^3\operatorname{Im}\psi(z,0)}{\partial z_j\partial z_t\partial\overline{z}_s}\Big|_{z=0}\right|^2b_0(0,0)^2\\&\quad+\frac{1}{2}\sum_{j,t,s=1}^n\frac{1}{|\lambda_j|\,|\lambda_t|\,|\lambda_s|}\frac{\partial^3\operatorname{Im}\psi(z,0)}{\partial\overline{z}_j\partial z_j\partial\overline{z}_s}\Big|_{z=0}\frac{\partial^3\operatorname{Im}\psi(z,0)}{\partial\overline{z}_t\partial z_t\partial z_s}\Big|_{z=0}b_0(0,0)^2\end{aligned}$$

$$+\frac{1}{12}\sum_{j,k,s=1}^{n}\frac{1}{|\lambda_j|\,|\lambda_t|\,|\lambda_s|}\left|\frac{\partial^3 \mathrm{Im}\,\psi(z,0)}{\partial z_j\partial z_t\partial z_s}\Big|_{z=0}\right|^2 b_0(0,0)^2. \tag{4.11}$$

From (4.4), (4.9), (4.10) and (4.11), we get the following

Theorem 4.3. *We have*

$$b_1(0,0)=|\lambda_1|^{-1}\cdots|\lambda_n|^{-1}\,\pi^n\Big(b_1(0,0)b_0(0,0)+b_0(0,0)b_1(0,0)+C_0\Big), \tag{4.12}$$

where C_0 *denote the right side of* (4.11).

For $A,B\in\mathscr{L}(\Lambda^{0,q}T^*_0(\mathbb{C}^n),\Lambda^{0,q}T^*_0(\mathbb{C}^n))$, we write $A\perp B$ if $(Au\mid Bu)=0$ for all $u\in\Lambda^{0,q}T^*_0(\mathbb{C}^n)$. For $F\in\mathscr{L}(\Lambda^{0,q}T^*_0(\mathbb{C}^n),\Lambda^{0,q}T^*_0(\mathbb{C}^n))$, we werit $\hat{F}$ to denote the component of A in the direction $d\overline{z}_1^{\wedge}d\overline{z}_1^{\wedge,*}\cdots d\overline{z}_q^{\wedge}d\overline{z}_q^{\wedge,*}$. More precisely, if

$$F=\alpha d\overline{z}_1^{\wedge}d\overline{z}_1^{\wedge,*}\cdots d\overline{z}_q^{\wedge}d\overline{z}_q^{\wedge,*}+G,\quad \alpha\in\mathbb{C},\quad G\perp d\overline{z}_1^{\wedge}d\overline{z}_1^{\wedge,*}\cdots d\overline{z}_q^{\wedge}d\overline{z}_q^{\wedge,*}, \tag{4.13}$$

then $\hat{F}=\alpha d\overline{z}_1^{\wedge}d\overline{z}_1^{\wedge,*}\cdots d\overline{z}_q^{\wedge}d\overline{z}_q^{\wedge,*}$. Now, we are ready to compute the constant c. We recall that c is given by (4.1). From (4.1), we know that

$$\hat{b}_1(0,0)=cb_0(0,0). \tag{4.14}$$

Note that $b_0(0,0)=b_0^0$. From (4.1), (4.12) and (4.14), we get

$$-cb_0(0,0)=|\lambda_1|^{-1}\cdots|\lambda_n|^{-1}\,\pi^n\hat{C}_0. \tag{4.15}$$

From (4.15) and (4.11), we can determine c. The computation is very straightforward. We omit the detail and we refer the reader to Section 4 in [10] for the details. We state our result

