Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **11** (2016), No. 2, pp. 343-357

LEAF-WISE HARMONIC MAPS OF MANIFOLDS WITH 2-DIMENSIONAL FOLIATIONS

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Abstract

We derive the expressions of harmonic non $\pm$ holomorphic maps of Riemann surfaces. We study the relationship between leaf-wise harmonic maps and harmonic maps. We investigate the Gauss-Bonnet theorem for leaf-wise harmonic maps of manifolds with 2-dimensional foliations.

1. Introduction

The theory of harmonic maps between Riemannian manifolds were first established by Eells and Sampson [11] in 1964. Afterwards, there are two reports by Eells and Lemaire [9, 10] about the developments of harmonic maps up to 1988. Chiang and Ratto also studied harmonic maps in [2]-[6]. Harmonic and biharmonic maps of manifolds with Riemannian foliations were investigated by Eells and Verjovsky [12], El Kacimi and Gomez [17], Konderak and Wolak [18], Chiang and Wolak [7], etc.

In this paper, we derive the expressions of harmonic non $\pm$ holomorphic maps between Riemann surfaces in Theorem 2.2. In section three, we study the relationship between leaf-wise harmonic maps and harmonic maps of foliated Riemannian manifolds. In the 1980s, Connes [8] and Ghys [16] studied the Gauss-Bonnet (type) theorem for compact manifolds with 2-dimensional foliations. Based on Theorem 2.2, we are able to construct a non-trivial

Received April 27, 2015 and in revised form June 4, 2015.
AMS Subject Classification: 58E20, 53C12, 58G11.
Key words and phrases: Leaf-wise harmonic map, harmonic map, Gauss-Bonnet theorem.
The author is supported by a Faculty Research Grant of the University of Mary Washington.

$\mathcal{F}_1$-harmonic measure on the domain globally so that we can prove Gauss-Bonnet (type) Theorem 4.1 for a leaf-wise harmonic map between compact manifolds with 2-dimensional foliations without singularities on leaves, which generalize the main theorems in [8] and [16] (cf. Corollary 4.2). If a leaf-wise harmonic map between compact manifolds with 2-dimensional foliations is with isolated singularities on some leaves, then we can not define a harmonic measure globally. Therefore, we study the Gauss-Bonnet theorem for a leaf-wise harmonic map differently using usual measure by considering the stationary indices of the singularities.

2. Harmonic Maps of Riemann Surfaces

Let $f : M \to N$ be a C^2 map between two Riemann surfaces M and N with Riemannian metrics g and h. With respect to isothermal local coordinate systems (U, z) and (V, ζ) on M and N, respectively, the given metrics are represented as $g = \lambda\, dz \otimes d\overline{z}$ and $h = \mu\, d\zeta \otimes d\overline{\zeta}$ for some $\lambda \in C^\infty(U)$ and $\mu \in C^\infty(V)$ with $\lambda > 0$ and $\mu > 0$. Let $D \subset M$ be a relatively compact domain. The *Dirichlet energy* functional $E_D : C^2(M, N) \to \mathbb{R}$ is given by

$$E_D(f) = \frac{1}{2} \int_D \|df\|^2\, dv_g,$$

where $\|df\| : M \to [0, +\infty)$ is the Hilbert-Schmidt norm of df and dv_g the canonical volume form on (M, g). Locally,

$$\|df\|^2 = \lambda^{-1}\, (\mu \circ f) \left\{ |w_z|^2 + |w_{\overline{z}}|^2 \right\},$$

where $w = \zeta \circ f$. Also if $z = x + iy$, then $dv_g = \lambda\, dx \wedge dy$ on U. A map $f \in C^2(M, N)$ is *harmonic* if for any relatively compact domain $D \subset M$ and any smooth 1-parameter variation $\{f_t\}_{|t|<\epsilon} \subset C^2(M, N)$ with $\text{supp}(V) \subset D$

$$\frac{d}{dt} \{E(f_t)\}_{t=0} = 0,$$

where V is the infinitesimal variation induced by $\{f_t\}_{|t|<\epsilon}$ (i.e. $V_p = (d_{p,0}F)$ $(\partial/\partial t)_{(p,0)}$ for any $p \in M$ and $F : M \times (-\epsilon, \epsilon) \to N$ is given by $F(p, t) = f_t(p)$ for any $p \in M$ and $|t| < \epsilon$).

