BV SOLUTIONS FOR HYPERBOLIC BALANCE LAWS WITH PERIODIC INITIAL DATA

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For Tai-Ping Liu, with admiration and affection, on the occasion of his 70th birthday

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Abstract

We construct spatially periodic solutions for systems of balance laws with partially dissipative source and show exponential decay as time goes to infinity.

1. Introduction

The root of the difficulties encountered in the analysis of nonlinear strictly hyperbolic systems

$$\partial_t U(x,t) + \partial_x F(U(x,t)) = 0 \tag{1.1}$$

of conservation laws, in one space dimension, lies in their distinctive feature of wave breaking: Even when the initial values

$$U(x,0) = U_0(x), \qquad -\infty < x < \infty, \tag{1.2}$$

are smooth, solutions to the Cauchy problem develop spontaneously singularities that propagate on as shock waves. Thus, one must resort to weak solutions. The state of the art is that when the total variation of U_0 is sufficiently small, there exists a unique admissible weak solution U of class BV to (1.1), (1.2), defined on the upper half-plane $(-\infty, \infty) \times [0, \infty)$.

BV solutions to the Cauchy problem were first constructed by Glimm [9], for systems that are genuinely nonlinear, via the celebrated random choice method. This approach was further developed by many authors, and most notably by Tai-Ping Liu (see [12, 13, 14] and the references therein), who among other things, replaced randomness with certainty, removed the restriction of genuine nonlinearity and established the long time behavior of solutions. These solutions can also be constructed by the front tracking method developed by the Italian School, headed by Bressan [3], and through the vanishing viscosity approach of Bianchini and Bressan [2]. In fact, the last approach provides the most general result. On the other hand, the random choice method exhibits remarkable flexibility and robustness for handling source terms and inhomogeneity in the system, boundary conditions etc.

The restriction that the total variation of U_0 must be small is essential, as there are cases of systems of three conservation laws [11] in which specially constructed initial data with large variation generate solutions with variation that explodes in finite time. Apparently, this pathology is widespread. In particular, one may not generally expect global existence of BV solutions with periodic initial values for systems of three or more conservation laws (in that respect, scalar conservation laws and genuinely nonlinear systems of two conservation laws are exceptional [10]).

The aim of this paper is to establish the existence and long time behavior of spatially periodic BV solutions to the Cauchy problem for systems of balance laws

$$\partial_t U(x,t) + \partial_x F(U(x,t)) + G(U(x,t)) = 0, \qquad (1.3)$$

under the following hypotheses:

(a) The *state vector* U takes values in a ball $\mathcal{B}_\rho$ in $\mathbb{R}^n$, centered at the origin, with radius ρ.

(b) The *flux* F is a given smooth function from $\mathcal{B}_\rho$ to $\mathbb{R}^n$. For any $U \in \mathcal{B}_\rho$, the Jacobian matrix DF(U) possesses real eigenvalues $\lambda_1(U) < \cdots < \lambda_n(U)$ and thereby linearly independent left (row) eigenvectors $L_1(U), \ldots, L_n(U)$ and right (column) eigenvectors $R_1(U), \ldots, R_n(U)$, normalized by $L_i(U)R_j(U) = \delta_{ij}$ (the Kronecker delta).

(c) The *source* G is also a given smooth function from $\mathcal{B}_\rho$ to $\mathbb{R}^n$, and it vanishes at the origin, $G(0) = 0$.

(d) Source and flux interact through the *Kawashima condition*

$$\mathrm{D}G(0)R_i(0) \neq 0, \qquad i = 1, \ldots, n. \qquad (1.4)$$

(e) The system is endowed with an entropy-entropy flux pair (η, q), normalized at the origin by $\eta(0) = 0$, $\mathrm{D}\eta(0) = 0$, and such that $\mathrm{D}^2\eta(U)$ is positive definite for $U \in \mathcal{B}_\rho$. Furthermore, the *entropy production* is positive semidefinite, in the sense

$$\mathrm{D}\eta(U)G(U) \geq \gamma|G(U)|^2, \qquad (1.5)$$

for $U \in \mathcal{B}_\rho$ and some positive constant γ.

