

# THE MATHEMATICAL THEORY OF SELF-SIMILAR BOUNDARY LAYERS FOR NONLINEAR HYPERBOLIC SYSTEMS WITH VISCOSITY AND CAPILLARITY

*Dedicated to Tai-Ping Liu on the Occasion of his 70th Birthday*

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## Abstract

We study the vanishing viscosity–capillarity limit under the assumption of self-similarity when the underlying nonlinear hyperbolic system of conservation laws is formulated as a boundary value problem on the half-line. We establish a uniform bound on the total variation of solutions for the corresponding viscous–capillary boundary Riemann problem, provided the capillarity coefficient does not exceed a critical threshold. This leads us to a convergence theorem, as well as an existence result for the boundary Riemann problem for systems with sufficiently small Riemann data and sufficiently small capillarity. Furthermore, allowing for a possibly large capillarity coefficient, we then derive an equation governing the boundary layer and we introduce the notion of “viscous–capillary set of admissible boundary states”, which, following Dubois and LeFloch, represents all possible boundary states arising in the vanishing viscosity–capillarity limit. This set may involve, both, classical (compressive) and nonclassical (undercompressive) shock layers, the latter being typically determined by a kineric relation associated with the problem.

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## 1. Introduction

Our general motivation is to investigate the structure of boundary layers associated with nonlinear hyperbolic systems, when vanishing viscosity and capillarity effects are taken into account. Our analysis covers the general regime when the boundary is characteristic, that is, one of the wave speeds may vanish. In the present paper, we address this problem first in the case of the Riemann problem associated with a scalar conservation law in one space dimension. That is, we consider weak solutions to the equation

$$\partial_t u + \partial_x f(u) = 0, \quad t > 0, x > 0, \quad (1.1)$$

with smooth prescribed flux  $f : \mathbb{R} \rightarrow \mathbb{R}$  and unknown  $u = u(t, x) \in \mathbb{R}$ , subjected to the boundary and initial conditions

$$\begin{aligned} u(t, 0) &= u_b, & t > 0, \\ u(0, x) &= u_0, & x > 0, \end{aligned} \quad (1.2)$$

the two constant values  $u_b, u_0$  are prescribed. In the second part of this paper, we will also treat general hyperbolic systems of conservation laws.

Solutions to hyperbolic conservation laws are generally discontinuous and are not uniquely determined by their initial data, unless some entropy criterion is added. In addition, since we consider the boundary value problem, an additional difficulty arises at the boundary and a layer is expected to develop (in a sequence of approximations to the above problem, say). As discussed in Dubois and LeFloch [5], the prescribed boundary condition at  $x = 0$  in (1.2) can not be achieved as stated, but must be *weakened*.

We rely here on extensive work by Joseph and LeFloch [7, 8, 9, 10] on the (boundary) Riemann problem with vanishing viscosity, as well as on a paper by LeFloch and Rohde [17] concerning the Riemann problem with viscosity and capillarity. The general methodology was introduced by Tzavaras [22] in order to cope with system and an artificial viscosity term and extended earlier works by Slemrod et al. [6, 20, 21]. Specifically, we construct here solutions  $u_{\varepsilon, \gamma} = u_{\varepsilon, \gamma}(x)$  to the boundary Riemann problem by adding

vanishing diffusion–dispersion terms, as follows:

$$\begin{aligned} -x u'_{\varepsilon,\gamma} + f(u_{\varepsilon,\gamma})' &= \varepsilon u''_{\varepsilon,\gamma} + \gamma \varepsilon^2 u'''_{\varepsilon,\gamma}, & x > 0, \\ u_{\varepsilon,\gamma}(0) &= u_b, & u_{\varepsilon,\gamma}(+\infty) = u_0, \end{aligned} \tag{1.3}$$

where  $\gamma$  is a fixed parameter and  $\varepsilon \rightarrow 0$ . For definiteness, we take  $\gamma > 0$  and, without loss of generality for our purpose, we assume  $\varepsilon \in (0, 1]$ . Hence, we treat here the regime when the diffusion  $\varepsilon u''_{\varepsilon,\gamma}$  is *in balance* with the dispersion  $\varepsilon^2 u'''_{\varepsilon,\gamma}$ . Observe that (1.3) is equivalent to adding a time–dependent diffusion–dispersion regularization to the right–hand side of (1.1), that is,

$$\partial_t u_{\varepsilon,\gamma} + \partial_x f(u_{\varepsilon,\gamma}) = \varepsilon t \partial_{xx} u_{\varepsilon,\gamma} + \gamma \varepsilon^2 t^2 \partial_{xxx} u_{\varepsilon,\gamma}.$$

Since this problem is directly motivated by models arising in fluid dynamics, especially models of complex fluid flows including viscosity and capillarity, it is natural to refer to the former term as “viscosity” and the latter as “capillarity”. The regularization (1.3) provides a selection of physically admissible solutions to the Riemann problem (1.1), and, of course, is also relevant for the general Cauchy problem; see LeFloch [14, 15, 16] for a background on diffusive-dispersive limits.

The self-similar strategy for the Riemann problem was investigated first by Dafermos [1, 2, 3] (without a boundary and without capillarity term), in order to develop an existence theory for the Riemann problem. We do not attempt to review the large literature on this subject but refer to [4, 15] for background and additional references.

An outline of this paper is as follows. In Section 2, we study a “linearized problem” which is used later to construct an integral form of the problem. In Section 3, we establish the existence of solutions and derive a uniform bound for the total variation. Section 4 is devoted to the analysis of the boundary set. Then, in Sections 5 and 6 we extend our analysis and conclusions to general systems of conservation laws. The analysis therein is more involved, since the nonlinear coupling within the system must be analyzed.

## 2. A Linear Second–Order Problem

It is convenient to introduce the notation  $\lambda(u) := f_u(u)$  for the wave speed function associated with the scalar conservation law. We consider a bounded interval  $[0, L]$  and begin by considering (1.3) in which the left–hand side  $-x u'_{\varepsilon, \gamma} + f(u_{\varepsilon, \gamma})' = (-x + \lambda(u_{\varepsilon, \gamma})) u'_{\varepsilon, \gamma}$ , the speed coefficient  $\lambda(u_{\varepsilon, \gamma})$  is replaced by a fixed function  $\lambda : [0, L] \rightarrow \mathbb{R}$  which is assumed to be smooth and bounded function and is defined in the interval  $[0, L]$ , which is yet to be specified. Then, we solve (1.3) in terms of  $\phi := u'_{\varepsilon, \gamma}/(u_0 - u_b)$ , (1.3) which becomes a second–order differential equation. We set

$$\begin{aligned} \lambda^M &:= \max_{[0, L]} \lambda, & \lambda_+^M &:= \max(0, \lambda^M), \\ \lambda^m &:= \min_{[0, L]} \lambda, & \lambda_+^m &:= \max(0, \lambda^m). \end{aligned}$$

We assume that the upper bound  $L$  is large enough so that

$$\lambda^M < L. \tag{2.1}$$

Consider the second–order differential equation

$$\gamma \varepsilon^2 \phi''(x) + \varepsilon \phi'(x) + (x - \lambda(x)) \phi(x) = 0, \quad x \in [0, L]. \tag{2.2}$$

As we will now show, our key assumption about the coefficient of this equation is that  $\gamma$  is sufficiently small so that

$$\mu(x) := \lambda(x) - x + \frac{1}{4\gamma} > 0, \quad x \in [0, L]. \tag{2.3}$$

In fact, provided the inequality

$$\gamma < \frac{1}{4(\lambda_+^M - \lambda^m)} \tag{2.4}$$

holds, we can then choose  $L$  sufficient close (but larger) to  $\lambda_+^M$  so that

$$\lambda^m - L + \frac{1}{4\gamma} > 0$$

is satisfied, which implies precisely (2.3).

Our first objective is to show the existence of fundamental solutions to (2.2), which will be unique up to a multiplicative constant and be normalized later on. Consider first the function  $p : [0, L] \rightarrow \mathbb{R}$  defined by

$$p(x) := -\frac{x - L}{2\gamma} - \int_x^L \sqrt{\frac{\mu(y)}{\gamma}} dy = \frac{1}{2\gamma} \int_x^L \left(1 - \sqrt{1 + 4\gamma(\lambda(y) - y)}\right) dy.$$

Observe that, clearly, in the limit  $\gamma \rightarrow 0$  we obtain

$$\lim_{\gamma \rightarrow 0} \left(\mu(x) - \frac{1}{4\gamma}\right) = \lambda(x) - x, \quad p(x) = \int_x^L (y - \lambda(y)) dy.$$

Hence,  $\mu$  plays the role, roughly speaking, of the wave speed  $\lambda$  (up to a linear shift) when the viscosity and capillarity are taken into account.

**Proposition 2.1.** *There exists a real  $\rho \in [\lambda_+^m, \lambda_+^M]$  such that  $p'(\rho) = 0$  if  $\rho > 0$  and, more precisely,*

- (i)  $p'(x) < 0, \quad x \in (\lambda_+^M, L],$
- (ii)  $p(x) \leq p(\rho), \quad x \in [0, L],$

and the behavior of  $p$  is further described as follows, with constants  $c, C > 0$  independent of  $\gamma$ ,

- (iii)  $p(x) - p(\rho) \geq -c|x - \rho|, \quad x \in [0, \lambda_+^M],$
- (iv)  $p(x) - p(\rho) \leq -C(x - \lambda_+^M)^2, \quad x \in [\lambda_+^M, L].$

**Proof.** 1. If  $x > \lambda_+^M$  we have  $\mu(x) < 1/(4\gamma)$  and, therefore,

$$p'(x) = -\frac{1}{2\gamma} + \sqrt{\frac{\mu(x)}{\gamma}},$$

which is negative. By continuity, the function  $p$  achieves its maximum value at some point of the interval  $[0, \lambda_+^M]$ , which we denote by  $\rho$ .

2. Let us consider first the case  $\rho > 0$ , that is, the maximum is an (interior) point of the interval  $(0, L)$ . Then, it follows that  $p'(\rho) = 0$  and we deduce that  $\frac{1}{2\gamma} = \sqrt{\frac{\mu(\rho)}{\gamma}}$ , and hence

$$p(x) - p(\rho) = \int_x^\rho \left(\sqrt{\frac{\mu(\rho)}{\gamma}} - \sqrt{\frac{\mu(y)}{\gamma}}\right) dy.$$

This identity is now used in order to derive the first estimate of interest.

For definiteness, we treat the interval  $[\rho, \lambda_+^M]$ , since the argument in the interval  $(0, \rho]$  is completely similar. Since for  $y \in [0, \lambda_+^M]$  one has

$$\mu(y) \geq \lambda^m - y + \frac{1}{4\gamma}, \quad \mu(\rho) \leq \lambda^M - \rho + \frac{1}{4\gamma},$$

we find that, for all  $x \in [\rho, \lambda_+^M]$ ,

$$\begin{aligned} p(x) - p(\rho) &= \int_{\rho}^x \left( \sqrt{\frac{\mu(y)}{\gamma}} - \sqrt{\frac{\mu(\rho)}{\gamma}} \right) dy \\ &\geq \frac{1}{2\gamma} \int_{\rho}^x \left( \sqrt{1 - 4\gamma(y - \lambda^m)} - \sqrt{1 + 4\gamma(\lambda^M - \rho)} \right) dy \\ &= \frac{1}{12\gamma^2} \left( (1 - 4\gamma(\rho - \lambda^m))^{3/2} - (1 - 4\gamma(x - \lambda^m))^{3/2} \right. \\ &\quad \left. - 6\gamma(x - \rho)(1 + 4\gamma(\lambda^M - \rho))^{1/2} \right) \end{aligned}$$

Since  $g = g(\rho) = (1 - 4\gamma(\rho - \lambda^m))^{3/2}$  is a concave function, this yields

$$\begin{aligned} p(x) - p(\rho) &\geq \frac{1}{12\gamma^2} (g(\rho) - (x - \rho)g'(\rho) - g(x)) \\ &\quad + \frac{1}{2\gamma} (x - \rho) \left( (1 + 4\gamma(\lambda^M - \rho))^{1/2} - (1 + 4\gamma(\lambda^M - \rho))^{1/2} \right) \\ &\geq -c(x - \rho) \end{aligned}$$

and

$$\begin{aligned} c &:= \frac{1}{2\gamma} \left( (1 + 4\gamma(\lambda^M - \rho))^{1/2} - (1 + 4\gamma(\lambda^m - \rho))^{1/2} \right) \\ &\geq \frac{2(\lambda^M - \lambda^m)}{(1 + 4\gamma(\lambda^M - \rho))^{1/2} + (1 + 4\gamma(\lambda^m - \rho))^{1/2}} \geq c' > 0, \end{aligned}$$

since  $1 + 4\gamma(\lambda^m - \rho) \leq 1$  and  $1 + 4\gamma(\lambda^M - \rho) \leq 1 + \lambda_+^M/(\lambda_+^M - \lambda^m)$ , so that our estimate is independent of the parameter  $\gamma$ . This completes the proof of (iii).

We now turn our attention to interval  $x \in [\lambda_+^M, L]$  and write

$$\begin{aligned} p(x) - p(\rho) &\leq p(x) - p(\lambda_+^M) = \frac{1}{2\gamma} \int_{\lambda_+^M}^x \left( -1 + \sqrt{1 + 4\gamma(\lambda(y) - y)} \right) dy \\ &\leq \frac{1}{2\gamma} \int_{\lambda_+^M}^x \left( -1 + \sqrt{1 + 4\gamma(\lambda^M - y)} \right) dy \\ &\leq \frac{1}{2} (\lambda_+^M - \lambda^M)^2 - \frac{1}{2} (x - \lambda^M)^2 \leq -\frac{1}{2} (x - \lambda_+^M)^2, \end{aligned}$$

where we used  $-1 + \sqrt{1 + \alpha} \leq \alpha/2$  for  $\alpha > -1$ .

3. It remains to consider the case  $\rho = 0$ , that is, the maximum is at the left-hand boundary of the interval. Then, it follows that  $p'(\rho) \leq 0$  and it is not difficult to check the inequalities above remain true. In this case, we can actually establish a better estimate than (iii), as follows:

$$\begin{aligned} p(x) - p(\rho) &= p(x) - p(0) = -\frac{x}{2\gamma} + \int_0^x \sqrt{\frac{\mu(y)}{\gamma}} dy \\ &\geq -\frac{x}{2\gamma} + \frac{1}{2\gamma} \int_0^x \sqrt{1 - 4\gamma(y - \lambda^m)} dy \\ &= \frac{1}{12\gamma^2} \left( (1 + 4\gamma\lambda^m)^{3/2} - (1 - 4\gamma(x - \lambda^m))^{3/2} - 6\gamma x \right) \end{aligned}$$

and thus, with the function  $g$  defined earlier,

$$\begin{aligned} p(x) - p(\rho) &\geq \frac{1}{12\gamma^2} \left( g(0) - xg'(0) - g(x) + 6\gamma x \left( (1 + 4\gamma\lambda^m)^{1/2} - 1 \right) \right) \\ &\geq \frac{1}{2\gamma} \left( (1 + 4\gamma\lambda^m)^{1/2} - 1 \right) x \geq 2 \min(0, \lambda^m)x, \quad x \in [0, \lambda_+^M]. \end{aligned}$$

The argument for (iv) is completely similar, and the proof of the proposition is now completed. □

We arrive at the main result of the present section.