Let ∇ and ∇^N be the Levi-Civita connections of (M, g) and (N, h), respectively. Let $h^f = f^*h$ (or $f^{-1}h$) and $\nabla^f = f^{-1}\nabla^N$ be the Riemannian bundle metric and connection in the pullback bundle $f^{-1}T(N) \to M$ induced by h and ∇^N. For $X \in \mathfrak{X}(M)$ we denote $f_*X \in C^\infty(f^{-1}T(N))$ the section given by $(f_*X)(p) = (d_pf)X_p$ for any $p \in M$. The *second fundamental form* of f is

$$\beta_f(X,Y) = \nabla^f_X f_*Y - f_*\nabla_X Y, \quad X, Y \in \mathfrak{X}(M).$$

The *tension field* of f is $\tau(f) = \mathrm{trace}_g\ (\beta_f) \in C(f^{-1}T(N))$. Locally

$$\tau(f) = \sum_{a=1}^{2} \beta_f(X_a, X_a),$$

where $X_1 = \lambda^{-1/2}\partial/\partial x$ and $X_2 = \lambda^{-1/2}\partial/\partial y$. The *first variation formula* (cf. [11]) is

$$\frac{d}{dt}\,\{E_D(f_t)\}_{t=0} = -\int_D h^f(V\,,\,\tau(f))\,dv_g\,.$$

Hence, a map $f \in C^2(M, N)$ is harmonic if $\tau(f) = 0$, i.e. locally

$$w_{z\overline{z}} + (\mu\circ f)^{-1}(\mu\circ f)_{\zeta\circ f}w_z w_{\overline{z}} = 0 \tag{2.1}$$

on U (the domains U and V of the local charts are tacitly chosen such that $f(U) \subset V$). Both the Dirichlet energy and harmonicity are known to be conformal invariants. We shall need the following lemma (cf. [13, 21]) to prove Theorem 2.2.

Lemma 2.1. *Let $f : M \to N$ be a harmonic map between two Riemann surfaces. Then the $(2, 0)$ component of f^*h is a holomorphic quadratic differential on M locally given by*

$$Q = \phi\, dz \otimes dz = \lambda\, w_z\, \overline{w}_z\, dz \otimes dz. \tag{2.2}$$

Moreover, $\phi = 0$ if and only if f is $\pm$ holomorphic. If f is harmonic non $\pm$ holomorphic, then $\phi(z) \neq 0$ and the zeros of w_z and $w_{\overline{z}}$ are isolated of finite order.

Proof. Differentiating $\phi = \lambda\, w_z\, \overline{w}_z$, one finds

$$\phi_{\overline{z}} = \lambda_w\, w_{\overline{z}}\, w_z\overline{w}_z + \lambda_{\overline{w}}\, \overline{w}_{\overline{z}}\, w_z\overline{w}_z + \lambda\, w_{z\overline{z}}\,\overline{w}_z + \lambda\,\overline{w}_{z\overline{z}}\, w_z,$$

where $\lambda = \mu \circ f$ and $w = \zeta \circ f$. It may be rewritten as $\phi_{\overline{z}} = \overline{w}_z H + w_z \overline{H}$, where H denotes the left hand side of (2.1). It is clear that f is $\pm$ holomorphic if and only if $\phi = 0$. Recall that a function $F : D \to \mathbb{C}$ defined on an open neighborhood $D \subset \mathbb{C}$ of the origin has a zero of *infinite order* at $z = 0$ if $F(z) = o(|z|^m)$ as $z \to 0$ for all $m \geq 0$. If w_z and $w_{\overline{z}}$ have zeros of infinite order, then $\phi = o(|z|^m)$ as $z \to 0$ for all $m \geq 0$. Therefore, $\phi = 0$, and so f must be a $\pm$ holomorphic map. Otherwise, if f is harmonic non $\pm$ holomorphic, then $\phi(z) \neq 0$ and the zeros of w_z and $w_{\bar{z}}$ are isolated of finite order. $\square$

Theorem 2.2. *If $f : M \to N$ is a harmonic non $\pm$ holomorphic map between two Riemann surfaces, then*

$$\begin{aligned} w(z) &= A\, z^m + o(|z|^m)\, \textit{for some } A \in \mathbb{C} \setminus \{0\} \textit{ and } m \geq 1, \textit{ or} && (2.3)\\ w(z) &= B\, \overline{z}^n + o(|z|^n)\, \textit{for some } B \in \mathbb{C} \setminus \{0\} \textit{ and } n \geq 1, \textit{ or} && (2.4)\\ w(z) &= C\, z^k + D\, \overline{z}^k + o(|z|^k)\, \textit{for some } C, D \in \mathbb{C} \setminus \{0\} \textit{ and } k \geq 1. && (2.5)\end{aligned}$$

