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A CLASS OF REPRESENTATIONS OF HECKE ALGEBRAS

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Abstract

A type of directed multigraph called a *W-digraph* is introduced to model the structure of certain representations of Hecke algebras, including those constructed by Lusztig and Vogan from involutions in a Weyl group. Building on results of Lusztig, a complete characterization of W-digraphs is given in terms of subdigraphs for dihedral parabolic subgroups. In addition, results are obtained relating graph-theoretic properties of W-digraphs (acyclicity, existence of sources or sinks, connectedness) to the structure of the corresponding H-module or its character.

0. Overview

Let W be a Weyl group with set of fundamental generators S and length function ℓ, let u be an indeterminate, and let H be the Hecke algebra of (W, S) over $\mathbb{Q}(u)$. (See the next section for a presentation of H.) Put $I = \{w \in W \mid w^{-1} = w\}$. In [7], Lusztig and Vogan construct an H-module M with basis $\{m_w \mid w \in I\}$ indexed by I, on which the generator T_s of H acts according to the rule

$$T_s m_w = \begin{cases} m_{sws} & \text{if } sw \neq ws,\ \ell(sw) > \ell(w), \\ (u^2-1)m_w + u^2 m_{sws} & \text{if } sw \neq ws,\ \ell(sw) < \ell(w), \\ u m_w + (u+1) m_{sw} & \text{if } sw = ws,\ \ell(sw) > \ell(w), \\ (u^2-u-1)m_w + (u^2-u)m_{sw} & \text{if } sw = ws,\ \ell(sw) < \ell(w), \end{cases}$$

for $s \in S$, $w \in I$. These expressions are given geometric interpretations in [7]: when u is replaced by a power q of a prime number, each coefficient in

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the expansion of $T_s m_w$ evaluates to the number of $\mathbb{F}_q$-rational points in a corresponding subset of a variety constructed from Borel subgroups in an algebraic group with Weyl group W. (See 1.1-1.6 of [7] for the details of this construction, and Lusztig's paper [6] for an extension to arbitrary Coxeter groups.)

The present work originated in the author's attempt to visualize the structure of the H-module M described above. A directed multigraph Γ can be constructed, with set of vertices $\{m_w \mid w \in I\}$, as follows. If $w \in I$, $s \in S$, $sw \neq ws$, and $\ell(w) < \ell(sw)$, then a solid edge $m_w \xrightarrow{s} m_{sws}$ is included in Γ, while if $sw = ws$ and $\ell(w) < \ell(sw)$, then a dashed edge $m_w \overset{s}{\dashrightarrow} m_{sw}$ is included. The result is an example of what will be called a W-digraph (see Definition 1.2). In broad terms, the notion of W-digraph is similar to the notion of W-graph introduced by Kazhdan and Lusztig in [4]: both give rise to graph-theoretic objects that encode the action of the generators T_s, $s \in S$, on an H-module. There are also combinatorial similarities: if a finite dimensional H-module M affords both a W-digraph Γ and a W-graph Ψ, then the number edges labeled $s \in S$ in Γ is equal to the multiplicity of the eigenvalue -1 of T_s on M (see Lemma 2.4(i)), which in turn is equal to the number of vertices with label including s in Ψ.

On the other hand, there are significant differences between the notions of W-digraph and W-graph, including the obvious structural differences: a W-digraph is directed rather than undirected, can have two different types of edges (corresponding to commutation relations in the motivating example above) rather than one type, and has generators labeling edges rather than scalars. The encodings of the actions of generators for W-digraphs and W-graphs are necessarily different. Further, the class of modules afforded by W-digraphs need not coincide with the class of modules afforded by W-graphs. When (W, S) is finite and $S \neq \emptyset$, not all H-modules are afforded by W-digraphs, whereas every H-module is afforded by a W-graph over a suitable field of scalars (Gyoja, [3]). Specifically, when $S \neq \emptyset$, the sign representation $T_s \mapsto -1$ is not afforded by a W-digraph (see Theorem 1.7(i)), but is afforded by the W-graph with a single vertex labeled S. In the other direction, an example can be given of an infinite (W, S) and corresponding

H-module that is afforded by a W-digraph but is not afforded by a W-graph (see Theorem 1.12 and Example 7.1).

1. Statement of Results

The Coxeter system (W, S) has a presentation of the form

$$W = \left\langle s \in S \mid (rs)^{n(r,s)} = e \text{ for } r, s \in S,\ n(r,s) < \infty \right\rangle,$$

where $n(s,s) = 1$ and $1 < n(r,s) = n(s,r) \leq \infty$ for $r, s \in S$, $r \neq s$. The Hecke algebra H has basis $\{T_w \mid w \in W\}$ satisfying

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ u^2 T_{sw} + (u^2 - 1) T_w & \text{if } \ell(sw) < \ell(w) \end{cases} \tag{1.1}$$

for $s \in S$. As a $\mathbb{Q}(u)$-algebra, H has generators $\{T_s \mid s \in S\}$ satisfying the relations

$$(T_s - u^2)(T_s + 1) = 0 \qquad \text{if } s \in S, \tag{1.2a}$$

$$\overbrace{T_s T_t \cdots}^{n} = \overbrace{T_t T_s \cdots}^{n} \qquad \text{if } s, t \in S,\ 1 < n = n(s,t) < \infty \tag{1.2b}$$

(where the factors in the products of (1.2b) are alternately T_s and T_t). Moreover,

$$T_x T_y = T_{xy} \qquad \text{if } \ell(xy) = \ell(x) + \ell(y).$$

Definition 1.1. Let S be a set. Let $\Gamma = (\mathscr{V}, \mathscr{E})$ be a directed multigraph with set of vertices $\mathscr{V} = \mathscr{V}(\Gamma)$ and set of edges $\mathscr{E} = \mathscr{E}(\Gamma)$ such that each edge is either solid or dashed and is labeled by an element of S, that is, has one of the forms

$$\alpha \xrightarrow{s} \beta \qquad \text{or} \qquad \alpha \overset{s}{\dashrightarrow} \beta$$

with $\alpha, \beta \in \mathscr{V}$, $s \in S$. Then Γ is an *S-labeled digraph* if Γ has no loops and, for all $s \in S$, every vertex of Γ occurs in exactly one edge labeled s.

Examples of S-labeled digraphs appear in Figures 1.1–1.2.

Let Γ be an S-labeled digraph. Let $M(\Gamma)$ be a vector space over $\mathbb{Q}(u)$ with basis $\mathscr{V}(\Gamma)$, and let $\mathrm{gl}(M(\Gamma))$ be the $\mathbb{Q}(u)$-algebra of all linear operators

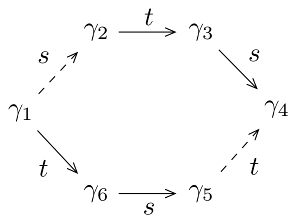


Figure 1.1 An $\{s,t\}$-labeled digraph.

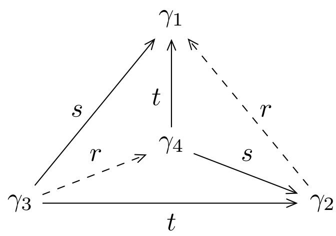


Figure 1.2 An $\{r,s,t\}$-labeled digraph.

on $M(\Gamma)$. For each $s \in S$, define $\tau_s \in \mathrm{gl}(M(\Gamma))$ as follows: if $\alpha \in \mathscr{V}(\Gamma)$, then

$$\tau_s(\alpha) = \begin{cases} \beta & \text{if } \alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma), \\ (u^2-1)\alpha + u^2\beta & \text{if } \alpha \xleftarrow{s} \beta \in \mathscr{E}(\Gamma), \\ u\alpha + (u+1)\beta & \text{if } \alpha \overset{s}{\dashrightarrow} \beta \in \mathscr{E}(\Gamma), \\ (u^2-u-1)\alpha + (u^2-u)\beta & \text{if } \alpha \overset{s}{\dashleftarrow} \beta \in \mathscr{E}(\Gamma). \end{cases} \tag{1.3}$$

Definition 1.2. An S-labeled digraph Γ is a *W-digraph* if the mapping $T_s \mapsto \tau_s$ extends to a representation of H, that is, a homomorphism of $\mathbb{Q}(u)$-algebras $\rho : H \to \mathrm{gl}\,(M(\Gamma))$.

Let $J \subseteq S$, so (W_J, J) is a Coxeter system with $W_J = \langle J \rangle$ the associated parabolic subgroup of W. For Γ an S-labeled digraph, denote by Γ_J the subdigraph with the same set of vertices obtained from Γ by removing all edges labeled by elements of $S \setminus J$. Thus Γ_J is a J-labeled digraph. If Γ is a W-digraph, then clearly Γ_J is a W_J-digraph. Conversely, because of the presentation (1.2a), (1.2b) it is also clear that Γ is a W-digraph if Γ_J is a W_J-digraph whenever $J \subseteq S$, $|J| \leq 2$. Note also that Γ is a W-digraph if and only if each connected component of Γ is a W-digraph.

In Figures 1.3–1.10 several J-labeled digraphs are given with $J = \{s,t\}$. These multigraphs have $2m$ vertices, with $m \geq 2$ except for Figures 1.9–1.10. Also, $s' = s$ if m is even, $s' = t$ if m is odd, t' is defined by $\{s',t'\} = \{s,t\}$, and any edge not shown has one of the forms $\alpha_{2j} \xrightarrow{s} \alpha_{2j+1}$, $\alpha_{2j-1} \xrightarrow{t} \alpha_{2j}$, $\beta_{2j-1} \xrightarrow{s} \beta_{2j}$, or $\beta_{2j} \xrightarrow{t} \beta_{2j+1}$.

Two S-labeled digraphs $\Gamma = (\mathscr{V}, \mathscr{E})$ and $\Gamma' = (\mathscr{V}', \mathscr{E}')$ are *isomorphic* if there is some bijection $\varphi : \mathscr{V} \to \mathscr{V}'$ such that for all $\alpha, \beta \in \mathscr{V}$ and $s \in S$, $\alpha \xrightarrow{s} \beta \in \mathscr{E}$ if and only if $\varphi(\alpha) \xrightarrow{s} \varphi(\beta) \in \mathscr{E}'$ and $\alpha \overset{s}{\dashrightarrow} \beta \in \mathscr{E}$ if and only if $\varphi(\alpha) \overset{s}{\dashrightarrow} \varphi(\beta) \in \mathscr{E}'$.

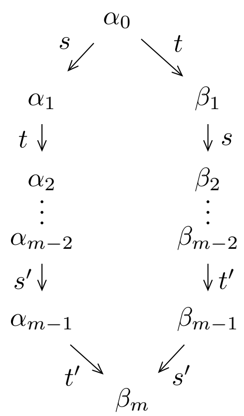


Figure 1.3

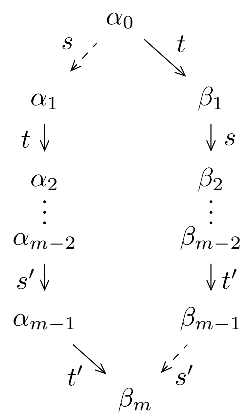


Figure 1.4

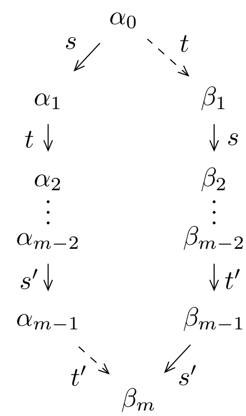
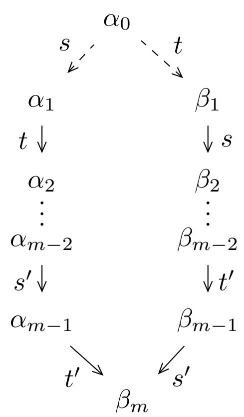


Figure 1.5



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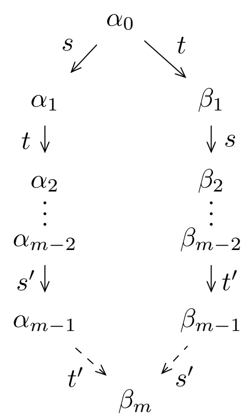

Figure 1.7

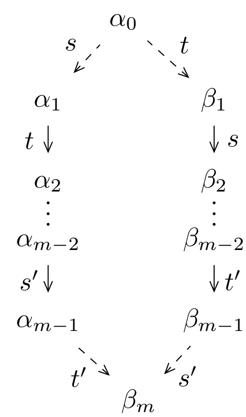

Figure 1.8

Theorem 1.3. *Let (W, S) be a Coxeter system. The following are equivalent.*

(a) Γ *is a W-digraph.*

(b) Γ *is an S-labeled digraph such that for all $s, t \in S$ with $1 < n = n(s,t) < \infty$, each connected component of Γ_J, $J = \{s,t\}$, is isomorphic to one of the J-labeled digraphs in Figures 1.3–1.10, with*

  (i) *$m \geq 2$ and m a divisor of n in Figure 1.3, Figure 1.4, or Figure 1.5,*

  (ii) *$m \geq 2$ and $2m-1$ a divisor of n in Figure 1.6 or Figure 1.7,*

  (iii) *$m \geq 2$ and $2m-2$ a divisor of n in Figure 1.8,*

  (iv) *$m = 1$ and $n \geq 2$ arbitrary in Figure 1.9 or Figure 1.10.*

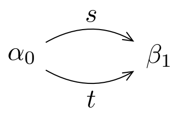

Figure 1.9

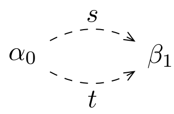

Figure 1.10

If Γ is an S-labeled digraph, let Γ_{rev} be the S-labled digraph obtained by reversing the direction of all edges of Γ while keeping their types and labels. For example, if Γ is as in Figure 1.6, then Γ_{rev} is isomorphic to the digraph in Figure 1.7. We can assume $M(\Gamma)$ and $M(\Gamma_{\text{rev}})$ are the same as vector spaces over $\mathbb{Q}(u)$ (since $\mathscr{V}(\Gamma) = \mathscr{V}(\Gamma_{\text{rev}})$), but the endomorphisms of $M(\Gamma)$ and $M(\Gamma_{\text{rev}})$ corresponding to an element of S are different.

Corollary 1.4. *If Γ is an S-labeled digraph, then Γ is a W-digraph if and only if Γ_{rev} is a W-digraph.*

For Γ an S-labeled digraph, let $\Gamma_{\to}$ be the S-labeled digraph obtained from Γ by replacing any dashed edge $\alpha \overset{s}{\dashrightarrow} \beta$ by the corresponding solid edge $\alpha \xrightarrow{s} \beta$. Let Γ_{dir} be the directed multigraph obtained by removing all labels from $\Gamma_{\to}$. Let Γ_{undir} be the (undirected) multigraph obtained from Γ_{dir} by replacing each directed edge $\alpha \longrightarrow \beta$ by an undirected edge α — β. For example, with Γ as in Figure 1.1, the associated graphs $\Gamma_{\to}$, Γ_{dir}, and

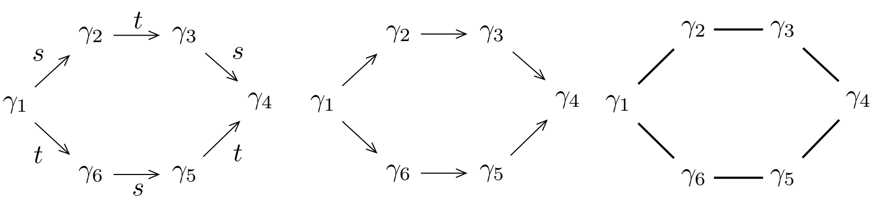

Figure 1.11 $\Gamma_{\to}$, Γ_{dir}, and Γ_{undir} for Γ as in Figure 1.1

Γ_{undir} are given in Figure 1.11. We say a vertex α of Γ is a *source* (*sink*) of Γ if α is a source (sink, respectively) in Γ_{dir}. We consider an empty path to be a directed circuit in any directed multigraph, and define Γ to be *acyclic* if Γ_{dir} is acyclic, that is, if there is no nonempty directed circuit in Γ_{dir}. Also, Γ is *connected* if Γ_{undir} is connected.