Theorem 4.4. *For* c *in* (4.1)*, we have*

$$c=\frac{1}{4}\Big(-\sum_{1\leq j,t\leq q}\frac{1}{|\lambda_j|\,|\lambda_t|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_t\partial z_t}(0)+\sum_{q+1\leq j,t\leq n}\frac{1}{|\lambda_j|\,|\lambda_t|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_t\partial z_t}(0)$$

$$-2\sum_{q+1\leq j\leq n,1\leq t\leq q}\frac{|\lambda_j|-|\lambda_t|}{|\lambda_j|\,|\lambda_t|\,(|\lambda_j|+|\lambda_t|)}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_t\partial z_t}(0)\Big)$$

$$-\frac{1}{2}\sum_{q+1\leq j\leq n,1\leq t,s\leq q}\frac{|\lambda_j|-|\lambda_t|}{|\lambda_j|\,|\lambda_t|\,(|\lambda_j|+|\lambda_t|)(|\lambda_s|+|\lambda_j|)}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_t}(0)\right|^2$$

$$+\frac{1}{2}\sum_{q+1\leq j,s\leq n,1\leq t\leq q}\frac{|\lambda_j|-|\lambda_t|}{|\lambda_j|\,|\lambda_t|\,(|\lambda_j|+|\lambda_t|)(|\lambda_s|+|\lambda_t|)}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_t}(0)\right|^2$$

$$+\frac{1}{4}\sum_{q+1\leq j,t\leq n,1\leq s\leq q}\Big(\frac{|\lambda_t|^2\,|\lambda_j|^2}{|\lambda_s|\,|\lambda_j|\,|\lambda_t|\,(|\lambda_t|+|\lambda_j|+|\lambda_s|)^2}(\frac{1}{|\lambda_j|+|\lambda_s|}+\frac{1}{|\lambda_t|+|\lambda_s|})^2$$

$$+\frac{1}{(|\lambda_t|+|\lambda_j|+|\lambda_s|)\,|\lambda_j|\,|\lambda_t|}\Big)\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_t}(0)\right|^2$$

$$+\frac{1}{4}\sum_{q+1\leq s\leq n,1\leq j,t\leq q}\Big(\frac{|\lambda_t|^2\,|\lambda_j|^2}{|\lambda_s|\,|\lambda_j|\,|\lambda_t|\,(|\lambda_t|+|\lambda_j|+|\lambda_s|)^2}(\frac{1}{|\lambda_j|+|\lambda_s|}+\frac{1}{|\lambda_t|+|\lambda_s|})^2$$

$$+\frac{1}{(|\lambda_t|+|\lambda_j|+|\lambda_s|)\,|\lambda_j|\,|\lambda_t|}\Big)\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_t}(0)\right|^2$$

$$-\frac{1}{4}\sum_{1\leq j,t,s\leq q}\frac{1}{|\lambda_j|\,|\lambda_t|\,|\lambda_s|}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_t}(0)\right|^2$$

$$-\frac{1}{4}\sum_{q+1\leq j,t,s\leq n}\frac{1}{|\lambda_j|\,|\lambda_t|\,|\lambda_s|}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_t}(0)\right|^2$$

$$+\frac{1}{2}\sum_{q+1\leq j,t\leq n,1\leq s\leq q}\frac{|\lambda_s|}{|\lambda_j||\lambda_t|(|\lambda_j|+|\lambda_s|)(|\lambda_t|+|\lambda_s|)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_t\partial\overline{z}_s}(0)$$

$$+\frac{1}{2}\sum_{1\leq j,t\leq q,q+1\leq s\leq n}\frac{|\lambda_s|}{|\lambda_j||\lambda_t|(|\lambda_j|+|\lambda_s|)(|\lambda_t|+|\lambda_s|)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_t\partial\overline{z}_s}(0)$$

$$-\frac{1}{2}\sum_{1\leq s,t\leq q,q+1\leq j\leq n}\frac{1}{(|\lambda_j|+|\lambda_t|)(|\lambda_j|+|\lambda_s|)\,|\lambda_t|}$$

$$\times\Big(\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_t\partial\overline{z}_s}(0)+\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_t\partial\overline{z}_s}(0)\Big)$$

$$-\frac{1}{2}\sum_{1\leq t\leq q,q+1\leq j,s\leq n}\frac{1}{(|\lambda_j|+|\lambda_t|)(|\lambda_t|+|\lambda_s|)\,|\lambda_j|}$$

$$\times\Big(\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_t\partial\overline{z}_s}(0)+\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_t\partial\overline{z}_s}(0)\Big).\tag{4.16}$$