**Theorem 2.2.** *The equation (2.2) admits a smooth solution  $\phi : [0, L] \rightarrow \mathbb{R}_+$  of the form*

$$\phi(x) = \frac{1 + \Phi(x)}{(4\gamma\mu(x))^{1/4}} e^{\frac{p(x)-p(\rho)}{\varepsilon}},$$

in which the remainder  $\Phi$  satisfies, for all  $x \in [0, L]$ ,

$$\begin{aligned} \|\Phi\|_{L^\infty(0,L)} + \frac{\varepsilon\sqrt{\gamma}}{2}\|\mu^{-1/2}\Phi'\|_{L^\infty(0,L)} &\leq \frac{\varepsilon}{4}\sqrt{\gamma}K, \\ K := \frac{5}{4}\|\mu^{-5/2}(\mu')^2\|_{L^1(0,L)} + \|\mu^{-3/2}\mu''\|_{L^1(0,L)}, \end{aligned} \quad (2.5)$$

provided the following condition on the parameter  $\varepsilon, \gamma$  hold

$$\frac{\varepsilon}{4}\sqrt{\gamma}K \leq 1. \quad (2.6)$$

In particular, for some constant  $C > 1$  independent of  $\varepsilon, \gamma$ , one has

$$(i) \quad 0 < \phi(x) \leq \left(\frac{4}{\gamma\mu(x)}\right)^{1/4}, \quad (ii) \quad \frac{\varepsilon}{C} \leq \int_0^L \phi(y) dy \leq C.$$

**Proof.** 1. The function  $H(x) := e^{\frac{x}{2\gamma\varepsilon}}\phi(x)$  is easily found to satisfy the equation

$$H''(x) - \frac{\mu(x)}{\gamma\varepsilon^2}H(x) = 0.$$

Since the coefficient  $\mu$  is bounded below by a positive constant on the interval  $[0, L]$ , Theorem 2.1 of Chapter 6 in [19] provides us with the existence of a solution  $\phi$  having the form stated in the theorem, together with the pointwise estimate

$$|\Phi(x)| + \frac{\varepsilon\sqrt{\gamma}}{2}\mu(x)^{-1/2}|\Phi'(x)| \leq e^{\frac{1}{2}TV_0^x(F)} - 1,$$

where  $F : [0, L] \rightarrow \mathbb{R}$  is defined by

$$\begin{aligned} F'(x) &= \varepsilon\sqrt{\gamma}\mu(x)^{-1/4}\left(\mu(x)^{-1/4}\right)'' \\ &= \varepsilon\sqrt{\gamma}\mu(x)^{-1/4}\left(\frac{5}{16}\mu(x)^{-9/4}|\mu'(x)|^2 - \frac{1}{4}\mu(x)^{-5/4}\mu''(x)\right), \end{aligned}$$

so that its total variation is bounded, as follows:

$$\begin{aligned} TV_0^x(F) &\leq \varepsilon\sqrt{\gamma}\int_0^x\left(\frac{5}{16}\mu(y)^{-5/2}|\mu'(y)|^2 + \frac{1}{4}\mu(y)^{-3/2}|\mu''(y)|\right)dy \\ &\leq \frac{\varepsilon}{4}\sqrt{\gamma}K. \end{aligned}$$

Observe that  $e^{A/2} \leq 1 + A$  for  $0 \leq A < 1$ , which motivates us to impose the condition (2.6).

2. Next, we note that

$$|\Phi(x)| \leq \frac{1}{2} TV_0^x \leq \frac{1}{2} \frac{\varepsilon}{4} \sqrt{\gamma} K \leq \frac{1}{2},$$

so that  $1 + \Phi(x) \geq \frac{1}{2}$  and the unknown function  $\phi(x)$  is positive. Furthermore, the upper bound in (i) follows from the property  $p(x) - p(\rho) \leq 0$ .

3. We now establish (ii). Provided  $(\varepsilon/4) \sqrt{\gamma} K < 1$ , we find  $\phi(x) > 0$  for all  $x \in [0, L]$ , and the coefficient in  $\phi$  satisfies

$$(4\gamma\mu(x))^{-1/4} \geq (1 + \lambda_+^M / (\lambda_+^M - \lambda^m))^{-1/4}$$

and, therefore, is bounded away from zero by a constant independent of  $\gamma$ . Hence, for some uniform  $C_1$ , we find

$$\int_0^L \phi(y) dy \geq C_1 \int_0^L e^{\frac{p(y)-p(\rho)}{\varepsilon}} dy,$$

but

$$\int_0^L e^{\frac{p(y)-p(\rho)}{\varepsilon}} dy \geq \int_0^{\lambda_+^M} e^{-c \frac{|\rho-y|}{\varepsilon}} dy = \varepsilon \int_{\frac{(-\rho)}{\varepsilon}}^{(\lambda_+^M - \rho)/\varepsilon} e^{-c|y|} dy \geq C_2 \varepsilon,$$

where, in the last inequality, we have used that  $\varepsilon$  is bounded above (and, actually,  $\varepsilon \in (0, 1]$ ). This establishes the lower bound.

For the upper bound, we note that  $(4\gamma\mu(x))^{-1/4} \leq (1 - 4\gamma(L - \lambda^m))^{-1/4}$  and thus is uniformly bounded in  $[0, L]$ . For some constant  $C_3 > 0$ , we obtain

$$\int_0^L \phi(y) dy \leq C_3 \int_0^L e^{\frac{p(y)-p(\rho)}{\varepsilon}} dy \leq C_3 L,$$

which completes the proof of Theorem 2.2. □

### 3. Viscous–Capillary Boundary Riemann Problem

We are now in a position to treat the problem of interest (1.3), associated

with prescribed data  $u_b, u_0$ . We write

$$I(u_b, u_0) := [\min(u_b, u_0), \max(u_b, u_0)]$$

and from the flux function  $f$  we determine the smallest and largest wave speeds

$$\begin{aligned} \lambda^M(u_b, u_0) &:= \max_{u \in I(u_b, u_0)} \lambda(u), & \lambda_+^M(u_b, u_0) &:= \max(0, \lambda^M(u_b, u_0)), \\ \lambda^m(u_b, u_0) &:= \min_{u \in I(u_b, u_0)} \lambda(u). \end{aligned}$$

**Theorem 3.1** (Viscous–capillary boundary Riemann problem). *Given boundary and initial data  $u_b, u_0 \in \mathbb{R}$ , a viscosity coefficient  $\varepsilon \in (0, 1]$ , and a (sufficiently small) capillarity coefficient  $\gamma$  satisfying*

$$\gamma < \frac{1}{4(\lambda_+^M(u_b, u_0) - \lambda^m(u_b, u_0))}, \quad (3.1)$$

*the boundary Riemann problem with viscosity and capillarity (1.3) admits a unique solution  $u_{\varepsilon, \gamma} = u_{\varepsilon, \gamma}(x)$  defined on some interval  $[0, L]$  with  $L > \lambda_+^M$ . This solution is smooth, strictly monotone, and satisfies*

$$\begin{aligned} \min(u_b, u_0) &\leq u_{\varepsilon, \gamma} \leq \max(u_b, u_0), \\ u_{\varepsilon, \gamma}(0) &= u_b, \quad u_{\varepsilon, \gamma}(x) = u_0, \quad x \in (\lambda_+^M, L], \\ TV_0^L(u_{\varepsilon, \gamma}) &= |u_0 - u_b|. \end{aligned} \quad (3.2)$$

*Furthermore, one has the uniform bound*

$$\varepsilon \|u'_{\varepsilon, \gamma}\|_{L^\infty(0, L)} + \gamma \varepsilon^2 \|u''_{\varepsilon, \gamma}\|_{L^\infty(0, L)} \leq C, \quad x \in [0, L], \quad (3.3)$$

*where the constant  $C$  is independent of  $\varepsilon, \gamma$ .*

Based on this result, we can next justify the limit  $\varepsilon \rightarrow 0$ , while  $\gamma$  is kept fixed.

**Theorem 3.2** (Boundary Riemann problem for the hyperbolic conservation law). *As  $\varepsilon \rightarrow 0$ , the solutions  $u_{\varepsilon, \gamma}$  given by Theorem 3.1 converge almost everywhere to a limiting function  $u_\gamma : [0, L] \rightarrow I(u_b, u_0) \subset \mathbb{R}$ , which is monotone and has bounded total variation less or equal to the prescribed*

jump, that is,

$$TV_0^L(u_\gamma) \leq |u_0 - u_b|, \quad (3.4)$$

and is a weak solution of the self-similar boundary Riemann problem (1.1)–(1.2). The solution also satisfies the following properties:  $u_\gamma(x) = u_0$  for  $x \in (\lambda_+^M, L]$ . If  $x \in \text{supp } u'_\gamma \cap (0, \lambda_+^M]$  is a point of continuity of  $u_\gamma$ , then

$$\lambda(u_\gamma(x)) = x, \quad (3.5)$$

which is the equation of a rarefaction wave. If  $x \in \text{supp } u'_\gamma \cap (0, \lambda_+^M]$  is a point of jump discontinuity, then  $u_\gamma$  satisfies the Lax shock admissibility inequalities [11, 12]:

$$\lambda(u_\gamma(x-)) \geq x \geq \lambda(u_\gamma(x+)). \quad (3.6)$$

Thanks to the monotonicity property established for the solutions  $u_{\varepsilon, \gamma}$ , we see that this limit can not contain nonclassical shocks, but only shocks satisfying the standard shock admissibility conditions.

**Proof.**[Proof of Theorem 3.1] Given a function  $v : [0, L] \rightarrow \mathbb{R}$  satisfying the boundary conditions

$$v(0) = u_b, \quad v(L) = u_0,$$

we can determine the corresponding function  $\phi[v] : [0, L] \rightarrow \mathbb{R}$  from Theorem 2.2: it has the form

$$\phi[v](x) = \frac{1 + \Phi[v](x)}{(4\gamma\mu(v(x)))^{1/4}} e^{\frac{p[v](x) - p[v](\rho)}{\varepsilon}},$$

in which the argument  $p[v]$  is given by

$$p[v](x) := -\frac{x-L}{2\gamma} + \gamma^{-1/2} \int_L^x \mu(v(y))^{1/2} dy$$

and the remainder term  $\Phi[v]$  satisfies the inequality (2.5), with obvious no-

tation. Then, we recover a “new” function  $w : [0, L] \rightarrow \mathbb{R}$  by setting

$$w(x) = u_b + (u_0 - u_b) \frac{\int_0^x \phi(y) dy}{\int_0^L \phi(y) dy}, \quad x \in [0, L],$$

which allows us to define a map  $T : v \mapsto T[v] = w$ .

Given the data  $u_b, u_0$ , we introduce the affine space of continuous functions with prescribed boundary values

$$\mathcal{E}(0, L) := \{v \in C[0, L] / \min(u_b, u_0) \leq v \leq \max(u_b, u_0)\}.$$

Observe that  $T$  maps  $\mathcal{E}$  into itself, and we can endow  $\mathcal{E}(0, L)$  with the uniform norm  $\|\cdot\|_{L^\infty(0, L)}$ . We want now to apply the Schauder fixed point theorem: it is clear that  $\mathcal{E}(0, L)$  is bounded, closed, and convex subset of  $C[0, L]$ ; thus, we only need to check that  $T$  is continuous and compact.

If  $v^n \rightarrow v$  is a converging sequence in  $\mathcal{E}(0, L)$ , then

$$\begin{aligned} |p[v](x) - p[v_n](x)| &\leq \gamma^{-1/2} \int_x^L \left| \mu(v(y))^{1/2} - \mu(v_n(y))^{1/2} \right| dy \\ &\leq C(L-x) \|\mu' \mu^{-1/2}\|_{L^\infty(0, L)} \|v - v_n\|_{L^\infty(0, L)}. \end{aligned}$$

Similarly, we can write

$$\begin{aligned} &\left| \frac{1 + \Phi[v](x)}{4\gamma\mu(v(x))^{1/4}} - \frac{1 + \Phi[v_n](x)}{4\gamma\mu(v_n(x))^{1/4}} \right| \\ &\leq \left( \frac{\|\Phi'\|_{L^\infty(0, L)}}{4\gamma\|\mu^{1/4}\|_{L^\infty(0, L)}} + \frac{1 + \|\Phi\|_{L^\infty(0, L)}}{16\gamma\|\mu^{5/4}\|_{L^\infty(0, L)}} \|\mu'\|_{L^\infty(0, L)} \right) \|v - v_n\|_{L^\infty(0, L)} \end{aligned}$$

and we deduce that  $T[v^n] \rightarrow T[v]$ ; hence, the map  $T$  is continuous.

Furthermore, let us check that  $T$  is compact. If  $\{v_n\}$  be a bounded sequence in  $\mathcal{E}(0, L)$ , we have the inequality

$$\begin{aligned} |p[v_n](x) - p[v_n](x')| &\leq C \|\mu' \mu^{-1/2}\|_{L^\infty(0, L)} \|v_n\|_{L^\infty(0, L)} |x - x'| \\ &\leq C |x - x'| \|\mu' \mu^{-1/2}\|_{L^\infty(0, L)} \sup_n \|v_n\|_{L^\infty(0, L)} \end{aligned}$$

shows that  $\{p[v_n]\}$  is equicontinuous. Obviously  $\{p[v_n]\}$  is bounded, hence,

by Ascoli’s theorem,  $\{p[v_n]\}$  is relatively compact. So,  $T$  is the composition of a compact operator and a continuous one and, thus, a compact operator. In conclusion, by Schauder’s fixed point theorem,  $T$  admits a fixed point which we denote by  $u_{\varepsilon,\gamma}$  and is easily checked to satisfy all of the conditions in the theorem. The corresponding function  $\phi$  as defined above is denoted by  $\phi_{\varepsilon,\gamma}$

To derive (3.3), we simply observe that

$$u'_{\varepsilon,\gamma} = (u_0 - u_b) \frac{\phi_{\varepsilon,\gamma}}{\int_0^L \phi_{\varepsilon,\gamma}(y) dy},$$

in which  $\int_0^L \phi_{\varepsilon,\gamma}(y) dy$  is bounded below by  $C\varepsilon$ . □

**Proof of Theorem 3.2.** 1. In view of Theorem 3.1, we can now take the limit  $\varepsilon \rightarrow 0$  in the family of solutions  $u_{\varepsilon,\gamma}$ . Since we have established a uniform bound on the total variation, Helly’s theorem allows us to extract a converging subsequence which converges to some limit  $u_\gamma : (0, L] \rightarrow \mathbb{R}$  which is also a function of bounded variation, i.e.

$$u_{\varepsilon,\gamma}(y) \rightarrow u_\gamma(y), \quad y \in (0, L].$$

Since  $u_{\varepsilon,\gamma}$  connect monotonically  $u_b$  to  $u_0$ , it is clear that the total variation of the limit  $u_\gamma$  is less or equal to the jump  $|u_0 - u_b|$ . Note that it can be smaller, however, due to the possible formation of a boundary layer at  $x = 0$ . We can also assume that the sequence of measures  $u'_{\varepsilon,\gamma}$  converges in the weak-star sense

$$u'_{\varepsilon,\gamma} \rightharpoonup u'_\gamma \quad \text{weakly.}$$

With obvious notation, we can also extract a converging subsequence so that  $\rho_{\varepsilon,\gamma} \rightarrow \rho_\gamma$  and in the uniform norm

$$\begin{aligned} p_{\varepsilon,\gamma}(y) - p_{\varepsilon,\gamma}(\rho_{\varepsilon,\gamma}) &\longrightarrow p_\gamma(x) - p_\gamma(\rho_\gamma) \\ &= \frac{1}{2\gamma} \int_x^{\rho_\gamma} \left( 1 - \sqrt{1 + 4\gamma(\lambda(u_\gamma(y)) - y)} \right) dy. \end{aligned}$$

Since  $u_{\varepsilon,\gamma}$  solves

$$-y u'_{\varepsilon,\gamma} + f(u_{\varepsilon,\gamma})' = \varepsilon u''_{\varepsilon,\gamma} + \gamma \varepsilon^2 u'''_{\varepsilon,\gamma}, \quad x > 0,$$

we immediately deduce that  $u_\gamma$  is a weak solution to the hyperbolic conservation law

$$-y u'_\gamma + f(u_\gamma)' = 0.$$

Moreover, the exponential decay properties yield that the boundary condition at  $u_0$  holds, and  $u$  is a solution of the boundary Riemann problem, except that the boundary condition at  $x = 0$  need not hold.

2. We now discuss the structure of this solution. Using the inequality  $|u'_{\varepsilon,\gamma}| \leq C \phi_{\varepsilon,\gamma}$  and the decay properties for the functions  $\phi_{\varepsilon,\gamma}$ , we see that for small  $\eta > 0$

$$u'_{\varepsilon,\gamma}(y) \rightarrow 0 \quad \text{uniformly in } y \in [\lambda^M + \eta, L].$$

Therefore,  $u$  is constant equal to  $u_0$  on the interval  $(\lambda^M, L]$ .

Now, we show that if  $y \in \text{supp } \varphi_\gamma$ , then

$$p_\gamma(x) \geq p_\gamma(y) = 0, \quad x \in (0, L).$$

This shows that the points in the support of  $\varphi_\gamma$  are global maxima for the function  $p_\gamma$ .

Namely, fix  $y$  and some  $\alpha > 0$  and consider the set

$$A = \{x \in [0, L] / p_\gamma(x) - p_\gamma(y) < -\alpha < 0\}.$$

The functions  $p_\gamma$  being (Lipschitz) continuous, we have either  $A = \emptyset$  or  $A$  contains a non-empty open interval. In the second case, we shall show that there is an open interval  $I$  containing  $y$  such that

$$\int_0^L \phi_\gamma \psi \, dx = 0$$

for every  $\psi$  with compact support in  $I$ . This will show that  $y \notin \text{supp } \phi_\gamma$ . Henceforth if  $y \in \text{supp } \phi_\gamma$ , we can only be in the first case  $A = \emptyset$  for every  $\alpha$  and thus  $p_\gamma(x) - p_\gamma(y) \geq 0$  for all  $x \in [0, L]$ , which is the desired conclusion.