Theorem 1.5. *If $n(s,t) < \infty$ for all $s, t \in S$ and Γ is a connected W-digraph, then the following hold.*

(i) Γ *can have at most one source and at most one sink.*

(ii) *If* Γ *has either a source or a sink, then* Γ *is acyclic.*

(iii) *If* (W, S) *is finite, then* Γ *has both a source and a sink, and so is acyclic.*

Corollary 1.6. *If* (W, S) *is finite, then any* W*-digraph is acyclic. Further, the number of sources (or sinks) in a finite* W*-digraph is equal to the number of its connected components.*

Let ind and sgn be the linear characters of H determined by $\text{ind}(T_w) = u_w = u^{2\ell(w)}$ and $\text{sgn}(T_w) = \varepsilon_w = (-1)^{\ell(w)}$ for $w \in W$, respectively. For λ a linear character of H and M an H-module, put $M_\lambda = \{v \in M \mid hv = \lambda(h)v$ for $h \in H\}$.

Theorem 1.7. *If* Γ *is a* W*-digraph and* $\mathscr{V}(\Gamma)$ *is finite, then the following hold.*

(i) *The number of connected components of* Γ *is equal to* $\dim M(\Gamma)_{\text{ind}}$.

(ii) *If* $n(s,t) < \infty$ *for all* $s, t \in S$*, then the number of acyclic connected components of* Γ *is equal to* $\dim M(\Gamma)_{\text{sgn}}$.

Theorem 1.8. *If* (W, S) *is finite,* $J \subseteq S$*, and* Γ *is a connected* W*-digraph, then* Γ_J *has at most* $|W : W_J|$ *connected components.*

Taking $J = \emptyset$ gives the following.

Corollary 1.9. *If* (W, S) *is finite and* Γ *is a connected* W*-digraph, then* $|\mathscr{V}(\Gamma)| \leq |W|$.

The bound in Corollary 1.9 is always attained: see Example 4.5.

Theorem 1.10. *Assume* $n(s,t) < \infty$ *for* $s, t \in S$ *and* Γ *is a connected* W*-digraph with a source or sink. Then for* $\alpha, \beta \in \mathscr{V}(\Gamma)$*, any two directed paths from* α *to* β *in* Γ *have the same number of edges.*

If Γ is a W-digraph and $\mathscr{V}(\Gamma)$ is finite, so $M(\Gamma)$ is finite dimensional, let χ_Γ be the character of H afforded by $M(\Gamma)$.

Theorem 1.11. *If* Γ *is a* W*-digraph and* $\mathscr{V}(\Gamma)$ *is finite, then the following hold.*

(i) *If* σ *is the automorphism of* $\mathbb{Q}(u)$ *determined by* ${}^\sigma u = -1/u$*, then* $\chi_{\Gamma_{rev}}(T_w) = {}^\sigma\chi_\Gamma(T_{w^{-1}}^{-1})$ *for* $w \in W$.

(ii) *If* $n(s,t) < \infty$ *for* $s, t \in S$ *and* Γ *is acyclic, then* $\chi_{\Gamma_{rev}}(T_w) = \varepsilon_w u_w \chi_\Gamma(T_w^{-1})$ *for* $w \in W$.

In the case of an affine Weyl group, the following holds.

Theorem 1.12. *If* (W_J, J) *is finite for proper subsets* J *of* S, Γ *is a finite, connected* W*-digraph, and* $M(\Gamma)$ *affords a* W*-graph (as defined in* [4]) *over* $\mathbb{Q}$, *then* Γ *is acyclic.*

The organization of this paper is as follows. Section 2 contains preliminary results, and Section 3 contains a proof of Theorem 1.3 and related results. Section 4 contains proofs of Theorems 1.5, 1.7, 1.8, and 1.10. Section 5 contains a proof of Theorem 1.11 and related results. Section 6 contains a proof of Theorem 1.12, and the last section has additional examples.

2. Preliminary Results

Assume that (W, S) is a Coxeter system and let Γ be an S-labeled digraph. Throughout this and later sections, the notation $x \le y$ is used to indicate the usual Bruhat order on W relative to S when $x, y \in W$. For any $s \in S$, we have

$$(\tau_s - u^2)(\tau_s + 1) = 0 \qquad \text{in } \mathrm{gl}(M), \tag{2.1}$$

where τ_s is as in (1.3) and $M = M(\Gamma)$ (see [6], 2.3). Indeed, suppose α is connected to β by an edge of Γ labeled s. Exchanging α, β if necessary, we can assume this edge is directed from α to β. By (1.3), τ_s leaves invariant the subspace with basis $\{\alpha, \beta\}$, and the matrix of τ_s acting on this subspace with respect to this basis is

$$\begin{pmatrix} 0 & u^2 \\ 1 & u^2 - 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} u & u^2 - u \\ u + 1 & u^2 - u - 1 \end{pmatrix}$$

according to whether $\alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma)$ or $\alpha \overset{s}{\dashrightarrow} \beta \in \mathscr{E}(\Gamma)$. In either case, the eigenvalues are u^2 and -1, and thus (2.1) holds. Hence Γ is a W-digraph if and only if

$$\overbrace{\tau_s \tau_t \cdots}^{n} = \overbrace{\tau_t \tau_s \cdots}^{n} \qquad \text{whenever } s, t \in S,\ 1 < n(s,t) < \infty. \tag{2.2}$$

Define $T_s^\circ \in H$ by

$$T_s^\circ = (u+1)^{-1}(T_s - u).$$

(This element is denoted $\mathring{T}_s$ in [6], 2.2.) By (1.2a), both T_s and T_s° are units in H, with inverses given by

$$T_s^{-1} = u^{-2}(T_s - (u^2-1)), \qquad (T_s^\circ)^{-1} = (u^2-u)^{-1}(T_s - (u^2-u-1)).$$

The terminology used in the next definition will be justified by the remarks after Lemma 2.2.

Definition 2.1. Let M be an H-module. Then a subset X of M *supports a* W*-digraph* if X is linearly independent over $\mathbb{Q}(u)$ and, for each $\alpha \in X$ and $s \in S$,

$$X \cap \{T_s\alpha, T_s^{-1}\alpha, T_s^\circ\alpha, (T_s^\circ)^{-1}\alpha\} \neq \emptyset.$$

Lemma 2.2. *If M is an H-module and $X \subseteq M$ supports a W-digraph, then the following hold.*

(i) *If $s \in S$ and $\alpha \in X$, then α, $T_s\alpha$, $T_s^{-1}\alpha$, $T_s^\circ\alpha$, $(T_s^\circ)^{-1}\alpha$ are distinct and X contains a unique element of $\{T_s\alpha, T_s^{-1}\alpha, T_s^\circ\alpha, (T_s^\circ)^{-1}\alpha\}$.*

(ii) *The subspace of M spanned by X is an H-submodule of M.*

Proof. Suppose $s \in S$ and $\alpha \in X$. Put $Y = \{T_s\alpha, T_s^{-1}\alpha, T_s^\circ\alpha, (T_s^\circ)^{-1}\alpha\}$. By (1.2a), there are unique $\gamma, \delta \in M$ such that

$$\alpha = \gamma + \delta, \qquad T_s\gamma = -\gamma, \qquad T_s\delta = u^2\delta.$$

Thus

$$T_s\alpha = -\gamma + u^2\delta, \qquad T_s^{-1}\alpha = -\gamma + \frac{1}{u^2}\delta,$$
$$T_s^\circ\alpha = -\gamma + \frac{u^2-u}{u+1}\delta, \qquad (T_s^\circ)^{-1}\alpha = -\gamma + \frac{u+1}{u^2-u}\delta.$$

Since X is linearly independent and X contains α and at least one element of Y, it follows that γ, δ are linearly independent over $\mathbb{Q}(u)$. Therefore α, $T_s\alpha$, $T_s^{-1}\alpha$, $T_s^\circ\alpha$, $(T_s^\circ)^{-1}\alpha$ are distinct. Also, since α, $T_s\alpha$, $T_s^{-1}\alpha$, $T_s^\circ\alpha$, $(T_s^\circ)^{-1}\alpha$ are all in span$\{\gamma,\delta\}$, X can contain at most one element of Y. Thus (i) holds. Further, since X contains two elements of span$\{\gamma,\delta\}$, span$X$

contains span $\{\gamma,\delta\}$ by dimension, and thus $T_s\alpha \in \text{span}X$. Since $\alpha \in X$ was arbitrary, we have $T_s\text{span}X \subseteq \text{span}X$. Thus $\text{span}X$ is an H-submodule of M since $s \in S$ was arbitrary, so (ii) holds. $\square$

If M is an H-module and $X \subseteq M$ supports a W-digraph, then we construct a directed multigraph Γ, as follows. If $\alpha, \beta \in X$ and $s \in S$, then

$$\alpha \xrightarrow{s} \beta \text{ is an edge of } \Gamma \text{ if } \beta = T_s\alpha,$$
$$\alpha \overset{s}{\dashrightarrow} \beta \text{ is an edge of } \Gamma \text{ if } \beta = T_s^\circ\alpha.$$

Then Γ is a well-defined S-labeled digraph by Lemma 2.2. Moreover, from the definition of T_s°, it is easily checked that H acts on $M_0 = \text{span}X$ according to

$$T_s\alpha = \tau_s(\alpha),$$

where τ_s is as in (1.3). Therefore Γ is indeed a W-digraph with associated H-module M_0.

Lemma 2.3. *Suppose X is a linearly independent subset of an H-module M. Then X supports a W-digraph if and only if for each $s \in S$, there exists a partition P_s of X such that, for all $U \in P_s$, there are $\alpha, \beta \in U$ such that $\alpha \neq \beta$, $U = \{\alpha, \beta\}$, and either $T_s\alpha = \beta$ or $T_s^\circ\alpha = \beta$.*

Proof. First suppose X supports a W-digraph. Let $s \in S$. For $\lambda \in X$, define $U_\lambda = \{\lambda, \mu\}$ where

$$X \cap \left\{T_s\lambda, T_s^{-1}\lambda, T_s^\circ\lambda, (T_s^\circ)^{-1}\lambda\right\} = \{\mu\}.$$

Then $\lambda \in X \cap \left\{T_s\mu, T_s^{-1}\mu, T_s^\circ\mu, (T_s^\circ)^{-1}\mu\right\}$, and so $U_\lambda = U_\mu$. By Lemma 2.2, $P_s = \{U_\lambda \mid \lambda \in X\}$ is a partition of X satisfying the conditions above: if $U = U_\lambda$ and $\mu = T_s\lambda$ or $\mu = T_s^\circ\lambda$, then take $\alpha = \lambda$, $\beta = \mu$, and otherwise take $\alpha = \mu$, $\beta = \lambda$.

Conversely, suppose for each $s \in S$, a partition P_s satisfying the conditions above exists. Let $\gamma \in X$. There is some $\delta \in X$ such that $U = \{\gamma, \delta\} \in P_s$. For this δ we either have $T_s^{\pm 1}\gamma = \delta$ or $(T_s^\circ)^{\pm 1}\gamma = \delta$. Thus X supports a W-digraph. $\square$

Lemma 2.4. *Suppose M is an H-module with basis X supporting a W-digraph Γ, $v = \sum_{\gamma \in X} \lambda_\gamma \gamma \in M$, and $s \in S$. Then the following hold.*

(i) $T_s v = u^2 v$ *if and only if* $\lambda_\beta = \lambda_\alpha$ *whenever* $\alpha \xrightarrow{s} \beta$ *or* $\alpha \overset{s}{\dashrightarrow} \beta$ *is an edge of* Γ.

(ii) $T_s v = -v$ *if and only if*

$$\lambda_\beta = \begin{cases} -u^{-2}\lambda_\alpha & \text{whenever } \alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma), \\ -(u+1)(u^2-u)^{-1}\lambda_\alpha & \text{whenever } \alpha \overset{s}{\dashrightarrow} \beta \in \mathscr{E}(\Gamma). \end{cases}$$

Proof. With P_s as in Lemma 2.3, M is the direct sum of the subspaces span $\{\alpha, \beta\}$, $\{\alpha, \beta\} \in P_s$. Also, T_s leaves each such subspace invariant, with eigenvalues u^2 and -1. Hence it suffices to show that for $\{\alpha, \beta\} \in P_s$, span $\{\alpha, \beta\}$ has a basis consisting of eigenvectors for T_s of the form $\lambda_\alpha \alpha + \lambda_\beta \beta$ with coefficients satisfying the relations of (i) and (ii). If $\alpha \xrightarrow{s} \beta$ is an edge of Γ, then

$$T_s(\alpha+\beta) \;=\; \beta + \big(u^2\alpha + (u^2-1)\beta\big) = u^2(\alpha+\beta)$$

and

$$T_s\left(\alpha - u^{-2}\beta\right) \;=\; \beta - u^{-2}\left(u^2\alpha + (u^2-1)\beta\right) = -\left(\alpha - u^{-2}\beta\right),$$

so the basis $\left\{\alpha+\beta, \alpha - u^{-2}\beta\right\}$ has the desired property. On the other hand, if $\alpha \overset{s}{\dashrightarrow} \beta$ is an edge of Γ, then

$$\begin{aligned} T_s(\alpha+\beta) \;&=\; (u\alpha + (u+1)\beta) + \big((u^2-u)\alpha + (u^2-u-1)\beta\big) \\ &=\; u^2(\alpha+\beta) \end{aligned}$$

and

$$\begin{aligned} &T_s\left(\alpha - (u+1)(u^2-u)^{-1}\beta\right) \\ &\quad=\; (u\alpha + (u+1)\beta) \\ &\qquad -(u+1)(u^2-u)^{-1}\left((u^2-u)\alpha + (u^2-u-1)\beta\right) \\ &\quad=\; -\left(\alpha - (u+1)(u^2-u)^{-1}\beta\right), \end{aligned}$$

so $\left\{\alpha+\beta, \alpha - (u+1)(u^2-u)^{-1}\beta\right\}$ is an appropriate basis. □

For the remainder of this section we assume $J = \{s,t\} \subseteq S$, $1 < n = n(s,t) < \infty$. For $0 \le k \le n$, define elements s_k, t_k of W_J by

$$s_k = \overbrace{\cdots sts}^{k}, \qquad t_k = \overbrace{\cdots tst}^{k},$$

with k factors in each product, alternately s and t. For example, $s_0 = e = t_0$, and $s_n = w_0 = t_n$ is the longest element of W_J. Define elements σ_k of H_J as follows:

$$\sigma_k = \sum_{\substack{w \in W_J \\ \ell(w)=k}} T_w.$$

Thus $\sigma_0 = T_e$, $\sigma_n = T_{w_0}$, and $\sigma_k = T_{s_k} + T_{t_k}$ for $0 < k < n$.

Lemma 2.5. *Suppose* $a_0 = \sigma_k + \sum_{\substack{w \in W_J \\ \ell(w)<k}} \gamma_w T_w \in H_J$, *where* $0 \le k < n$ *and* $\gamma_w \in \mathbb{Q}(u)$ *for* $w \in W_J$. *Suppose further that for* $0 \le j \le n-k$, $\overline{S}_j \in \{T_s, T_s^\circ\}$ *and* $\overline{T}_j \in \{T_t, T_t^\circ\}$. *Put* $b_0 = a_0$, *and define* $a_1, \dots, a_{n-k}, b_1, \dots, b_{n-k}$ *by*

$$a_{j+1} = \begin{cases} \overline{S}_j a_j & \text{if } j \text{ is even,} \\ \overline{T}_j a_j & \text{if } j \text{ is odd,} \end{cases} \quad \text{and} \quad b_{j+1} = \begin{cases} \overline{T}_j b_j & \text{if } j \text{ is even,} \\ \overline{S}_j b_j & \text{if } j \text{ is odd} \end{cases}$$

for $0 \le j < n-k$. *Then* $X = \{a_0, a_1, \dots, a_{n-k-1}, b_1, b_2, \dots, b_{n-k}\}$ *is linearly independent. Moreover, if* $a_{n-k} = b_{n-k}$, *then* X *supports a* W_J*-digraph and* $L = \mathrm{span} X$ *is a left ideal of* H_J.