Thus (1.3) is a strictly hyperbolic system of balance laws, admitting $U \equiv 0$ as an equilibrium solution. Admissible weak solutions must satisfy the entropy condition

$$\partial_t \eta(U(x,t)) + \partial_x q(U(x,t)) + \mathrm{D}\eta(U(x,t))G(U(x,t)) \leq 0. \qquad (1.6)$$

In view of (1.5) and (1.6), the source is dissipative, so it should be expected that the demise of spatially periodic *BV* solutions may not take place in its presence. Indeed, it is natural to conjecture that dissipation will drive solutions to equilibrium, as time tends to infinity.

Situations, allowed by (1.5), in which the damping effect of the source may become inactive at points where G vanishes will be averted by the synergy between flux and source, manifested in the Kawashima condition (1.4). Notice that (1.4) simply states that the system

$$\partial_t Z + \mathrm{D}F(0)\partial_x Z + \mathrm{D}G(0)Z = 0, \qquad (1.7)$$

resulting from linearizing (1.3) about the origin, does not admit solutions representing undamped propagating fronts, in the form $Z(x,t) = u(x - \lambda_i(0)t)R_i(0)$.

The main result of this paper is the following

Theorem 1.1. *Consider the Cauchy problem* (1.3), (1.2), *for a system satisfying the assumptions* (a)−(e), *listed above, and initial data U_0 of locally bounded variation that are periodic:*

$$U_0(x - 1) = U_0(x + 1), \qquad -\infty < x < \infty. \qquad (1.8)$$

There are positive constants σ_0 and δ_0, $\sigma_0 < \rho$, such that if

$$\sup_{[-1,1)} |U_0(\cdot)| = \sigma, \qquad (1.9)$$

$$TV_{[-1,1)}U_0(\cdot) = \delta, \qquad (1.10)$$

with $\sigma < \sigma_0$ and $\delta < \delta_0$, then there exists an admissible BV solution U on $(-\infty, \infty) \times [0, \infty)$. Furthermore,

$$\int_{-1}^{1} |U(x,t) - U_\infty|dx \leq a\sigma e^{-\nu t}, \qquad 0 \leq t < \infty, \qquad (1.11)$$

$$TV_{[-1,1)}U(\cdot, t) \leq (b\sigma + \beta\delta)e^{-\mu t}, \qquad 0 \leq t < \infty, \qquad (1.12)$$

for postive constants a, b, β, μ, ν, independent of U_0, and some equilibrium state $U_\infty \in \mathcal{B}_\rho$, with $G(U_\infty) = 0$.

The proof of the above proposition will occupy the remainder of the paper. In Section 2, starting out from the premise that an admissible BV weak solution exists, we establish the estimate (1.11). Section 3 outlines the construction of the BV solution and demonstrates (1.12).

Systems governing relaxation phenomena typically result from the coupling of conservation laws with balance laws and accordingly appear in the form

$$\begin{cases} \partial_t V + \partial_x P(V, W) = 0 \\ \\ \partial_t W + \partial_x Q(V, W) + C(V, W)W = 0, \end{cases} \qquad (1.13)$$

which fits naturally in the scheme presented above. In fact, a version of Theorem 1.1 for systems (1.13), under the additional constraint that the spatial mean of V vanishes, is found in [7].

2. Decay of Solutions in L^1

The aim here is to demonstrate how (1.5) in conjunction with the Kawashima condition (1.4) induce the exponential decay of admissible weak solutions, in $L^1(-1, 1)$.

The first step is to rewrite the system (1.3) in a form that identifies and exposes the component of the state vector that is directly affected by the damping action of the source.

Casting aside the case of a nonsingular $DG(0)$, which is relatively easy, let us assume that its null space has dimension $k, 1 \leq k < n$. Thus $DG(0) = S^{-1}JS$, where S is a nonsingular $n \times n$ matrix while J is a $n \times n$ matrix in the form

$$J = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}, \tag{2.1}$$

where C is a nonsingular $\ell \times \ell$ matrix, $\ell = n - k$. Passing from U to the new state vector $\hat{U} = SU$, with projections V and W on $\mathbb{R}^k$ and $\mathbb{R}^\ell$, reduces (1.3) to a system in the form

$$\begin{cases} \partial_t V + \partial_x P(V, W) + X(V, W) = 0 \\ \partial_t W + \partial_x Q(V, W) + CW + Y(V, W) = 0, \end{cases} \tag{2.2}$$

where X and Y are functions that vanish, along with their first derivatives, at the origin.