5. The end of the proof of Theorem 1.3

In this section, we will use the same notations as in section 1. Let

$$F : \Lambda^{1,0}T(\mathbb{C}^n) \to \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p_1,q_1}T^*(\mathbb{C}^n)$$

and

$$T : \Lambda^{1,0}T(\mathbb{C}^n)) \to \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p_2,q_2}T^*(\mathbb{C}^n)$$

be linear operaors, where $p_1, q_1, p_2, q_2 \in \mathbb{Z}$, $p_1, q_1, p_2, q_2 \geq 0$. We write $F = (F_{j,k})_{j,k=1}^n$, $F_{j,k} \in \Lambda^{p_1,q_1}T^*(\mathbb{C}^n)$, $1 \leq j,k \leq n$, and $T = (T_{j,k})_{j,k=1}^n$, $T_{j,k} \in \Lambda^{p_2,q_2}T^*(\mathbb{C}^n)$, $j,k = 1,\ldots,n$, as in (1.17). $TF : \Lambda^{1,0}T(\mathbb{C}^n) \to \Lambda^{1,0}T(\mathbb{C}^n) \otimes \Lambda^{p_1+p_2,q_1+q_2}T^*(\mathbb{C}^n)$ is the linear operator defined by $TF\frac{\partial}{\partial z_k} = \sum_{s,j=1}^n \frac{\partial}{\partial z_j} \otimes (T_{j,s} \wedge F_{s,k})$, $k = 1,\ldots,n$. We have $TF = (a_{j,k})_{j,k=1}^n$. We can compute

$$a_{j,k} = \sum_{s=1}^n T_{j,s} \wedge F_{s,k} \in \Lambda^{p_1+p_2,q_1+q_2}T^*(\mathbb{C}^n), \quad j,k = 1,\ldots,n. \tag{5.1}$$

For $p \in \mathbb{C}^n$, we may assume that $p = 0$ and $\phi(z) = \sum_{j=1}^n \lambda_j |z_j|^2 + O(|z|^3)$ near 0 and as before we suppose that $\lambda_j < 0$, $j = 1,\ldots,q$, and $\lambda_j > 0$, $j = q+1,\ldots,n$. Let $M_\phi : C^\infty(\mathbb{C}^n; \Lambda^{1,0}T(\mathbb{C}^n)) \to C^\infty(\mathbb{C}^n; \Lambda^{1,0}T(\mathbb{C}^n))$ be as in (1.20). We recall that $M_\phi = \left(\frac{\partial^2 \phi}{\partial \overline{z}_j \partial z_k}\right)_{j,k=1}^n$. We have

$$U_j = \frac{\partial}{\partial z_j} \text{ at } 0, \quad j = 1,\ldots,n, \tag{5.2}$$

where U_j, $j = 1,\ldots,n$, are given by (1.22). Moreover, we have

$$(\frac{\partial}{\partial z_j} \mid \frac{\partial}{\partial z_k})_{|\phi|} = (\frac{\partial}{\partial \overline{z}_j} \mid \frac{\partial}{\partial \overline{z}_k})_{|\phi|} = \delta_{j,k} |\lambda_j| \text{ at } 0, \quad j,k = 1,\ldots,n, \tag{5.3}$$

$$(dz_j \mid dz_k)_{|\phi|} = (d\overline{z}_j \mid d\overline{z}_k)_{|\phi|} = \delta_{j,k} \frac{1}{|\lambda_j|} \text{ at } 0, \quad j,k = 1,\ldots,n. \tag{5.4}$$

Here $\delta_{j,k} = 1$ if $j = k$, $\delta_{j,k} = 0$ if $j \neq k$. For the definition of the Hermitian metrix $(\ \mid\ )_{|\phi|}$, see the discussion after (1.22).

Put $M_\phi^{-1}(z)=(b_{j,k}(z))_{j,k=1}^n$. We claim that

$$b_{j,k}(z)=\frac{\delta_{j,k}}{\lambda_j}-\frac{1}{\lambda_j\lambda_k}\sum_{s=1}^n\Big(\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0)z_s+\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial\overline{z}_s}(0)\overline{z}_s\Big)+O(|z|^2), \tag{5.5}$$