Proof. If $1 \le j \le n-k$ and a_j is expressed as a linear combination of $\{T_w \mid w \in W_J\}$, then the unique $w \in W$ of maximal length such that T_w appears with nonzero coefficient is given by

$$w = \begin{cases} s_j s_k = s_{j+k} & \text{if } k \text{ is even,} \\ s_j t_k = t_{j+k} & \text{if } k \text{ is odd.} \end{cases}$$

Similarly, if $1 \le j \le n-k$, then the unique $w \in W_J$ of maximal length such that T_w appears with nonzero coefficient in b_j is given by

$$w = \begin{cases} t_j t_k = t_{j+k} & \text{if } k \text{ is even,} \\ t_j s_k = s_{j+k} & \text{if } k \text{ is odd.} \end{cases}$$

Thus X is linearly independent.

Suppose $a_{n-k} = b_{n-k}$. If $n-k$ is even, then the partitions

$$P_s = \{\{a_0, a_1\}, \{b_1, b_2\}, \dots, \{b_{n-k-1}, b_{n-k}\}\},$$
$$P_t = \{\{b_0, b_1\}, \{a_1, a_2\}, \dots, \{a_{n-k-1}, a_{n-k}\}\}$$

satisfy the conditions of Lemma 2.3. On the other hand, if $n-k$ is odd, then the partitions

$$P_s = \{\{a_0, a_1\}, \{b_1, b_2\}, \ldots, \{a_{n-k-1}, a_{n-k}\}\},$$
$$P_t = \{\{b_0, b_1\}, \{a_1, a_2\}, \ldots, \{b_{n-k-1}, b_{n-k}\}\}$$

satisfy the conditions of Lemma 2.3. Thus X supports a W_J-digraph, and $L = \mathrm{span}X$ is a left ideal of H_J by Lemma 2.2(ii). □

For $d \geq 0$, define a polynomial $p_d(u) \in \mathbb{Q}[u]$ as follows: $p_0(u) = 1$, and for $d > 0$,

$$p_d(u) = 1 + 2\sum_{i=1}^{d-1}(-u^2)^i + (-u^2)^d.$$

Thus $p_1(u) = 1 - u^2$, $p_2(u) = 1 - 2u^2 + u^4$, $p_3(u) = 1 - 2u^2 + 2u^4 - u^6$. Let $y \in W_J$. A straightforward induction argument based on 2.0.b and 2.0.c of [4] shows

$$u^{2\ell(y)}T_{y^{-1}}^{-1} = T_y + \sum_{x<y} p_{\ell(y)-\ell(x)}(u)T_x. \tag{2.3}$$

For $0 \leq j \leq n$, we define elements $\widetilde{\varphi}_j$, $\widetilde{\eta}_j$, $\widetilde{\gamma}_j$, $\widetilde{\delta}_j$ of H_J, as follows:

$$\begin{cases} \widetilde{\varphi}_j = \displaystyle\sum_{i=0}^{j} p_{j-i}(u)\sigma_i \\ \quad = \sigma_j + (1-u^2)\sigma_{j-1} + (1-2u^2+u^4)\sigma_{j-2} + \cdots \\ \qquad\qquad + (1-2u^2+2u^4 \mp \cdots + 2(-u^2)^{j-1} + (-u^2)^j)\sigma_0, \\ \widetilde{\eta}_j = \widetilde{\varphi}_j + u\widetilde{\varphi}_{j-1} + u^2\widetilde{\varphi}_{j-2} + \cdots + u^j\widetilde{\varphi}_0, \\ \widetilde{\gamma}_j = \widetilde{\varphi}_j - u\widetilde{\varphi}_{j-1} + u^2\widetilde{\varphi}_{j-2} \mp \cdots + (-u)^j\widetilde{\varphi}_0, \\ \widetilde{\delta}_j = \dfrac{1}{2}(\widetilde{\eta}_j + \widetilde{\gamma}_j). \end{cases}$$

These elements are used in describing the constructions of Lusztig in the next section.

If $0 < j < n$, then by (2.3) we have

$$\widetilde{\varphi}_j = T_{s_j} + T_{t_j} + (1-u^2)\sigma_{j-1} + (1-2u^2+u^4)\sigma_{j-2} + \cdots$$
$$+ (1 - 2u^2 \pm \cdots + 2(-u^2)^{j-1} + (-u^2)^j)\sigma_0$$

$$= u^{2j}T^{-1}_{s_j^{-1}} + T_{t_j} = u^{2j}T^{-1}_{t_j^{-1}} + T_{s_j}. \tag{2.4}$$

Lemma 2.6. *If* $0 < j \leq k$ *and* $j + k \leq n$*, then*

$$T_{s_k^{-1}}\widetilde{\varphi}_j = u^{2j}T_{s^{-1}_{k-j}} + T_{s^{-1}_{k+j}} \qquad \textit{and} \qquad T_{t_k^{-1}}\widetilde{\varphi}_j = u^{2j}T_{t^{-1}_{k-j}} + T_{t^{-1}_{k+j}}.$$

Proof. Note $j < n$, so (2.4) applies to $\widetilde{\varphi}_j$. Define $s^*, t^* \in \{s, t\}$ by $s_k = s^*_j s_{k-j}$ and $\{s^*, t^*\} = \{s, t\}$. Then

$$\begin{aligned} T_{s_k^{-1}}\widetilde{\varphi}_j &= T_{s^{-1}_{k-j}} T_{{s^*}_j^{-1}} \left(u^{2j} T^{-1}_{{s^*}_j^{-1}} + T_{t^*_j} \right) \\ &= u^{2j} T_{s^{-1}_{k-j}} + T_{s^{-1}_{k-j}} T_{{s^*}_j^{-1}} T_{t^*_j} = u^{2j} T_{s^{-1}_{k-j}} + T_{s^{-1}_{k+j}} \end{aligned}$$

since $s^{-1}_{k-j}{s^*}_j^{-1}t^*_j = s^{-1}_{k+j}$ and $\ell(s^{-1}_{k-j}) + \ell({s^*}_j^{-1}) + \ell(t^*_j) = \ell(s^{-1}_{k+j})$. Thus the first equation holds. The second follows by applying the automorphism $T_s \leftrightarrow T_t$ of H_J to the first. $\square$

Lemma 2.7. *If* $0 \leq j \leq k$ *and* $j + k \leq n$*, then*

$$\begin{aligned} &\text{(i)} \quad T_{s_k^{-1}}\widetilde{\eta}_j = \sum_{i=0}^{2j} u^i T_{s^{-1}_{k+j-i}} \qquad \textit{and} \quad T_{t_k^{-1}}\widetilde{\eta}_j = \sum_{i=0}^{2j} u^i T_{t^{-1}_{k+j-i}}, \\ &\text{(ii)} \quad T_{s_k^{-1}}\widetilde{\gamma}_j = \sum_{i=0}^{2j} (-u)^i T_{s^{-1}_{k+j-i}} \quad \textit{and} \quad T_{t_k^{-1}}\widetilde{\gamma}_j = \sum_{i=0}^{2j} (-u)^i T_{t^{-1}_{k+j-i}}, \\ &\text{(iii)} \quad T_{s_k^{-1}}\widetilde{\delta}_j = \sum_{i=0}^{j} u^{2i} T_{s^{-1}_{k+j-2i}} \qquad \textit{and} \quad T_{t_k^{-1}}\widetilde{\delta}_j = \sum_{i=0}^{j} u^{2i} T_{t^{-1}_{k+j-2i}}. \end{aligned}$$

Proof. Observe that (i) holds when $k = 0$ because $\widetilde{\eta}_0 = T_e$. Assume $\ell > 0$ and (i) holds when $0 \leq k < \ell$. Suppose $0 \leq j \leq \ell$ and $j + \ell \leq n$. If $j \leq \ell - 1$, then

$$T_{s_\ell^{-1}}\widetilde{\eta}_j = T_s T_{t^{-1}_{\ell-1}} \widetilde{\eta}_j = T_s \sum_{i=0}^{2j} u^i T_{t^{-1}_{\ell+j-i-1}} = \sum_{i=0}^{2j} u^i T_{s^{-1}_{\ell+j-i}}$$

by the induction hypothesis. On the other hand, if $j = \ell$, then

$$\begin{aligned} T_{s_\ell^{-1}}\widetilde{\eta}_\ell &= T_{s_\ell^{-1}}(\widetilde{\varphi}_\ell + u\widetilde{\eta}_{\ell-1}) = T_{s_\ell^{-1}}\widetilde{\varphi}_\ell + uT_sT_{t_{\ell-1}^{-1}}\widetilde{\eta}_{\ell-1} \\ &= u^{2\ell}T_e + T_{s_{2\ell}^{-1}} + uT_s\sum_{i=0}^{2\ell-2} u^i T_{t_{2\ell-i-2}^{-1}} \\ &= u^{2\ell}T_e + T_{s_{2\ell}^{-1}} + \sum_{i=0}^{2\ell-2} u^{i+1}T_{s_{2\ell-(i+1)}^{-1}} = \sum_{\ell=0}^{2\ell} u^\ell T_{s_{2\ell-\ell}^{-1}} \end{aligned}$$

by Lemma 2.6. Thus the first equation of (i) holds in the case $k = \ell$. The second equation of (i) follows in the case $k = \ell$ by applying the automorphism $T_s \leftrightarrow T_t$ to the first equation. Thus (i) holds by induction.

Let ζ be the automorphism of $\mathbb{Q}(u)$ determined by $\zeta(u) = -u$. Extend ζ to a semilinear automorphism of H_J by defining $\zeta(\sum \alpha_w T_w) = \sum \zeta(\alpha_w)T_w$. Then $\zeta(\widetilde{\eta}_m) = \widetilde{\gamma}_m$, and so the formulas of (ii) are obtained by applying ζ to the formulas of (i). Finally, (iii) follows by averaging the formulas of (i) and (ii). $\square$

3. Proof of Theorem 1.3

We begin this section by outlining constructions due to Lusztig ([6], 2.4–2.10) of H_J-modules with bases supporting W_J-digraphs when W_J is a finite dihedral group. The arguments given here, which differ somewhat from those in [6], are included for the sake of completeness.

Assume $J = \{s,t\}$, $1 < n = n(s,t) < \infty$. When arguing that the $\{s,t\}$-labeled digraphs in Figures 1.3–1.10 are W_J-digraphs, we may as well assume $n = m$ for Figures 1.3–1.5, $n = 2m-1$ for Figures 1.6–1.7, and $n = 2m-2$ in Figure 1.8. Indeed, if n, n' are positive integers and n divides n', then

$$\overbrace{\tau_s\tau_t\cdots}^{n} = \overbrace{\tau_t\tau_s\cdots}^{n} \quad \text{implies} \quad \overbrace{\tau_s\tau_t\cdots}^{n'} = \overbrace{\tau_t\tau_s\cdots}^{n'}.$$

Put $s' = s$ if m is even, $s' = t$ if m is odd, and define t' by $\{s',t'\} = \{s,t\}$. We consider cases.

Case 1. Figure 1.3, $n = m \geq 2$.

Define $\mu_0 = T_e$ and

$$\begin{cases} \mu_1 = T_s\mu_0, \mu_2 = T_t\mu_1, \cdots, \mu_{m-1} = T_{s'}\mu_{m-2}, \mu_m = T_{t'}\mu_{m-1}, \\ \mu_1' = T_t\mu_0, \mu_2' = T_s\mu_1', \cdots, \mu_{m-1}' = T_{t'}\mu_{m-2}', \mu_m = T_{s'}\mu_{m-1}'. \end{cases}$$

Then

$$\mu_m = T_{s_m} = T_{t_m} = \mu_m',$$

and so by Lemma 2.5 $X = \{\mu_0, \mu_1, \ldots, \mu_{m-1}, \mu_1', \mu_2', \ldots, \mu_m'\} = \{T_w \mid w \in W_J\}$ supports a W_J-digraph. This W_J-digraph is isomorphic to the J-labeled digraph of Figure 1.3 via $\mu_j \leftrightarrow \alpha_j$ for $0 \le j \le m-1$, $\mu_j' \leftrightarrow \beta_j$ for $1 \le j \le m$.

Case 2. Figure 1.4, $n = m \ge 2$.

Let $\nu_0 = T_e$, and define

$$\begin{cases} \nu_1 = T_s^\circ\nu_0, \nu_2 = T_t\nu_1, \ldots, \nu_{m-1} = T_{s'}\nu_{m-2}, \nu_m = T_{t'}\nu_{m-1}, \\ \nu_1' = T_t\nu_0, \nu_2' = T_s\nu_1', \ldots, \nu_{m-1}' = T_{t'}\nu_{m-2}', \nu_m' = T_{s'}^\circ\nu_{m-1}'. \end{cases}$$

Then

$$\begin{aligned} \nu_m &= T_{t_{m-1}}T_s^\circ = (u+1)^{-1}T_{t_{m-1}}(T_s - u) = (u+1)^{-1}\left(T_{s_m} - uT_{t_{m-1}}\right) \\ &= (u+1)^{-1}\left(T_{t_m} - uT_{t_{m-1}}\right) = (u+1)^{-1}\left(T_{s'}T_{t_{m-1}} - uT_{t_{m-1}}\right) \\ &= (u+1)^{-1}(T_{s'} - u)T_{t_{m-1}} = T_{s'}^\circ T_{t_{m-1}} = \nu_m', \end{aligned}$$

so $X = \{\nu_0, \nu_1, \ldots, \nu_{m-1}, \nu_1', \nu_2', \ldots, \nu_m'\}$ is linearly independent, so is a basis for H_J, and supports a W_J-digraph by Lemma 2.5. This W_J-digraph is isomophic to the J-labeled digraph of Figure 1.4 via $\nu_j \leftrightarrow \alpha_j$ for $0 \le j \le m-1$, $\nu_j' \leftrightarrow \beta_j$ for $1 \le j \le m$.

Case 3. Figure 1.5, $n = m \ge 2$.

Interchanging s and t in the argument given for the previous case shows that the J-labeled multigraph in Figure 1.5 is a W_J-digraph.

Case 4. Figure 1.6, $n = 2m-1$, $m \ge 2$.

Define an element η_0 of H_J by

$$\eta_0 = \widetilde{\eta}_{m-1} = \widetilde{\varphi}_{m-1} + u\widetilde{\varphi}_{m-2} + u^2\widetilde{\varphi}_{m-3} + \cdots + u^{m-1}\widetilde{\varphi}_0.$$

Suppose m is even. Then by part (i) of Lemma 2.7,

$$
\begin{aligned}
T_{t_{m-1}}(T_s-u)\eta_0 &= T_{s_m}\widetilde{\eta}_m - uT_{t_{m-1}}\widetilde{\eta}_{m-1} = T_{t_m^{-1}}\widetilde{\eta}_{m-1} - uT_{t_{m-1}^{-1}}\widetilde{\eta}_{m-1}\\
&= \sum_{i=0}^{2m-2} u^i T_{t_{2m-i-1}^{-1}} - u\sum_{i=0}^{2m-2} u^i T_{t_{2m-i-2}^{-1}}\\
&= T_{t_{2m-1}^{-1}} - u^{2m-1} = T_{w_0} - u^n.
\end{aligned}
$$

On the other hand, if m is odd, then

$$
\begin{aligned}
T_{t_{m-1}}(T_s-u)\eta_0 &= T_{s_m}\widetilde{\eta}_{m-1} - uT_{t_{m-1}}\widetilde{\eta}_{m-1} = T_{s_m^{-1}}\widetilde{\eta}_{m-1} - uT_{s_{m-1}^{-1}}\widetilde{\eta}_{m-1}\\
&= \sum_{i=0}^{2m-2} u^i T_{s_{2m-i-1}^{-1}} - u\sum_{i=0}^{2m-2} u^i T_{s_{2m-i-2}^{-1}}\\
&= T_{s_{2m-1}^{-1}} - u^{2m-1} = T_{w_0} - u^n.
\end{aligned}
$$

Hence

$$T_{t_{m-1}}(T_s-u)\eta_0 = T_{w_0} - u^n = T_{s_{m-1}}(T_t-u)\eta_0,$$

where the second equation follows by applying the automorphism $T_s \leftrightarrow T_t$ to the first. Hence if we define

$$
\begin{cases}
\eta_1 = T_s^{\circ}\eta_0, \eta_2 = T_t\eta_1, \ldots, \eta_{m-1} = T_{s'}\eta_{m-2}, \eta_m = T_{t'}\eta_{m-1},\\
\eta_1' = T_t^{\circ}\eta_0, \eta_2' = T_s\eta_1', \ldots, \eta_{m-1}' = T_{t'}\eta_{m-2}', \eta_m' = T_{s'}\eta_{m-1}',
\end{cases}
$$

then

$$\eta_m = (u+1)^{-1}(T_{w_0} - u^n) = \eta_m'.$$

Therefore $X = \left\{\eta_0, \eta_1, \eta_2, \ldots, \eta_{m-1}, \eta_1', \eta_2', \ldots, \eta_{m-1}', \eta_m'\right\}$ is a basis for a left ideal in H_J and X supports a W_J-digraph by Lemma 2.5. This W_J-digraph is isomorphic to the J-labeled digraph in Figure 1.6 via $\eta_j \leftrightarrow \alpha_j$ for $0 \le j \le m-1$, $\eta_j' \leftrightarrow \beta_j$ for $1 \le j \le m$.