3. First, we conclude the proof of the theorem as follows. Fix  $y$  and use

that  $u$  has bounded variation to write:

$$\lim_{x \rightarrow y \pm} \frac{p_\gamma(x) - p_\gamma(y)}{x - y} = \lim_{x \rightarrow y \pm} \frac{1}{x - y} \int_y^x \left( -\frac{1}{2\gamma} + \sqrt{\frac{\mu(u_\gamma(z))}{\gamma}} \right) dz.$$

Therefore, we find

$$p'_\gamma(y) = -\frac{1}{2\gamma} + \sqrt{\frac{\mu(u_\gamma(y))}{\gamma}} \quad \text{if } y \text{ is a point of continuity of } u_\gamma,$$

while the left- and right-derivative exist at a point of jump of  $u_\gamma$  and

$$p'_\gamma(y \pm) = -\frac{1}{2\gamma} + \sqrt{\frac{\mu(u_\gamma(y \pm))}{\gamma}} \quad \text{if } y \text{ is a point of jump of } u_\gamma.$$

Suppose now that  $y \in \text{supp } u'_\gamma \cap [\lambda_+^m, \lambda_+^M]$ , thus  $y \in \text{supp } \phi_\gamma$  and, using the above claim,

$$p_\gamma(x) \geq p_\gamma(y),$$

so  $\frac{p_\gamma(x) - p_\gamma(y)}{x - y} \geq 0$  when  $x - y \geq 0$ , while  $\frac{p_\gamma(x) - p_\gamma(y)}{x - y} \leq 0$  when  $x - y \leq 0$ . This leads us to, both,  $-\frac{1}{2\gamma} + \sqrt{\frac{\mu(u_\gamma(y \pm))}{\gamma}} \geq 0$  and  $-\frac{1}{2\gamma} + \sqrt{\frac{\mu(u_\gamma(y \pm))}{\gamma}} \leq 0$ . This implies  $y - \lambda(u_\gamma(y+)) \geq 0$  and  $y - \lambda(u_\gamma(y-)) \leq 0$ . Of course, the equality holds when  $u_\gamma$  is continuous at the point  $y$ .

4. We can determine an interval  $I = (y - \delta, y + \delta)$  in which

$$0 < \phi_{\varepsilon, \gamma}(s) \leq \frac{e^{-\frac{\alpha}{2\varepsilon}}}{\text{meas } (A)} \rightarrow 0.$$

Since  $p_{\varepsilon, \gamma}$  is continuous, we have

$$\begin{aligned} |p_\gamma(\theta) - p_\gamma(y)| &< \frac{\alpha}{6}, & \theta \in (y - \delta, y + \delta), \\ |p_{\varepsilon, \gamma}(\theta) - p_\gamma(\theta)| &< \frac{\alpha}{6}, & \theta \in [0, \lambda_+^M]. \end{aligned}$$

Thus for all  $\theta \in (y - \delta, y + \delta)$  and  $x \in A$

$$\begin{aligned} &p_{\varepsilon, \gamma}(x) - p_{\varepsilon, \gamma}(\theta) \\ &\leq p_\gamma(x) - p_\gamma(y) + |p_\gamma(y) - p_\gamma(\theta)| + |p_{\varepsilon, \gamma}(\theta) - p_\gamma(\theta)| + |p_{\varepsilon, \gamma}(x) - p_\gamma(x)| \end{aligned}$$

$$\leq -\frac{\alpha}{2} < 0.$$

That is, the inequality defining the set  $A$  remains valid for the functions  $p_{\varepsilon,\gamma}$  and in a uniform neighborhood of  $y$ .

Returning to the definition, for some constant  $C > 0$  and for all  $\theta \in (y - \delta, y + \delta)$ ,

$$\begin{aligned} 0 < \phi_{\varepsilon,\gamma}(\theta) &\leq \frac{C}{\int_A \exp(-\frac{1}{\varepsilon}(p_{\varepsilon,\gamma}(x) - p_{\varepsilon,\gamma}(\theta))) dx} \\ &\leq \frac{C}{\exp(\frac{\alpha}{2\varepsilon}) \text{meas}(A)} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Thus  $\langle \phi_{\varepsilon,\gamma}, \psi \rangle \rightarrow 0$  and  $\langle \phi_\gamma, \psi \rangle = 0$  for all  $\psi$  compactly supported in  $I$ .  $\square$

#### 4. The Viscous–Capillarity Set of Admissible Boundary Values

##### Derivation of the layer equation

In this section, we rigorously derive the equation describing the boundary layer which, in general, arises near  $x = 0$  in the solutions  $u_{\varepsilon,\gamma}$  to the boundary Riemann problem. Our objective is thus to establish a relation between the prescribed boundary data, that is,  $u_b$ , and the trace  $u_\gamma(0+)$  of the solution  $u_\gamma$  to the boundary Riemann problem constructed in Theorem 3.2. More precisely, we are going to establish that this layer is governed by the following ordinary differential problem with unknown  $V_\gamma = V_\gamma(y)$ :

$$\begin{aligned} \gamma V_\gamma'' + V_\gamma' &= f(V_\gamma) - f(V_{\gamma,\infty}), \quad y \in \mathbb{R}_+, \\ V_\gamma(0) &= u_b, \quad V_\gamma(+\infty) = V_{\gamma,\infty}, \end{aligned} \tag{4.1}$$

in which  $V_{\gamma,\infty}$  is expected to be closely related to, but need not coincide with,  $u_\gamma(0+)$ .

The smallness assumption made on the capillarity coefficient is not necessary in the present analysis, which therefore does cover the possibility of *nonclassical shocks*. So, we proceed here under the assumption that the total variation of a sequence of viscous–capillary boundary Riemann solutions  $u_{\varepsilon,\gamma}$  is uniformly bounded (which is valid for small  $\gamma$ , at least), and now we perform the corresponding boundary layer analysis. As we are going to see,

the structure of this layer is much richer than the one obtained by adding viscosity, only.

**Theorem 4.3** (Boundary layer equation for the vanishing viscosity–capillarity limit). *The trace  $u_\gamma(0+)$  of the boundary Riemann solution constructed in Theorem 3.2 satisfies the following property. There exists  $V_{\gamma,\infty} \in \mathbb{R}$  and a smooth function  $V_\gamma : [0, \infty) \rightarrow \mathbb{R}$  which satisfies the boundary layer problem (4.1) and the following jump relation at infinity:*

$$f(V_{\gamma,\infty}) = f(u_\gamma(0)). \quad (4.2)$$

**Proof.** We follow the argument in [8] and consider an arbitrary sequence  $\xi_\varepsilon > 0$  such that

$$\xi_\varepsilon = o(\varepsilon).$$

Define the function  $V_{\varepsilon,\gamma}(y) = u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y)$  for all  $y > 0$ . Since  $u_{\varepsilon,\gamma}$  is uniformly bounded and of uniformly bounded total variation, the functions  $V_{\varepsilon,\gamma}$  are also bounded and of uniformly bounded total variation. So there exists a function  $V_\gamma = V_\gamma(y)$  of bounded total variation defined on the interval  $[0, \infty)$  and there exist two constants  $V_{\gamma,0}, V_{\gamma,\infty}$  such that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} V_{\varepsilon,\gamma}(y) &= V_\gamma(y), & y > 0 \\ V_\gamma(0+) &= V_{\gamma,0}, & \lim_{y \rightarrow +\infty} V_\gamma(y) = V_{\gamma,\infty}. \end{aligned} \quad (4.3)$$

To check that, in fact,  $V_0 = u_b$ , we note that

$$|V_\gamma(0) - u_b| = \lim_{y \rightarrow 0+} |V(y) - u_b| = \lim_{y \rightarrow 0} \lim_{\varepsilon \rightarrow 0+} |u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y) - u_b|$$

and

$$|u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y) - u_b| \leq \int_0^{\xi_\varepsilon + \varepsilon y} |u'_{\varepsilon,\gamma}(s)| ds \leq \frac{C}{\varepsilon} \int_0^{\xi_\varepsilon + \varepsilon y} ds = C(y + \xi_\varepsilon/\varepsilon),$$

where we used  $|u'_{\varepsilon,\gamma}| \leq C\varepsilon$ . Since  $\xi_\varepsilon = o(\varepsilon)$ , we deduce that  $V_\gamma(0) = u_b$ .

Next we derive the boundary layer equation (4.1). Integrating (1.3) from

some point  $a$  to  $\xi_\varepsilon + \varepsilon y$ , we get

$$\begin{aligned} & \varepsilon u'_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y) - \varepsilon u'_{\varepsilon,\gamma}(a) + \gamma \varepsilon^2 u''_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y) - \gamma \varepsilon^2 u''_{\varepsilon,\gamma}(a) \\ &= -(\xi_\varepsilon + \varepsilon y)u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y) + f(u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y)) \\ & \quad + au_{\varepsilon,\gamma}(a) - f(u_{\varepsilon,\gamma}(a)) + \int_a^{\xi_\varepsilon + \varepsilon y} u_{\varepsilon,\gamma}(s) ds. \end{aligned}$$

After integration with respect to  $a \in (0, \delta)$ , this identity becomes

$$\begin{aligned} & \frac{d}{dy} (u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y)) - \frac{\varepsilon}{\delta} \int_0^\delta u'_{\varepsilon,\gamma}(a) da + \gamma \frac{d^2}{dy^2} (u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y)) - \frac{\gamma \varepsilon^2}{\delta} \int_0^\delta u''_{\varepsilon,\gamma}(a) da \\ &= -(\xi_\varepsilon + \varepsilon y)u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y) + f(u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y)) \\ & \quad + \frac{1}{\delta} \int_0^\delta (au_{\varepsilon,\gamma}(a) - f(u_{\varepsilon,\gamma}(a))) da + \frac{1}{\delta} \int_0^\delta \int_a^{\xi_\varepsilon + \varepsilon y} u_{\varepsilon,\gamma}(s) ds da. \end{aligned}$$

We now integrate with respect to  $y$ , starting at 0:

$$\begin{aligned} & u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y) - u_{\varepsilon,\gamma}(\xi_\varepsilon) - \frac{\varepsilon}{\delta} y \int_0^\delta u'_{\varepsilon,\gamma}(a) da + \gamma \frac{d}{dy} (u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon y)) \\ & \quad - \gamma \frac{d}{dy} (u_{\varepsilon,\gamma}(\xi_\varepsilon)) - \gamma \varepsilon^2 y \int_0^\delta u''_{\varepsilon,\gamma}(a) da \\ &= \int_0^y \left\{ -(\xi_\varepsilon + \varepsilon x)u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon x) + f(u_{\varepsilon,\gamma}(\xi_\varepsilon + \varepsilon x)) \right\} dx \\ & \quad + \frac{y}{\delta} \int_0^\delta (au_{\varepsilon,\gamma}(a) - f(u_{\varepsilon,\gamma}(a))) da \\ & \quad + \frac{1}{\delta} \int_0^y \int_0^\delta \int_a^{\xi_\varepsilon + \varepsilon x} u_{\varepsilon,\gamma}(s) ds da dx. \end{aligned} \tag{4.4}$$

Next, letting  $\varepsilon \rightarrow 0$  in (4.4) and using (4.3) and (4.4), we arrive at

$$\begin{aligned} & V_\gamma(y) - u_b + \gamma V'_\gamma(y) - \gamma V'_\gamma(0) \\ &= \int_0^y f(V_\gamma(x)) dx + \frac{y}{\delta} \int_0^\delta (a u_\gamma(a) - f(u_\gamma(a))) da + \frac{y}{\delta} \int_0^\delta \int_a^0 u_\gamma(s) ds da \end{aligned}$$

for all  $\delta, y > 0$ . Next when  $\delta \rightarrow 0$ , it follows that

$$V_\gamma(y) - u_b + \gamma V'_\gamma(y) - \gamma V'_\gamma(0) = \int_0^y f(V_\gamma(x)) dx - y f(u_\gamma(0+)),$$

which is equivalent to the first equation in (4.1).

Integrating the equation (4.1) from  $n$  to  $n + 1$ , it follows that

$$\int_n^{n+1} f(V_\gamma(x)) dx - f(u_\gamma(0+)) = V_\gamma(n + 1) - V_\gamma(n) + \gamma V'_\gamma(n + 1) - \gamma V'_\gamma(n). \tag{4.5}$$

Since  $V_\gamma$  and  $V'_\gamma$  have bounded total variation and  $V_\gamma$  converges to  $V_{\gamma,\infty}$  at infinity, we have

$$\begin{aligned} \int_n^{n+1} |f(V_\gamma(x)) - f(V_{\gamma,\infty})| dx &\leq \int_n^{n+1} |V_\gamma(x) - V_{\gamma,\infty}| dx \\ &\leq C TV_n^{n+1}(V_\gamma) + C |V_\gamma(n) - V_{\gamma,\infty}|, \end{aligned}$$

in which the upper bound vanishes in the limit  $n \rightarrow +\infty$ . Therefore letting  $n$  tend to  $+\infty$  in (4.5), we obtain  $f(V_{\gamma,\infty}) = f(u_\gamma(0+))$ , which is the condition (4.2) stated in the theorem.  $\square$

**Notion of set of admissible boundary states**

Following Dubois and LeFloch’s approach based on sets of admissible boundary values [5], we define the *set of admissible boundary states*

$$\Phi_\gamma(u_b) := \{V_\infty / \text{There exists a solution } V_\gamma : [0, +\infty) \rightarrow \mathbb{R} \text{ to the boundary problem (4.1)}\}.$$

and now determine this set under various circumstances. One may not be able to establish directly that  $u_\gamma(0+)$  coincides with  $V_{\gamma,\infty}$  but, yet, we may conclude that sufficient information is deduced from the boundary layer analysis in the sense that existence and uniqueness is recovered at the level of the boundary Riemann problem. We consider the cases when the boundary is non-characteristic or the capillarity is vanishing.

**Increasing flux**

Suppose that

$$\lambda(u) > 0 \quad \text{for all } u \in \mathbb{R}. \tag{4.6}$$

The condition (4.6) implies that the flux-function  $f$  is one-to-one on the interval  $[0, \infty)$  and hence the condition (4.2) is equivalent to saying  $u_\gamma(0+) =$

$V_{\gamma,\infty}$ . It is also elementary to check that the viscous-capillary equation admits *no solution*  $V_\gamma = V_\gamma(y)$  except the trivial one  $V_\gamma(y) = u_b$ , hence

$$\Phi_\gamma(u_b) = \{u_b\}.$$

This concludes the proof that the Riemann problem (1.3) admits a weak solution and the boundary condition must be imposed in the strong sense

$$u(t, 0) = u_b. \quad (4.7)$$

### Decreasing flux

Suppose that

$$\lambda(u) < 0 \quad \text{for all } u \in \mathbb{R}. \quad (4.8)$$

The discussion is similar to the one in the increasing case, except that now the boundary value is not achieved by the limiting solution. It is elementary to see that any  $V_{\gamma,\infty}$  can now be achieved, that is, there exists a boundary layer connecting  $u_b$  to any  $V_{\gamma,\infty}$ , so that

$$\Phi_\gamma(u_b) = \mathbb{R}. \quad (4.9)$$

The Riemann problem (1.3) admits a weak solution and no boundary condition is necessary at  $y = 0$ .

### Viscous boundary layer

The case  $\alpha = 0$  was treated in [13] (see also the references therein and [8]) and leads to the following result:

$$\Phi_0(u_b) = \{u_b\} \cup \left\{ V_\infty / \frac{f(V_\infty) - f(k)}{V_\infty - k} < 0, \quad k \in I(u_b, V_\infty) \right\}.$$

In particular, we have the following two special cases:

- Case of a strictly convex function  $f$  tending to infinity at infinity and normalized so that  $f(0) = f'(0) = 0$ . Provided we normalize the boundary data so that  $u_b \geq 0$ , it follows that

$$\Phi_0(u_b) = (-\infty, u'_b) \cup \{u_b\},$$

where  $u'_b \leq 0$  is defined by  $f(u'_b) = f(u_b)$ .

- Case of a flux  $f$  having a single inflection point and tending to infinity at infinity and satisfying, say with  $f(0) = 0$  and  $f'(0) < 0$ , having a maximum at some  $u_M^- < 0$  and a minimum at  $u_m^+ > 0$ . Let us define  $u_M^+ > 0$  and  $u_m^- < 0$  by the condition  $f(u_m^-) = f(u_m^+)$  and  $f(u_M^-) = f(u_M^+)$ . Depending on the value of the boundary data  $u_b$  with respect to  $u_m^- < u_M^- < 0 < u_m^+ < u_M^+$ , we obtain the following boundary set (omitting certain isolated values):

- If  $u_b < u_m^-$ , then  $\Phi_0(u_b) = \{u_b\}$ .
- If  $u_b \in (u_m^-, u_M^-)$ , then  $\Phi_0(u_b) = \{u_b\} \cup [u_b^*, u_m^+]$ , where  $u_b^*$  is characterized by  $f(u_b) = f(u_b^*)$  and  $f'(u_b^*) < 0$ .
- If  $u_b \in (u_M^-, u_m^+)$ , then  $\Phi_0(u_b) = [u_M^-, u_m^+]$ .
- If  $u_b \in (u_m^+, u_M^+)$ , then  $\Phi_0(u_b) = [u_M^-, u_b^*]$ , where  $u_b^*$  is characterized by  $f(u_b) = f(u_b^*)$  and  $f'(u_b^*) < 0$ .
- If  $u_b > u_M^+$ , then  $\Phi_0(u_b) = \{u_b\}$ .