Proof. Let $p \in M$ and let (U, z) and (V, ζ) be the isothermal local coordinate systems on M and N such that $p \in U$, $f(U) \subset V$, $z(p) = 0$ and $\zeta(a) = 0$ where $a = f(p)$. Since $f : M \to N$ is harmonic and g, h are analytic, f is analytic (cf. [11]). Therefore, we may expand $w = \zeta \circ f$ in a power series

$$w(z) = \sum_{i,j=0}^{\infty} a_{ij} z^i \overline{z}^j, \quad (a_{ij} \in \mathbb{C}), \tag{2.6}$$

which converges in a neighborhood of $z = 0$. We first have $a_{00} = 0$ (due to $\zeta(a) = 0$). Because $w_z, w_{\overline{z}}$ and $w_{z\overline{z}}$ are analytic (as w is analytic), and w_z and $w_{\overline{z}}$ have isolated zeros of finite order by Lemma 2.1, we may choose a sufficiently small neighborhood of p (denoted again by U) such that it avoids all zeros of w_z and $w_{\overline{z}}$. Then we can rewrite (2.1) as

$$(\mu \circ f)^{-1}\, (\mu \circ f)_{\zeta \circ f} = -w_{z\overline{z}} / \, (w_z w_{\overline{z}}) \tag{2.7}$$

on U. As $\mu^{-1}\, \mu_\zeta$ is analytic at a, it may be expanded in a power series

$$\mu^{-1}\, \mu_\zeta = \sum_{k,\ell=0}^{\infty} b_{kl} \zeta^k \overline{\zeta}^\ell \quad (b_{k\ell} \in \mathbb{C}), \tag{2.8}$$

which converges in a neighborhood of $\zeta = 0$. We assume that the complex coordinate ξ on the codomain of $v = \mu^{-1}\mu_\zeta$ is chosen such that $\xi(v(a)) = 0$, it implies that $b_{00} = 0$. Substituting (2.6) and (2.7) into (2.8), it yields to $a_{11} = 0$. If $a_{10} \neq 0$, $a_{01} = 0$, then $w(z)$ is in (2.3); if $a_{10} = 0, a_{01} \neq 0$, then $w(z)$ is in (2.4) ; if $a_{10} \neq 0$, $a_{01} \neq 0$, then $w(z)$ is in (2.5). If $a_{10} = a_{01} = 0$, then we derive $a_{21} = a_{12} = 0$ by (2.7), and $w(z)$ can be expressed in (2.3), (2.4) or (2.5), etc. ☐

As a corollary to Theorem 2.2, it was shown by Eells and Wood [13] as follows.

Corollary 2.3. *If $f : M \to N$ is harmonic non $\pm$ holomorphic between Riemann surfaces, then*

$$\begin{aligned} w_z &= Ez^{m-1} + o(|z|^{m-1}), \text{ for some } m \geq 1 \text{ and complex number } E \neq 0; \\ w_{\bar{z}} &= F\bar{z}^{n-1} + o(|z|^{n-1}), \text{ for } n \geq 1 \text{ and } F \neq 0. \end{aligned}$$

3. Leaf-wise Harmonic Maps

Let $\mathcal{F}$ be a foliation on a Riemannian n-manifold (M, g). Then $\mathcal{F}$ is defined by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}_{i\in I}$ modeled on a q-manifold N_0 such that

(1) $\{U_i\}_{i\in I}$ is an open covering of M;

(2) $f_i : U_i \to N_0$ are submersions with connected fibres;

(3) $g_{ij} : N_0 \to N_0$ are local diffeomorphisms of N_0 with $f_i = g_{ij} f_j$ on $U_i \cap U_j$.

The connected components of the trace of any leaf of $\mathcal{F}$ on U_i consist of the fibres of f_i. The open subsets $N_i = f_i(U_i) \subset N_0$ form a q-manifold $N = \amalg N_i$, which can be considered as a transverse manifold of the foliation $\mathcal{F}$. The pseudogroup $\mathcal{H}_N$ of local diffeomorphisms of N generated by g_{ij} is called the holonomy pseudogroup of the foliated manifold $(M, \mathcal{F})$ defined by the cocycle $\mathcal{U}$. If the foliation $\mathcal{F}$ is Riemannian for the Riemannian metric g, then it induces a Riemannian metric $\bar{g}$ on N such that the submersions f_i are Riemannian submersions and the elements of the holonomy group are isometries.

Let $\phi : U \to \mathbf{R}^p \times \mathbf{R}^q$, $\phi = (\phi^1, \phi^2) = (x_1, \ldots, x_p, y_1, \ldots, y_q)$ be an adapted chart on a foliated manifold $(M, \mathcal{F})$. Then on U the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_p}$ span the bundle $T\mathcal{F}$ tangent to the leaves of the foliation $\mathcal{F}$, the equivalence classes of $\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_q}$ denoted by $\frac{\bar{\partial}}{\partial y_1}, \ldots, \frac{\bar{\partial}}{\partial y_q}$, span the normal bundle $N(M, \mathcal{F}) = TM/T\mathcal{F}$, which is isomorphic to the subbundle $T\mathcal{F}^{\perp}$. Please see more details about foliations in [19, 23].

We study the relationship between leaf-wise harmonic maps and harmonic maps between foliated Riemannian manifolds as follows.