$j,k=1,\ldots,n$, near $0\in\mathbb{C}^n$. In fact, if we put $\widetilde{B}=\Big(\widetilde{b}_{j,k}\Big)_{j,k=1}^n$, $\widetilde{b}_{j,k}=\frac{\delta_{j,k}}{\lambda_j}-\frac{1}{\lambda_j\lambda_k}\sum_{s=1}^n\Big(\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0)z_s+\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial\overline{z}_s}(0)\overline{z}_s\Big)$. Then $M_\phi\widetilde{B}=C$, $C=(c_{j,k})_{j,k=1}^n$,

$$\begin{aligned}c_{j,k}&=\sum_{t=1}^n\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_t}(z)\widetilde{b}_{t,k}(z)\\&=\sum_{t=1}^n\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_t}(z)\Big(\frac{\delta_{t,k}}{\lambda_t}-\frac{1}{\lambda_t\lambda_k}\sum_{s=1}^n\big(\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_k\partial z_s}(0)z_s+\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_k\partial\overline{z}_s}(0)\overline{z}_s\big)\Big)\\&=\sum_{t=1}^n\Big(\lambda_t\delta_{j,t}+\sum_{s=1}^n\big(\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial z_s}(0)z_s+\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_s}(0)\overline{z}_s\big)\Big)\\&\quad\times\Big(\frac{\delta_{t,k}}{\lambda_t}-\frac{1}{\lambda_t\lambda_k}\sum_{s=1}^n\big(\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_k\partial z_s}(0)z_s+\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_k\partial\overline{z}_s}(0)\overline{z}_s\big)\Big)+O(|z|^2)\\&=\delta_{j,k}+O(|z|^2).\end{aligned}\tag{5.6}$$

Thus, $\widetilde{B}=M_\phi^{-1}(z)+O(|z|^2)$. (5.5) follows.

Let $Q:\Lambda^{1,0}T(\mathbb{C}^n)\to\Lambda^{1,0}T^*(\mathbb{C}^n)\otimes\Lambda^{1,0}T(\mathbb{C}^n)$ be as in (1.26). We write $Q=(Q_{j,k})_{j,k=1}^n$, $Q_{j,k}(z)=\sum_{s=1}^n\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial z_s}(0)q_{j,k,s}dz_s$ at 0, where

$$\begin{aligned}q_{j,k,s}\ =\ &\frac{|\lambda_k|\,\delta_j(k)+|\lambda_s|\,\delta_j(s)}{|\lambda_j|+|\lambda_k|\,\delta_j(k)+|\lambda_s|\,\delta_j(s)}-\delta_j(k)\delta_j(s)\\&\times\frac{|\lambda_k|^2\,|\lambda_s|^2}{(|\lambda_j|+|\lambda_k|+|\lambda_s|)^2}\Big(\frac{1}{|\lambda_j|+|\lambda_k|}+\frac{1}{|\lambda_j|+|\lambda_s|}\Big)^2.\end{aligned}\tag{5.7}$$

We recall that the definition of $\delta_k(j)$ is given by (1.25). Put $\overline{\partial}M_\phi^{-1}=(d_{j,k})_{j,k=1}^n$. From (5.5), we see that

$$d_{j,k}=-\sum_{s=1}^n\frac{1}{\lambda_j\lambda_k}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_k\partial\overline{z}_s}(0)d\overline{z}_s \tag{5.8}$$

at 0. Put $\overline{\partial}M_\phi^{-1}Q=(f_{j,k})_{j,k=1}^n$. From (5.1), (5.8), we see that

$$f_{k,j}=-\sum_{1\leq t,s,u\leq n}\frac{1}{\lambda_k\lambda_t}q_{t,j,u}\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_t\partial z_j\partial z_u}(0)d\overline{z}_s\wedge dz_u \tag{5.9}$$

at 0. As in section 1, put $e_j=\frac{1}{\sqrt{|\lambda_j|}}U_j$, $j=1,\ldots,n$. From (5.9), it is not difficult to see that

$$<(\overline{\partial}M_\phi^{-1}Qe_j\mid e_k)_{|\phi|},\overline{e}_j\wedge e_k>(0)=-\sum_{1\leq t\leq n}\frac{1}{\lambda_k\lambda_t\left|\lambda_j\right|}q_{t,j,k}\left|\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_j}(0)\right|^2, \tag{5.10}$$