Case 5. Figure 1.7, $n = 2m-1$, $m \ge 2$.

Put

$$\gamma_0 = \widetilde{\gamma}_{m-1} = \widetilde{\varphi}_{m-1} - u\widetilde{\varphi}_{m-2} + u^2\widetilde{\varphi}_{m-3} \pm \cdots + (-u)^{m-1}\widetilde{\varphi}_0.$$

If m is even, then part (ii) of Lemma 2.7 gives

$$\begin{aligned}
&(T_{t'}-u)T_{s_{m-1}}\gamma_0\\
&\quad=(T_s-u)T_{s_{m-1}}\widetilde{\gamma}_{m-1}=T_{s_m}\widetilde{\gamma}_{m-1}-T_{s_{m-1}}\widetilde{\gamma}_{m-1}\\
&\quad=T_{t_m^{-1}}\widetilde{\gamma}_{m-1}-T_{s_{m-1}^{-1}}\widetilde{\gamma}_{m-1}=\sum_{i=0}^{2m-2}(-u)^iT_{t_{2m-i-1}^{-1}}-u\sum_{i=0}^{2m}(-u)^iT_{s_{2m-i-2}^{-1}}\\
&\quad=\sum_{w\in W}(-u)^{n-\ell(w)}T_w.
\end{aligned}$$

On the other hand, if m is odd, then

$$\begin{aligned}
&(T_{t'}-u)T_{s_{m-1}}\gamma_0\\
&\quad=(T_t-u)T_{s_{m-1}}\widetilde{\gamma}_{m-1}=T_{s_m}\widetilde{\gamma}_{m-1}-uT_{s_{m-1}}\widetilde{\gamma}_{m-1}\\
&\quad=T_{s_m^{-1}}\widetilde{\gamma}_{m-1}-uT_{t_{m-1}^{-1}}\widetilde{\gamma}_{m-1}=\sum_{i=0}^{2m-2}(-u)^iT_{s_{2m-i-1}^{-1}}-u\sum_{i=0}^{2m-2}(-u)^iT_{t_{2m-i-2}^{-1}}\\
&\quad=\sum_{w\in W}(-u)^{n-\ell(w)}T_w.
\end{aligned}$$

Therefore

$$(T_{t'}-u)T_{s_{m-1}}\gamma_0=\sum_{w\in W}(-u)^{n-\ell(w)}T_w=(T_{s'}-u)T_{t_{m-1}}\gamma_0,$$

with the second equation following from the first by applying the automorphism $T_s\leftrightarrow T_t$. Hence if we put

$$\begin{cases}\gamma_1=T_s\gamma_0,\gamma_2=T_t\gamma_1,\ldots,\gamma_{m-1}=T_{s'}\gamma_{m-2},\gamma_m=T_{t'}^{\circ}\gamma_{m-1},\\ \gamma_1'=T_t\gamma_0,\gamma_2'=T_s\gamma_1',\ldots,\gamma_{m-1}'=T_{t'}\gamma_{m-2}',\gamma_m'=T_{s'}^{\circ}\gamma_{m-1}',\end{cases}$$

then

$$\gamma_m=(u+1)^{-1}\sum_{w\in W}(-u)^{n-\ell(w)}T_w=\gamma_m'.$$

Thus $X=\{\gamma_0,\gamma_1,\ldots,\gamma_{m-1},\gamma_1',\gamma_2',\ldots,\gamma_{m-1}',\gamma_m'\}$ is a basis for a left ideal of H_J supporting a W_J-digraph. Moreover, this W_J-digraph is isomorphic to the digraph of Figure 1.7 via $\gamma_j\leftrightarrow\alpha_j$ for $0\le j\le m-1$, $\gamma_j'\leftrightarrow\beta_j$ for $1\le j\le m$.

Case 6. Figure 1.8, $n=2m-2$, $m\ge 2$.

Define

$$\delta_0 = \widetilde{\delta}_{m-2} = \frac{1}{2}(\widetilde{\eta}_{m-2} + \widetilde{\gamma}_{m-2}).$$

If m is even, then by part (iii) of Lemma 2.7 we have

$$\begin{aligned}(T_{t'} - u)T_{t_{m-2}}(T_s - u)\delta_0 &= (T_{s_m} - uT_{s_{m-1}} - uT_{t_{m-1}} + u^2 T_{t_{m-2}})\widetilde{\delta}_{m-2} \\ &= (T_{t_m^{-1}} - uT_{s_{m-1}^{-1}} - uT_{t_{m-1}^{-1}} + u^2 T_{s_{m-2}^{-1}})\widetilde{\delta}_{m-2} \\ &= \sum_{i=0}^{m-2} u^{2i} T_{t_{2m-2-2i}^{-1}} - u \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-3-2i}^{-1}} \\ &\qquad - u \sum_{i=0}^{m-2} u^{2i} T_{t_{2m-3-2i}^{-1}} + u^2 \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-4-2i}^{-1}} \\ &= \sum_{w\in W} (-u)^{n-\ell(w)} T_w.\end{aligned}$$

On the other hand, if m is odd then

$$\begin{aligned}(T_{t'} - u)T_{t_{m-2}}(T_s - u)\delta_0 &= (T_{s_m} - uT_{s_{m-1}} - uT_{t_{m-1}} + u^2 T_{t_{m-2}})\widetilde{\delta}_{m-2} \\ &= (T_{s_m^{-1}} - uT_{t_{m-1}^{-1}} - uT_{s_{m-1}^{-1}} + u^2 T_{t_{m-2}^{-1}})\widetilde{\delta}_{m-2} \\ &= \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-2-2i}^{-1}} - u \sum_{i=0}^{m-2} u^{2i} T_{t_{2m-3-2i}^{-1}} \\ &\qquad - u \sum_{i=0}^{m-2} u^{2i} T_{s_{2m-3-2i}^{-1}} + u^2 \sum_{i=0}^{m-2} u^{2i} T_{t_{2m-4-2i}^{-1}} \\ &= \sum_{w\in W} (-u)^{n-\ell(w)} T_w.\end{aligned}$$

Therefore

$$(T_{t'} - u)T_{t_{m-2}}(T_s - u)\delta_0 = \sum_{w\in W} (-u)^{n-\ell(w)} T_w = (T_{s'} - u)T_{s_{m-2}}(T_t - u)\delta_0,$$

with the second equation following from the first by applying the automorphism $T_s \leftrightarrow T_t$. Thus if we define

$$\begin{cases}\delta_1 = T_s^{\circ}\delta_0, \delta_2 = T_t\delta_1, \ldots, \delta_{m-2} = T_{s'}\delta_{m-1}, \delta_m = T_{t'}^{\circ}\delta_{m-1}, \\ \delta_1' = T_t^{\circ}\delta_0, \delta_2' = T_s\delta_1', \ldots, \delta_{m-2}' = T_{t'}\delta_{m-1}', \delta_m' = T_{s'}^{\circ}\delta_{m-1}',\end{cases}$$

then

$$\delta_m = (u+1)^{-2} \sum_{k=0}^{n} (-u)^{n-k} \sigma_k = \delta'_m.$$

Hence $X = \{\delta_0, \delta_1, \dots, \delta_{m-1}, \delta'_1, \delta'_2, \dots, \delta'_m\}$ is a basis for a left ideal of H_J that supports a W_J-digraph. This W_J-digraph is isomorphic to the J-labeled multigraph of Figure 1.8 via $\delta_j \leftrightarrow \alpha_j$ for $0 \le j \le m-1$, $\delta'_j \leftrightarrow \beta_j$ for $1 \le j \le m$.

Case 7. Figure 1.9 or Figure 1.10, $m = 1$, $n \ge 2$ arbitrary.

Suppose Γ is one of the J-labeled digraphs of Figures 1.9–1.10. Then with $M = \operatorname{span}\{\alpha_0, \beta_1\}$, T_s and T_t induce the same operator $\tau_s = \tau_t$ on M. Thus the relation (2.2) holds automatically, and so Γ is a W_J-digraph.

From the constructions above, it follows that (b) implies (a) in Theorem 1.3. To establish the converse, we can reduce to the case $S = J = \{s,t\}$, $1 < n = n(s,t) < \infty$, $W = W_J$, $H = H_J$, and need only show that any connected W-digraph Γ is isomorphic to one of the J-labeled digraphs of Figures 1.3–1.10, with m and n satisfying the appropriate divisibility conditions.

Let X be the set of vertices of Γ, and let $M = \operatorname{span} X$ be the associated H-module. If $\alpha \in X$, then $X \subseteq H\alpha$ because Γ is connected, so $|X| = \dim M = \dim H\alpha \le \dim H = 2n$. Moreover, $|X|$ is even by Lemma 2.3. Since every vertex of Γ is contained in exactly $|S| = 2$ edges, it follows that Γ_{undir} is a simple cycle of size $2m$, where $1 \le m \le n$.

Let γ_0 be any vertex of Γ. Number the remaining vertices $\gamma_1, \gamma_2, \dots, \gamma_{2m-1}$ in such a way that Γ has an edge from γ_{i-1} to γ_i or from γ_i to γ_{i-1} for $1 \le i \le 2m-1$. Put $\gamma_{2m} = \gamma_0$, so Γ also has an edge from γ_{2m-1} to γ_{2m} or from γ_{2m} to γ_{2m-1}. We consider the subscript j in γ_j as an integer modulo $2m$.

Recall the linear characters $\lambda_1 = \text{ind}, \lambda_2 = \text{sgn} : H \to \mathbb{Q}(u)$ of H are determined by

$$\lambda_1(T_s) = \lambda_1(T_t) = u^2 \quad \text{and} \quad \lambda_2(T_s) = \lambda_2(T_t) = -1.$$

If n is even, there are two additional linear characters $\lambda_3, \lambda_4 : H \to \mathbb{Q}(u)$ given by

$$\lambda_3(T_s) = u^2, \lambda_3(T_t) = -1 \quad \text{and} \quad \lambda_4(T_s) = -1, \lambda_4(T_t) = u^2.$$

It is known that $H_{\mathbb{C}} = \mathbb{C}(u) \otimes_{\mathbb{Q}(u)} H$ is split semisimple over $\mathbb{C}(u)$, any irreducible representation of $H_{\mathbb{C}}$ of dimension greater than 1 is two-dimensional, and the eigenvalues of T_s and T_t in any two-dimensional irreducible representation are -1 and u^2 (see [5], or [2], 8.3). Let m_1, m_2, m_3, m_4 be the number of summands in a direct sum decomposition of $M_{\mathbb{C}} = \mathbb{C}(u) \otimes_{\mathbb{Q}(u)} M$ into irreducible modules that afford λ_1, λ_2, λ_3, λ_4, respectively (with $m_3 = m_4 = 0$ if n is odd), and let N be the number of two-dimensional irreducible summands. With P_s as in Lemma 2.3, T_s has eigenvalues -1 and u^2 on each subspace span$\{\alpha, \beta\}$, $\{\alpha, \beta\} \in P_s$, and thus T_s has a total of m eigenvalues -1 and m eigenvalues u^2 on M. Since the same is true of T_t, we must have

$$m_1 + m_3 + N = m_1 + m_4 + N = m_2 + m_3 + N = m_2 + m_4 + N = m,$$

and so $m_1 = m_2$ and $m_3 = m_4$. By Lemma 2.4, the unique one-dimensional subspace M_1 of M that affords the character λ_1 is spanned by $v_1 = \sum_{i=1}^{2m} \gamma_i$, and thus $m_1 = 1$. Hence $m_2 = 1$, so there is a unique one-dimensional subspace M_2 of M affording λ_2. Let $v_2 = \sum_{i=1}^{2m} \zeta_i \gamma_i$ be a nonzero element of M_2. By Lemma 2.4, we have

$$\zeta_i = \begin{cases} -\frac{1}{u^2}\zeta_{i-1} & \text{if } \gamma_{i-1} \overset{s}{\longrightarrow} \gamma_i \text{ or } \gamma_{i-1} \overset{t}{\longrightarrow} \gamma_i \text{ is an edge of } \Gamma, \\ -u^2\zeta_{i-1} & \text{if } \gamma_{i-1} \overset{s}{\longrightarrow} \gamma_i \text{ or } \gamma_{i-1} \overset{t}{\longrightarrow} \gamma_i \text{ is an edge of } \Gamma, \\ -\frac{u+1}{u^2-u}\zeta_{i-1} & \text{if } \gamma_{i-1} \overset{s}{\dashrightarrow} \gamma_i \text{ or } \gamma_{i-1} \overset{t}{\dashrightarrow} \gamma_i \text{ is an edge of } \Gamma, \\ -\frac{u^2-u}{u+1}\zeta_{i-1} & \text{if } \gamma_{i-1} \overset{s}{\dashrightarrow} \gamma_i \text{ or } \gamma_{i-1} \overset{t}{\dashrightarrow} \gamma_i \text{ is an edge of } \Gamma. \end{cases}$$

for $1 \leq i \leq 2m$. If $m = 1$, then it follows that the edge joining γ_0 and γ_1 labeled s must have the same type and direction as the edge joining γ_0 and γ_1 labeled t, and so Γ is isomorphic to one of the J-labeled digraphs of Figure 1.9–1.10. We assume $m \geq 2$ for the remainder of the proof, so there is a unique edge joining γ_i to γ_{i-1} for $1 \leq i \leq 2m$. Further,

$$\zeta_0 = \zeta_{2m} = \zeta_0 \prod_{i=1}^{2m} \frac{\zeta_i}{\zeta_{i-1}},$$

and so $\prod_{i=1}^{2m}(\zeta_i/\zeta_{i-1}) = 1$. It follows that the number of edges of type $\gamma_{i-1} \longrightarrow \gamma_i$ (labeled either s or t) is equal to the number of edges of type $\gamma_{i-1} \longrightarrow \gamma_i$, $1 \le i \le 2m$, and the number of edges of type $\gamma_{i-1} \dashrightarrow \gamma_i$ is equal to the number of edges of type $\gamma_{i-1} \dashrightarrow \gamma_i$, $1 \le i \le 2m$.