In order to avoid cumbersome notation, we shall establish the desired estimate (1.11) under certain simplifying assumptions. To begin with, we assume that (1.3) is already in the form (2.2), i.e., $S = I$, so $U = (V, W)$. Next, we note that the presence of the convex entropy η implies that the system is symmetrizable. Let us then assume that actually the system is symmetric. Thus, upon using subscripts V and W to denote derivatives with respect to V and W, we have $P_V^\top = P_V$, $Q_W^\top = Q_W$, $P_W^\top = Q_V$. Moreover, the entropy is $\eta = |V|^2 + |W|^2$. Finally, in order to secure that (1.5) holds, we assume that $X_V(V, W) = 0$ and $Y(0, W) = 0$, in which case the system assumes the form

$$\begin{cases} \partial_t V + \partial_x P(V, W) + X(W) = 0 \\ \partial_t W + \partial_x Q(V, W) + C(V, W)W = 0, \end{cases} \tag{2.3}$$

with $X(0) = 0$, $X_W(0) = 0$ and the $\ell \times \ell$ matrix $C(0,0)$ is positive definite.

Let us now assume that $U = (V, W)$ is a BV weak solution of (2.3) on a strip $(-\infty, \infty) \times [0, T]$, with initial conditions $U_0 = (V_0, W_0)$, satisfying (1.8) and (1.9), which is admissible in that the entropy condition (1.6) holds.

We introduce the following notation: For $t \in [0, T]$,

$$\bar{V}(t) = \frac{1}{2} \int_{-1}^{1} V(x, t) dx, \qquad \bar{W}(t) = \frac{1}{2} \int_{-1}^{1} W(x, t) dx, \qquad (2.4)$$

$$\bar{P}(t) = \frac{1}{2} \int_{-1}^{1} P(V(x, t), W(x, t)) dx, \qquad (2.5)$$

$$\bar{X}(t) = \frac{1}{2} \int_{-1}^{1} X(W(x, t)) dx, \qquad (2.6)$$

noting that

$$\partial_t \bar{V}(t) = -\bar{X}(t). \qquad (2.7)$$

In what follows, c stands for a generic positive constant that is independent of σ and T. The aim is to show

$$\int_{-1}^{1} [|V(x, t) - \bar{V}(t)|^2 + |W(x, t)|^2] dx \leq c\sigma^2 e^{-2\nu t}, \qquad (2.8)$$

with a positive constant ν, independent of T.

This will be achieved by constructing two functionals, $\mathcal{J}\{V, W\}$ and $\mathcal{H}\{V, W\}$, on $L^2((-1, 1); \mathbb{R}^k) \times L^2((-1, 1); \mathbb{R}^\ell)$, with the following properties:

$$\mathcal{H}\{V, W\} \geq 2\nu \mathcal{J}\{V, W\} \geq c \int_{-1}^{1} [|V(x) - \bar{V}|^2 + |W(x)|^2] dx, \qquad (2.9)$$

$$\frac{d}{dt} \mathcal{J}\{V(\cdot, t), W(\cdot, t)\} + \mathcal{H}\{V(\cdot, t), W(\cdot, t)\} \leq 0. \qquad (2.10)$$

The starting point is the inequality

$$\frac{d}{dt} \int_{-1}^{1} [|V - \bar{V}|^2 + |W|^2] dx$$

$$\leq -2 \int_{-1}^{1} [V^\top X(W) - \bar{V}^\top \bar{X} + W^\top C(V, W) W] dx, \qquad (2.11)$$

easily verified by integrating (1.6), with respect to x, over $(-1, 1)$ and using (2.7). Notice that (2.11) is in the form (2.10) but it fulfills the requirement (2.9) only partially, as a refection of the fact that the source is merely partially dissipative. We thus need a supplementary estimate, reflecting the synergy between source and flux, manifested in the Kawashima condition (1.4). To that end, we follow a procedure, which originated in [15] and has since been applied to a number of related situations, e.g. [6, 7].