### Viscous–capillary boundary set

A full analysis of the boundary layer with viscosity and capillarity is not realistic, since even the traveling wave solutions are understood only under certain conditions on the flux and require a rather technical analysis; see [15]. Consequently, we propose to follow Dubois and LeFloch [5] who observed that, for scalar equations, the boundary set based on viscosity can be equivalently determined from the Riemann problem on the real line. We thus determine here the boundary layer set which is based on the nonclassical solutions to the Riemann problem described in [15], while conjecturing that it should coincide with the one defined from the boundary layer equations—except for certain exceptional values so that it is convenient to look at the closure of this set.

We assume that the flux  $f : \mathbb{R} \rightarrow \mathbb{R}$  is concave/convex in the following sense:

$$u f''(u) > 0 \quad (u \neq 0), \quad \lim_{u \rightarrow \pm\infty} f'(u) = +\infty.$$

Denote by  $\varphi^\sharp : \mathbb{R} \rightarrow \mathbb{R}$  the tangent function characterized by

$$f' \circ \varphi^\sharp(u) = \frac{f \circ \varphi^\sharp(u) - f(u)}{\varphi^\sharp(u) - u} \quad (u \neq 0).$$

Recall that an analysis of traveling wave solutions to the viscosity–capillarity model allows one to define a kinetic function  $\varphi^\sharp_\gamma : \mathbb{R} \rightarrow \mathbb{R}$  satisfying [15]

$$u \varphi^{-\sharp}(u) < u \varphi^b_\gamma(u) \leq u \varphi^\sharp_\gamma(u), \quad u \in \mathbb{R},$$

where  $\varphi^{-\sharp} : \mathbb{R} \rightarrow \mathbb{R}$  denotes the inverse of the function  $\varphi^\sharp$ . To this kinetic function we associate its companion  $\varphi^\sharp_\gamma : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $u \varphi^b < u \varphi^\sharp_\gamma$  for  $u \neq 0$  and

$$\frac{f \circ \varphi^\sharp_\gamma(u) - f(u)}{\varphi^\sharp_\gamma(u) - u} = \frac{f \circ \varphi^b_\gamma(u) - f(u)}{\varphi^b_\gamma(u) - u} \quad (u \neq 0).$$

We also assume  $f'(0) < 0$  and introduce  $u_m^- < u_M^- < 0 < u_m^+ < u_M^+$  as in the previous subsection.

The structure of the boundary set in presence of nonclassical shocks induced by viscosity and capillarity is much more involved. For definiteness, we begin with the (most interesting) case that  $u_b < 0$  with  $f'(u_b) > 0$  (entering boundary data) and its kinetic state  $\varphi^b_\gamma(u_b)$  has  $f' \circ \varphi^b_\gamma(u_b) > 0$  (entering kinetic image). To the boundary data we associate  $u_b^*$  defined by the conditions

$$f(u_b) = f(u_b^*), \quad f'(u_b^*) < 0.$$

To the kinetic image we also associate  $\varphi^b(u_b)^*$  by the conditions

$$f \circ \varphi^b_\gamma(u_b)^* = f \circ \varphi^b_\gamma(u_b), \quad f' \circ \varphi^b_\gamma(u_b)^* < 0.$$

We then distinguish between two cases, as follows:

- When  $\varphi^\sharp_\gamma(u_b) < \varphi^b_\gamma(u_b)^*$ , we find

$$\overline{\Phi_\gamma(u_b)} = \{u_b\} \cup \{\varphi^b_\gamma(u_b)\} \cup [u_b^*, \varphi^b(u_b)^*].$$

- When  $\varphi^b_\gamma(u_b)^* < \varphi^\sharp(u_b)$ , we find

$$\Phi_\gamma(u_b) = \{u_b\}.$$

Hence, it appears that the shock set *does depend* upon the kinetic function and, therefore, the shock layer equation can be thought of as a “composite” of classical as well as nonclassical shock layer equations.

## 5. Systems with Diffusion and Dispersion

### Formulation of the problem

Our method also applies to the boundary and initial value problem for a nonlinear hyperbolic system of conservation laws [4, 15, 18]

$$\partial_t u + \partial_x f(u) = 0, \quad (5.1)$$

where  $u = u(x, t) \in B(u_*, \delta_0)$  is the unknown, and  $B(u_*, \delta_0)$  denotes the open ball with center  $u_*$  and radius  $\delta_0$ , and  $f : B(u_*, \delta_0) \rightarrow R^N$  is a smooth mapping, such that  $A(u) := Df(u)$  admits  $N$  real and distinct eigenvalues

$$\lambda_1(u) < \dots < \lambda_N(u).$$

We denote by  $l_j(u)$  and  $r_j(u)$  corresponding basis of left and right eigenvectors normalised so that for all  $u \in B(u_*, \delta_0)$ ,

$$l_i(u) \cdot r_k(u) = \delta_{ik}$$

It is well known that weak solutions are not uniquely determined by boundary and initial data. In this paper we consider the simplest initial boundary value problem namely the Riemann initial boundary value problem. One very successful approach is to study the self similar zero-diffusion method. In the present paper, we consider a different regularization namely the self similar zero-diffusion-dispersion method. So, consider (5.1) in the quarter-plane  $x > 0, t > 0$  and assume that the data at  $x = 0$  and at  $t = 0$  are two constants,  $u_I$  and  $u_B (\in B(u_*, \delta_0))$ , respectively. Let us continue to denote the self-similar variable  $\frac{x}{t}$  by  $x$  itself.

$$-x u'_{\varepsilon, \gamma} + f(u_{\varepsilon, \gamma})' = \varepsilon u''_{\varepsilon, \gamma} + \gamma \varepsilon^2 u'''_{\varepsilon, \gamma}, \quad x > 0, \quad (5.2)$$

$$\lim_{x \rightarrow \infty} u_{\varepsilon, \gamma}(x) = u_I, \quad (5.3)$$

$$u_{\varepsilon, \gamma}(0) = u_B. \quad (5.4)$$

**The linearized problem**

In this section we study a linearized version of the problem (5.2)-(5.4). Given data  $u_I, u_B \in B(u_*, \delta)$ , the unknown function  $u_{\varepsilon, \gamma}$  will take its values in  $B(u_*, C_* \delta)$  for  $C_* \delta < \delta_0$ . We assume that (5.1) is strict hyperbolic at  $u_*$  but the characteristic fields are not necessarily genuinely nonlinear nor linearly degenerate. Further we assume that the boundary  $x = 0$  is not characteristic. Thus we assume that for  $\delta_0$  small enough, the eigenvalues satisfies

$$\lambda_1^m < \lambda_1^M < \lambda_2^m < \dots < \lambda_p^M < 0 < \lambda_{p+1}^m < \dots < \lambda_{N-1}^M < \lambda_N^m < \lambda_N(u) < \lambda_N^M,$$

for  $u \in B(u_*, \delta_0)$ . Since  $Df(u)$  depends smoothly upon  $u$ , one can ensure that  $\lambda_k^M - \lambda_k^m = O(\delta_0)$ .

Given  $u_I, u_B \in B(u_*, \delta)$  for some  $\delta < \delta_0$ , we are going to construct a solution  $u^\varepsilon$  of (5.2)-(5.4) having uniformly bounded variation, i.e.,

$$TV(u_{\varepsilon, \gamma}) := \int_0^\infty |u'_{\varepsilon, \gamma}(x)| dx = O(1).$$

We shall henceforth write  $u_{\varepsilon, \gamma}$  as  $u$  itself. We set

$$u'(x) = \sum_{k=1}^N a_k(x) r_k(u(x)),$$

where  $a_k$  ( $k = 1, \dots, N$ ) satisfy  $a_k(x) = l_k(u(x))$ .  $u'(x)$  and are determined by the system

$$\gamma \varepsilon^2 a_j'' + \varepsilon a_j' + (x - \lambda_j(u)) a_j = D_1(a) + D_2(a, a'), \tag{5.5}$$

where

$$\begin{aligned} D_k^1(a) &= -l_k(u) \cdot \sum_{j,i=1}^N a_i a_j (Dr_i(u) \cdot r_j(u)), \\ D_k^2(a, a') &= -l_k(u) \cdot \left( \sum_{k,i} (2a_i a_j' + a_i' a_j) (Dr_i \cdot r_j)(u) \right. \\ &\quad \left. + \sum_{k,i,l=1}^N a_i a_k a_l D(Dr_i \cdot r_k) r_l(u) \right). \end{aligned} \tag{5.6}$$

Now we look for solutions of (5.5) having an asymptotic expansion of the form

$$a_i(x) = \tau_i \phi_i(x) + \theta_i(x), \quad x \in [0, L] \tag{5.7}$$

The vector  $\tau = (\tau_1, \dots, \tau_N)$  encodes the wave-strengths and  $\theta_i$  is of second-order with respect to  $\tau$ .

From (5.5) it then follows that  $\phi_i$  satisfy the decoupled homogeneous equations

$$\gamma \varepsilon^2 \phi_i'' + \varepsilon \phi_i' + (x - \lambda_i(u)) \phi_i = 0, \quad i = 1, \dots, N, \tag{5.8}$$

whereas  $\theta_i$  satisfy the following coupled system of N inhomogeneous equations

$$\gamma \varepsilon^2 \theta_i'' + \varepsilon \theta_i' + (x - \theta_i(u)) \theta_i = \varepsilon D_i^1(a) + \gamma \varepsilon^2 D_i^2(a, a'), \quad i = 1, \dots, N. \tag{5.9}$$

Let us therefore consider the linear equation

$$\gamma \varepsilon^2 \phi_i'' + \varepsilon \phi_i' + (x - \lambda_i(x)) \phi_i = 0.$$

We have already seen one of the solutions, we shall call it  $\phi_i$ , for the above second order equation in Theorem 2.2. The other solution is described below. We write

$$\begin{aligned} q_i(x) &= \frac{L-x}{2\gamma} + \int_x^L \sqrt{\frac{\mu_i(y)}{\gamma}} dy \\ &= \frac{1}{2\gamma} \int_x^L \left( 1 + \sqrt{1 + 4\gamma(\lambda_i(y) - y)} \right) dy. \end{aligned}$$

**Theorem 5.1.** *The linearized equation admits a smooth solution  $\check{\psi}_i$  defined in  $[0, L]$  satisfying the following asymptotic formula*

$$\check{\psi}_i(x) = \frac{1 + \check{\Psi}_i(x)}{(4\gamma\mu_i(x))^{\frac{1}{4}}} e^{\frac{q_i(x) - q_i(\rho_i)}{\varepsilon}}, \tag{5.10}$$

where

$$|\check{\Psi}_i(x)| + \frac{\varepsilon\sqrt{\gamma}}{2\mu_i(x)^{\frac{1}{2}}} |\check{\Psi}_i'(x)| \leq \varepsilon\sqrt{\gamma}K, \tag{5.11}$$

with  $K, \rho_i$  as chosen in Theorem 2.2.

Let us introduce the notation

$$\check{\phi}_i(x) := -\frac{\check{\psi}_i(x)}{\text{Det}(\hat{\phi}_i, \check{\psi}_i)(x)}, \quad \hat{\psi}_i(x) := \frac{\hat{\phi}_i(x)}{\text{Det}(\hat{\phi}_i, \check{\psi}_i)(x)}, \tag{5.12}$$

where  $\text{Det}(\hat{\phi}_i, \check{\psi}_i) := \hat{\phi}_i \check{\psi}'_i - \hat{\phi}'_i \check{\psi}_i$ . The following theorem, which describes an asymptotic form for  $\check{\phi}_i$  and  $\hat{\psi}_i$ , will also be of importance later.

**Theorem 5.2.** *Assuming that*

$$K \leq C \frac{\gamma^{\frac{1}{2}}}{\varepsilon},$$

one has

$$\text{Det}(\hat{\phi}_i, \check{\psi}_i)(x) = \text{Det}(\hat{\phi}_i, \check{\psi}_i)(\rho_i) e^{\frac{\rho_i - x}{\gamma \varepsilon}}, \quad x \in [0, L]$$

and, for some constant  $C > 1$ ,

$$\frac{1}{C \gamma \varepsilon} \leq |\text{Det}(\hat{\phi}_i, \check{\psi}_i)(\rho_i)| \leq \frac{C}{\gamma \varepsilon}.$$

Also up to constant multiplication factors, the functions  $\check{\phi}_i, \hat{\psi}_i$  have the form

$$\check{\phi}_i(x) = \gamma \varepsilon \frac{1 + \check{\Phi}_i(x)}{(\gamma \mu_i(x))^{\frac{1}{4}}} e^{\frac{p_i(\rho_i) - p_i(x)}{\varepsilon}}, \quad \hat{\psi}_i(x) = \gamma \varepsilon \frac{1 + \hat{\Psi}_i(x)}{(\gamma \mu_i(x))^{\frac{1}{4}}} e^{\frac{q_i(\rho_i) - q_i(x)}{\varepsilon}} \tag{5.13}$$

with  $\hat{\Psi}_i, \check{\Phi}_i$  satisfying the bounds as in Theorems 2.2 and 5.1.

Now given a continuous and bounded source  $S : [0, L] \rightarrow \mathbb{R}$ , let us consider the non-homogeneous equation

$$\theta''(x) + \frac{1}{\gamma \varepsilon} \theta'(x) + \frac{1}{\gamma \varepsilon^2} (x - \lambda(x)) \theta(x) = S(x), \quad x \in [0, L]. \tag{5.14}$$

Then it can be proved that the general bounded solution  $\theta$  of the above equation can be represented as

$$\theta(x) = \check{\psi}(x) \int_0^x \hat{\psi}(y) S(y) dy + \hat{\phi}(x) \int_c^x \check{\phi}(y) S(y) dy, \tag{5.15}$$

where  $c$  is an arbitrary non-negative constant.

**Wave interaction estimates**

To begin with let us fix some  $i = 1, 2, \dots, N$  and some smooth function  $v : [0, L] \rightarrow \mathcal{B}(u_*, \delta_0)$  satisfying the condition

$$|v'(x)| \leq \frac{C}{\varepsilon}, \quad x \in [0, L]. \tag{5.16}$$

Let us then consider the linearized equation

$$\gamma\varepsilon^2\phi_i'' + \varepsilon\phi_i' + (x - \lambda_i(v))\phi_i = 0$$

and denote by  $\mu_i, p_i, \hat{\phi}_i, q_i, \check{\psi}_i, \check{\phi}_i$  and  $\hat{\psi}_i$  the corresponding solutions. More specifically we have

$$\begin{aligned} \mu_i(x) &= \lambda_i(x) - x + \frac{1}{4\gamma}, \\ p_i(x) &= \frac{L-x}{2\gamma} + \int_L^x \sqrt{\frac{\mu_i(y)}{\gamma}} dy, \quad q_i(x) = \frac{L-x}{2\gamma} + \int_x^L \sqrt{\frac{\mu_i(y)}{\gamma}} dy, \end{aligned} \tag{5.17}$$

where the coefficients  $\mu_i$  satisfy the bounds

$$\begin{aligned} \gamma\mu_i(x) &\geq c > 0, \quad x \in [0, L], \\ |\mu_i'(x)| &\leq \frac{C}{\varepsilon}, \quad x \in [0, L]. \end{aligned} \tag{5.18}$$

Let  $p_i(x, \rho) := p_i(x) - p_i(\rho)$  and  $q_i(x, \rho) := q_i(x) - q_i(\rho)$ . Then upto constant multiplying factors, we have

$$\begin{aligned} \hat{\phi}_i(x) &= \frac{1 + \hat{\Phi}_i(x)}{(\gamma\mu_i(x))^{\frac{1}{4}}} e^{\frac{p_i(x, \rho_i)}{\varepsilon}}, & \check{\phi}_i(x) &= \gamma\varepsilon \frac{1 + \check{\Phi}_i(x)}{(\gamma\mu_i(x))^{\frac{1}{4}}} e^{-\frac{p_i(x, \rho_i)}{\varepsilon}}, \\ \hat{\psi}_i(x) &= \gamma\varepsilon \frac{1 + \hat{\Psi}_i(x)}{(\gamma\mu_i(x))^{\frac{1}{4}}} e^{-\frac{q_i(x, \rho_i)}{\varepsilon}}, & \check{\psi}_i(x) &= \frac{1 + \check{\Psi}_i(x)}{(\gamma\mu_i(x))^{\frac{1}{4}}} e^{\frac{q_i(x, \rho_i)}{\varepsilon}}, \end{aligned} \tag{5.19}$$

where  $\rho_i$  is as defined before. Further we assume that the constant  $K$  appearing in Theorem 2.2 satisfies the condition  $0 < K \leq \frac{\gamma^{\frac{1}{2}}}{2\varepsilon}$  which further leads to the fact that for  $x \in [0, L]$ , we have

$$|\hat{\Phi}_i(x)|, |\check{\Phi}_i(x)|, |\hat{\Psi}_i(x)|, |\check{\Psi}_i(x)| \leq \frac{\gamma}{2} \leq \frac{1}{2},$$

for all  $i = 1, 2, \dots, N$ . Similar inequalities hold for their derivatives as well.