Next, we compute the coefficient κ_j of γ_j in the expression for $T_sT_t\gamma_j$ as a linear combination of $\{\gamma_1, \ldots, \gamma_{2m}\}$. These coefficients are given in Table 3.1, which is organized according to the types of edges joining γ_j to the adjacent vertices δ, ε in Γ. (Either $\delta = \gamma_{j-1}$ and $\varepsilon = \gamma_{j+1}$ or $\delta = \gamma_{j+1}$ and $\varepsilon = \gamma_{j-1}$: the coefficient κ_j is the same in either case.) The entries of

Table 3.1

edges in Γ	coefficient κ_j	edges in Γ	coefficient κ_j
$\delta \xrightarrow{s} \gamma_j \xrightarrow{t} \varepsilon$	0	$\delta \xleftarrow{s} \gamma_j \xrightarrow{t} \varepsilon$	0
$\delta \xrightarrow{s} \gamma_j \overset{t}{\dashrightarrow} \varepsilon$	$u(u^2-1)$	$\delta \xleftarrow{s} \gamma_j \overset{t}{\dashrightarrow} \varepsilon$	0
$\delta \overset{s}{\dashrightarrow} \gamma_j \xrightarrow{t} \varepsilon$	0	$\delta \overset{s}{\dashleftarrow} \gamma_j \xrightarrow{t} \varepsilon$	0
$\delta \overset{s}{\dashrightarrow} \gamma_j \overset{t}{\dashrightarrow} \varepsilon$	$u(u^2-u-1)$	$\delta \overset{s}{\dashleftarrow} \gamma_j \overset{t}{\dashrightarrow} \varepsilon$	u^2
$\delta \xleftarrow{s} \gamma_j \xleftarrow{t} \varepsilon$	0	$\delta \xrightarrow{s} \gamma_j \xleftarrow{t} \varepsilon$	$(u^2-1)^2$
$\delta \xleftarrow{s} \gamma_j \overset{s}{\dashleftarrow} \varepsilon$	0	$\delta \xrightarrow{s} \gamma_j \overset{t}{\dashleftarrow} \varepsilon$	$(u^2-1)(u^2-u-1)$
$\delta \overset{s}{\dashleftarrow} \gamma_j \xleftarrow{t} \varepsilon$	$u(u^2-1)$	$\delta \overset{s}{\dashrightarrow} \gamma_j \xleftarrow{t} \varepsilon$	$(u^2-1)(u^2-u-1)$
$\delta \overset{s}{\dashleftarrow} \gamma_j \overset{t}{\dashleftarrow} \varepsilon$	$u(u^2-u-1)$	$\delta \overset{s}{\dashrightarrow} \gamma_j \overset{t}{\dashleftarrow} \varepsilon$	$(u^2-u-1)^2$

this table are easily verified. For example, if $\gamma_{j-1} \overset{s}{\dashleftarrow} \gamma_j \overset{t}{\dashrightarrow} \gamma_{j+1}$ are edges in Γ, then

$$T_sT_t\gamma_j = T_s(u\gamma_j + (u+1)\gamma_{j+1}) = u(u\gamma_j + (u+1)\gamma_{j-1}) + (u+1)T_s\gamma_{j+1},$$

and so $\kappa_j = u^2$ since $T_s\gamma_{j+1} \in \operatorname{span}\{\gamma_{j+1}, \gamma_{j+2}\}$. On the other hand, if Γ has edges $\gamma_{j+1} \xrightarrow{s} \gamma_j \overset{t}{\dashrightarrow} \gamma_{j-1}$, then

$$T_sT_t\gamma_j = T_s(u\gamma_{j+1} + (u+1)\gamma_{j-1}) = u(u^2\gamma_{j+1} + (u^2-1)\gamma_j) + (u+1)T_s\gamma_{j-1},$$

and so $\kappa_j = u(u^2-1)$ because $T_s\gamma_{j-1} \in \operatorname{span}\{\gamma_{j-1}, \gamma_{j-2}\}$.

From Table 3.1 we see that the constant term in the trace $\operatorname{tr}(T_sT_t) = \sum_{j=1}^{2m} \kappa_j$ is equal to the number of sinks in Γ. However, T_sT_t has values u^4 and 1 under λ_1 and λ_2, respectively, and value $-u^2$ under both λ_3 and λ_4 if n is even. Also, T_sT_t has eigenvalues of the form $e^{\imath\theta}u^2$, $e^{-\imath\theta}u^2$ in any two-dimensional irreducible representation of $H_{\mathbb{C}}$, where $e^{\imath\theta}$ is a complex nth root of unity ([5], Theorem 2, or [2], Theorem 8.3.1). Therefore the constant term of $\mathrm{tr}(T_sT_t)$ is $m_2 = 1$. Hence Γ has a unique sink β, and so also a unique source α.

Renumbering the vertices if necessary, we can assume that $\gamma_0 = \alpha$. Since Γ has a unique sink β and the number of edges of type $\gamma_{i-1} \xrightarrow{s} \gamma_i$ is equal to the number of edges of type $\gamma_{i-1} \longrightarrow \gamma_i$, $1 \leq i \leq 2m$, and the number of edges of type $\gamma_{i-1} \dashrightarrow \gamma_i$ is equal to the number of edges of type $\gamma_{i-1} \dashrightarrow \gamma_i$, $1 \leq i \leq 2m$, it follows that $\beta = \gamma_m$ is opposite to α.

Renumbering the vertices if needed, we can assume that γ_0 and γ_1 are connected by an edge labeled s. Define $\gamma'_j = \gamma_{2m-j}$ for $0 \leq j \leq m$, so $\beta = \gamma'_m$. Then $\Gamma_{\rightarrow}$ has the form shown in Figure 3.1. (Since each edge of $\Gamma_{\rightarrow}$ arises from either a solid or a dashed edge in Γ, there are 2^{2m} possible J-labeled digraphs Γ with this configuration.)

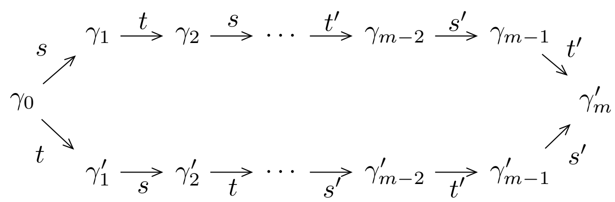

Figure 3.1 $\Gamma_{\rightarrow}$

From the discussion above, we know that the number of edges in Γ of type $\gamma_{i-1} \dashrightarrow \gamma_i$ (labeled either s or t), $1 \leq i \leq m$, is equal to the number of edges of type $\gamma'_{i-1} \dashrightarrow \gamma'_i$, $1 \leq i \leq m$. Also, from the description of the eigenvalues of T_sT_t above, $\mathrm{tr}(T_sT_t)$ must be an even function of u. Let N_1 be the number of edges of the form $\xi \dashrightarrow \omega$ with ξ not a source, that is, $\xi \neq \alpha = \gamma_0$, and let N_2 be the number of edges $\xi \dashrightarrow \omega$ with ω a sink, that is, $\omega = \beta = \gamma'_m$. Then from Table 3.1, the coefficient of u^3 in $\mathrm{tr}(T_sT_t)$ is $N_1 - N_2$, and hence $N_1 = N_2$. Therefore any edge of type $\xi \dashrightarrow \omega$ that does not begin at γ_0 must end at γ'_m. Hence Γ is isomorphic to one of the J-labeled digraphs in Figures 1.3–1.8 via $\gamma_j \leftrightarrow \alpha_j$, $0 \leq j \leq m-1$, $\gamma'_j \leftrightarrow \beta_j$, $1 \leq j \leq m$.

Finally, let τ_s and τ_t be as in (1.3), and let $\widetilde{A}_s(u), \widetilde{A}_t(u)$ be the $(2m) \times (2m)$ matrices over $\mathbb{Q}[u]$ representing τ_s and τ_t with respect to the basis X for M. Put $A_s = \widetilde{A}_s(1)$, $A_t = \widetilde{A}_t(1)$. Then

$$A_s^2 = I = A_t^2, \qquad \overbrace{\cdots A_tA_s}^{n} = \overbrace{\cdots A_sA_t}^{n},$$

by (2.1), (2.2), and so $s \mapsto A_s$, $t \mapsto A_t$ extends to a representation of groups $W_J \to \mathrm{GL}(2m, \mathbb{Q})$. One checks that the characteristic polynomial of the matrix $A_{st} = A_s A_t$ representing st is as given in Table 3.2. Hence the order

Table 3.2

W-digraph	characteristic polynomial of A_{st}
Figure 1.3, Figure 1.4, Figure 1.5	$(x^m - 1)^2$
Figure 1.6, Figure 1.7	$(x-1)(x^{2m-1} - 1)$
Figure 1.8	$(x-1)^2(x^{m-1}+1)^2$

of A_{st} as an element of $\mathrm{GL}(2m, \mathbb{Q})$ is m in the case of Figures 1.3–1.5, $2m-1$ in the case of Figures 1.6–1.7, and $2m-2$ in the case of Figure 1.8. Since this order must divide n, the proof is complete.

Proof of Corollary 1.4. Let (W, S) be a Coxeter system, and let Γ be an S-labeled digraph. For $J = \{s, t\}$ and $1 < n < \infty$, denote by $\mathcal{F}_{J,n}$ the collection of all J-labeled digraphs C of Figures 1.3–1.10 for which $m = |\mathscr{V}(C)|/2$ satisfies the divisibility conditions of Theorem 1.3. Then Γ is a W-digraph if and only if whenever $J = \{s, t\} \subseteq S$ with $1 < n = n(s,t) < \infty$, each connected component of Γ_J is isomorphic to an element of $\mathcal{F}_{J,n}$. It is easily seen that $\mathcal{F}_{J,n}$ is invariant under $C \mapsto C_{\mathrm{rev}}$. Also, C is a connected component of Γ_J if and only if C_{rev} is a connected component of $(\Gamma_J)_{\mathrm{rev}}$. The assertion of the corollary follows. □

Let $w \mapsto w^*$ be an involutory automorphism of (W, S), and let $I_* = \{x \in W \mid x^* = x^{-1}\}$ be the set of twisted involutions of W. By Lusztig [6], Theorem 0.1, there is an H-module M_* with basis $X = \{m_w \mid w \in I_*\}$ and H-action determined by

$$T_s m_w = \begin{cases} m_{sws^*} & \text{if } sw \neq ws^* > w, \\ (u^2-1)m_w + u^2 m_{sws^*} & \text{if } sw \neq ws^* < w, \\ u m_w + (u+1) m_{sw} & \text{if } sw = ws^* > w, \\ (u^2-u-1)m_w + (u^2-u)m_{sw} & \text{if } sw = w^* < w. \end{cases}$$

(Here $<$ and $>$ refer to the usual Bruhat order on (W, S).) The basis X for M_* then affords the W-digraph Γ_* defined by

$$m_w \xrightarrow{s} m_{sws^*} \in \mathscr{E}(\Gamma_*) \iff sw \neq ws^* > w$$

and

$$m_w \overset{s}{\dashrightarrow} m_{sw} \in \mathscr{E}(\Gamma_*) \quad \Longleftrightarrow \quad sw = ws^* > w.$$

Theorem 3.1. *Let (W, S) be finite with longest element w_0, and let Γ_* be the W-digraph corresponding to the involutory automorphism $w \mapsto w^*$ of (W, S). Then $(\Gamma_*)_{rev}$ is isomorphic to the W-digraph $\Gamma_\#$ corresponding to the automorphism $w \mapsto w^\# = w_0 w^* w_0$ of (W, S) via the bijection sending $m_x \in \mathscr{V}(\Gamma_*)$ to $m_{xw_0} \in \mathscr{V}(\Gamma_\#)$.*

Proof. Observe $w_0^* = w_0$ since $w \mapsto w^*$ preserves lengths. Suppose $x \in I_*$, so $x^* = x^{-1}$. Then $xw_0 \in I_\#$ because

$$(xw_0)^\# = w_0(xw_0)^* w_0 = w_0 x^* w_0 w_0 = w_0 x^{-1} = (xw_0)^{-1}.$$

Likewise, if $xw_0 \in I_\#$, then $x \in I_*$. If $x, y \in I_*$ and $s \in S$, then

$$\begin{aligned} m_x \xrightarrow{s} m_y \in \mathscr{E}(\Gamma_*) &\Longleftrightarrow x < y = sxs^* \\ &\Longleftrightarrow yw_0 < xw_0 = (sys^*)w_0 = s(yw_0)(w_0 s^* w_0) = s(yw_0)s^\# \\ &\Longleftrightarrow m_{yw_0} \xrightarrow{s} m_{xw_0} \in \mathscr{E}(\Gamma_\#), \end{aligned}$$

and

$$\begin{aligned} m_x \overset{s}{\dashrightarrow} m_y \in \mathscr{E}(\Gamma_*) &\Longleftrightarrow x < y = sx = xs^* \\ &\Longleftrightarrow yw_0 < xw_0 = (sy)w_0 = s(yw_0) = (ys^*)w_0 = (yw_0)s^\# \\ &\Longleftrightarrow m_{yw_0} \overset{s}{\dashrightarrow} m_{xw_0} \in \mathscr{E}(\Gamma_\#). \end{aligned}$$

Therefore $(\Gamma_*)_{\text{rev}}$ is isomorphic to $\Gamma_\#$ via the bijection $m_x \mapsto m_{xw_0}$ on vertices. □

Corollary 3.2. *If w_0 is central in W, then $(\Gamma_*)_{rev}$ is isomorphic to Γ_*.*

Example 3.3. Suppose $W = \langle r, s, t \rangle$ with $n(r, s) = n(s, t) = 3$, $n(r, t) = 2$ and $w^* = w$ for $w \in W$, so I_* is the set of involutions in W (including e). The corresponding W-digraph Γ_* is shown in Figure 3.2. (The vertices are labeled x rather than m_x for $x \in I_*$.) If $w \mapsto w^\#$ is the nonidentity graph automorphism of W, the corresponding W-digraph $\Gamma_\#$ is as shown in Figure 3.3. Note $(\Gamma_*)_{\text{rev}} \cong \Gamma_\#$.

Example 3.4. Suppose $W = \langle r, s, t \rangle$ with $n(r, s) = 3$, $n(s, t) = 4$, $n(r, t) = 2$. With $w \mapsto w^* = w$, I_* is the set of involutions of W. The corresponding W-digraph Γ_* takes the form shown in Figure 3.4. Note $(\Gamma_*)_{\text{rev}} \cong \Gamma_*$.

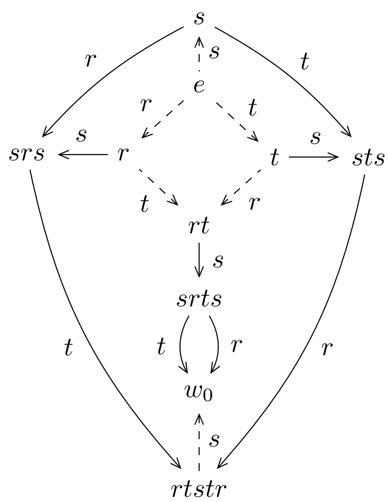

Figure 3.2 Γ_* for $W(A_3)$

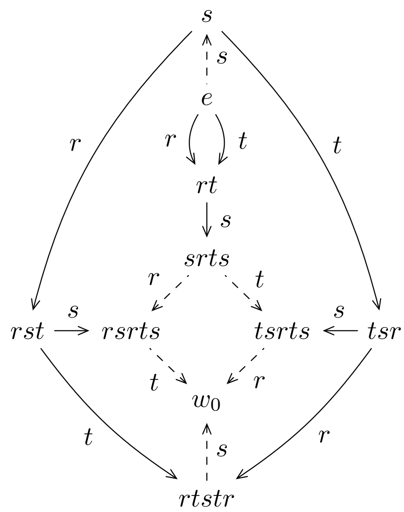

Figure 3.3 $\Gamma_\#$ for $W(A_3)$

4. The Proofs of Theorems 1.5, 1.7, 1.8, and 1.10

Throughout this section (W, S) is a Coxeter system and Γ is a W-digraph. If $\varepsilon = \alpha \overset{s}{\longrightarrow} \beta \in \mathscr{E}(\Gamma)$ or $\varepsilon = \alpha \overset{s}{\dashrightarrow} \beta \in \mathscr{E}(\Gamma)$, then we call $\alpha \overset{s}{\longrightarrow} \beta \in \mathscr{E}(\Gamma_\to)$ the *image* of ε in $\Gamma_\to$. Clearly there is a directed path from α to β in Γ if and only if there is a directed path from α to β in $\Gamma_\to$.