The first remark is that, when the system (1.3) is in the form (2.2), $Q_V(0,0)K \neq 0$, for any eigenvector K of the $k \times k$ matrix $P_V(0,0)$, since otherwise $R = \binom{K}{0}$ would be an eigenvector of $DF(0)$ that violates (1.4). As shown in [15], this in turn implies that there exists a skew-symmetric $k \times k$ matrix Ω such that $\Omega P_V(0,0)$ is positive definite on the null space of $Q_V(0,0)$.

We now introduce the "potential" function of V:

$$\Phi(x,t) = \int_0^x [V(y,t) - \bar{V}(t)]dy - \frac{1}{2}\int_0^1 \int_0^z V(y,t)dydz, \qquad (2.12)$$

which has been normalized so that

$$\Phi(-1,t) = \Phi(1,t), \qquad (2.13)$$

$$\int_{-1}^1 \Phi(x,t)dx = 0, \qquad (2.14)$$

$$\partial_x \Phi(x,t) = V(x,t) - \bar{V}(t), \qquad (2.15)$$

$$\partial_t \Phi(x,t) = -P(V(x,t), W(x,t)) + \bar{P}(t) + \mathcal{A}\{X(\cdot,t)\}, \qquad (2.16)$$

where

$$\mathcal{A}\{X(\cdot,t)\} = -\int_0^x [X(W(y,t)) - \bar{X}(t)]dy + \frac{1}{2}\int_{-1}^1 \int_0^z X(W(y,t))dydz. (2.17)$$

In what follows, we use the symbols $\hat{Q}, \hat{P}_V, \hat{P}_W, \hat{Q}_V, \hat{Q}_W$ and $\hat{C}$ to denote the matrix-valued functions of t resulting from evaluating $Q(V,W)$, $P_V(V,W), P_W(V,W), Q_V(V,W), Q_W(V,W)$ and $C(V,W)$ at $(\bar{V}(t), 0)$. Upon using (2.12), (2.15), (2.16) and the system (2.3), one easily verifies the following balance laws:

$$\partial_t(\Phi^\top \Phi) + \partial_x(\Phi^\top \hat{P}_V \Phi) = -2\Phi^\top(P - \bar{P}) + 2\Phi^\top \hat{P}_V(V - \bar{V}) + 2\Phi^\top \mathcal{A}, \quad (2.18)$$

$$\partial_t[-2\Phi^\top \hat{P}_W \hat{C}^{-1}(W - \bar{W})] + \partial_x[-2\Phi^\top \hat{P}_W \hat{C}^{-1}(Q - \hat{Q})]$$

$$= 2(P - \bar{P})^\top \hat{P}_W \hat{C}^{-1}(W - \bar{W}) - 2(V - \bar{V})^\top \hat{P}_W \hat{C}^{-1}(Q - \hat{Q})$$

$$-2\mathcal{A}^\top \hat{P}_W \hat{C}^{-1}(W - \bar{W}) + 2\Phi^\top \hat{P}_W \hat{C}^{-1}\partial_t\bar{W}$$

$$+2\Phi^\top \hat{P}_W \hat{C}^{-1}CW - 2\Phi^\top \partial_t(\hat{P}_W \hat{C}^{-1})(W - \bar{W}), \tag{2.19}$$

$$\partial_t[\Phi^\top \Omega(V - \bar{V})] + \partial_x[\Phi^\top \Omega(P - \bar{P})]$$

$$= 2(V - \bar{V})^\top \Omega(P - \bar{P}) + \mathcal{A}^\top \Omega(V - \bar{V}) - \Phi^\top \Omega X. \tag{2.20}$$

We add (2.18) and (2.19) and integrate the resulting equation, with respect to x, over $(-1, 1)$. Next, we multiply (2.20) by some positive number r and integrate, with respect to x, over $(-1, 1)$. Finally, we multiply (2.11) by a positive number ω. Upon adding together the three integrals, we arrive at an inequality in the form (2.10), with

$$\mathcal{J} = \int_{-1}^{1}[\omega|V - \bar{V}|^2 + \omega|W|^2 + \Phi^\top \Phi - 2\Phi^\top \hat{P}_W \hat{C}^{-1}(W - \bar{W}) + r\Phi^\top \Omega(V - \bar{V})]dx. \tag{2.21}$$