Theorem 3.1. *Suppose that $(M_1, \mathcal{F}_1)$ and $(M_2, \mathcal{F}_2)$ are two foliated Riemannian manifolds such that $\mathcal{F}_1$ is minimal and $\mathcal{F}_2$ is totally geodesic. If $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is leaf-wise harmonic and transversally harmonic, then f is harmonic.*

Proof. Since the definitions of transversally harmonic map and harmonic map are local, we consider open subsets $U_i \subset M_i$ and Riemannian submersions $\phi_i : U_i \to \bar{U}_i$, $i = 1, 2$, such that the foliations on U_i are fibres of the submersions $\phi_i : U_i \to \bar{U}_i$ and $f(U_1) \subset U_2$. Then there exists the unique map $\bar{f} : \bar{U}_1 \to \bar{U}_2$ such that

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & U_2 \\ {\scriptstyle \phi_1}\big\downarrow & & {\scriptstyle \phi_2}\big\downarrow \\ \bar{U}_1 & \xrightarrow{\bar{f}} & \bar{U}_2 \end{array}$$

Diagram 3.1.

commutes, where the vertical maps are Riemannian submersions. A map $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ between two foliated Riemannian manifolds is *transversally harmonic* if and only if $\bar{f} : \bar{U}_1 \to \bar{U}_2$ is harmonic locally (cf. [18, 7]).

On a manifold M with a foliation $\mathcal{F}$, we can have another topology and smooth structure to take as open subsets of the set M open subsets of leaves (cf. [20]). Then the leaves of $\mathcal{F}$ are connected components in this topology and the set M carries a differentiable structure compatible with this topology; we denote this manifold by $M_{\mathcal{F}}$. Moreover, a smooth map $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is foliated iff it induces a smooth map $\hat{f} : M_{\mathcal{F}_1} \to M_{\mathcal{F}_2}$. The map f is *leaf-wise harmonic* if $\tau(\hat{f}) = 0$.

If X is a vector tangent to a foliated manifold, denote $\mathcal{H}(X)$ and $\mathcal{V}(X)$ the orthogonal to and tangent to the leaves, respectively. Let B_i denote the second fundamental form and H_i denote the mean curvature vector fields of leaves of $\mathcal{F}_i$, $i = 1, 2$, respectively. In Diagram 3.1, considering the vertical maps ϕ_1 and ϕ_2 are Riemannian submersions, we can apply [26] (eq. (6.10)) and obtain

$$\tau(f) = \tau(\bar{f}) + trace_{T\mathcal{F}_1} f^* B_2 - f_* H_1 + \tau(\hat{f}). \tag{3.1}$$

Since $\mathcal{F}_1$ is minimal and $\mathcal{F}_2$ is totally geodesic, the second and third terms vanish. If f is transversally harmonic ($\tau(f)^H = \tau(\bar{f}) = 0$) and leaf-wise harmonic ($\tau(f)^V = \tau(\hat{f}) = 0$), then it implies $\tau(f) = 0$. □

In particular, if $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is leaf-wise harmonic and transversally harmonic between two manifolds with 2-dimensional foliations such that $\mathcal{F}_1$ is minimal and $\mathcal{F}_2$ is totally geodesic, then f is harmonic.

We review a small part of the main results of Garnett [14, 15], which is useful to study Gauss-Bonnet type theorem. Let $\mathcal{F}$ be any foliation on a compact manifold M equipped with a Riemannian metric g on its tangent bundle. We may assume that $\mathcal{F}$ and the Riemannian metric are of class C^3. We can use the Laplace operators of the leaves to construct a global operator $\triangle^{\mathcal{F}}$ defined on functions $u : M \to \mathbb{R}$ that are C^2 along the leaves:

$$\triangle^{\mathcal{F}} u(x) = \triangle_{L(x)} u_{|L(x)}(x), \tag{3.2}$$

where $L(x)$ is the leaf through x and $\triangle_{L(x)}$ is the Laplace operator of the Riemannian manifold $L(x)$ with respect to its Riemannian metric induced by g.

Definition 3.2. A measure μ on (M, g) is called *$\mathcal{F}$-harmonic*, if for every continuous function $u : M \to R$ which is C^2 along the leaves, the integral $\int \triangle^{\mathcal{F}} u \, d\mu$ is zero.

Theorem 3.3 ([14, 15])**.**

(1) *A compact foliated manifold (M, g) always admits a non-trivial $\mathcal{F}$-harmonic measure.*

(2) *A measure μ is $\mathcal{F}$-harmonic if and only if in any disintegrated open set, μ can be disintegrated as a transversal sum of leaf measure, where every leaf*

measure is a positive harmonic function times the Riemannian volume of the leaf.

(3) *The measure on M obtained by combination of a transverse invariant measure and the volume along the leaves is always $\mathcal{F}$-harmonic. These special $\mathcal{F}$-harmonic measures are "completely invariant measures."*

(4) *Suppose that μ is an $\mathcal{F}$-harmonic measure such that for μ almost every point x, the universal covering space $\tilde{L}(x)$ of $L(x)$ has no non-constant positive harmonic functions. Then μ is completely invariant.*

Notice that the positive harmonic function changes by a positive multiplicative constant when one changes the distinguished open set. Also, the harmonic functions are constants if and only if the measure μ is the combination of a transverse invariant measure and the volume along the leaves.