Let us consider the normalized function

$$\phi_i(x) := \frac{\hat{\phi}_i(x)}{I(\hat{\phi}_i)}, \text{ where } I(\hat{\phi}_i) = \int_0^L \hat{\phi}_i(y) dy, \quad i = 1, 2, \dots, N.$$

Let us define a weight function  $\omega_\gamma$  by

$$\omega_\gamma(x) = \frac{1 + x + \lambda_{1+}^m}{\gamma}, \quad x \in [0, L].$$

**Proposition 5.3.** *There exist constants  $C, c > 0$ , independent of  $\varepsilon$  and  $\gamma$ , such that for all  $i = 1, 2, \dots, N$  and  $x \in [0, L]$*

$$0 < \phi_i \leq \frac{C}{\varepsilon}, \tag{5.20}$$

$$|\phi'_i(x)| \leq \frac{C}{\varepsilon} \omega_\gamma(x) \phi_i(x), \tag{5.21}$$

$$\phi_i(x) \leq \frac{C}{\varepsilon} e^{-\frac{\varepsilon}{\varepsilon}(x-\lambda_{i+}^M)^2}, \quad x \in [\lambda_{i+}^M, L] \tag{5.22}$$

**Proof.** We have already seen that  $0 < \hat{\phi}_i(x) < \frac{2}{(4\gamma\mu_i(x))^{\frac{1}{4}}}$  and  $\frac{\varepsilon}{C} \leq \int_0^L \hat{\phi}_i(x) \leq C$  which imply that

$$0 < \phi_i(x) < \frac{2C}{\varepsilon(4\gamma\mu_i(x))^{\frac{1}{4}}}.$$

Now using the fact that  $\gamma\mu_i(x) \geq c > 0$  (see (5.18)), we have  $0 \leq \phi_i(x) \leq \frac{C}{\varepsilon}$ .

For  $x \in [\lambda_{i+}^M, L]$ , we have

$$\phi_i(x) = \frac{\hat{\phi}_i(x)}{\int_0^L \hat{\phi}_i(y) dy} \leq \frac{C}{\varepsilon} \hat{\phi}_i(x) \leq \frac{C}{\varepsilon} \frac{1 + \hat{\Phi}_i(x)}{(4\gamma\mu(x))^{\frac{1}{4}}} e^{\frac{p_i(x) - p_i(\rho_i)}{\varepsilon}} \leq \frac{C}{\varepsilon} e^{-\frac{\varepsilon}{\varepsilon}(x-\lambda_{i+}^M)^2}$$

and therefore

$$\phi_i(x) \leq \frac{C}{\varepsilon} e^{-\frac{\varepsilon}{\varepsilon}(x-\lambda_{i+}^M)^2}, \quad x \in [\lambda_{i+}^M, L]$$

which proves (5.22).

Differentiating  $\phi_i(x)$ , we get  $\phi'_i(x) = \frac{\hat{\phi}'_i(x)}{I(\hat{\phi}_i)}$  and hence

$$\frac{\phi'_i(x)}{\phi_i(x)} = \frac{\hat{\phi}'_i(x)}{\hat{\phi}_i(x)} = \frac{\hat{\Phi}'_i(x)}{1 + \hat{\Phi}_i(x)} + \frac{p'_i(x)}{\varepsilon} - \frac{\mu'_i(x)}{4\mu_i(x)} := I_1 + I_2 + I_3.$$

To estimate  $I_2$ , we begin by noting that  $p'_i(x) = \frac{1}{2\gamma}(-1 + \sqrt{1 + 4\gamma(\lambda_i(x) - x)})$  and hence

$$|p'_i(x)| \leq \frac{1}{2\gamma} | -1 + \sqrt{1 + 4\gamma(\lambda_i(x) - x)} |.$$

Using the fact that  $4\gamma(\lambda_i(x) - x) > -1$ ,  $x \in [0, L]$  and the inequality  $| -1 + \sqrt{1 + \alpha} | \leq |\alpha|$ ,  $\alpha > -1$ , we then obtain

$$|p'_i(x)| \leq \frac{1}{2\gamma} |4\gamma(\lambda_i(x) - x)| \leq 2(|x| + |\lambda_i|) \leq C\omega_\gamma(x).$$

$I_3$  can be estimated as

$$|I_3| = \left| \frac{\mu'_i(x)}{4\mu_i(x)} \right| \leq \frac{C}{\varepsilon} \leq \frac{C}{\varepsilon} \omega_\gamma(x).$$

Let us next estimate  $I_1$ . First we note that  $\frac{1}{1 + \hat{\Phi}_i(x)} \leq 2$ . Also, we have

$$|\hat{\Phi}'_i(x)| \leq Ck\sqrt{\mu_i} \leq \frac{C}{\varepsilon} \sqrt{\gamma\mu_i}.$$

Now using the inequality  $-1 + \sqrt{1 + \alpha} \leq \frac{\alpha}{2}$ ,  $\alpha > -1$ , we have

$$\begin{aligned} \sqrt{\gamma\mu_i} &= \frac{1}{2} \sqrt{1 + 4\gamma(\lambda_i(x) - x)} \leq \frac{1}{2} + \gamma(\lambda_i(x) - x) \\ &\leq 1 + |x| + |\lambda_i(x)| \quad (\text{using } \gamma < 1) \\ &\leq (1 + |x| + |\lambda_i(x)|) \frac{1}{\gamma} \leq C\omega_\gamma(x). \end{aligned}$$

Hence we get  $|I_i| \leq \frac{C}{\varepsilon} \omega_\gamma(x)$ . □

Observe in passing that for some constants  $0 < C_1 < C_2$ , the following relations hold:

$$C_1(1 + \gamma\omega_\gamma) \leq \sqrt{\gamma\mu_i} \leq C_2(1 + \gamma\omega_\gamma).$$

Henceforth, we shall assume that, for all  $x \in [0, L]$ ,

$$\mu_i(x) = \lambda_i(x) - x + \frac{1}{4\gamma} \geq \frac{1}{8\gamma}. \tag{5.23}$$

We note that this is a slightly stronger condition on  $\gamma$  than that we had assumed till now. Let us fix a number  $c_i$  such that

$$\begin{cases} c_i \in (\lambda_i^m, \lambda_i^M), & \text{if } i \geq p + 1, \\ c_i = 0, & \text{if } i \leq p, \end{cases}$$

and consider the interaction terms defined, for  $i, j, k, l = 1, \dots, N$ , by

$$\begin{aligned} F_{ijk}^1(x) &= \frac{\hat{\phi}_i(x)}{\gamma\varepsilon} \int_{c_i}^x \check{\phi}_i \phi_j \phi_k \, dy, \\ F_{ijk}^{2,1}(x) &= \hat{\phi}_i(x) \int_{c_i}^x \check{\phi}_i \phi_j \phi'_k \, dy, \quad F_{ijkl}^{2,2}(x) = \hat{\phi}_i(x) \int_{c_i}^x \check{\phi}_i \phi_j \phi_k \phi_l \, dy, \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} G_{ijk}^1(x) &= \frac{\check{\psi}_i(x)}{\gamma\varepsilon} \int_0^x \hat{\psi}_i \phi_j \phi_k \, dy, \\ G_{ijk}^{2,1}(x) &= \check{\psi}_i(x) \int_0^x \hat{\psi}_i \phi_j \phi'_k \, dy, \quad G_{ijkl}^{2,2}(x) = \check{\psi}_i(x) \int_0^x \hat{\psi}_i \phi_j \phi_k \phi_l \, dy. \end{aligned} \tag{5.25}$$

With the above notation, we have the following result.

**Theorem 5.4.** *For some fixed, sufficiently small,  $\delta_0$  and  $\gamma_0$ , there exists a uniform constant  $C > 0$  such that for any smooth function  $v : [0, L] \rightarrow \mathcal{B}(u_*, \delta_0)$  satisfying the condition*

$$|v'(x)| \leq \frac{C}{\varepsilon}, \quad x \in [0, L],$$

one has (for all  $x \in [0, L]$  and all  $i, j, k, l$ )

$$|F_{ijk}^1(x)| + |F_{ijk}^{2,1}(x)| + |F_{ijkl}^{2,2}(x)| \leq C \sum_{m=1}^N \phi_m(x), \tag{5.26}$$

$$|G_{ijk}^1(x)| + |G_{ijk}^{2,1}(x)| + |G_{ijkl}^{2,2}(x)| \leq C\gamma \sum_{m=1}^N \phi_m(x). \tag{5.27}$$

**Proof.** To begin with we note that an application of the Cauchy-Schwartz inequality leads to

$$\begin{aligned} \sum_{i,j,k,l=1}^N |F_{ijk}^1(x)| + |F_{ijk}^{2,1}(x)| + |F_{ijkl}^{2,2}(x)| &\leq C \sum_{i,j=1}^N \tilde{F}_{ij}(x), \\ \sum_{i,j,k,l=1}^N |G_{ijk}^1(x)| + |G_{ijk}^{2,1}(x)| + |G_{ijkl}^{2,2}(x)| &\leq C\gamma \sum_{i,j=1}^N \tilde{G}_{ij}(x), \end{aligned}$$

where

$$\begin{aligned} \tilde{F}_{ij}(x) &= \frac{\hat{\phi}_i(x)}{\gamma\varepsilon} \int_{c_i}^x \check{\phi}_i(y)\phi_j^2(y)(1 + \gamma\omega_\gamma(y)) dy, \\ \tilde{G}_{ij}(x) &= \frac{\check{\psi}_i(x)}{\gamma\varepsilon} \int_0^x \hat{\psi}_i(y)\phi_j^2(y)(1 + \gamma\omega_\gamma(y)) dy. \end{aligned} \tag{5.28}$$

Hence it would be enough to prove the required estimates (5.26)-(5.27) for  $\tilde{F}_{ij}$  and  $\tilde{G}_{ij}$  respectively.

**Estimating  $\tilde{G}_{ij}$**

Let us define  $D_{ij}(y) := -q_i(y, \rho_i) + 2p_j(y, \rho_j)$ . Then for all  $x \in [0, L]$ , it follows using (5.23) that

$$\begin{aligned} D'_{ij}(x) = -q'_i(x) + 2p'_j(x) &= \frac{1}{2\gamma} + \sqrt{\frac{\mu_i(x)}{\gamma}} - \frac{1}{\gamma} + 2\sqrt{\frac{\mu_j(x)}{\gamma}} \\ &= -\frac{1}{2\gamma} + \sqrt{\frac{\mu_i(x)}{\gamma}} + 2\sqrt{\frac{\mu_j(x)}{\gamma}} \\ &\geq -\frac{1}{2\gamma} + \frac{3}{2\gamma\sqrt{2}} \geq \frac{1}{2\gamma} > 0. \end{aligned} \tag{5.29}$$

Also since the maximal value of  $1 + \gamma\omega_\gamma$  in the interval  $[0, L]$  is bounded, it can also be seen that

$$1 + \gamma\omega_\gamma(x) \leq C\gamma D'_{ij}(x) \tag{5.30}$$

Then, we have

$$\begin{aligned}
 |\tilde{G}_{ij}(x)| &\leq C(\gamma\mu_i(x))^{-\frac{1}{4}} e^{\frac{q_i(x,\rho_i)}{\varepsilon}} \\
 &\quad \times \int_0^x (\gamma\mu_i(y))^{-\frac{1}{4}} e^{-\frac{q_i(y,\rho_i)}{\varepsilon}} \left( \frac{e^{\frac{p_j(y,\rho_j)}{\varepsilon}}}{(\gamma\mu_j(y))^{\frac{1}{4}} I(\hat{\phi}_j)} \right)^2 (1 + \gamma\omega_\gamma(y)) \, dy \\
 &\leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{e^{\frac{q_i(x,\rho_i)}{\varepsilon}}}{I(\hat{\phi}_j)^2} \int_0^x e^{-\frac{q_i(y,\rho_i)}{\varepsilon}} e^{\frac{2p_j(y,\rho_j)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} \, dy \\
 &\leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{e^{\frac{q_i(x,\rho_i)}{\varepsilon}}}{I(\hat{\phi}_j)^2} \int_0^x e^{\frac{D_{ij}(y)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} \, dy.
 \end{aligned}$$

Now, we obtain

$$\begin{aligned}
 G_{ij}(x) &:= (1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{e^{\frac{q_i(x,\rho_i)}{\varepsilon}}}{I(\hat{\phi}_j)^2} \int_0^x e^{\frac{D_{ij}(y)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} \, dy \\
 &= (1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{\phi_j(x)^2}{\hat{\phi}_j(x)^2} e^{\frac{q_i(x,\rho_i)}{\varepsilon}} \int_0^x e^{\frac{D_{ij}(y)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} \, dy \\
 &\leq C(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \phi_j(x)^2 e^{-\frac{D_{ij}(x)}{\varepsilon}} \int_0^x e^{\frac{D_{ij}(y)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} \, dy \\
 &\leq C(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \phi_j(x)^2 e^{-\frac{D_{ij}(x)}{\varepsilon}} \\
 &\quad \times \int_0^x \frac{1 + \gamma\omega_\gamma(y)}{D'_{ij}(y)} D'_{ij}(y) e^{\frac{D_{ij}(y)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{3}{2}} \, dy \\
 &\leq C\gamma\varepsilon(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \phi_j(x)^2 e^{-\frac{D_{ij}(x)}{\varepsilon}} \int_0^x \frac{D'_{ij}(y)}{\varepsilon} e^{\frac{D_{ij}(y)}{\varepsilon}} \, dy,
 \end{aligned}$$

since  $(1 + \gamma\omega_\gamma(y))^{-\frac{3}{2}} \leq 1$ , so that

$$\begin{aligned}
 G_{ij}(x) &\leq C\gamma\varepsilon(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \phi_j(x)^2 e^{-\frac{D_{ij}(x)}{\varepsilon}} e^{\frac{D_{ij}(x)}{\varepsilon}} \\
 &\quad - C\gamma\varepsilon(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \phi_j(x)^2 e^{-\frac{D_{ij}(x)}{\varepsilon}} e^{\frac{D_{ij}(0)}{\varepsilon}} \\
 &\leq \underbrace{C\gamma\varepsilon(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \phi_j(x)^2}_{P_1} - C\gamma\varepsilon(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \phi_j(x)^2 e^{\frac{1}{\varepsilon}[D_{ij}(0) - D_{ij}(x)]} \\
 &\leq P_1
 \end{aligned}$$

To estimate  $P_1$ , we note that

$$\begin{aligned} \varepsilon\phi_j(x)^2 &= \varepsilon \frac{\hat{\phi}_j(x)^2}{I(\hat{\phi}_j)^2} = \frac{\varepsilon}{I(\hat{\phi}_j)} \frac{\hat{\phi}_j(x)^2}{I(\hat{\phi}_j)} \\ &\leq C\phi_j(x)\hat{\phi}_j(x) \leq C\phi_j(x) \frac{1}{(\gamma\mu_j(x))^{\frac{1}{4}}} \quad [\text{since } p_j(x, \rho_j) \leq 0] \\ &\leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}}\phi_j(x) \end{aligned} \tag{5.31}$$

and, therefore,

$$|P_1| \leq C\gamma\phi_j(x).$$

Hence we have  $|G_{ij}(x)| \leq C\gamma\phi_j(x)$ . and therefore

$$|\tilde{G}_{ij}(x)| \leq C\gamma\phi_j(x).$$

**Estimating  $\tilde{F}_{ij}$**

We begin by observing that

$$\tilde{F}_{ij}(x) \leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{e^{\frac{p_i(x, \rho_i)}{\varepsilon}}}{I(\hat{\phi}_j)^2} \int_{c_i}^x e^{\frac{-p_i(y, \rho_i) + 2p_j(y, \rho_j)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy$$

Then using the fact that  $p_i(x, \rho) - p_i(y, \rho) = p_i(x, y)$  we have

$$\begin{aligned} \tilde{F}_{ij}(x) &\leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{e^{\frac{2p_j(x, \rho_j)}{\varepsilon}}}{I(\hat{\phi}_j)^2} \\ &\quad \times \int_{c_i}^x e^{\frac{[-p_i(y, \rho_i) + p_i(x, \rho_i) + 2p_j(y, \rho_j) - 2p_j(x, \rho_j)]}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy \\ &\leq C(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_j(x)^2}{I(\hat{\phi}_j)^2} \int_{c_i}^x e^{\frac{p_i(x, y) - 2p_j(x, y)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy. \end{aligned}$$

Let us define the term  $E_{ij}(x, y) := p_i(x, y) - 2p_j(x, y)$  and define

$$F_{ij}(x) := (1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_j(x)^2}{I(\hat{\phi}_j)^2} \int_{c_i}^x e^{\frac{1}{\varepsilon}E_{ij}(x, y)} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy.$$

Thus it is enough to estimate the term  $F_{ij}$ .