Upon expanding the functions $Q - \hat{Q}$ and $P - \bar{P}$ about the state $(\bar{V}, 0)$ and recalling (2.7) and (2.14), we can write $\mathcal{H}$ associated with the above $\mathcal{J}$ as

$$\mathcal{H} = \int_{-1}^{1}\{\omega W^\top CW + 2(V - \bar{V})^\top \hat{Q}_V^\top \hat{C}^{-1}\hat{Q}_V(V - \bar{V})$$

$$-2(W - \bar{W})^\top \hat{P}_W^\top \hat{C}^{-1}(W - \bar{W})$$

$$+2(V - \bar{V})^\top[\hat{P}_W \hat{C}^{-1}\hat{Q}_W - \hat{P}_V \hat{P}_W \hat{C}^{-1}](W - \bar{W})$$

$$+2r(V - \bar{V})^\top \Omega\hat{P}_V(V - \bar{V}) + 2r(V - \bar{V})^\top \Omega\hat{P}_W(W - \bar{W})$$

$$+O(\rho)[|V - \bar{V}|^2 + |W|^2]\}dx. \tag{2.22}$$

When ρ is sufficiently small, the matrix $\Omega\hat{P}_V$ is positive definite on the null space of $\hat{Q}_V$. On the other hand, the positive semidefinite matrix $\hat{Q}_V^\top \hat{C}^{-1}\hat{Q}_V$ is positive definite on the complementary space. Hence, for r sufficiently small, the matrix $\hat{Q}_V^\top \hat{C}^{-1}\hat{Q}_V + r\Omega\hat{P}_V$ is positive definite on $\mathbb{R}^k$. It follows that, for ρ and r small and ω large, $\mathcal{J}$ and $\mathcal{H}$ satisfy the conditions (2.9), which establishes (2.8).

From (2.8) and Schwarz's inequality,

$$\int_{-1}^{1}[|V(x,t) - \bar{V}(t)| + |W(x,t)|]dx \leq c\sigma e^{-\nu t}. \qquad (2.23)$$

When the solution is global, $T = \infty$, it follows from (2.7) and (2.8) that

$$|\bar{V}(t) - V_{\infty}| \leq c\sigma^2 e^{-2\nu t}, \qquad 0 \leq t < \infty, \qquad (2.24)$$

for some $V_{\infty} \in \mathbb{R}^k$. In that case, combining (2.23) with (2.24), we arrive at (1.11), with $U_{\infty} = (V_{\infty}, 0)$.

3. *BV* Solutions

We now return to the original, more compact, form (1.3) of our system and construct *BV* solutions with initial data U_0 satisfying (1.8), (1.9) and (1.10), for σ and δ sufficiently small.

It is a straightforward, though tedious, exercise to construct a local admissible *BV* solution U to the Cauchy problem (1.3), (1.2), on some strip $(-\infty, \infty) \times [0, T]$, by combining the random choice method, for dealing with the flux, with an operator splitting scheme, to account for the effect of the source (see [8]). Because of the local dependence property of hyperbolic systems, the random choice method, which is traditionally employed for obtaining global solutions under initial data of small total variation over $(-\infty, \infty)$, is equally effective for constructing local solutions with initial data of small local variation. Indeed, as shown in [8], for systems in the form (1.3), the local variation of the solution at time $t \in [0, T]$ is bounded by

$$TV_{[-1,1)}U(\cdot, t) \leq ce^{-\kappa t}TV_{[-1-\lambda t, 1+\lambda t]}U_0(\cdot), \qquad (3.1)$$

where λ is an upper bound of the characteristic speeds $|\lambda_i(U)|$. The exponent κ marks the effect of the source.

In the absence of a source term, (3.1) holds with $\kappa = 0$, but this estimate eventually breaks down, as the variation of the initial data over the domain $[-1 - \lambda t, 1 + \lambda t]$ of dependence grows linearly with time. The situation is aggravated in the presence of a source for which κ in (3.1) is negative. On the contrary, in the case of a source associated with a positive κ, the exponential

decay of $e^{-\kappa t}$ offsets the linear growth of the variation of the initial values over the domain of dependence and thus the local variation of the solution stays small, uniformly in time. This would guarantee the existence and exponential decay of global BV solutions to the Cauchy problem (1.3), (1.2) when U_0 satisfies the periodicity condition (1.8) and its variation δ per period is sufficiently small.