4. Gauss-Bonnet Theorem

Let $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ be a foliated map between two foliated Riemannian manifolds. When both foliated Riemannian manifolds M_1 and M_2 are considered as disjoint unions of leaves, the definition of a leaf-wise harmonic map in section two is equivalent to the following definition of a leaf-wise harmonic map between two manifolds with 2-dimensional foliations. A map $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ between two foliated Riemannian manifolds is a leaf-preserving map if it has the property that $df(T\mathcal{F}_1) \subset T\mathcal{F}_2$. A map $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a *leaf-wise harmonic map* between two manifolds with 2-dimensional foliations, if $f : (M_1, \mathcal{F}_1 \to (M_2, \mathcal{F}_2)$ is a foliated leaf-preserving map between two manifolds with 2-dimensional foliations which sends a 2-dimensional leaf L_1 of $\mathcal{F}_1$ into a 2-dimensional leaf L_2 of $\mathcal{F}_2$, as f restricted to each leaf, still denoted by $f : L_1 \to L_2$, is harmonic.

Suppose that M_1 and M_2 are disjoint unions of leaves. Let $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ be a leaf-wise harmonic map between two compact Riemannian manifolds with 2-dimensional foliations, which sends a 2-dimensional leaf L_1 of $\mathcal{F}_1$ into a 2-dimensional leaf L_2 of $\mathcal{F}_2$, as f restricted to each leaf, still denoted by $f : L_1 \to L_2$, is harmonic. The Jacobian $J = \frac{\lambda}{\sigma}(|w_z|^2 - |w_{\bar{z}}|^2)$ of a harmonic map $f : L_1 \to L_2$ may be positive, or negative, or zero by Theorem 2.2. A point $p \in D$ is a *singular point* iff the Jacobian J vanishes at p. A point p is a *stationary point* iff $w_z = w_{\bar{z}} = 0$ (since $df(x) = w_z dz + w_{\bar{z}} d\bar{z} = 0$,

z is the local coordinate of p). When J is zero, f may have isolated singularities (i.e., isolated stationary points), or non-isolated singularities. Wood [24] studied the singularities of harmonic maps between surfaces, and Smith [22] constructed a harmonic non $\pm$ holomorphic map from a torus into a sphere, which exhibited collapsed lines. We shall not consider this degenerated case here. From now on, we assume that $f : L_1 \to L_2$ is a harmonic map with isolated stationary points.

Since $f : M_1 \to M_2$ is a leaf-preserving harmonic map with isolated stationary points, the pull-back metric on a pull-back leaf $L(x) \subset f^{-1}L(f(x))$ is

$$\begin{aligned} f^*h &= \lambda w_z \bar{w}_z dz^2 + \lambda w_z \bar{w}_{\bar{z}} dz d\bar{z} + \lambda w_{\bar{z}} \bar{w}_z d\bar{z} dz + \lambda w_{\bar{z}} \bar{w}_{\bar{z}} d\bar{z}^2 \\ &= [\lambda w_z \bar{w}_z + \lambda(|w_z|^2 + |w_{\bar{z}}|^2|) + \lambda w_{\bar{z}} \bar{w}_{\bar{z}}] dx^2 + [2i\lambda w_z \bar{w}_z - 2i\lambda w_{\bar{z}} \bar{w}_{\bar{z}}] dx dy \\ &\quad + [-\lambda w_z \bar{w}_z + \lambda(|w_z|^2 + |w_{\bar{z}}|^2) - \lambda w_{\bar{z}} \bar{w}_{\bar{z}}] dy^2 \qquad (4.1) \\ &= [2Re\, \phi(z) + \lambda(|w_z|^2 + |w_{\bar{z}}|^2)] dx^2 - 4Im\, \phi(z) dx dy \\ &\quad + [-2Re\, \phi(z) + \lambda(|w_z|^2 + |w_{\bar{z}}|^2)] dy^2, \end{aligned}$$

where $Re\phi$ and $Im\phi$ are harmonic functions on $L(x) - S$, and $S = \{x \in L(x) | w_z = w_{\bar{z}} = 0\}$ is a set of finite isolated stationary points ($L(x)$ is assumed compact).