Let H_0 be the 0-Hecke algebra of (W, S) (see [8], or [1], Chapter IV, §2, Exercise 23, with $\lambda_s = -1$, $\mu_s = 0$ for $s \in S$). Thus H_0 is an associative algebra over $\mathbb{Q}$ with generating set $\{a_s \mid s \in S\}$ satisfying the presentation

$$a_s^2 = -a_s$$

for $s \in S$ and

$$\overbrace{a_s a_t a_s \cdots}^{n(s,t)} = \overbrace{a_t a_s a_t \cdots}^{n(s,t)}$$

if $s, t \in S$, $n(s,t) < \infty$. Also, H_0 has basis $\{a_w \mid w \in W\}$ with a_e the identity element of H_0 and

$$a_s a_w = \begin{cases} a_{sw} & \text{if } sw > w, \\ -a_w & \text{if } sw < w. \end{cases} \tag{4.1}$$

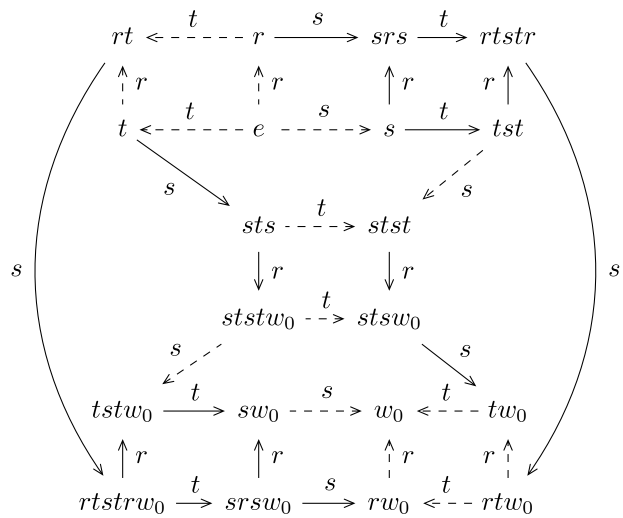

Figure 3.4 Γ_* for $W(B_3)$

It follows that for $x, y \in W$, there is $z \in W$ such that

$$a_x a_y = \pm a_z \quad \text{and} \quad \max\{\ell(x), \ell(y)\} \leq \ell(z),$$

and

$$a_x a_y = a_{xy} \iff \ell(x) + \ell(y) = \ell(xy).$$

If (W, S) is finite and w_0 is the longest element of W, then

$$a_w a_{w_0} = (-1)^{\ell(w)} a_{w_0} = a_{w_0} a_w$$

for $w \in W$.

Let $M = M(\Gamma)$ be the module afforded by Γ, so M has basis $X = \mathscr{V}(\Gamma)$ over $\mathbb{Q}(u)$. Let M_0 be the $\mathbb{Q}$-subspace of M with basis X. For $s \in S$, define a $\mathbb{Q}$-linear operator $(\tau_s)_0$ on M_0 by

$$(\tau_s)_0(\alpha) = \begin{cases} \beta & \text{if } \alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma_\rightarrow), \\ -\alpha & \text{if } \alpha \text{ is a sink in } \Gamma_s. \end{cases}$$

Notice that by (1.3), $(\tau_s)_0(\alpha)$ can be obtained by replacing the coefficients of the image $\tau_s(\alpha)$ expressed as a linear combination of the elements of X

with their values at $u = 0$. Since in $\mathrm{gl}(M)$ we have

$$(\tau_s - u^2)(\tau_s + 1) = 0 \qquad \text{and} \qquad \overbrace{\tau_s\tau_t\tau_s\cdots}^{n(s,t)} = \overbrace{\tau_t\tau_s\tau_t\cdots}^{n(s,t)},$$

it follows that in $\mathrm{gl}(M_0)$ we have

$$((\tau_s)_0))^2 = -(\tau_s)_0 \qquad \text{and} \qquad \overbrace{(\tau_s)_0(\tau_t)_0(\tau_s)_0\cdots}^{n(s,t)} = \overbrace{(\tau_t)_0(\tau_s)_0(\tau_t)_0\cdots}^{n(s,t)}$$

if $n(s,t) < \infty$. Hence $a_s \mapsto (\tau_s)_0$ defines a representation $\rho_0 : H_0 \to \mathrm{gl}(M_0)$, giving M_0 the structure of an H_0-module. In particular, for $\alpha \in \mathscr{V}(\Gamma)$,

$$a_s\alpha = \begin{cases} \beta & \text{if } \alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma_\to), \\ -\alpha & \text{if } \alpha \text{ is a sink in } \Gamma_{\{s\}}. \end{cases} \tag{4.2}$$

Lemma 4.1. *Assume (W,S) is a Coxeter system, Γ is a W-digraph, $X = \mathscr{V}(\Gamma)$, and H_0 acts on M_0 as described above. Then the following hold.*

(i) *If $\alpha \in X$ and $w \in W$, then $a_w\alpha \in X$ or $-a_w\alpha \in X$.*

(ii) *If $\beta \in X$, then there exists some $w \in W$ such that $\beta = \pm a_w\alpha$ if and only if there is a directed path from α to β in Γ.*

Proof. Since $a_w = a_{s_1}a_{s_2}\cdots a_{s_\ell}$ if $s_1s_2\ldots s_\ell$ is a reduced expression for $w \in W$ by (4.1), an easy induction argument based on (4.2) establishes (i).

For (ii), we can argue with $\Gamma_\to$ in place of Γ. Suppose $\beta \in X$ and there is some directed path

$$\gamma_0 \xrightarrow{s_1} \gamma_1 \xrightarrow{s_2} \gamma_2 \xrightarrow{s_3} \cdots \xrightarrow{s_{k-1}} \gamma_{k-1} \xrightarrow{s_k} \gamma_k \tag{4.3}$$

in $\Gamma_\to$ with $\gamma_0 = \alpha$, $\gamma_k = \beta$. Define $y \in W$ by $\pm a_y = a_{s_k}a_{s_{k-1}}\cdots a_{s_2}a_{s_1}$. Then

$$\pm a_y\alpha = a_{s_k}a_{s_{k-1}}\cdots a_{s_2}a_{s_1}\gamma_0 = \gamma_k = \beta.$$

Conversely, assume $\beta = \pm a_w\alpha \in X$, where $w \in W$. Let $w = t_kt_{k-1}\cdots t_2t_1$ be a reduced expression for w as a product of generators $t_k, \ldots, t_1 \in S$, so $a_w = a_{t_k}\cdots a_{t_2}a_{t_1}$ by (4.1). Put $\delta_0 = \alpha$ and $\delta_j = a_{t_j}\delta_{j-1}$ for $1 \le j \le k$, so $\beta = \pm\delta_k$. By (i), there are $\varepsilon_j \in \{-1, 1\}$ such that $\alpha_j = \varepsilon_j\delta_j \in X$ for

$0 \le j \le k$. Then for $1 \le j \le k$, $\alpha_{j-1} \ne \alpha_j$ if and only if $\alpha_{j-1} \xrightarrow{t_j} \alpha_j \in \mathscr{E}(\Gamma_\to)$. If $0 < j_1 < j_2 < \cdots < j_\ell$ are the values of j, $1 \le j \le k$, for which $\alpha_{j-1} \ne \alpha_j$, then

$$\alpha_0 \xrightarrow{t_{j_1}} \alpha_{j_1} \xrightarrow{t_{j_2}} \alpha_{j_2} \xrightarrow{t_{j_3}} \cdots \xrightarrow{t_{t_{\ell-1}}} \alpha_{j_{\ell-1}} \xrightarrow{t_{j_\ell}} \alpha_{j_\ell}$$

is a directed path in $\Gamma_\to$ from α to β. Thus (ii) holds. □

Lemma 4.2. *Assume (W, S) is a Coxeter system, Γ is a W-digraph, $X = \mathscr{V}(\Gamma)$, and H_0 acts on M_0 as in (4.2). For $\omega \in X$, the following are equivalent.*

(i) *ω is a sink in Γ.*

(ii) *$a_s\omega = -\omega$ for all $s \in S$.*

(iii) *$a_w\omega = (-1)^{\ell(w)}\omega$ for all $w \in W$.*

Moreover, if (W, S) is finite, then (i)–(iii) are equivalent to

(iv) *$\omega = \pm a_{w_0}\alpha$ for some $\alpha \in X$.*

Proof. If ω is a sink in Γ, then $a_s\omega = -\omega$ for all $s \in S$ by (4.2). Thus (i) implies (ii).

Assume $a_s\omega = -\omega$ for all $s \in S$ and $w \in W$ has reduced expression $w = s_1 s_2 \cdots s_k$. Then by (4.1),

$$a_w\omega = a_{s_1}a_{s_2}\cdots a_{s_k}\omega = (-1)^k\omega = (-1)^{\ell(w)}\omega.$$

Hence (ii) imples (iii).

Suppose $a_w\omega = (-1)^{\ell(w)}\omega$ for all $w \in W$. Then $a_s\omega = -\omega$ for all $s \in S$, and thus ω must be a sink in Γ by (4.2). Hence (iii) implies (i).

Suppose (W, S) is finite and $\omega = \varepsilon a_{w_0}\alpha$, where $\alpha \in X$ and $\varepsilon \in \{-1, 1\}$. Then

$$a_s\omega = a_s(\varepsilon a_{w_0}\alpha) = \varepsilon(a_s a_{w_0})\alpha = -\varepsilon a_{w_0}\alpha = -\omega$$

for any $s \in S$, and so ω is a sink in Γ. Conversely, if ω is a sink in Γ, then $a_{w_0}\omega = (-1)^N\omega = \pm\omega$ by (4.2), where $N = \ell(w_0)$, and thus $\omega = \pm a_{w_0}\omega$. Hence (i) and (iv) are equivalent. □

Define relations $\equiv_{s,t}$ and $\equiv$ on the set of directed paths in Γ as follows. If π_1 and π_2 are directed paths in Γ and $s, t \in S$ satisfy $1 < n(s,t) < \infty$, then $\pi_1 \equiv_{s,t} \pi_2$ if there is some connected component C of $\Gamma_{\{s,t\}}$ such that π_1 and π_2 both pass through the source σ and the sink ω of C, and π_2 can be obtained from π_1 by replacing one of the directed paths from σ to ω in $\Gamma_{\{s,t\}}$ by the other. Let $\equiv$ be the equivalence relation on directed paths in Γ generated by the relations $\equiv_{s,t}$ for $s, t \in S$, $1 < n(s,t) < \infty$. Similar relations, also denoted $\equiv_{s,t}$ and $\equiv$, can be defined for directed paths in $\Gamma_{\rightarrow}$. It is clear that two directed paths in Γ are in the same $\equiv$ equivalence class if and only if their images in $\Gamma_{\rightarrow}$ are in the same $\equiv$ equivalence class.

For $\alpha \in \mathscr{V}(\Gamma)$, denote by $[\alpha, \infty)$ the set of all $\beta \in \mathscr{V}(\Gamma)$ such that there exists a directed path in Γ from α to β. Clearly if $\beta \in [\alpha, \infty)$ and $\gamma \in [\beta, \infty)$, then $\gamma \in [\alpha, \infty)$. For $\beta \in [\alpha, \infty)$, let $\mu(\alpha, \beta)$ be the minimum number of edges in a directed path from α to β (with $\mu(\alpha, \alpha) = 0$).

Lemma 4.3. *Suppose (W, S) is a Coxeter system such that $n(s,t) < \infty$ for all $s, t \in S$, and Γ is a W-digraph with source σ.*

(i) *If $\alpha \in [\sigma, \infty)$, then any two directed paths from σ to α are in the same $\equiv$-equivalence class.*

(ii) *If $\alpha \in [\sigma, \infty)$, $\zeta \in \mathscr{V}(\Gamma)$, and $\alpha \in [\zeta, \infty)$, then $\zeta \in [\sigma, \infty)$.*

(iii) *If Γ is connected, then $\mathscr{V}(\Gamma) = [\sigma, \infty)$.*

Proof. We can argue with $\Gamma_{\rightarrow}$ in place of Γ. We prove (i) and (ii) simultaneously by induction on $\mu(\sigma, \alpha)$. If $\mu(\sigma, \alpha) = 0$, then $\alpha = \sigma$, so (i) holds because the only directed path from σ to σ is the empty path because σ is a source. Also, if $\alpha = \sigma \in [\gamma, \infty)$, then there is a directed path from γ to σ, so the path must be empty and $\gamma = \sigma \in [\sigma, \infty)$, and thus (ii) holds.

Suppose $\mu(\sigma, \alpha) = k > 0$ and (i) and (ii) hold with β in place of α whenever $\beta \in [\sigma, \infty)$ and $\mu(\sigma, \beta) < k$. Let π_1 be some directed path from σ to α with k edges, and let π_2 be an arbitrary directed path from σ to α. For $j = 1, 2$, let $\varepsilon_j \in \mathscr{E}(\Gamma_{\rightarrow})$ be the last edge of π_j and let ρ_j be the remainder of the path π_j, so $\pi_j = \rho_j \varepsilon_j$, where juxtaposition indicates concatination of paths. Thus ε_1 takes the form $\beta \xrightarrow{s} \alpha$ for some $s \in S$ and $\beta \in \mathscr{V}(\Gamma)$ with $\beta \in [\sigma, \infty)$ and $\mu(\sigma, \beta) = k - 1$. Also, ε_2 has the form $\gamma \xrightarrow{t} \alpha$ for some $t \in S$, $\gamma \in \mathscr{V}(\Gamma)$. If $t = s$, then $\gamma = \beta$, so $\rho_1 \equiv \rho_2$ by (i) applied to β, and thus $\pi_1 \equiv \pi_2$ as desired. Suppose $t \neq s$. Let τ be the source of the connected component C of $(\Gamma_\rightarrow)_{\{s,t\}}$ whose sink is α. (Note that C has a unique source by the classification of possible connected components of $\Gamma_{\{s,t\}}$ given in Theorem 1.3.) Let ν_1 (ν_2) be the directed path in C from τ to β (γ, respectively), so $\nu_1\varepsilon_1 \equiv_{s,t} \nu_2\varepsilon_2$. Since $\beta \in [\tau, \infty)$, we have $\tau \in [\sigma, \infty)$ by (ii) applied to β, and so there is some directed path ρ from σ to τ in $\Gamma_\rightarrow$. (See Figure 4.1, in which edges represent directed paths in $\Gamma_\rightarrow$.) We have

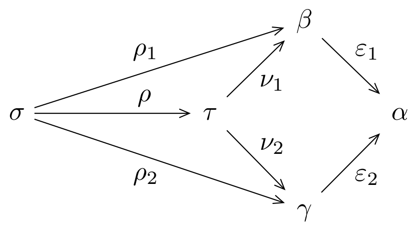

Figure 4.1

$\rho_1 \equiv \rho\nu_1$ by (i) applied to β, and thus $\rho\nu_1$ has $k-1$ edges. Since ν_1 and ν_2 have the same number of edges, it follows that $\rho\nu_2$ also has $k-1$ edges, and thus $\mu(\sigma,\gamma) \leq k-1$. Hence by (i) applied to γ, we also have $\rho\nu_2 \equiv \rho_2$. Thus

$$\begin{aligned}\pi_1 = \rho_1\varepsilon_1 &\equiv (\rho\nu_1)\varepsilon_1 = \rho(\nu_1\varepsilon_1)\\ &\equiv \rho(\nu_2\varepsilon_2) = (\rho\nu_2)\varepsilon_2 \equiv \rho_2\varepsilon_2 = \pi_2.\end{aligned}$$

Therefore (i) holds for α.