The sign of κ in (3.1) is tied to properties of the matrices

$$A(U) = L(U)\mathrm{D}G(U)R(U), \qquad (3.2)$$

where $L(U)$ is the $n \times n$ matrix with rows the left eigenvectors $L_1(U), \ldots,$ $L_n(U)$ of $\mathrm{D}F(U)$ and $R(U)$ is the $n \times n$ matrix with columns the right eigenvectors $R_1(U), \ldots, R_n(U)$ of $\mathrm{D}F(U)$. (Recall the normalization $L(U)R(U) = I$.) Indeed, as shown in [1, 4, 8], when $A(0)$ is diagonally dominant,

$$A_{ii}(0) > \sum_{j \neq i} |A_{ji}(0)|, \qquad i = 1, \ldots, n, \qquad (3.3)$$

then (3.1) holds with $\kappa > 0$.

Unfortunately, the diagonal dominance condition generally fails in the class of systems (1.3) considered in this paper. Indeed, as we saw in Section 2, our systems may be reduced to the form (2.2), indicating that the damping action of the source does not affect directly all of the equations, as (3.3) would require. We shall surmount this obstacle by performing a transformation in the state vector that redistributes the damping more equitably among the equations of the system. This approach has been tried successfully in [5, 6, 7].

The first step is to note that, even though (3.3) may fail, our assumptions (1.4) and (1.5) at least guarantee the weaker condition

$$A_{ii}(0) > 0, \qquad i = 1, \ldots, n. \qquad (3.4)$$

Indeed, by (1.5) the function

$$\theta(U) = \mathrm{D}\eta(U)G(U) - \gamma|G(U)|^2 \qquad (3.5)$$

is minimized at $U = 0$, and hence the Hessian matrix

$$\mathrm{D}^2\theta(0) = \mathrm{D}^2\eta(0)\mathrm{D}G(0) + \mathrm{D}G(0)^\top\mathrm{D}^2\eta(0) - 2\gamma\mathrm{D}G(0)^\top\mathrm{D}G(0) \qquad (3.6)$$

is positive semidefinite. The entropy satisfies the well-known symmetry condition

$$\mathrm{D}^2\eta(U)\mathrm{D}F(U) = \mathrm{D}F(U)^\top \mathrm{D}^2\eta(U). \tag{3.7}$$

Multiplying (3.7), from the left, by $R_i(U)^\top$ yields

$$R_i(U)^\top \mathrm{D}^2\eta(U)\mathrm{D}F(U) = \lambda_i(U)R_i(U)^\top \mathrm{D}^2\eta(U), \tag{3.8}$$

which shows that $R_i^\top \mathrm{D}^2\eta$ is collinear to L_i, and specifically

$$R_i(U)^\top \mathrm{D}^2\eta(U) = [R_i(U)^\top \mathrm{D}^2\eta(U)R_i(U)]L_i(U). \tag{3.9}$$

We now multiply (3.6), from the left by $R_i(0)^\top$, and from the right by $R_i(0)$. Using (3.8), for $U = 0$, and since $\mathrm{D}^2\theta(0)$ is positive semidefinite, we deduce

$$[R_i(0)^\top \mathrm{D}^2\eta(0)R_i(0)]A_{ii}(0) \geq \gamma |\mathrm{D}G(0)R_i(0)|^2, \tag{3.10}$$

whence (3.4) follows by the Kawashima condition (1.4).

We now return to the solution U of (1.3), (1.2), on the strip $(-\infty, \infty) \times [0, T]$. For $t \in [0, T]$, we set

$$\bar{U}(t) = \frac{1}{2}\int_{-1}^{1} U(x, t)dx, \qquad \bar{G}(t) = \frac{1}{2}\int_{-1}^{1} G(U(x, t))dx, \tag{3.11}$$

noting that

$$\partial_t \bar{U}(t) + \bar{G}(t) = 0. \tag{3.12}$$

Next we introduce the functions

$$\Psi(x, t) = \int_0^x [U(y, t) - \bar{U}(t)]dy - \frac{1}{2}\int_{-1}^{1}\int_0^z [U(y, t) - \bar{U}(t)]dydz, \tag{3.13}$$