Theorem 4.1. *Suppose that $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ is a leaf-wise harmonic map between compact manifolds with 2-dimensional foliations (i.e. manifolds foliated by Riemann surfaces) without singularities on leaves. Denote by $\kappa(x)$ the Gauss curvature of the pull-back leaf through x with respect to the pull-back metric $g_1 = f^*h$. If the set of spherical leaves is harmonic measure-negligible, then $\int \kappa(x)\, d\mu$ is non-positive.*

Proof. The given map $f : (M_1, \mathcal{F}_1, g) \to (M_2, \mathcal{F}_2, h)$ is a leaf-wise harmonic map between compact manifolds foliated by Riemann surfaces without singularities on leaves. Let h be the restriction of the Riemannian metric to $T\mathcal{F}_2$, the pull-back metric $g_1 = f^*h$ does not vanish ($S = \emptyset$) on the pull-back leaf $L(x)$. It follows from (4.1) that (1) if the restricted map of each full-back leaf is holomorphic (resp. anti-holomorphic), then (4.1) reduces to $g_1 = f^*h = \lambda |w_z|^2 dz d\bar{z}$ (resp. $\lambda |w_{\bar{z}}|^2 dz d\bar{z}$) which is Riemannian and hermitian on each pull-back leaf $L(x)$ of $\mathcal{F}_1$. (2) If the restricted map of each leaf is harmonic non $\pm$holomorphic, then by Theorem 2.2, $g_1 = f^*h$ is Riemannian, but not necessarily hermitian on each full-back leaf $L(x)$ of $\mathcal{F}_1$. We can use the Laplace operators of the pull-back leaves to construct a global operator $\triangle^{\mathcal{F}_1}$ defined on functions $u : M_1 \to \mathbb{R}$ that are C^2 along the leaves:

$$\triangle^{\mathcal{F}_1} u(x) = \triangle_{L(x)} u(x)_{|L(x)}, \tag{4.2}$$

where $L(x)$ is the pull-back leaf of $\mathcal{F}_1$ through x, $L(f(x))$ is the leaf $\mathcal{F}_2$ through $y = f(x)$, and $\triangle_{L(x)}$ is the Laplace operator of the leaf $L(x)$ (viewed as a Riemannian manifold $(L(x), g_1)$). By Theorem 3.3, there exists a non-trivial $\mathcal{F}_1$-harmonic measure μ on M_1 with respect to g_1, and so $\int \triangle^{\mathcal{F}_1} u d\mu$ is zero, or equivalently,

$$\int \triangle_{L(x)} u_{|L(x)}(x) d\mu = 0, \tag{4.3}$$

for every continuous function $u : M_1 \to \mathbb{R}$ which is C^2 along the $\mathcal{F}_1$-leaves.

Let $\kappa(x)$ be the Gaussian curvature at x of the pull-back leaf $L(x)$ through x with respect to the pull-back metric $g_1 = f^*h$. The technique is to change the pull-back metric conformally along the pull-back leaf of $\mathcal{F}_1$ to generate a new metric of constant negative curvature, which can be fulfilled by a well-known fact that the harmonicity of f between Riemann surfaces is conformally invariant by Eells-Sampson [11]. Recall that the pull-back metric $g_1 = f^*h$ is Riemannian and non-vanishing on the pull-back leaf $L(x)$ through x. By the uniformization theorem, there are three possibilities: (a) $L(x)$ is a sphere; (b) the universal covering space $\tilde{L}$ of L is conformal equivalent to the Euclidean plane $\mathbb{R}^2$; (c) the universal covering space $\tilde{L}$ of L is conformal equivalent to the Poincaré disc $\mathbb{D}^2$. One can divide M_1 into three $\mathcal{F}_1$-saturated sets: $M_1 = B_1 \cup B_2 \cup B_3$, where B_1 (resp. B_2, B_3) is the set of points such that $\tilde{L}$ is conformally equivalent to S^2 (resp. $\mathbb{R}^2$, $\mathbb{D}^2$). For simplicity, one may assume that either $\tilde{L}$ is conformally equivalent to $\mathbb{R}^2$ for μ-almost every x or $\tilde{L}$ is conformally equivalent to $\mathbb{D}^2$ for μ-almost every x. Since the set of spherical leaves is measure-negligible, one can write μ as μ_2 on B_2 plus μ_3 on B_3. If one shows that $\int \kappa \, d\mu_2$ and $\int \kappa \, d\mu_3$ are non-positive, then the theorem follows by linearity. Firstly, if $\tilde{L}(x)$ is conformal equivalent to $\mathbb{R}^2$ for μ almost every x, then the proof of the theorem is similar to Connes' proof [8], since the harmonic measure μ is completely invariant by Theorem 3.3 (4).

Secondly, we show that if $\tilde{L}(x)$ is conformal equivalent to $\mathbb{D}^2$ for μ almost every x, then the theorem holds. Note that the conformal equivalence between $\tilde{L}$ and $\mathbb{D}^2$ is unique up to the isometries of $\mathbb{D}^2$. Moreover, the pull-back of the Poincare metric of $\mathbb{D}^2$ is a well-defined metric on $\tilde{L}$ which is invariant by the covering transformations of the covering $\tilde{L} \to L$. Therefore, there is a unique smooth function $u : L \to \mathbb{R}$ such that the metric $\exp(2u)g_1$ is complete and has curvature -1. If $g_1 = f^*h$ is a metric on a surface L with curvature $\kappa(x)$, then it is known that the curvature $\kappa_1(x)$ of the metric $\exp(2u)g_1$ is given by