First let us consider the case when  $i = j$ . Then  $E_{ii}(x, y) = -p_i(x, y)$

and

$$\begin{aligned} F_{ii}(x) &= (1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_i^2(x)}{I(\hat{\phi}_i)^2} \int_{c_i}^x e^{\frac{1}{\varepsilon}E_{ii}(x,y)}(1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy \\ &= (1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_i^2(x)}{I(\hat{\phi}_i)^2} \int_{c_i}^x e^{-\frac{1}{\varepsilon}p_i(x,y)}(1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy, \end{aligned}$$

which implies

$$\begin{aligned} |F_{ii}(x)| &\leq C \frac{(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} e^{\frac{2}{\varepsilon}p_i(x,\rho_i)}}{(\gamma\mu_i(x))^{\frac{1}{2}} I(\hat{\phi}_i)^2} \left| \int_{c_i}^x e^{-\frac{1}{\varepsilon}p_i(x,y)}(1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy \right| \\ &\leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{e^{\frac{1}{\varepsilon}p_i(x,\rho_i)}}{I(\hat{\phi}_i)^2} \left| \int_{c_i}^x e^{\frac{1}{\varepsilon}p_i(y,\rho_i)}(1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy \right| \\ &\leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{e^{\frac{1}{\varepsilon}p_i(x,\rho_i)}}{I(\hat{\phi}_i)} \frac{\left| \int_{c_i}^x e^{\frac{1}{\varepsilon}p_i(y,\rho_i)}(1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy \right|}{I(\hat{\phi}_i)} \\ &\leq C\phi_i(x) \int_0^L \phi_i(y) dy \leq C\phi_i(x). \end{aligned}$$

Next we consider the cases  $i < j$  and skip the case  $i > j$  since the proofs are analogous. For  $\alpha > 0$ , let us define

$$\begin{aligned} \Lambda_{ij}^m &= \min_{x \in [0, L]} \Lambda_{ij}(x) - \alpha\delta_0, \\ \Lambda_{ij}^M &= \sup_{x \in [0, L]} \Lambda_{ij}(x) + \alpha\delta_0, \end{aligned} \tag{5.32}$$

where  $\Lambda_{ij}(x) := 2\lambda_j(x) - \lambda_i(x)$ . Hence for  $\alpha$  sufficiently small,

$$\begin{aligned} \lambda_j^M &< \Lambda_{ij}^m \leq \Lambda_{ij}(x) \leq \Lambda_{ij}^M < L, \quad x \in [0, L] \\ \Lambda_{ij}^M - \Lambda_{ij}^m &= \mathcal{O}(\delta_0). \end{aligned} \tag{5.33}$$

Let us choose a point  $a_{ij} \in (\lambda_j^M, \Lambda_{ij}^m)$  such that

$$a_{ij} = \Lambda_{ij}^m - \kappa,$$

where  $\kappa > 0$  is sufficiently small. In case  $\Lambda_{ij}^m < 0$ , we shall choose  $a_{ij} = 0$ .

**Case  $i < j$  and  $j \geq p + 1$ .**

**Case 1:**  $0 < \lambda_i < \lambda_j$ . Let us briefly collect some properties of  $E_{ij}(x, y)$  to be used later. For a fixed  $\rho \in [0, L]$ , for any  $x \in [0, \lambda_j^m]$ , using the fact that  $\mu_i(x) \leq \mu_j(x)$ , it follows that

$$\begin{aligned} \partial_1 E_{ij}(x, \rho) &= \frac{1}{2\gamma} - 2\sqrt{\frac{\mu_j(x)}{\gamma}} + \sqrt{\frac{\mu_i(x)}{\gamma}} \\ &\leq -p'_j(x) \leq -\omega_\gamma(x). \end{aligned}$$

Now for  $x \in [0, L]$ , using the expansion (with respect to  $\gamma$ )

$$\sqrt{1 + 4\gamma(\lambda_i(x) - x)} = 1 + 2\gamma(\lambda_i(x) - x) + \mathcal{O}(\gamma^2),$$

we obtain

$$\partial_1 E_{ij}(x, \rho) = 2(x - (2\lambda_j(x) - \lambda_i(x))) + \mathcal{O}(\gamma).$$

Using these we find that  $\partial_1 E_{ij}(x, \rho)$  is strictly less than a negative constant, when  $x \leq a_{ij}$  and is negative (positive) for  $x < \Lambda_{ij}^m$  ( $x > \Lambda_{ij}^M$ ). Therefore, for each  $\rho \in [0, L]$ ,  $E_{ij}(x, \rho)$  achieves its minimum at a point  $\rho_{ij} \in [\Lambda_{ij}^m, \Lambda_{ij}^M]$ .

It can then be easily checked that

$$E_{ij}(x, \rho_{ij}) \geq 0, \quad x \in [0, L].$$

Also when  $\Lambda_{ij}^M < 0$ , it follows that

$$E_{ij}(x, 0) \geq 0.$$

**Let us suppose that**  $x \in [0, a_{ij}]$ . Using the fact  $E_{ij}(x, y) = E_{ij}(x, \rho_{ij}) - E_{ij}(y, \rho_{ij})$ ,

$$\begin{aligned} |F_{ij}(x)| &\leq C(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_j^2(x)}{I(\hat{\phi}_j)^2} e^{\frac{E_{ij}(x, \rho_{ij})}{\varepsilon}} \\ &\quad \times \left| \int_{c_i}^x \frac{\partial_1 E_{ij}(y, \rho_{ij})}{\partial_1 E_{ij}(y, \rho_{ij})} e^{-\frac{E_{ij}(y, \rho_{ij})}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} dy \right|. \end{aligned}$$

Similarly as in the case of  $D'_{ij}$  (see (5.30)), we can show that

$$\left| \frac{1 + \gamma\omega_\gamma(x)}{\partial_1 E_{ij}(x, \rho_{ij})} \right| \leq C, \quad x \in [0, a_{ij}].$$

Using this, we find that

$$\begin{aligned} |F_{ij}(x)| &\leq C(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_j^2(x)}{I(\hat{\phi}_j)^2} e^{\frac{E_{ij}(x, \rho_{ij})}{\varepsilon}} \\ &\quad \times \left| \int_{c_i}^x \partial_1 E_{ij}(y, \rho_{ij}) e^{-\frac{E_{ij}(y, \rho_{ij})}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{3}{2}} dy \right| \\ &\leq C\varepsilon(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_j^2(x)}{I(\hat{\phi}_j)^2} e^{\frac{E_{ij}(x, \rho_{ij})}{\varepsilon}} \left( e^{-\frac{E_{ij}(x, \rho_{ij})}{\varepsilon}} + e^{-\frac{E_{ij}(c_i, \rho_{ij})}{\varepsilon}} \right) \\ &= C(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \left( \varepsilon \frac{\hat{\phi}_j^2(x)}{I(\hat{\phi}_j)^2} + \frac{\varepsilon(1 + \gamma\omega_\gamma(x))^{-1}}{I(\hat{\phi}_j)^2} e^{\frac{E_{ij}(x, c_i) + 2p_j(x, \rho_j)}{\varepsilon}} \right) \\ &:= C(1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} (T_1 + T_2). \end{aligned} \tag{5.34}$$

The term  $T_1$  can be estimated as in the case of  $P_1$  above (see (5.31)). For the term  $T_2$  we proceed as follows

$$\begin{aligned} T_2 &\leq C\varepsilon \frac{(1 + \gamma\omega_\gamma(x))^{-1}}{I(\hat{\phi}_j)^2} e^{\frac{E_{ij}(x, c_i) + 2p_j(x, \rho_j) - p_i(x, \rho_i)}{\varepsilon}} I(\hat{\phi}_i) (1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_j(x)}{I(\hat{\phi}_i)} \\ &\leq C\varepsilon \frac{(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}}}{I(\hat{\phi}_j)^2} e^{\frac{E_{ij}(x, c_i) + 2p_j(x, \rho_j) - p_i(x, \rho_i)}{\varepsilon}} I(\hat{\phi}_i) \phi_i(x). \end{aligned} \tag{5.35}$$

Now , we have

$$\begin{aligned} &E_{ij}(x, c_i) + 2p_j(x, \rho_j) - p_i(x, \rho_i) \\ &= p_i(x, c_i) - 2p_j(x, c_i) + 2p_j(x, \rho_j) - p_i(x, \rho_i) \\ &= p_i(\rho_i, c_i) - 2p_j(\rho_j, c_i) - 2p_j(\rho_i, c_i) + 2p_j(\rho_i, c_i) \\ &= E_{ij}(\rho_i, c_i) + 2p_j(\rho_i, \rho_j) \end{aligned}$$

and therefore

$$T_2 \leq (1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \frac{C}{\varepsilon} e^{\frac{E_{ij}(\rho_i, c_i) + 2p_j(\rho_i, \rho_j)}{\varepsilon}} \phi_i(x).$$

Thus it will be enough if we can prove that  $E_{ij}(\rho_i, c_i) + 2p_j(\rho_i, \rho_j)$  is strictly

less than a negative number.

Now using the fact that for  $x, \rho \in [\lambda_i^m, L]$ , the partial derivative  $|\partial_1 E_{ij}(x, \rho)|$  is bounded by a positive constant that is independent of  $\gamma$  and the fact that  $\rho_i, c_i \in [\lambda_i^m, \lambda_i^M]$ , we have

$$|E_{ij}(\rho_i, c_i)| \leq \max_{x \in [\lambda_i^m, \lambda_i^M]} \{|\partial_1 E_{ij}(x, c_i)|\} |\rho_i - c_i|,$$

which implies  $E_{ij}(\rho_i, c_i) = \mathcal{O}(\delta_0)$ .

On the other hand, using the fact that for  $x \in [0, L]$ ,  $\sqrt{1 + 4\gamma(\lambda_l(x) - x)} = 1 + 2\gamma(\lambda_l(x) - x) + \mathcal{O}(\gamma^2)$ , we obtain

$$\begin{aligned} p_j(\rho_i, \rho_j) &= \frac{\rho_j - \rho_i}{2\gamma} - \frac{1}{2\gamma} \int_{\rho_i}^{\rho_j} \sqrt{1 + 4\gamma(\lambda_j(y) - y)} \, dy \\ &= - \int_{\rho_i}^{\rho_j} (\lambda_j(y) - y) \, dy + \mathcal{O}(\gamma) \\ &= - \int_{\rho_i}^{\lambda_j^m} (\lambda_j(y) - y) \, dy - \int_{\lambda_j^m}^{\rho_j} (\lambda_j(y) - y) \, dy + \mathcal{O}(\gamma) \\ &\leq - \int_{\rho_i}^{\lambda_j^m} (\lambda_j^m - y) \, dy + \mathcal{O}(\delta_0) + \mathcal{O}(\gamma) \\ &\leq -\frac{1}{2}(\lambda_j^m - \lambda_i^M)^2 + \mathcal{O}(\delta_0) + \mathcal{O}(\gamma). \end{aligned}$$

Therefore choosing  $\gamma, \delta_0$  small enough, we find that  $E_{ij}(\rho_i, c_i) + 2p_j(\rho_i, \rho_j)$  is dominated by the strictly negative term  $-\frac{1}{2}(\lambda_j^m - \lambda_i^M)^2$ , which proves the required estimate on  $T_2$ , that is,

$$T_2 \leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \phi_i(x).$$

Hence it follows that

$$|F_{ij}(x)| \leq C\phi_i(x).$$

**Let us suppose that**  $x \in [a_{ij}, L]$ . For this case we proceed as follows:

$$\begin{aligned} |F_{ij}(x)| &= (1 + \gamma\omega_\gamma(x))^{\frac{1}{2}} \frac{\hat{\phi}_j^2(x)}{I(\hat{\phi}_j)^2} \int_{c_i}^x e^{\frac{E_{ij}(x,y)}{\varepsilon}} (1 + \gamma\omega_\gamma(y))^{-\frac{1}{2}} \, dy \\ &\leq \frac{\phi_j(x)}{I(\hat{\phi}_j)} e^{\frac{p_j(x, \rho_j) + E_{ij}(x, \rho_{ij})}{\varepsilon}} \int_{c_i}^x e^{-\frac{E_{ij}(y, \rho_{ij})}{\varepsilon}} \, dy. \end{aligned}$$

Now using the properties of  $E_{ij}(y, \rho_{ij})$ , it can be easily shown that  $\int_{c_i}^x e^{-\frac{E_{ij}(y, \rho_{ij})}{\varepsilon}} dy$  is bounded. Therefore it is enough to estimate

$$|F_{ij}(x)| \leq \frac{C}{\varepsilon} \phi_j(x) e^{\frac{P_{ij}(x)}{\varepsilon}},$$

where

$$P_{ij}(x) := p_j(x, \rho_j) + p_i(x, \rho_{ij}) - 2p_j(x, \rho_{ij}).$$

In particular we shall now prove that  $P_{ij}(x)$  is strictly less than a negative constant.

$$\begin{aligned} P_{ij}(x) &= p_j(\rho_{ij}, \rho_j) + p_i(x, \rho_{ij}) - p_j(x, \rho_{ij}) \\ &= \frac{\rho_j - \rho_{ij}}{2\gamma} + \int_{\rho_j}^{\rho_{ij}} \sqrt{\frac{\mu_j(y)}{\gamma}} dy + \int_{\rho_{ij}}^x \left( \sqrt{\frac{\mu_i(y)}{\gamma}} - \sqrt{\frac{\mu_j(y)}{\gamma}} \right) dy \\ &= \frac{\rho_j - \rho_{ij}}{2\gamma} + \frac{1}{2\gamma} \int_{\rho_j}^{\rho_{ij}} \sqrt{1 + 4\gamma(\lambda_j(y) - y)} dy \\ &\quad + \frac{1}{2\gamma} \int_{\rho_{ij}}^x \left( \sqrt{1 + 4\gamma(\lambda_i(y) - y)} - \sqrt{1 + 4\gamma(\lambda_j(y) - y)} \right) dy \\ &= \left( \int_{\rho_j}^{\rho_{ij}} (\lambda_j(y) - y) dy + \int_{\rho_{ij}}^x (\lambda_i(y) - \lambda_j(y)) dy \right) + \mathcal{O}(\gamma) \\ &:= \tilde{P}_{ij}(x) + \mathcal{O}(\gamma). \end{aligned}$$

Now using the fact that  $\lambda_i(y) < \lambda_j(y)$  we find that  $\tilde{P}_{ij}$  takes its maximum at  $a_{ij}$ . In what follows we prove that  $\tilde{P}_{ij}(a_{ij})$  is less than a negative constant, which in turn would prove the required estimate.

$$\begin{aligned} \tilde{P}_{ij}(a_{ij}) &\leq \int_{\rho_j}^{\rho_{ij}} (\lambda_j(y) - y) dy + \int_{\rho_{ij}}^{a_{ij}} (\lambda_i(y) - \lambda_j(y)) dy + \mathcal{O}(\gamma) \\ &\leq - \int_{\rho_j}^{\rho_{ij}} (y - \lambda_j^M) dy + (\rho_{ij} - a_{ij})(\lambda_j^M - \lambda_i^m) + \mathcal{O}(\gamma) \\ &\leq -\frac{1}{2}((\rho_{ij} - \lambda_j^M)^2 - (\rho_j - \lambda_j^M)^2) + (\rho_{ij} - a_{ij})(\lambda_j^M - \lambda_i^m) + \mathcal{O}(\gamma) \\ &= -\frac{1}{2}(\rho_{ij} - \lambda_j^M)^2 + (\rho_{ij} - a_{ij})(\lambda_j^M - \lambda_i^m) + \mathcal{O}(\delta_0) + \mathcal{O}(\gamma). \end{aligned}$$

Now choosing  $\gamma, \delta_0$  and  $\kappa$  (as in the definition of  $a_{ij}$ ) small enough, we obtain that  $\tilde{P}_{ij}(a_{ij}) < -C$  with  $C$  strictly positive, whence the desired estimate for  $F_{ij}$  follows.

**Case 2:**  $\lambda_i < 0 < \lambda_j$ .

*Let us suppose that*  $x \in [0, a_{ij}]$ . Proceeding as in Case 1, we deduce (5.34) and (5.35) corresponding to this case as well. The estimate for  $T_1$  is again obtained as before. For  $T_2$  proceeding as in the previous case, we have

$$\begin{aligned} E_{ij}(x, c_i) + 2p_j(x, \rho_j) - p_i(x, \rho_i) &= E_{ij}(\rho_i, c_i) + 2p_j(\rho_i, \rho_j) \\ &= 2p_j(0, \rho_j), \text{ using the fact that } \rho_i = 0 = c_i. \end{aligned}$$

But  $2p_j(0, \rho_j)$  is strictly negative ( $p_j$  attains its maximum at  $\rho_j$  and is strictly increasing in  $[0, \lambda_j^m]$ ) and therefore

$$\begin{aligned} T_2 &\leq \frac{C}{\varepsilon} (1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} e^{2p_j(0, \rho_j)} \phi_i(x) \\ &\leq C(1 + \gamma\omega_\gamma(x))^{-\frac{1}{2}} \phi_i(x), \end{aligned}$$

whereby it follows again that  $|F_{ij}(x)| \leq C\phi_i(x)$ .