$$\Gamma(x, t) = \int_0^x [G(U(y, t)) - \bar{G}(t)]dy. \tag{3.14}$$

After a short calculation,

$$\partial_x \Psi(x, t) = U(x, t) - \bar{U}(t), \tag{3.15}$$

$$\partial_t \Psi(x,t) = -F(U(x,t)) + \bar{F}(t) - \Gamma(x,t) + \bar{\Gamma}(t), \qquad (3.16)$$

where

$$\bar{F}(t) = \frac{1}{2} \int_{-1}^{1} F(U(x,t))dx, \qquad \bar{\Gamma}(t) = \frac{1}{2} \int_{-1}^{1} \Gamma(x,t)dx. \qquad (3.17)$$

We replace U by the new state vector $\hat{U}$, defined by

$$\hat{U}(x,t) = U(x,t) - \bar{U}(t) - N\Psi(x,t), \qquad (3.18)$$

where N is a $n \times n$ matrix to be specified below. Upon setting

$$\Theta(x,t) = \bar{U}(t) + N\Psi(x,t) \qquad (3.19)$$

and eliminating U, between (1.3) and (3.18), we arrive at

$$\partial_t \hat{U}(x,t) + \partial_x \hat{F}(\hat{U}(x,t),x,t) + \hat{G}(\hat{U}(x,t),x,t) = 0, \qquad (3.20)$$

with

$$\hat{F}(\hat{U},x,t) \;=\; F(\hat{U} + \Theta(x,t)) - F(\Theta(x,t)), \qquad (3.21)$$
$$\hat{G}(\hat{F},x,t) \;=\; G(\hat{U} + \Theta(x,t)) - \bar{G}(t) - N[F(\hat{U} + \Theta(x,t)) - \bar{F}(t)]$$
$$+\mathrm{D}F(\Theta(x,t))N[\hat{U} + \Theta(x,t) - \bar{U}(t)] - N[\Gamma(x,t) - \bar{\Gamma}(t)].(3.22)$$

We regard $\Theta, \Gamma, \bar{U}, \bar{F}, \bar{G}$ and $\bar{\Gamma}$ as known functions, and view (3.20) as an inhomogeneous system of balance laws for the state vector $\hat{U}$. A comparison between the matrices $A(U)$, defined by (3.2) and associated with (1.3), and

$$\hat{A}(\hat{U},x,t) = \hat{L}(\hat{U},x,t)\mathrm{D}\hat{G}(\hat{U},x,t)\hat{R}(U,x,t), \qquad (3.23)$$

associated with (3.20), reveals the advantage of dealing with this new system in the place of the original one. Indeed, after a short calculation,

$$\hat{A}_{ij}(0,x,t) = A_{ij}(\Theta(x,t)) + [\lambda_i(\Theta(x,t)) - \lambda_j(\Theta(x,t))]\Delta_{ij}(\Theta(x,t)), \quad (3.24)$$

where

$$\Delta(\Theta) = L(\Theta)NR(\Theta). \qquad (3.25)$$

So long as U stays confined in the ball $\mathcal{B}_\rho$, with ρ sufficiently small, it is easy to make $\hat{A}(0, x, t)$ diagonally dominant, with the help of (3.24) and (3.4), by a judicious selection of the matrix N. For example, one may choose $N = R(0)\hat{\Delta}L(0)$, where $\hat{\Delta}$ is the $n \times n$ matrix with entries $\hat{\Delta}_{ii} = 0$, for $i = 1, \ldots, n$, and

$$\hat{\Delta}_{ij} = \frac{A_{ij}(0)}{\lambda_i(0) - \lambda_j(0)}, \qquad \text{for } i \neq j, \qquad (3.26)$$

which renders an $\hat{A}(0, x, t)$ with principal diagonal entries $\hat{A}_{ii}(0, x, t)$ close to $A_{ii}(0)$ and off diagonal entries $\hat{A}_{ij}(0, x, t), i \neq j$, close to zero.