$j,k=1,\ldots,n$. From (5.10) and (5.7), it is straight forward to see that

Proposition 5.1. *We have that*

$$\begin{aligned}&<(\overline{\partial}M_\phi^{-1}Qe_j\mid e_k)_{|\phi|},\overline{e}_j\wedge e_k>(0)\\&=\sum_{1\leq t\leq q}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_j}(0)\right|^2\\&\times\Big(\frac{|\lambda_j|+|\lambda_k|}{|\lambda_j|+|\lambda_k|+|\lambda_t|}-\frac{|\lambda_j|^2\,|\lambda_k|^2}{(|\lambda_j|+|\lambda_k|+|\lambda_t|)^2}\big(\frac{1}{|\lambda_j|+|\lambda_t|}+\frac{1}{|\lambda_k|+|\lambda_t|}\big)^2\Big),\end{aligned} \tag{5.11}$$

where $q+1\leq j,k\leq n$,

$$\begin{aligned}&<(\overline{\partial}M_\phi^{-1}Qe_j\mid e_k)_{|\phi|},\overline{e}_j\wedge e_k>(0)\\&=\sum_{q+1\leq t\leq n}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_j}(0)\right|^2\\&\times\Big(\frac{|\lambda_j|+|\lambda_k|}{|\lambda_j|+|\lambda_k|+|\lambda_t|}-\frac{|\lambda_j|^2\,|\lambda_k|^2}{(|\lambda_j|+|\lambda_k|+|\lambda_t|)^2}\big(\frac{1}{|\lambda_j|+|\lambda_t|}+\frac{1}{|\lambda_k|+|\lambda_t|}\big)^2\Big),\end{aligned} \tag{5.12}$$

where $1\leq j,k\leq q$,

$$\begin{aligned}&<(\overline{\partial}M_\phi^{-1}Qe_j\mid e_k)_{|\phi|},\overline{e}_j\wedge e_k>(0)\\&=-\sum_{1\leq t\leq q}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\frac{|\lambda_j|}{|\lambda_t|+|\lambda_j|}\left|\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_j}(0)\right|^2\\&\quad+\sum_{q+1\leq t\leq n}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\frac{|\lambda_k|}{|\lambda_t|+|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_j}(0)\right|^2,\end{aligned} \tag{5.13}$$

where $q+1\leq j\leq n$, $1\leq k\leq q$, *and*

$$\begin{aligned}
&< (\overline{\partial}M_\phi^{-1}Qe_j \mid e_k)_{|\phi|}, \overline{e}_j\wedge e_k > (0)\\
&= \sum_{1\leq t\leq q}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\frac{|\lambda_k|}{|\lambda_t|+|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_j}(0)\right|^2\\
&\quad -\sum_{q+1\leq t\leq n}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\frac{|\lambda_j|}{|\lambda_t|+|\lambda_j|}\left|\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_t\partial\overline{z}_j}(0)\right|^2,
\end{aligned}\tag{5.14}$$

where $1\leq j\leq q$, $q+1\leq k\leq n$.

Let $\Theta_\phi : C^\infty(\mathbb{C}^n;\,\Lambda^{1,0}T(\mathbb{C}^n))\rightarrow C^\infty(\mathbb{C}^n;\,\Lambda^{1,1}T^*(\mathbb{C}^n)\otimes\Lambda^{1,0}T(\mathbb{C}^n))$ be as in (1.24). We recall that $\Theta_\phi=\left(\overline{\partial}\theta_{j,k}\right)_{j,k=1}^n=(\Theta_{j,k})_{j,k=1}^n$, where $\theta=h^{-1}\partial h=(\theta_{j,k})_{j,k=1}^n$, $h=\left(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k}\right)_{j,k=1}^n$. It is not difficult to see that

$$\Theta_{j,k}=\frac{1}{\lambda_j}\overline{\partial}\partial(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_k})-\sum_{t=1}^n\frac{1}{\lambda_t\lambda_j}\overline{\partial}(\frac{\partial^2\phi}{\partial\overline{z}_j\partial z_t})\wedge\partial(\frac{\partial^2\phi}{\partial\overline{z}_t\partial z_k}),\tag{5.15}$$