*Let us next suppose that*  $x \in [a_{ij}, L]$ . The proof in this case is analogous to that corresponding to the previous one and hence we omit the proof.

**Case**  $i < j \leq p$ .

Then we have  $c_i = 0, \rho_i = 0 = \rho_j$ . We can then write

$$\begin{aligned} |F_{ij}(x)| &\leq \frac{\phi_j(x)}{I(\hat{\phi}_j)} e^{\frac{p_j(x,0) + E_{ij}(x,0)}{\varepsilon}} \int_0^x e^{-\frac{E_{ij}(y,0)}{\varepsilon}} dy \\ &\leq \frac{C}{\varepsilon} \phi_j(x) e^{\frac{P_{ij}(x)}{\varepsilon}} \int_0^x e^{-\frac{E_{ij}(y,0)}{\varepsilon}} dy, \end{aligned}$$

where  $P_{ij}(x) := p_j(x, 0) + E_{ij}(x, 0)$ . Observe here that choosing  $\alpha$  sufficiently small, we can always make  $\Lambda_{ij}^M \neq 0$ .

Let us first consider the case when  $\Lambda_{ij}^M < 0$ . We have

$$\begin{aligned} P_{ij}(x) &= p_i(x, 0) - p_j(x, 0) = (p_i(x) - p_i(0)) - (p_j(x) - p_j(0)) \\ &= \frac{L-x}{2\gamma} + \int_L^x \sqrt{\frac{\mu_i(y)}{\gamma}} dy - \frac{L}{2\gamma} - \int_L^0 \sqrt{\frac{\mu_i(y)}{\gamma}} dy \\ &\quad - \left( \frac{L-x}{2\gamma} + \int_L^x \sqrt{\frac{\mu_j(y)}{\gamma}} dy - \frac{L}{2\gamma} - \int_L^0 \sqrt{\frac{\mu_j(y)}{\gamma}} dy \right) \\ &= \int_0^x \left( \sqrt{\frac{\mu_i(y)}{\gamma}} - \sqrt{\frac{\mu_j(y)}{\gamma}} \right) dy \leq 0, \end{aligned}$$

since  $\mu_i(y) \leq \mu_j(y)$ . Now  $\partial_1 E_{ij}(x, 0) = 2(x - (2\lambda_j(x) - \lambda_i(x))) + \mathcal{O}(\gamma)$  and hence choosing  $\gamma$  to be sufficiently small, we obtain

$$\begin{aligned} -\partial_1 E_{ij}(x, 0) &= 2((2\lambda_j(x) - \lambda_i(x)) - x) + \mathcal{O}(\gamma) \\ &\leq 2\Lambda_{ij}^M + \mathcal{O}(\gamma) \\ &\leq -C_1, \text{ where } C_1 \text{ is a positive constant.} \end{aligned}$$

Integrating this from 0 to  $y$ , and using the fact that  $E_{ij}(0, 0) = 0$ , we obtain

$$-E_{ij}(y, 0) \leq -C_1 y$$

and since  $P_{ij}$  is nonpositive, we further have  $P_{ij}(x) - E_{ij}(y, 0) \leq -C_1 y$ .

Therefore, we obtain

$$\begin{aligned} |F_{ij}(x)| &\leq \frac{C}{\varepsilon} \phi_j(x) \int_0^L e^{\frac{P_{ij}(x) - E_{ij}(y, 0)}{\varepsilon}} dy \\ &\leq \frac{C}{\varepsilon} \phi_j(x) \int_0^L e^{-\frac{C_1 y}{\varepsilon}} dy \\ &\leq C \phi_j(x) \frac{1 - e^{-\frac{C_1 L}{\varepsilon}}}{-\frac{C_1}{\varepsilon}} \leq C \phi_j(x) (1 - e^{-\frac{C_1 L}{\varepsilon}}) \end{aligned}$$

and hence we obtain

$$|F_{ij}(x)| \leq C \phi_j(x).$$

Next let us consider the case  $\Lambda_{ij}^M > 0$ . We have

$$\begin{aligned}
 P_{ij}(x) - E_{ij}(y, 0) &= \int_y^x \left( \sqrt{\frac{\mu_i(s)}{\gamma}} - \sqrt{\frac{\mu_j(s)}{\gamma}} \right) ds + p_j(y, 0) \\
 &\leq p_j(y, 0),
 \end{aligned}$$

using the fact that  $\mu_i(s) < \mu_j(s)$  and  $y \leq x$ . Therefore it is enough to estimate the term

$$\frac{C}{\varepsilon} \phi_j(x) \int_0^x e^{\frac{p_j(y,0)}{\varepsilon}} dy.$$

We obtain

$$\begin{aligned}
 p_j(y, 0) &= -\frac{y}{2\gamma} + \int_0^y \sqrt{\frac{\mu_j(s)}{\gamma}} ds \\
 &= -\frac{y}{2\gamma} + \frac{1}{2\gamma} \int_0^y \sqrt{1 + 4\gamma(\lambda_j(s) - s)} ds \\
 &= -\frac{y}{2\gamma} + \frac{1}{2\gamma} \int_0^y [1 + 2\gamma(\lambda_j(s) - s) + \mathcal{O}(\gamma^2)(\lambda_j(s) - s)^2] ds \\
 &= \frac{1}{2\gamma} \int_0^y [2\gamma(\lambda_j(s) - s) + \mathcal{O}(\gamma^2)(\lambda_j(s) - s)^2] ds \\
 &< -C_1 y,
 \end{aligned}$$

choosing  $\gamma$  sufficiently small. Therefore, we find

$$\begin{aligned}
 \frac{C}{\varepsilon} \phi_j(x) \int_0^x e^{\frac{p_j(y,0)}{\varepsilon}} dy &\leq \frac{C}{\varepsilon} \phi_j(x) \int_0^L e^{-\frac{C_1 y}{\varepsilon}} dy \\
 &= C \phi_j(x) (1 - e^{-\frac{C_1 L}{\varepsilon}}) \leq C \phi_j(x)
 \end{aligned}$$

and therefore  $|F_{ij}(x)| \leq C \phi_j(x)$ . This completes the proof of the theorem. □

### 6. Existence Theory for Systems with Diffusion and Dispersion

We now establish the existence of a solution for the boundary Riemann problem (5.2)-(5.4) with diffusion and dispersion. Throughout this section  $\varepsilon > 0$  is a given parameter; all the estimates below are uniform in the limit  $\varepsilon \rightarrow 0$ . We follow the previous works of LeFloch and Rohde [17], and Joseph

and LeFloch [8, 9, 10] and since the proof of the theorems are straightforward adaptation of these papers, we omit all the proofs.

The first step is to analyze the coupled system

$$\gamma\varepsilon^2 a_k'' + \varepsilon a_k' + (x - \lambda_k(v))a_k = \frac{1}{\gamma\varepsilon} D_k^1(a) + D_k^2(a, a'), \tag{6.1}$$

where

$$\begin{aligned} D_k^1(a) &= -l_k(v) \cdot \sum_{j,i=1}^N a_i a_j (Dr_i(v) \cdot r_j(v)), \\ D_k^2(a, a') &= -l_k(v) \cdot \left( \sum_{k,i} (2a_i a_j' + a_i' a_j) (Dr_i \cdot r_j)(v) \right. \\ &\quad \left. + \sum_{k,i,l=1}^N a_i a_k a_l D(Dr_i \cdot r_k) r_l(v) \right), \end{aligned} \tag{6.2}$$

for a fixed function  $v : [0, \infty) \rightarrow B(u_*, C_* \delta)$ .

We are given the boundary value  $u_B \in B(u_*, \delta)$  and, instead of using a right-end state  $u_I$ , we first describe the Riemann solutions using a “wave strength” vector  $\tau \in R^N$ . The coefficients  $a_k^\varepsilon$  are sought in the form of an asymptotic expansion in the wave strength:

$$a_k^\varepsilon(x; v, \tau) = \tau_k \varphi_k^\varepsilon(x; v) + \theta_k^\varepsilon(x; v, \tau), \tag{6.3}$$

where  $\tau = (\tau_1, \dots, \tau_N) \in B(0, \delta_1)$ , the ball in  $R^N$  having center 0 and radius  $\delta_1 > 0$  and  $\varphi$  are solutions of the homogeneous system constructed in in the previous section. The remainder  $\theta_k^\varepsilon(x; v, \tau)$  in (6.3) is sought to be second-order in  $\tau$ . In view of (6.1), the coefficients  $x \mapsto \theta_k^\varepsilon(x; v)$  must satisfy the coupled system ( $k = 1, \dots, N$ )

$$\theta_k^{\varepsilon''} + \frac{1}{\gamma\varepsilon} \theta_k^{\varepsilon'} + \frac{1}{\gamma\varepsilon^2} (x - \lambda_k(v)) \theta_j^\varepsilon = \frac{D_k^i(a)}{\gamma\varepsilon} + D_k^2(a, a'). \tag{6.4}$$

Using an equivalent integral equation for  $\theta = (\theta_1, \dots, \theta_N)$ , we have

$$\theta_k(x) = \psi_k(x) \int_0^x \psi_k(y) S_k(\theta)(y) dy + \varphi_k(x) \int_{c_k}^x \varphi_k(y) S_k(\theta)(y) dy, \tag{6.5}$$

where

$$S_k(\theta) = \frac{1}{\gamma\varepsilon} D_k^1(a) + D_k^2(a, a'), \quad a = (a_1, \dots, a_N),$$

$$a_k^\varepsilon(x; v, \tau) = \tau_k \varphi_k^\varepsilon(x; v) + \theta_k^\varepsilon(x; v, \tau),$$

and by a straightforward generalization of [17], we get the following theorem.

**Theorem 6.1.** *There exist constants  $\delta, \delta_1, C_*, C > 0$  with the following property. For all  $\varepsilon > 0$  and every continuous function  $v : [0, \infty) \rightarrow B(u_*, C_* \delta)$  and for every  $\tau \in B(0, \delta_1)$ , there exists a unique solution  $(\theta_1^\varepsilon, \dots, \theta_N^\varepsilon)$  of (6.4) satisfying (6.5) with following estimates,*

$$|\theta_k^\varepsilon(\cdot; v, \tau)| + \frac{\varepsilon}{\omega_\gamma} |\theta_k^\varepsilon(\cdot; v, \tau)'| \leq C |\tau|^2 \sum_{j=1}^N \varphi_j^\varepsilon(\cdot; v), \quad k = 1, \dots, N. \quad (6.6)$$

Furthermore for  $|\tau|, |\sigma| \leq \delta_1$  we have the continuous dependence estimate

$$|\theta_k^\varepsilon(\cdot; v, \tau) - \theta_k^\varepsilon(\cdot; v, \sigma)| + \frac{\varepsilon}{\omega_\gamma} |\theta_k^\varepsilon(\cdot; v, \tau)' - \theta_k^\varepsilon(\cdot; v, \sigma)'|$$

$$\leq C \delta_1 |\tau - \sigma| \sum_{j=1}^N \varphi_j^\varepsilon(\cdot; v), \quad k = 1, \dots, N. \quad (6.7)$$

Next we construct solution to the nonlinear problem. For each  $u_B$  and each vector of wave strengths  $\tau$ , we construct a solution  $u^\varepsilon$  of the Riemann problem (5.2)-(5.4) with diffusion and dispersion.

**Theorem 6.2.** *There exist  $\delta, \delta_1, C_*, C > 0$  with the following property. For all  $\varepsilon > 0$  and all  $u_B \in B(u_*, \delta)$  and  $\tau \in B(0, \delta_1)$ , the boundary Riemann problem (5.2)-(5.4) admits a solution  $x \mapsto u^\varepsilon(x; \tau) \in B(u_*, C_* \delta)$  leaving from  $u_B = u^\varepsilon(0; \tau)$  and reaching some state  $u_I := u^\varepsilon(+\infty; \tau)$ , with*

$$u^{\varepsilon'} = \sum_{k=1}^N a_k^\varepsilon r_k(u^\varepsilon), \quad (6.8)$$

$$a_k^\varepsilon(x; \tau) = \tau_k \varphi_k^\varepsilon(x; \tau) + \theta_k^\varepsilon(x; \tau), \quad |\theta_k^\varepsilon(x; \tau)| \leq C |\tau|^2 \sum_{j=1}^N \varphi_j^\varepsilon(x; \tau),$$

$$\varphi_k^\varepsilon(x; \tau) = \frac{e^{\frac{(p_k^\varepsilon(x; \tau) - p_k^\varepsilon(\rho_k; \tau))}{\varepsilon}}}{\int_0^\infty e^{\frac{(p_k^\varepsilon(x; \tau) - p_k^\varepsilon(\rho_k; \tau))}{\varepsilon}} dx}, \quad p_k^\varepsilon(x; \tau) = -\frac{x}{2\gamma} + \int_0^x \sqrt{\frac{\mu_k(y)}{\gamma}} dy, \quad (6.9)$$

$$\mu_k(y) = \lambda_k(u^\varepsilon(y; \tau)) - x + \frac{1}{4\gamma}.$$

In particular this implies that  $u^\varepsilon$  has uniformly bounded total variation, indeed

$$|u^{\varepsilon'}| \leq O(1) |\tau| \sum_{j=1}^N \varphi_j^\varepsilon \tag{6.10}$$

and, thus,

$$TV(u^\varepsilon) \leq O(1) |\tau|. \tag{6.11}$$

We now consider the Riemann problem with both ends  $u_B$  and  $u_I$  fixed.

**Theorem 6.3** (The Riemann problem with diffusion and dispersion). *There exist  $\delta, C_*, C > 0$  with the following property. For every  $\varepsilon > 0$ ,  $u_B, u_I \in B(u_*, \delta)$ , the Riemann problem (5.2)-(5.4) admits a solution  $x \mapsto u^\varepsilon(x)$  connecting  $u_B = u^\varepsilon(0)$  to  $u_I = u^\varepsilon(+\infty)$  and satisfying  $u^\varepsilon(x) \in B(u_*, C_* \delta)$  for all  $x > 0$ . It satisfies also the expansion (6.8)-(6.9) for some  $\tau = \tau^\varepsilon$  with*

$$\frac{1}{C} |\tau^\varepsilon| \leq |u_I - u_B| \leq C |\tau^\varepsilon|. \tag{6.12}$$

Furthermore,  $u^\varepsilon$  is of uniformly bounded total variation and satisfies (6.10)-(6.11).

The proof of this theorem relies on the invertibility of the mapping

$$S_\varepsilon : \tau = (\tau_1, \dots, \tau_N) \in B(0, \delta_1) \rightarrow S_\varepsilon(v, \tau) \in B(u_*, C_* \delta)$$

defined for each function  $v$  by

$$S_\varepsilon(v, \tau) = u_B + \sum_{k=1}^N \int_0^\infty (\tau_k \varphi_k^\varepsilon(\cdot; v) + \theta_k^\varepsilon(\cdot; v, \tau)) r_k(v) dx,$$

the right-hand side being defined by Theorem 6.2 from the data  $u_B, \tau$  and  $v$ . We state the result, the proof is similar to LeFloch and Rohde [17] and is omitted.

**Proposition 6.4.** *There exist  $\delta, \delta_1, C_*, C > 0$  with the following property. For all  $\varepsilon > 0$ ,  $u_B \in B(u_*, \delta)$  and each function  $v : [0, \infty) \rightarrow B(u_*, C_* \delta)$ , we have the following.*

For each  $u_I \in B(u_*, \delta)$ , there exists a unique solution  $\tau \in B(0, \delta_1)$  of the equation

$$S_\varepsilon(v, \tau) = u_I.$$

The mapping  $S_\varepsilon$  is thus locally invertible and  $S_\varepsilon^{-1}$  is uniformly bounded in  $\varepsilon$  in the sense that

$$|S_\varepsilon^{-1}(u_I)| \leq C |u_I - u_B|.$$

## 7. Analysis of Boundary Layers and Convergence Results

In this section we study the structure of the limiting solution  $u = \lim_{\varepsilon \rightarrow 0} u^\varepsilon$ . This analysis rely on the estimate  $u^\varepsilon$  obtained in the previous section. Our objective is to describe the boundary layer that generally arises in the solution at  $x = 0$ . We refer to Joseph and LeFloch [8] for a discussion of this problem in the general framework of  $L^\infty$  solutions. First of all, as an easy consequence of Theorem 6.3 we obtain the following result when  $\varepsilon \rightarrow 0$ .