After diagonal dominance of $\hat{A}(0, x, t)$ has been secured, one may bound the variation of the solution to (3.23) by applying estimates derived in [8], which yield

$$
\begin{aligned}
TV_{[-1,1)}\hat{U}(\cdot, t) \leq\ & ce^{-\kappa t} TV_{[-1-\lambda t, 1+\lambda t]}\hat{U}_0(\cdot) \\
& + c \int_0^t e^{-\kappa(t-\tau)} TV_{[-1-\lambda(t-\tau), 1+\lambda(t-\tau)]}\Theta(\cdot, \tau) d\tau \\
& + c \int_0^t e^{-\kappa(t-\tau)} TV_{[-1-\lambda(t-\tau), 1+\lambda(t-\tau)]}\Gamma(\cdot, \tau) d\tau. \ (3.27)
\end{aligned}
$$

When U_0 satisfies (1.8), (1.9) and (1.10),

$$TV_{[-1-\lambda t, 1+\lambda t]}\hat{U}_0(\cdot) \leq (\delta + c\sigma)(2\lambda t + 4). \qquad (3.28)$$

Furthermore, by (2.23),

$$\int_{-1}^1 |U(x, t) - \bar{U}(t)| dx \leq c\sigma e^{-\nu t}, \qquad 0 \leq t \leq T, \qquad (3.29)$$

and this in turn implies

$$
\begin{aligned}
TV_{[-1-\lambda(t-\tau), 1+\lambda(t-\tau)]}\Theta(\cdot, \tau) &\leq c\sigma[2\lambda(t-\tau) + 4]e^{-\nu \tau}, &\qquad (3.30) \\
TV_{[-1-\lambda(t-\tau), 1+\lambda t(t-\tau)]}\Gamma(\cdot, \tau) &\leq c\sigma[2\lambda(t-\tau) + 4]e^{-\nu \tau}. &\qquad (3.31)
\end{aligned}
$$

Upon combining (3.27), (3.28), (3.30) and (3.31), we arrive at the estimate (1.12), with positive constants b, β and μ independent of T. Thus, if σ and δ are sufficiently small, we may extend U into a global solution, defined on

the entire upper half-plane $(-\infty, \infty) \times [0, \infty)$ and satisfying the estimates (1.11) and (1.12).

References

1. D. Amadori and G. Guerra, Uniqueness and continuous dependence for systems of balance laws with dissipation, *Nonlinear Anal.*, **49** (2002), 987-1014.

2. S. Bianchini and A. Bressan, Vanishing viscosity solutions of nonlinear hyperbolic systems, *Ann. of Math.*, **161** (2005), 223-342.

3. A. Bressan, *Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem,* Oxford University Press, Oxford 2000.

4. C. C. Christoforou, Hyperbolic systems of balance laws via vanishing viscosity, *J. Differential Equations*, **221** (2006), 470-541.

5. C. M. Dafermos, Hyperbolic systems of balance laws with weak dissipation, *J. Hyperbolic Differ. Equ.*, **3** (2006), 507-527.

6. C. M. Dafermos, *BV* solutions for hyperbolic systems of balance laws, with relaxation, *J. Differential Equations*, **255** (2013), 2521-2533.

7. C. M. Dafermos, Periodic *BV* solutions of hyperbolic balance laws with dissipative source, *J. Math. Anal. Appl.*, **428** (2015), 405-413.

8. C. M. Dafermos and L. Hsiao, Hyperbolic systems of balance laws with inhomogeneity and dissipation, *Indiana Univ. Math. J.*, **31** (1982), 471-491.

9. J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.*, **18** (1965), 697-715.

10. J. Glimm and P. D. Lax, Decay of solutions of systems of nonlinear hyperbolic conservation laws, *Mem. Amer. Math. Soc.*, **101** (1970).

11. H. K. Jenssen, Blowup for systems of conservation laws, *SIAM J. Math. Anal.*, **31** (2000), 894-908.

12. T.-P. Liu, The deterministic version of the Glimm scheme, *Comm. Math. Phys.*, **57** (1977), 135-148.

13. T.-P. Liu, Quasilinear hyperbolic systems, *Comm. Math. Phys.*, **68** (1979), 141-172.

14. T.-P. Liu, Admissible solutions of hyperbolic conservation laws, *Mem. Amer. Math. Soc.*, **30** (1981).

15. T. Ruggeri and D. Serre, Stability of constant equilibrium state for dissipative balance laws systems with a convex entropy, *Quart. Appl. Math.*, **62** (2004), 163-179.