$j,k=1,\ldots,n$. From (5.15), it is straight forward to see that

Proposition 5.2. *We have that*

$$\begin{aligned}
&< (\Theta_\phi e_j \mid e_k)_{|\phi|}, \overline{e}_j\wedge e_k > (0)\\
&= \frac{1}{|\lambda_j|\,|\lambda_k|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0)+\sum_{t=1}^q\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2\\
&\quad -\sum_{t=q+1}^n\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2,
\end{aligned}\tag{5.16}$$

where $q+1\leq j,k\leq n$,

$$\begin{aligned}
&< (\Theta_\phi e_j \mid e_k)_{|\phi|}, \overline{e}_j\wedge e_k > (0)\\
&= -\frac{1}{|\lambda_j|\,|\lambda_k|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0)-\sum_{t=1}^q\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2\\
&\quad +\sum_{t=q+1}^n\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2,
\end{aligned}\tag{5.17}$$

where $1 \leq j, k \leq q$,

$$\begin{aligned}
&< (\Theta_\phi e_j \mid e_k)_{|\phi|}, \overline{e}_j \wedge e_k > (0) \\
&= -\frac{1}{|\lambda_j|\,|\lambda_k|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0) - \sum_{t=1}^{q}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2 \\
&\quad + \sum_{t=q+1}^{n}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2, \qquad (5.18)
\end{aligned}$$

where $q+1 \leq j \leq n$, $1 \leq k \leq q$, *and*

$$\begin{aligned}
&< (\Theta_\phi e_j \mid e_k)_{|\phi|}, \overline{e}_j \wedge e_k > (0) \\
&= \frac{1}{|\lambda_j|\,|\lambda_k|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0) + \sum_{t=1}^{q}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2 \\
&\quad - \sum_{t=q+1}^{n}\frac{1}{|\lambda_t|\,|\lambda_j|\,|\lambda_k|}\left|\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_t\partial\overline{z}_k}(0)\right|^2, \qquad (5.19)
\end{aligned}$$

where $1 \leq j \leq q$, $q+1 \leq k \leq n$.

As in section 1, define

$$R = \Theta_\phi - (\overline{\partial} M_\phi^{-1})Q : \Lambda^{1,0}T(\mathbb{C}^n) \to \Lambda^{1,1}T^*(\mathbb{C}^n) \otimes \Lambda^{1,0}T(\mathbb{C}^n).$$

From (5.11)–(5.14) and (5.16)–(5.17), it is not difficult to see that

$$\begin{aligned}
&\frac{1}{4}\sum_{j,k=1}^{n}(1+\delta_j(k)\frac{|\lambda_j|-|\lambda_k|}{|\lambda_j|+|\lambda_k|} < (Re_j \mid e_k)_{|\phi|}, \overline{e}_j \wedge e_k > (0) \\
&= \frac{1}{4}\Big(-\sum_{1\leq j,k\leq q}\frac{1}{|\lambda_j|\,|\lambda_t|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0) + \sum_{q+1\leq j,k\leq n}\frac{1}{|\lambda_j|\,|\lambda_k|}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0) \\
&\quad -2\sum_{q+1\leq j\leq n, 1\leq k\leq q}\frac{|\lambda_j|-|\lambda_k|}{|\lambda_j|\,|\lambda_k|\,(|\lambda_j|+|\lambda_k|)}\frac{\partial^4\phi}{\partial\overline{z}_j\partial z_j\partial\overline{z}_k\partial z_k}(0)\Big) \\
&\quad -\frac{1}{2}\sum_{q+1\leq j\leq n, 1\leq k,s\leq q}\frac{|\lambda_j|-|\lambda_k|}{|\lambda_j|\,|\lambda_k|\,(|\lambda_j|+|\lambda_k|)(|\lambda_s|+|\lambda_j|)}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_k}(0)\right|^2 \\
&\quad +\frac{1}{2}\sum_{q+1\leq j,s\leq n, 1\leq k\leq q}\frac{|\lambda_j|-|\lambda_k|}{|\lambda_j|\,|\lambda_k|\,(|\lambda_j|+|\lambda_k|)(|\lambda_s|+|\lambda_k|)}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_k}(0)\right|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{q+1\leq j,k\leq n,1\leq s\leq q}\Big(\frac{|\lambda_k|^2\,|\lambda_j|^2}{|\lambda_s|\,|\lambda_j|\,|\lambda_k|\,(|\lambda_k|+|\lambda_j|+|\lambda_s|)^2}\big(\frac{1}{|\lambda_j|+|\lambda_s|}\\
&+\frac{1}{|\lambda_k|+|\lambda_s|}\big)^2+\frac{1}{(|\lambda_k|+|\lambda_j|+|\lambda_s|)\,|\lambda_j|\,|\lambda_k|}\Big)\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_k}(0)\right|^2\\
&+\frac{1}{4}\sum_{q+1\leq s\leq n,1\leq j,k\leq q}\Big(\frac{|\lambda_k|^2\,|\lambda_j|^2}{|\lambda_s|\,|\lambda_j|\,|\lambda_k|\,(|\lambda_k|+|\lambda_j|+|\lambda_s|)^2}\big(\frac{1}{|\lambda_j|+|\lambda_s|}+\frac{1}{|\lambda_k|+|\lambda_s|}\big)^2\\
&+\frac{1}{(|\lambda_k|+|\lambda_j|+|\lambda_s|)\,|\lambda_j|\,|\lambda_k|}\Big)\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_k}(0)\right|^2\\
&-\frac{1}{4}\sum_{1\leq j,k,s\leq q}\frac{1}{|\lambda_j|\,|\lambda_k|\,|\lambda_s|}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_k}(0)\right|^2\\
&-\frac{1}{4}\sum_{q+1\leq j,k,s\leq n}\frac{1}{|\lambda_j|\,|\lambda_k|\,|\lambda_s|}\left|\frac{\partial^3\phi}{\partial\overline{z}_s\partial z_j\partial z_k}(0)\right|^2. \qquad (5.20)
\end{aligned}
$$