$$\kappa_1 = \exp(-2u)(\kappa - \triangle u), \tag{4.4}$$

where $u : L \to \mathbb{R}$ is any smooth function, and $\triangle$ is the Laplace operator with respect to the metric g_1. In [16], it showed that: Let $B \subset M_1$ be a closed $\mathcal{F}_1$-saturated set of non-spherical leaves, and let $u : B \to \mathbb{R} \cup \{-\infty\}$ be a map. If $\tilde{L}$ is conformally equivalent to D^2, $u_{|L(x)}$ is the unique function so that $\exp(2u_{|L(x)})g_{1|L(x)}$ is complete and has curvature -1. Then u is upper semi-continuous and smooth along leaves of $\mathcal{F}_1$. Furthermore, the gradient $\nabla^{\mathcal{F}_1}$ along the leaves is bounded on B.

Thus we have $-1 = \exp(-2u(x)(\kappa(x) - \triangle^{\mathcal{F}_1} u(x))$, if we apply (4.4) leaf by leaf to the above defined function u. It implies that $\kappa(x) = \triangle^{\mathcal{F}_1} u(x) - \exp(2u(x))$. This formula holds μ almost everywhere, since we assume that $u(x) \neq -\infty$ almost everywhere (if $\tilde{L}$ is conformally equivalent to $\mathbb{R}^2$, $u(x) = -\infty$). Note that u is upper semi-continuous and bounded on B, and so $\exp(2u(x))$ is a positive bounded function on B. Therefore, $\triangle^{\mathcal{F}_1}$ is also bounded since κ is a continuous function. Hence, $\exp(2u)$ and $\triangle^{\mathcal{F}_1}$ are μ-integrable, and we arrive at

$$\int \kappa \, d\mu = \int \triangle^{\mathcal{F}_1} u d\mu - \int \exp(2u) d\mu \leq \int \triangle^{\mathcal{F}_1} u \, d\mu,$$

where u is continuous and smooth along the leaves. Consequently, the integral $\int \triangle^{\mathcal{F}_1} u \, d\mu$ is zero (since μ is a $\mathcal{F}_1$-harmonic measure), and we conclude the result. $\square$

In particular, take f as an identity map $id : (M, \mathcal{F}) \to (M, \mathcal{F})$ in Theorem 4.1 such that $\mathcal{F}$ admits a transverse invariant measure. Then the following corollary (1) is a main theorem obtained by Connes in [8].

Similarly, let f be an identity map $id : (M, \mathcal{F}) \to (M, \mathcal{F})$ such that M admits a $\mathcal{F}$-harmonic measure. Then (2) is a main theorem proved by Ghys [16], which generalizes Connes' theorem since a transverse invariant measure is a special $\mathcal{F}$-harmonic measure by Theorem 3.3.

Corollary 4.2. (1) *Let $\mathcal{F}$ be an oriented 2-dimensional foliation on a compact manifold M which admits a transverse invariant measure. Denote by $\kappa(x)$ the Gaussian curvature of the leaf $L(x)$ through x. If the set of spherical leaves is transverse invariant measure-negligible, then $\int kd\mu$ is non-positive.* (2) *Let $\mathcal{F}$ be an oriented 2-dimensional foliation on a compact manifold M. Choose a Riemannian metric on the tangent bundle of $\mathcal{F}$ such that μ is an $\mathcal{F}$-harmonic measure. If the set of spherical leaves is μ-negligible, then $\int kd\mu$ is non-positive.*

Suppose that $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a leaf-wise harmonic map between compact manifolds with 2-dimensional foliations (Riemann surfaces) with isolated stationary points on some leaves (assumed compact). Then the pull-back metric $g_1 = f^*h$ vanishes at finite isolated stationary points on some pull-back leaves. Therefore, we can not define an $\mathcal{F}_1$-harmonic measure μ on M_1 globally with respect to g_1. In this case, we deal with the Gauss-Bonnet formula using usual measure differently from the preceding case. We consider the restriction of f to leaves (Riemann surfaces), still denoted by $f : L_1 \to L_2$, is harmonic. Let D be a compact smooth domain bounded by a piece-wise smooth curve γ in the leaf L_1, z be a local coordinate at a point $p \in D$, and w be a local coordinate at the image point $a = f(p)$. In terms of the local coordinate $w = re^{i\phi}$, let $h = \lambda dw d\bar{w}$ be a hermitian metric on L_2. Set $\lambda = \mu^2$ and

$$\Theta = -d\phi + i(\partial - \bar{\partial})log\mu. \tag{4.5}$$

Then we have $d\Theta = \kappa\,\Omega$, where $\Omega = \frac{i}{2}\mu^2 dw \wedge d\bar{w}$ is the area element, and

$$\kappa = -(\frac{4}{\mu^2})\frac{\partial^2 log\mu}{\partial w \partial \bar{w}}. \tag{4.6}$$

is the Gaussian curvature with respect to the hermitian metric $h = \mu^2 dw d\bar{w}$ on L_2.