Now suppose $\alpha \in [\delta, \infty)$. Let ψ be a directed path from δ to α. Write $\psi = \psi_0\varepsilon_0$, where $\varepsilon_0 \in \mathscr{E}(\Gamma_\rightarrow)$ is the last edge of ψ, so ε_0 has the form $\phi \xrightarrow{r} \alpha$ for some $r \in S$, $\phi \in \mathscr{V}(\Gamma)$. If $r = s$, then $\beta = \phi \in [\delta, \infty)$, and hence $\delta \in [\sigma, \infty)$ by (ii) applied to β. Assume $r \neq s$. Let κ be the source of the connected component of $(\Gamma_\rightarrow)_{\{r,s\}}$ whose sink is α. (See Figure 4.2, in which the edges represent directed paths in $\Gamma_\rightarrow$.) There exists some directed path from σ to κ by (ii) applied to β. The argument given above for γ applies to

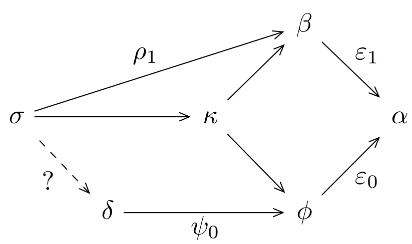

Figure 4.2

show that $\mu(\sigma,\phi) \le k-1$. Since $\phi \in [\delta,\infty)$, it follows that $\delta \in [\sigma,\infty)$ by (ii) applied to ϕ. Hence (ii) holds for α, so the proof of (i) and (ii) is complete.

Finally, suppose Γ is connected. For $\alpha \in \mathscr{V}(\Gamma)$, let $\delta(\alpha)$ be the minimal number of edges in a path in Γ_{undir} from σ to α. We prove $\alpha \in [\sigma,\infty)$ for all $\alpha \in \mathscr{V}(\Gamma)$ by induction on $\delta(\alpha)$. If $\delta(\alpha)=0$, then $\alpha=\sigma \in [\sigma,\infty)$. Suppose $\delta(\alpha)=\ell>0$ and $\gamma \in [\sigma,\infty)$ whenever $\gamma \in \mathscr{V}(\Gamma)$ and $\delta(\gamma)<\ell$. Let $\beta \longrightarrow \alpha$ be the last edge of a path in Γ_{undir} from σ to α of length ℓ, so $\delta(\beta)=\ell-1$ and $\beta \in [\sigma,\infty)$. If $\alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma_{\rightarrow})$ for some $s \in S$, then $\alpha \in [\sigma,\infty)$ because $\beta \in [\sigma,\infty)$ and $\alpha \in [\beta,\infty)$. On the other hand, if $\alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma_{\rightarrow})$, then $\beta \in [\alpha,\infty)$, and so $\alpha \in [\sigma,\infty)$ by (ii) applied to β. Hence $\alpha \in [\sigma,\infty)$ for all $\alpha \in \mathscr{V}(\Gamma)$, and therefore $\mathscr{V}(\Gamma)=[\sigma,\infty)$. This completes the proof. $\square$

Example 4.4. Let $W=W(A_3)=\langle r,s,t\rangle$, with $n(r,s)=3=n(s,t)$, $n(r,t)=2$, and let Γ be as in Figure 4.3. The directed paths $\alpha_2 \xrightarrow{s} \alpha_3 \xrightarrow{r} \beta_3$ and $\alpha_2 \xrightarrow{t} \beta_2 \xrightarrow{s} \beta_3$ from α_2 to β_3 are not in the same $\equiv$-equivalence class (even though adjoining the edge $\alpha_1 \xrightarrow{r} \alpha_2$ to both does produce two equivalent paths). Therefore the conclusion of Lemma 4.3(i) does not apply to arbitrary directed paths in a W-digraph.

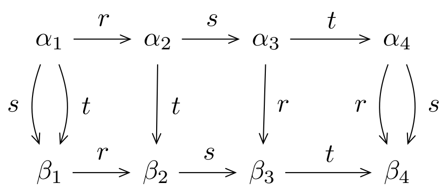

Figure 4.3 Digraph for Example 4.4

We now prove Theorems 1.5, 1.7, 1.8, and 1.10.

Proof of Theorem 1.5. Assume $n(s,t)<\infty$ for all $s,t \in S$ and Γ is a connected W-digraph. Since Γ_{rev} is also a connected W-digraph by Corollary 1.4, it is enough to prove the assertions involving sources. Suppose σ is a source of Γ. Then $\mathscr{V}(\Gamma)=[\sigma,\infty)$ by Lemma 4.3 (iii). Hence if $\gamma \ne \sigma$ is a vertex of Γ, there must be some nonempty directed path in Γ from σ to γ, and so γ cannot be a source. Thus σ is the unique source of Γ, so part (i) of the theorem holds.

Suppose Γ has source σ but is not acyclic. Let $\alpha \in \mathscr{V}(\Gamma)$ be contained in a nonempty directed circuit ρ in Γ. Since $\alpha \in [\sigma,\infty)$, there is some directed path π from σ to α. Then the directed paths π and $\pi\rho$ from σ to α are in different $\equiv$ equivalence classes because their lengths are different, contradicting Lemma 4.3 (i). Thus part (ii) of the theorem holds.

Finally, assume (W, S) is finite. By Lemma 4.2, Γ_{rev} has a sink. Thus Γ has a source, so part (iii) of the theorem holds. □

Proof of Theorem 1.7. Assume $\mathscr{V}(\Gamma)$ is finite. For a linear character λ of H, it is easily seen that $M(\Gamma)_\lambda$ is the direct sum of $M(C)_\lambda$ as C ranges over the connected components of Γ. By Lemma 2.4(i), if C is a connected component of Γ, then $v \in M(C)_{\mathrm{ind}}$ if and only if v is a scalar multiple of $\sum_{\alpha\in\mathscr{V}(C)} \alpha$. Thus (i) holds.

Suppose now that $n(s,t) < \infty$ for $s, t \in S$. Let C be a connected component of Γ. Assume C is acyclic. Then since $\mathscr{V}(C)$ is finite, there must be a source σ in C, and this source is unique by Theorem 1.5(i). Assign to each solid edge in C the weight $-1/u^2$, and to each dashed edge in C assign the weight $-(u+1)/(u^2-u)$. For $\alpha \in \mathscr{V}(C)$, let μ_α be the product of the weights of the edges of any directed path from σ to α in C: μ_α is well-defined by Lemma 4.3(i) since such products are constant on $\equiv$-equivalence classes. If $\alpha \xrightarrow{s} \beta$ is an edge of C, then $\mu_\beta = -\mu_\alpha/u^2$, while if $\alpha \overset{s}{\dashrightarrow} \beta$ is an edge of C, then $\mu_\beta = -(u+1)\mu_\alpha/(u^2-u)$. By Lemma 2.4(ii), $v \in M(C)_{\mathrm{sgn}}$ if and only if v is a scalar multiple of $\sum_{\alpha\in\mathscr{V}(C)} \mu_\alpha\alpha$. Therefore $\dim M(C)_{\mathrm{sgn}} = 1$.

Conversely, suppose $v = \sum_{\alpha\in\mathscr{V}(C)} \nu_\alpha\alpha \in M(C)_{\mathrm{sgn}}$ is nonzero. Since at least one of the coefficients ν_α is nonzero and C is connected, all of the coefficients ν_α are nonzero by Lemma 2.4(ii). Assume

$$\gamma_0 \xrightarrow{s_1} \gamma_1 \xrightarrow{s_2} \gamma_2 \xrightarrow{s_3} \cdots \xrightarrow{s_{k-1}} \gamma_{k-1} \xrightarrow{s_k} \gamma_k$$

is a directed path in $C_\to$, where $k > 0$, $s_1, s_2, \ldots, s_k \in S$. For $1 \le j \le k$ we have

$$\nu_{\gamma_j} = \begin{cases} -\dfrac{1}{u^2}\nu_{\gamma_{j-1}} & \text{if } \gamma_{j-1} \xrightarrow{s_j} \gamma_j \in \mathscr{E}(C), \\ -\dfrac{u+1}{u^2-u}\nu_{\gamma_{j-1}} & \text{if } \gamma_{j-1} \overset{s_j}{\dashrightarrow} \gamma_j \in \mathscr{E}(C), \end{cases}$$

If $\gamma_0 = \gamma_k$, then

$$\nu_{\gamma_0} = \nu_{\gamma_k} = \nu_{\gamma_0} \prod_{j=1}^{k} \frac{\nu_{\gamma_j}}{\nu_{\gamma_{j-1}}},$$

and therefore $\prod_{j=1}^{k}(\nu_{\gamma_j}/\nu_{\gamma_{j-1}}) = 1$, which is impossible since the product is equal to $\pm u^{-2(k-j)}(u+1)^j(u^2-u)^{-j}$ for some j with $0 \leq j \leq k$. Therefore C is acyclic, so the proof of (ii) is complete. □

Proof of Theorem 1.8. Assume (W, S) is a finite Coxeter system, Γ is a connected W-digraph, and $J \subseteq S$. Let

$$\Gamma_J = \bigcup_{i \in I} C_i$$

be the decomposition of Γ_J into its connected components, indexed by some set I. Let $\sigma_i \in \mathscr{V}(\Gamma)$ be the source of C_i. If σ is the source of Γ, then in $M(\Gamma)_0$ we have $\sigma_i = \pm a_{x(i)}\sigma$ for some $x(i) \in W$ by Lemma 4.3(iii) and Lemma 4.1(ii). Since σ_i is the source of C_i, we have $a_s\sigma_i \neq -\sigma_i$ for $s \in J$, so $a_s a_{x(i)} \neq -a_{x(i)}$, and thus $sx(i) > x(i)$, for all $s \in J$. Hence $x(i)$ is in the set of distinguished right coset representatives $X_J = \{w \in W \mid sw > w \text{ for } s \in J\}$ of W_J in W. Since $\sigma_i \neq \sigma_j$ when $i \neq j$ are in I, $i \mapsto x(i)$ is an injection from I into X_J. □

Example 4.5. Let (W, S) be a Coxeter system. Let Γ be the W-digraph defined by $\mathscr{V}(\Gamma) = W$ and $x \xrightarrow{s} y \in \mathscr{E}(\Gamma)$ if and only if $x < sx = y$ for $x, y \in W$, $s \in S$. Then the H-module $M(\Gamma)$ afforded by Γ is isomorphic to the left regular module H. Note that Γ is connected since if $w = s_k s_{k-1} \cdots s_1$ is a reduced expression for $w \in W$ and $x_j = s_j s_{j-1} \cdots s_1$ for $0 \leq j \leq k$ (with $x_0 = e$), then

$$x_0 \xrightarrow{s_1} x_1 \xrightarrow{s_2} \cdots \xrightarrow{s_{k-1}} x_{k-1} \xrightarrow{s_k} x_k$$

is a directed path from e to w in Γ. When (W, S) is finite, this example shows that the bound in Corollary 1.9 is always attained.

Proof.[Proof of Theorem 1.10] Suppose $n(s,t) < \infty$ for all $s, t \in S$ and Γ is a connected W-digraph with source σ. (The case in which Γ has a sink follows by applying the same reasoning to Γ_{rev}.) Let π_1, π_2 be two directed paths in Γ from α to β. Let ρ be some directed path from σ to α: such a path exists by Lemma 4.3 (iii). By Lemma 4.3 (i), the directed paths $\rho\pi_1$ and $\rho\pi_2$ from σ to β are in the same $\equiv$ equivalence class, and thus have the same number of edges. Hence π_1 and π_2 have the same number of edges. □

5. The Proof of Theorem 1.11

Let σ be an automorphism of $\mathbb{Q}(u)$, and let M be a vector space over $\mathbb{Q}(u)$. Let ${}^{\sigma}M$ be the vector space over $\mathbb{Q}(u)$ that has the same additive group as M and scalar multiplication $(\alpha, v) \mapsto \alpha *_{\sigma} v$ given by

$$\alpha *_{\sigma} v = ({}^{\sigma}\alpha)v,$$

where the scalar multiplication on the right hand side is that of M. It is clear that if $Y \subseteq M$, then Y is a basis (subspace) of M if and only if Y is a basis (subspace, respectively) of ${}^{\sigma}M$. Moreover, $\mathrm{gl}(M) = \mathrm{gl}({}^{\sigma}M)$ since if $\varphi : M \to M$ is an additive mapping, then

$$\varphi(\alpha *_{\sigma} v) = \alpha *_{\sigma} \varphi(v) \iff \varphi(({}^{\sigma}\alpha)v) = ({}^{\sigma}\alpha)\varphi(v)$$

for $\alpha \in \mathbb{Q}(u)$, $v \in M$.

Lemma 5.1. *Let (W, S) be a Coxeter system, and let $M = M(\Gamma)$ be the H-module afforded by the W-digraph Γ. Let σ be the automorphism of $\mathbb{Q}(u)$ determined by ${}^{\sigma}u = -1/u$. For $s \in S$, let $\tau_s \in \mathrm{gl}(M)$ be the operator $v \mapsto T_s v$. Then $T_s \mapsto \tau_s^{-1} \in \mathrm{gl}({}^{\sigma}M)$ extends to a representation $H \to \mathrm{gl}({}^{\sigma}M)$. Moreover, as a basis for the H-module ${}^{\sigma}M$, $X = \mathscr{V}(\Gamma)$ supports the W-digraph Γ_{rev}.*

Proof. Let $s \in S$. Since $(\tau_s - u^2)(\tau_s + 1) = 0$ in $\mathrm{gl}(M)$, we have $(\tau_s - u^{-2})(\tau_s + 1) = 0$ in $\mathrm{gl}({}^{\sigma}M)$, and thus $(\tau_s^{-1} - u^2)(\tau_s^{-1} + 1) = 0$ in $\mathrm{gl}({}^{\sigma}M)$. Also, if $s, t \in S$ and $1 < n(s,t) < \infty$, then

$$\overbrace{\tau_s^{-1}\tau_t^{-1}\cdots}^{n(s,t)} = (\overbrace{\cdots\tau_t\tau_s}^{n(s,t)})^{-1} = (\overbrace{\cdots\tau_s\tau_t}^{n(s,t)})^{-1} = \overbrace{\tau_t^{-1}\tau_s^{-1}\cdots}^{n(s,t)}.$$

Therefore $T_s \mapsto \tau_s^{-1}$ extends to a representation $H \to \mathrm{gl}({}^{\sigma}M)$.

Now suppose $\alpha, \beta \in X$, $s \in S$. If $\alpha \xrightarrow{s} \beta$ is an edge of Γ, then one checks that

$$\tau_s^{-1}(\alpha) = (u^2 - 1) *_\sigma \alpha + u^2 *_\sigma \beta \quad \text{and} \quad \tau_s^{-1}(\beta) = \alpha$$

in ${}^\sigma M$. On the other hand, if $\alpha \overset{s}{\dashrightarrow} \beta$ is an edge of Γ, then

$$\tau_s^{-1}(\alpha) = (u^2 - u - 1) *_\sigma \alpha + (u^2 - u) *_\sigma \beta \quad \text{and} \quad \tau_s^{-1}(\beta) = (u+1) *_\sigma \alpha + u *_\sigma \beta$$

in ${}^\sigma M$. These relations show that the basis X for ${}^\sigma M$ supports the W-digraph Γ_{rev}, so the proof is complete. ☐

For a matrix A over $\mathbb{Q}(u)$, denote by ${}^\sigma A$ the matrix obtained by applying the automorphism σ of $\mathbb{Q}(u)$ to each entry of A.

Corollary 5.2. *Suppose Γ is a W-digraph, $X = \mathscr{V}(\Gamma)$ is finite, and σ is the automorphism of $\mathbb{Q}(u)$ determined by ${}^\sigma u = -1/u$. Let ρ and ρ_{rev} be the matrix representations relative to the basis X for the actions of H on $M = M(\Gamma)$ and ${}^\sigma M$ according to the W-digraphs Γ and Γ_{rev}, respectively. Then*

$$\rho_{rev}(T_w) = {}^\sigma \rho(T_{w^{-1}}^{-1}) \tag{5.1}$$

for $w \in W$.