From (5.7), we can check that

$$
(Qe_j \mid e_j)=\sum_{s=1}^{n}\delta_j(s)\frac{|\lambda_s|}{|\lambda_j|\,(|\lambda_j|+|\lambda_s|)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)dz_s
$$

at 0. From this, it is straight forward to see that

$$
\begin{aligned}
&-\sum_{j,k=1}^{n}\delta_j(k)\frac{|\lambda_j|}{|\lambda_j|+|\lambda_k|}\mathrm{Re}\left((Qe_j \mid e_j) \mid \partial M_\phi e_k \mid e_k\right)_{|\phi|}(0)\\
&= -\frac{1}{2}\sum_{1\leq s,k\leq q,q+1\leq j\leq n}\frac{1}{(|\lambda_j|+|\lambda_k|)(|\lambda_j|+|\lambda_s|)\,|\lambda_k|}\\
&\quad\times\Big(\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_k\partial\overline{z}_s}(0)+\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_k\partial\overline{z}_s}(0)\Big)\\
&\quad-\frac{1}{2}\sum_{1\leq k\leq q,q+1\leq j,s\leq n}\frac{1}{(|\lambda_j|+|\lambda_k|)(|\lambda_k|+|\lambda_s|)\,|\lambda_j|}\\
&\quad\times\Big(\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_k\partial\overline{z}_s}(0)+\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_k\partial\overline{z}_s}(0)\Big) \qquad (5.21)
\end{aligned}
$$

and

$$
\frac{1}{2}\Big|\sum_{j=1}^{n}(Qe_j \mid e_j)^2_{|\phi|}\Big|
$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{q+1\leq j,k\leq n,1\leq s\leq q}\frac{|\lambda_s|}{|\lambda_j|\,|\lambda_k|\,(|\lambda_j|+|\lambda_s|)(|\lambda_k|+|\lambda_s|)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\\
&\qquad\times\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_k\partial\overline{z}_s}(0)\\
&\quad+\frac{1}{2}\sum_{1\leq j,k\leq q,q+1\leq s\leq n}\frac{|\lambda_s|}{|\lambda_j|\,|\lambda_k|\,(|\lambda_j|+|\lambda_s|)(|\lambda_k|+|\lambda_s|)}\frac{\partial^3\phi}{\partial\overline{z}_j\partial z_j\partial z_s}(0)\\
&\qquad\times\frac{\partial^3\phi}{\partial\overline{z}_k\partial z_k\partial\overline{z}_s}(0). \qquad (5.22)
\end{aligned}
$$

Combining (5.22), (5.21) and (5.20) with (4.16), Theorem 1.3 follows.

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