Proposition 4.3 ([5])**.** *If $f : D \subset L_1 \to L_2$ is a harmonic non $\pm$ holomorphic map between compact Riemann surfaces of Jacobian $J \geq 0$ (resp. $J \leq 0$) with isolated stationary points, then*

$$2\pi\chi(D) + \int_\gamma \Theta + 2\pi \sum_{i=1}^{n} (n(p_i,\, a) - 1) = \int_{f(D)} \kappa\, \Omega, \tag{4.7}$$

where $\chi(D)$ is the Euler characteristic of D, $n(p_i, a) - 1$ (resp. $-n(p_i,\, a) + 1$), $i = 1, \ldots, n$, are the stationary indices of f. (Note that if $f : L_1 \to L_2$ is $\pm$ holomorphic, then f is automatically of $J \geq 0$ (resp. $J \leq 0$) with isolated stationary points and (4.7) holds (cf. [1, 25]).

Recall that $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a leaf-wise harmonic map between compact manifolds foliated by Riemann surfaces of Jacobian $J \geq 0$ (resp. $J \leq 0$) with isolated stationary points on some leaves. We consider the restriction of f to compact Riemann surfaces, still denoted by $f : L_1 \to L_2$, is harmonic. Let $p_1, \ldots, p_n$ be a set of finite isolated stationary points of f at a point a in a leaf L_2 of $\mathcal{F}_2$. (1) If $p_1, \ldots, p_n$ all lie in the same pull-back leaf $L_j \subset f^{-1}L_2$ for some j, we consider the harmonic map restricted to the pull-back leaf $f : D_j \subset L_j \to L_2$ of $J \geq 0$ (resp. $J \leq 0$) with isolated stationary points. Thus by Proposition 4.3 we have

$$2\pi\chi(D_j) + \int_\gamma \Theta + 2\pi \sum_{i=1}^{n} (n(p_i,\, a) - 1) = \int_{f(D_j)} \kappa\, \Omega, \tag{4.8}$$

where D_j is a compact smooth domain bounded by a piece-wise smooth curve γ_j in the leaf L_j. (2) If $p_1, \ldots, p_n$ lie in some different pull-back leaves, say $L_1, \ldots, L_k$, such that $p_1, \ldots, p_{n_1} \in L_1, \ldots, p_{n_k+1}, \ldots, p_{n_k} \in L_k$, we assume that each pull-back leaf is a compact Riemann surface without boundary. Then by Proposition 4.3 we have

$$2\pi\chi(L_j) + 2\pi s(L_j) = \int_{f(L_j)} \kappa\, \Omega,\ j = 1, \ldots, k, \tag{4.9}$$

where $s(L_j) = \sum_{i=1}^{n_j} (n(p_i,\, a) - 1)$ (resp. $\sum_{i=1}^{n_j} (-n(p_i, a) + 1)$ is the stationary indices of f in L_j. Thus we obtain

$$2\pi \sum_{L_j \in \mathcal{F}_1} \chi(L_j) + 2\pi \sum_{L_j \in \mathcal{F}_1} s(L_j) = \sum_{L_j \in \mathcal{F}_1} \int_{f(L_j)} \kappa\, \Omega. \tag{4.10}$$

Theorem 4.4. *Suppose that $f : (M_1, \mathcal{F}_1) \to (M_2, \mathcal{F}_2)$ is a leaf-wise harmonic map between compact manifolds with 2-dimensional foliations of Jacobian $J \geq 0$ (resp. $J \leq 0$) with isolated stationary points on finite compact leaves.* (1) *If the stationary points lie in the same pull-back leaf L_j for some j, then* (4.8) *holds.* (2) *If the stationary points lie in some different compact pull-back leaves $L_1, \ldots, L_k$ without boundaries, then*

$$2\pi\chi(\mathcal{F}_1) + 2\pi s(\mathcal{F}_1) = \sum_{L_j \in \mathcal{F}_1} \int_{f(L_j)} \kappa\,\Omega, \tag{4.11}$$

where $\chi(\mathcal{F}_1) = \sum_{L_j \in \mathcal{F}_1} \chi(L_j)$ and $s(\mathcal{F}_1) = \sum_{L_j \in \mathcal{F}_1} s(L_j)$.

In the above theorem, if L_j is an n_j-sheet covering of L_2 and $\int_{f(L_j)} \kappa\,\Omega = 2\pi n_j \chi(L_2)$, then (4.10) yields to

$$\chi(L_j) + s(L_j) = n_j \chi(L_2). \tag{4.12}$$

Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ have finite compact leaves, and a leaf $L_j \in \mathcal{F}_1$ is an n_j-sheet covering of a leaf $L_2 \in \mathcal{F}_2$, then we obtain

$$\chi(\mathcal{F}_1) + s(\mathcal{F}_1) = \sum_{L_j \in \mathcal{F}_1} n_j \chi(L_2). \tag{4.13}$$

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