Proof. From the proof of Lemma 5.1, we have $\rho_{\text{rev}}(T_s) = {}^\sigma \rho(T_s^{-1})$ for $s \in S$. The assertion follows since if $w \in W$ has reduced expression $w = s_1 \cdots s_k$, then $T_w = T_{s_1} T_{s_2} \cdots T_{s_k}$ and $T_{w^{-1}}^{-1} = T_{s_1}^{-1} T_{s_2}^{-1} \cdots T_{s_k}^{-1}$. ☐

Next assume $n(s,t) < \infty$ for $s, t \in S$, Γ is an acyclic W-digraph, and $\mathscr{V}(\Gamma)$ is finite. For $\alpha \in X = \mathscr{V}(\Gamma)$, let σ_α be the source in the connected component of Γ containing α, and let $\mu(\alpha)$ be the number of edges in a directed path from σ_α to α. (Thus $\mu(\alpha)$ is well-defined by Lemma 4.3(i).) Put $\varepsilon_\alpha = (-1)^{\mu(\alpha)}$ for $\alpha \in X$, and define $X' = \{\varepsilon_\alpha \alpha \mid \alpha \in X\}$. Let ρ' be the matrix representation afforded by $M(\Gamma)$ with basis X', and let ρ_{rev} be the matrix representation corresponding to $M(\Gamma_{\text{rev}})$ with basis X.

Lemma 5.3. *If $n(s,t) < \infty$ for $s, t \in S$, Γ is an acyclic W-digraph, $\mathscr{V}(\Gamma)$ is finite, and ρ_{rev} and ρ' are defined as above, then*

$$\rho_{rev}(T_w) = \varepsilon_w u_w \rho'(T_w^{-1})^T \qquad \textit{for } w \in W. \tag{5.2}$$

Proof. Let $s \in S$. Suppose $\alpha \xrightarrow{s} \beta \in \mathscr{E}(\Gamma)$, so also $\beta \xrightarrow{s} \alpha \in \mathscr{E}(\Gamma_{\mathrm{rev}})$. Thus in $M(\Gamma_{\mathrm{rev}})$ we have

$$T_s\alpha = (u^2-1)\alpha + u^2\beta \qquad \text{and} \qquad T_s\beta = \alpha,$$

so the matrix of T_s acting on the subspace with basis $\{\alpha, \beta\}$ is

$$\begin{pmatrix} u^2-1 & 1 \\ u^2 & 0 \end{pmatrix}.$$

On the other hand, $\varepsilon_\beta = -\varepsilon_\alpha$ and $u_sT_s^{-1} = T_s - (u^2-1)$, so in $M(\Gamma)$ we have

$$\begin{aligned} \varepsilon_s u_s T_s^{-1}\varepsilon_\alpha\alpha &= -\varepsilon_\alpha\left(T_s - (u^2-1)\right)\alpha = -\varepsilon_\alpha\left(\beta - (u^2-1)\alpha\right) \\ &= (u^2-1)\varepsilon_\alpha\alpha + \varepsilon_\beta\beta \end{aligned}$$

and

$$\begin{aligned} \varepsilon_s u_s T_s^{-1}\varepsilon_\beta\beta &= -\varepsilon_\beta\left(T_s - (u^2-1)\right)\beta \\ &= -\varepsilon_\beta\left((u^2-1)\beta + u^2\alpha - (u^2-1)\beta\right) \\ &= u^2\varepsilon_\alpha\alpha, \end{aligned}$$

so the matrix of $\varepsilon_s u_s T_s^{-1}$ acting on the subspace with basis $\{\varepsilon_\alpha\alpha, \varepsilon_\beta\beta\}$ is

$$\begin{pmatrix} u^2-1 & u^2 \\ 1 & 0 \end{pmatrix}.$$

Now suppose that $\alpha \overset{s}{\dashrightarrow} \beta \in \mathscr{E}(\Gamma)$. Then in $M(\Gamma_{\mathrm{rev}})$ we have

$$T_s\alpha = (u^2-u-1)\alpha + (u^2-u)\beta$$

and

$$T_s\beta = (u+1)\alpha + u\beta,$$

so the matrix of T_s acting on the subspace with basis $\{\alpha, \beta\}$ is

$$\begin{pmatrix} u^2-u-1 & u+1 \\ u^2-u & u \end{pmatrix}.$$

In $M(\Gamma)$ we have

$$\begin{aligned}\varepsilon_s u_s T_s^{-1}\varepsilon_\alpha\alpha &= -\varepsilon_\alpha\left(T_s-(u^2-1)\right)\alpha\\ &= -\varepsilon_\alpha\left(u\alpha+(u+1)\beta-(u^2-1)\alpha\right)\\ &= (u^2-u-1)\varepsilon_\alpha\alpha+(u+1)\varepsilon_\beta\beta\end{aligned}$$

and

$$\begin{aligned}\varepsilon_s u_s T_s^{-1}\varepsilon_\beta\beta &= -\varepsilon_\beta\left(T_s-(u^2-1)\right)\beta\\ &= -\varepsilon_\beta\left((u^2-u-1)\beta+(u^2-u)\alpha-(u^2-1)\beta\right)\\ &= (u^2-u)\varepsilon_\alpha\alpha+\varepsilon_\beta\beta,\end{aligned}$$

so the matrix of $\varepsilon_s u_s T_s^{-1}$ acting on the subspace with basis $\{\varepsilon_\alpha\alpha, \varepsilon_\beta\beta\}$ is

$$\begin{pmatrix} u^2-u-1 & u^2-u\\ u+1 & u\end{pmatrix}.$$

To summarize, (5.2) holds when $w = s \in S$. The general case follows since if $w \in W$ has reduced expression $w = s_1 s_2 \cdots s_k$, then $T_w = T_{s_1}T_{s_2}\cdots T_{s_k}$. □

Proof of Theorem 1.11. Part (i) of the theorem follows by taking traces in (5.1). Since χ_Γ coincides with the character afforded by the matrix representation ρ', part (ii) of the theorem follows by taking traces in (5.2). □

6. The proof of Theorem 1.12

Proof. Suppose (W_J, J) is finite for proper subsets J of S and Γ is a finite, connected W-digraph. Suppose further that $M(\Gamma)$ is isomorphic to the module $M(\Psi)$ afforded by a W-graph Ψ for (W,S) (in the sense of [4]), and that Γ is not acyclic. For x in the set of vertices $\mathscr{V}(\Psi)$ of Ψ, let $I_x \subseteq S$ be the associated set of generators. For $\beta \in \mathscr{V}(\Gamma)$, let $\mathrm{In}(\beta)$ be the set of $s \in S$ such that Γ as an edge of the form $\alpha \xrightarrow{s} \beta$ or $\alpha \overset{s}{\dashrightarrow} \beta$ for some $\alpha \in \mathscr{V}(\Gamma)$. Let $\chi_\Gamma = \chi_\Psi$ be the character of H afforded by $M(\Gamma)$ or $M(\Psi)$. Put

$$N_\Gamma(J) = \left|\{\beta \in \mathscr{V}(\Gamma) \mid \mathrm{In}(\beta) = J\}\right|, \qquad N_\Psi(J) = \left|\{x \in \mathscr{V}(\Psi) \mid I_x = J\}\right|$$

for $J \subseteq S$. Since $M(\Gamma)_{\mathrm{ind}} \cong M(\Psi)_{\mathrm{ind}}$ is one-dimensional by Theorem 1.7(i), we must have $N_\Psi(\emptyset) > 0$. Also, since Γ is not acyclic, $M(\Gamma)_{\mathrm{sgn}} = \{0\} = M(\Psi)_{\mathrm{sgn}}$ by Theorem 1.7(ii), and therefore $N_\Psi(S) = 0$. Further, Γ has no sink by Theorem 1.5, and so also $N_\Gamma(S) = 0$. For $w \in W$, let $J(w)$ be the minimal $J \subseteq S$ such that $w \in W_J$. Then

$$\chi_\Psi(T_w)|_{u=0} = \varepsilon_w \left|\{x \in \mathscr{V}(\Psi) \mid J(w) \subseteq I_x\}\right|.$$

Since $I_x \neq S$ for $x \in \mathscr{V}(\Psi)$, we have

$$\begin{aligned} 0 < N_\Psi(\emptyset) &= \sum_{x \in \mathscr{V}(\Psi)} \sum_{w \in W_{I_x}} \varepsilon_w = \sum_{w \in W, J(w) \neq S} \varepsilon_w \left|\{x \in \mathscr{V}(\Psi) \mid J(w) \subseteq I_x\}\right| \\ &= \sum_{w \in W, J(w) \neq S} \chi_\Psi(T_w)|_{u=0}\,, \end{aligned}$$

with the sums finite by assumption. On the other hand, if $J(w) \neq S$, then $\Gamma_{J(w)}$ is acyclic by Theorem 1.5(iii), so if $\mathscr{V}(\Gamma)$ is ordered in a way consistent with directed paths in $\Gamma_{J(w)}$, then the matrix representing T_w acting on $M(\Gamma)$, when evaluated at $u = 0$, is triangular. Moreover, the nonzero diagonal entries of this matrix are all equal to ε_w, occurring in positions corresponding to those $\beta \in \mathscr{V}(\Gamma)$ such that $J(w) \subseteq \mathrm{In}(\beta)$. Since $\mathrm{In}(\beta) \neq S$ for $\beta \in \mathscr{V}(\Gamma)$ and $\chi_\Gamma = \chi_\Psi$, it follows that

$$\begin{aligned} 0 < \sum_{w \in W, J(w) \neq S} \chi_\Gamma(T_w)|_{u=0} &= \sum_{w \in W, J(w) \neq S} \varepsilon_w \left|\{\beta \in \mathscr{V}(\Gamma) \mid J(w) \subseteq \mathrm{In}(\beta)\}\right| \\ &= \sum_{\beta \in \mathscr{V}(\Gamma)} \sum_{w \in W_{\mathrm{In}(\beta)}} \varepsilon_w = N_\Gamma(\emptyset), \end{aligned}$$

and so Γ has a source. Therefore Γ is acyclic by Theorem 1.5. $\square$

7. Additional Examples

Let (W, S) be a Coxeter system, let $\gamma \mapsto \overline{\gamma}$ be the automorphism of $\mathbb{Q}(u)$ determined by $\overline{u} = u^{-1}$, and let $h \mapsto \overline{h}$ be the ring automorphism $\sum_{w \in W} \gamma_w T_w \mapsto \sum_{w \in W} \overline{\gamma_w}\, T_{w^{-1}}^{-1}$ of H. Following Lusztig [6], define a *bar operator* on an H-module M to be an additive bijection $\varphi : M \to M$ such that

$$\varphi(hv) = \overline{h}\,\varphi(v) \qquad \text{for } h \in H,\ v \in M. \tag{7.1}$$

Let Γ_* be the W-digraph associated with an involutory automorphism $w \mapsto w^*$ of (W, S), as described before Theorem 3.1. Lusztig has shown that $M(\Gamma_*)$ admits a unique bar operator that fixes the source of Γ_* ([6], Theorem 0.2). It can be shown that if (W, S) is finite, then any H-module admits a bar operator. However, there need not be a bar operator if (W, S) is infinite, as the next example shows.

Example 7.1. With $W = W(\widetilde{A}_2) = \langle r, s, t\rangle$, let Γ be as in Figure 7.1, and put $w = tsr$. Suppose a bar operator φ exists on $M(\Gamma)$. Let α be the vertex in the lower left corner of Figure 7.1, so $T_{tsr}\alpha = T_t T_s T_r \alpha = \alpha$. Then $\overline{T_{tsr}}\varphi(\alpha) = \varphi(\alpha)$, so $\varphi(\alpha) = T_{rst}\varphi(\alpha)$ is a fixed point of T_{rst}. However, one checks that the characteristic polynomial of T_{rst} acting on $M(\Gamma)$ is $(\lambda^2 + 1)(\lambda^2 - u^6)(\lambda - u^6)^2$, so a contradiction is obtained. Also, Γ provides an

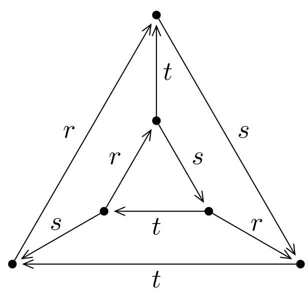

Figure 7.1 W-digraph for Example 7.1

example in which the equation of Theorem 1.11(ii) fails: with $y = w^{-1} = rst$, one checks that $\chi_{\Gamma_{\rm rev}}(T_y) = {}^{\sigma}\chi_\Gamma(T_{y^{-1}}^{-1}) = 2$ and $\varepsilon_y u_y \chi_\Gamma\left(T_y^{-1}\right) = -2$. Moreover, $M(\Gamma)$ does not afford a W-graph by Theorem 1.12.

Even if (W, S) is finite and Γ is connected, there may not exist a bar operator on $M(\Gamma)$ that fixes the source of Γ, as the next example shows.

Example 7.2. Let $W = W(B_3) = \langle r, s, t\rangle$, with $n(r, s) = 3$, $n(r, t) = 2$, $n(s, t) = 4$. Let Γ be the W-digraph of Figure 7.2. Suppose $M = M(\Gamma)$ admits a source-fixing bar operator $\varphi : M \to M$. Since v_0 is the source of Γ and $v_4 = T_r T_s v_0$, we have

$$\begin{aligned}\varphi(v_4) &= \overline{T_r}\,\overline{T_s}\,v_0 = u^{-4}(T_r - (u^2-1))(T_s - (u^2-1))v_0 \\ &= u^{-4}\left(v_4 - (u^2-1)v_2 - (u^2-1)v_8 + (u^2-1)^2 v_0\right).\end{aligned} \tag{7.2}$$

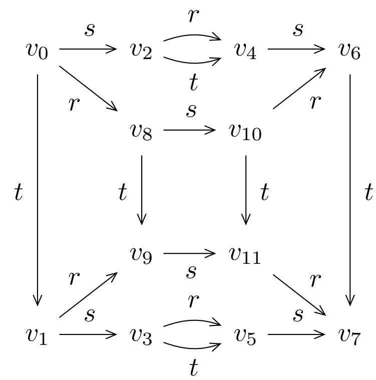

Figure 7.2 Digraph for Example 7.2

On the other hand, $v_4 = T_t T_s v_0$, so we also have

$$\begin{aligned}\varphi(v_4) &= \overline{T_t}\,\overline{T_s}\,v_0 = u^{-4}(T_t - (u^2-1))(T_s - (u^2-1))v_0 \\ &= u^{-4}\left(v_4 - (u^2-1)v_2 - (u^2-1)v_1 + (u^2-1)^2 v_0\right).\end{aligned} \tag{7.3}$$

Since (7.2) and (7.3) cannot simultaneously hold, a contradiction is reached. Thus M does not admit a source-fixing bar operator.

Let (W,S) be finite, and let Γ be a finite W-digraph. By Theorem 1.11 we have $\chi_\Gamma|_{u=-1} = \mathrm{sgn}_W \cdot \chi_\Gamma|_{u=1}$. Thus if (W,S) has no connected components with exceptional characters in the sense of Gyoja [3], then $\chi_\Gamma|_{u=1} = \chi_\Gamma|_{u=-1}$ is self-associated, that is, $\chi_\Gamma|_{u=1} = \mathrm{sgn}_W \cdot \chi_\Gamma|_{u=1}$. In particular, if (W,S) has no component of type H_3, H_4, E_7, or E_8, then $\chi_\Gamma|_{u=1}$ is self-associated. Our final example shows that if (W,S) has an exceptional character, then $\chi_\Gamma|_{u=1}$ need not be self-associated.

Example 7.3. Let $W = W(H_3) = \langle r,s,t\rangle$ with $n(r,s) = 3$, $n(s,t) = 5$, $n(r,t) = 2$. The W-digraph Γ of Figure 7.3 affords the non-self-associated character $\chi_\Gamma|_{u=1} = 1_W + \mathrm{sgn}_W + \chi_{4'}$, where $\chi_{4'}$ is the irreducible character of degree 4 with value -4 at the longest element of W. Then $\chi_{\Gamma_{\mathrm{rev}}}|_{u=1} = \mathrm{sgn}_W \cdot \chi_\Gamma|_{u=1} \neq \chi_\Gamma|_{u=1}$.

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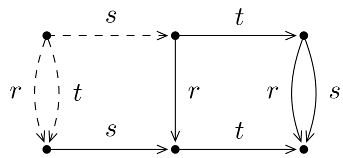

Figure 7.3 Digraph for Example 7.3

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