**Theorem 7.1** (Existence theory for nonlinear hyperbolic systems). *There exist  $\delta, C_*, C > 0$  such that the following property holds. For every  $u_B, u_I \in B(u_*, \delta)$ , the solution  $u^\varepsilon$  (or at least a subsequence) of the boundary Riemann problem (5.2)-(5.4) converges pointwise to a weak solution  $x \mapsto u(x)$  of*

$$-x u' + f(u)' = 0,$$

which is a function of bounded variation connecting some value  $u(0+)$  to  $u_I = u(+\infty)$  and satisfying  $u(x) \in B(u_*, C_* \delta)$  for all  $x > 0$ . Moreover, we have

$$u(x) = u_I \quad \text{for all } x > \lambda_N^M$$

and

$$TV(u) \leq C |u_I - u_B|.$$

Therefore the condition (5.3) at infinity holds for the limiting function. However the condition (5.4) does not pass to the limit in general due to the formation of boundary layers near  $x = 0$  and must be relaxed and expressed in the weak form.

To determine the value  $u(0+)$ , we analyze the boundary layer near 0. The same analysis as in the scalar case leads to the following result for the boundary layer.

**Theorem 7.2** (The boundary layer for systems with diffusion and dispersion). *The trace  $u(0+)$  of the Riemann solution constructed in Theorem 7.1 satisfies the following property. There exist a vector  $V_\infty$  and a smooth function  $y \geq 0 \mapsto V(y)$  such that*

$$\begin{aligned} \gamma V''' + V''(y) &= f(V(y))', \\ V(0) &= u_B, \quad \lim_{y \rightarrow \infty} V(y) = V_\infty, \\ \lim_{y \rightarrow \infty} V'(y) &= \lim_{y \rightarrow \infty} V''(y) = 0 \end{aligned} \tag{7.1}$$

and

$$f(V_\infty) = f(u(0+)). \tag{7.2}$$

Let us now introduce the admissible set based on the diffusive-dispersive regularization:

$$\Phi_\gamma(u_B) := \left\{ V_\infty / \text{There exists a solution } V_\gamma : [0, +\infty) \rightarrow \mathbb{R} \right. \\ \left. \text{to the boundary problem (7.1)-(7.2)} \right\}. \tag{7.3}$$

Then we claim that the trace  $u(0+)$  of the Riemann solution constructed in Theorem 7.1 belongs to this set. This follows because the flux-function  $f$  is locally one-to-one and the condition (7.2) is equivalent to saying

$$u(0+) = V_\infty.$$

In other words, the solution satisfies the boundary condition

$$u(0+) \in \Phi_\gamma(u_B).$$

Now, it is important to check that, in some sense, the boundary set  $\Phi_\gamma(u_B)$  is not too large, which is the purpose of the following section.

### Analysis of diffusive-dispersive boundary layer

One integration of (7.1) gives after using the boundary condition

$$\begin{aligned}\gamma V'' + V' &= f(V) - f(V_\infty), \\ V(0) &= u_B, & V(\infty) &= V_\infty, \\ \lim_{y \rightarrow \infty} V'(y) &= \lim_{y \rightarrow \infty} V''(y) = 0.\end{aligned}\tag{7.4}$$

Setting  $W = V'$ , we can write the second order system (7.4) as the first order system

$$\begin{aligned}V' &= W, \\ \gamma W' &= f(V) - f(V_\infty) - W,\end{aligned}\tag{7.5}$$

with boundary conditions

$$\begin{aligned}V(0) &= u_B, & \lim_{y \rightarrow \infty} V(y) &= V_\infty, \\ \lim_{y \rightarrow \infty} V'(y) &= \lim_{y \rightarrow \infty} W(y) = 0.\end{aligned}\tag{7.6}$$

Note that  $(V, W) = (V_\infty, 0)$  is a stationary point for the system (7.5) and we are interested in the set of  $V_\infty \in R^N$  for which it has a solution satisfying (7.6).

Setting  $Z = V - V_\infty$ , we write our system

$$\begin{aligned}Z' &= W, \\ \gamma W' &= Df(V_\infty)Z - W + [f(Z + V_\infty) - f(V_\infty) - Df(v_\infty)Z],\end{aligned}\tag{7.7}$$

with boundary conditions

$$Z(0) = u_B - V_\infty, \quad \lim_{y \rightarrow \infty} Z(y) = \lim_{y \rightarrow \infty} W(y) = 0.\tag{7.8}$$

Now  $(Z, W) = (0, 0)$  is the critical point of the system (7.7). We write the system as linear part at  $(0, 0)$  and the quadratic part namely

$$\begin{pmatrix} Z' \\ W' \end{pmatrix} = \begin{pmatrix} 0 & Id \\ \frac{1}{\gamma} Df(V_\infty) & -\frac{1}{\gamma} Id \end{pmatrix} \begin{pmatrix} Z \\ W \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\gamma} [g(Z, W)] \end{pmatrix},\tag{7.9}$$

where

$$g(Z, W) = f(Z + V_\infty) - f(V_\infty) - Df(v_\infty)Z = O(\|(Z, W)\|^2).$$

First we analyse the eigenvalues and eigenvectors of the matrix

$$A(V_\infty, \gamma) = \begin{pmatrix} 0 & Id \\ \frac{1}{\gamma}Df(V_\infty) & -\frac{1}{\gamma}Id \end{pmatrix}. \tag{7.10}$$

**Proposition 7.3.** *A(V<sub>∞</sub>, γ) has 2N distinct eigenvalues μ<sub>k</sub><sup>±</sup> with corresponding basis of left and right eigenvectors L<sub>k</sub><sup>±</sup>, R<sub>k</sub><sup>±</sup>, k = 1, 2, . . . , N given by*

$$\begin{aligned} \mu_k^-(V_\infty, \gamma) &= \frac{-1 - (1 + 4\gamma\lambda_k(V_\infty))^{1/2}}{2\gamma}, & k = 1, 2, \dots, N, \\ \mu_k^+(V_\infty, \gamma) &= \frac{-1 + (1 + 4\gamma\lambda_k(V_\infty))^{1/2}}{2\gamma}, & k = 1, 2, \dots, N, \end{aligned} \tag{7.11}$$

$$\begin{aligned} R_k^\pm(V_\infty, \gamma) &= (r_k(V_\infty), \mu_k^\pm(V_\infty, \gamma)r_k(V_\infty)), \\ L_k^\pm(V_\infty, \gamma) &= \pm\alpha_k(-\mu_k^\mp(V_\infty, \gamma)l_k(V_\infty), l_k(V_\infty)), \end{aligned} \tag{7.12}$$

where

$$\alpha_k = \frac{\gamma}{(1 + 4\gamma\lambda_k(V_\infty))^{1/2}}. \tag{7.13}$$

The eigenvalues μ<sub>k</sub><sup>-</sup>(V<sub>∞</sub>, γ), k = 1, 2, . . . , N and μ<sub>k</sub><sup>+</sup>(V<sub>∞</sub>, γ) (k = 1, 2, . . . , p) are negative and μ<sub>k</sub><sup>+</sup>(V<sub>∞</sub>, γ) (k = p+1, . . . , N) are positive. Further as γ ≈ 0,

$$\mu_k^-(V_\infty, \gamma) \approx -\frac{1}{\gamma}, \quad \mu_k^+(V_\infty, \gamma) \approx \lambda_k(V_\infty). \tag{7.14}$$

**Proof.** We note that μ is an eigenvalue of the matrix A(V<sub>∞</sub>, γ) with eigenvector (R<sub>1</sub>, R<sub>2</sub>), R<sub>1</sub>, R<sub>2</sub> ∈ R<sup>N</sup> iff

$$-\mu R_1 + R_2 = 0, \quad \frac{1}{\gamma}Df(V_\infty)R_1 - \frac{1}{\gamma}R_2 - \lambda R_2 = 0.$$

This system is equivalent to

$$Df(V_\infty)R_1 = (1 + \mu\gamma)\mu R_1, \quad R_2 = \mu R_1.$$

This means that  $\mu(1 + \gamma\mu)$  is an eigenvalue  $\lambda_k(V_\infty)$  of  $Df(V_\infty)$  with  $R_1 = r_k(V_\infty)$ ,  $k = 1, 2, \dots, N$ . So  $\mu$  satisfies the equation

$$\gamma\mu^2 + \mu - \lambda_k(V_\infty) = 0, \quad (7.15)$$

and with corresponding right eigenvectors

$$(R_1, R_2) = (r_k(V_\infty), \mu r_k(V_\infty)). \quad (7.16)$$

Left eigenvector  $(L_1, L_2)$ ,  $L_1, L_2 \in R^N$  corresponding to the eigenvalue satisfies the equation

$$\frac{1}{\gamma} Df(V_\infty)^T L_2 = \mu L_1, \quad L_1 - \frac{1}{\gamma} L_2 = \mu L_2$$

which leads to

$$Df(V_\infty)^T L_2 = (1 + \mu\gamma)\mu L_2, \quad L_1 = \left(\frac{1}{\gamma} + \mu\right)L_2$$

This relation says that up to a scalar multiple the left eigenvector corresponding to the eigenvalue  $\mu$  is of the form

$$(L_1, L_2) = \left(\left(\frac{1}{\gamma} + \mu\right)l_k(V_\infty), l_k(V_\infty)\right), \quad (7.17)$$

where  $\mu$  is a solution to (7.15). Solving (7.15) for  $\mu_k$  we get

$$\mu_k^\pm = \frac{-1 \pm (1 + 4\gamma\lambda_k(V_\infty))^{1/2}}{2\gamma}.$$

Also we have

$$\frac{1}{\gamma} + \mu_k^\pm = \frac{1 \pm (1 + 4\gamma\lambda_k(V_\infty))^{1/2}}{2\gamma} = -\mu_k^\mp.$$

Thus  $A(V_\infty)$  has  $2N$  distinct eigenvalues

$$\begin{aligned} \mu_k^- &= \frac{-1 - (1 + 4\gamma\lambda_k(V_\infty))^{1/2}}{2\gamma}, & k = 1, 2, \dots, N, \\ \mu_k^+ &= \frac{-1 + (1 + 4\gamma\lambda_k(V_\infty))^{1/2}}{2\gamma}, & k = 1, 2, \dots, N. \end{aligned} \quad (7.18)$$

Also up to a scalar multiple, the right and the left eigenvectors corresponding to  $\mu_k^\pm, k = 1, 2, \dots, N$  are

$$R_k^\pm = (r_k(V_\infty), \mu_k^\pm r_k(V_\infty)),$$

and

$$L_k^\pm = (-\mu_k^\mp l_k(V_\infty), l_k(V_\infty))$$

Now, we have

$$\begin{aligned} L_k^+ \cdot R_k^- &= L_k^- \cdot R_k^+ = L_k^\pm \cdot R_j^\pm = 0, & j \neq k, j, k = 1, 2, \dots, N, \\ L_k^\pm \cdot R_k^\pm &= \pm \frac{(1 + 4\gamma\lambda_k(V_\infty))^{1/2}}{\gamma}. \end{aligned}$$

We normalize them so that  $L_k^\pm \cdot R_j^\pm = \delta_{kj}$ . This leads to the choice of normalization factor  $\alpha_k$ , stated in the proposition. Since  $\lambda_k(V_\infty) < 0$  for  $k = 1, 2, \dots, p$  and  $\lambda_k(V_\infty) > 0$  for  $k = (p + 1), \dots, N$ , it easily follows that  $\mu_k^-(V_\infty), k = 1, 2, \dots, n$  and  $\mu_k^+(V_\infty), k = 1, 2, \dots, p$  are negative and  $\mu_k^+, k = p + 1, \dots, N$  are positive. The asymptotic form of  $\mu_k$  as  $\gamma$  goes to 0 follows from the formula. This, therefore, completes the proof of the theorem. □

In studying the system (7.7)-(7.8), it is convenient to write it in terms of the components in the directions of the eigenvectors of  $A(V_\infty, \gamma)$ . Thus we decompose  $(Z, W) = (V - V_\infty, W)$  with respect the basis given by (7.12) in the previous theorem.

$$\begin{pmatrix} V - V_\infty \\ W \end{pmatrix} = \sum_{k=1}^N L_k^+ \cdot \begin{pmatrix} V - V_\infty \\ W \end{pmatrix} \cdot R_k^+ + \sum_{k=1}^N L_k^- \cdot \begin{pmatrix} V - V_\infty \\ W \end{pmatrix} \cdot R_k^-. \tag{7.19}$$

Let us denote

$$a_k^\pm(y, \gamma) := \pm \alpha_k(V_\infty) l_k(V_\infty) \cdot [\mp \mu_k^\mp(V_\infty)(V(y, \gamma) - V_\infty) + W(y, \gamma)]. \tag{7.20}$$

Then

$$\begin{pmatrix} V - V_\infty \\ W \end{pmatrix} = \sum_{k=1}^N a_k^+(y, \gamma) R_k^+ + \sum_{k=1}^N a_k^-(y, \gamma) R_k^-. \tag{7.21}$$

The system (7.9) in  $a = (a_1^+, a_2^+, \dots, a_N^+, a_1^-, a_2^-, \dots, a_N^-)$  becomes

$$\begin{aligned} a_k^+{}'(y, \gamma) &= \mu_k^+(V_\infty, \gamma)a_k^+(y, \gamma) + g_k^+(a(y, \gamma)), & k = 1, 2, \dots, N, \\ a_k^-{}'(y, \gamma) &= \mu_k^-(V_\infty, \gamma)a_k^-(y, \gamma) + g_k^-(a(y, \gamma)), & k = 1, 2, \dots, N, \end{aligned} \tag{7.22}$$

$$\begin{aligned} g_k^+(a) &= -(1 + 4\gamma\lambda_k(V_\infty))^{-1/2}l_k(V_\infty)\bar{g}(a) = O(\|a\|^2), \quad k = 1, \dots, N, \\ g_k^-(a) &= (1 + 4\gamma\lambda_k(V_\infty))^{-1/2}l_k(V_\infty)\bar{g}(a) = O(\|a\|^2), \quad k = 1, \dots, N, \\ \bar{g}(a) &= g(Z, W) = f(Z + V_\infty) - f(V_\infty) - Df(v_\infty)Z, \quad Z = V - V_\infty. \end{aligned} \tag{7.23}$$

We remark that the estimate for  $g_k^\pm$  in (7.23) is uniform in  $\gamma \approx 0$  and  $\gamma > 0$  and we used the fact that  $(Z, W) = (V - V_\infty, W)$  can be written in terms of  $a$ . From (7.21) we have,

$$l_k(V_\infty) \cdot (V(y) - V_\infty) = (1 + 4\gamma\lambda_k(V_\infty))^{-1/2}(a_k^+(y, \gamma) + a_k^-(y, \gamma)) \tag{7.24}$$

$$l_k(V_\infty) \cdot W = (1 + 4\gamma\lambda_k(V_\infty))^{1/2}(\mu_k^-(V_\infty, \gamma)a_k^-(y, \gamma) - \mu_k^+(V_\infty, \gamma)a_k^-(y, \gamma))$$

which gives

$$\begin{aligned} (V(y) - V_\infty) &= \sum_{k=1}^N (1 + 4\gamma\lambda_k(V_\infty))^{1/2}(a_k^+(y, \gamma) + a_k^-(y, \gamma))r_k(V_\infty) \\ W &= \sum_{k=1}^N (1 + 4\gamma\lambda_k(V_\infty))^{1/2}(\mu_k^-(V_\infty, \gamma)a_k^-(y, \gamma) - \mu_k^+(V_\infty, \gamma)a_k^-(y, \gamma))r_k(V_\infty). \end{aligned} \tag{7.25}$$

The boundary conditions (7.8) becomes

$$\begin{aligned} a_k^\pm(0) &= \pm\alpha_k(V_\infty)l_k(V_\infty) \cdot [\mp\mu_k^\mp(V_\infty)(u_B - V_\infty) + W(0)] \\ \lim_{y \rightarrow \infty} a_k^\pm(y) &= 0, \quad k = 1, 2, \dots, N \end{aligned} \tag{7.26}$$

We observe that  $W(0)$  is not prescribed and depends on  $u_B$ . In fact  $W(y)$  depends on  $u_B, V_\infty$  and  $\gamma$  through the relation  $W = V'$ . With these observations in the next theorem we analyze the structure of boundary layer.

**Theorem 7.4** (A property of the boundary layer set). *There exists  $\gamma_0 > 0$  such that for any  $0 < \gamma < \gamma_0$ , the set  $\Phi_\gamma(u_B)$  contains the point  $u_B$  and near  $u_B$ , it is a  $p$  dimensional manifold whose tangent plane at  $u_B$  is given by  $u_B + \text{Span of } \{r_k(u_B), k = 1, 2, \dots, p\}$ .*

**Proof.** First we observe that  $(0, 0)$  is a hyperbolic stationary point for the system (7.22) with  $N + p$  negative eigenvalues and  $N - p$  positive eigenvalues. So by standard theory of hyperbolic points of systems, near  $(0, 0)$  the system has a stable invariant manifold of dimension  $N + p$  whose tangent space at  $(0, 0)$  is generated by  $R_k^-(V_\infty, \gamma), k = 1, 2, \dots, N, R_k^+(V_\infty, \gamma), k = 1, 2, \dots, p$ . So the initial value problem has a solution vanishing at infinity iff the initial data for  $a_k^\pm(0, \gamma)$  lies in this stable manifold.

Let us note that the standard way to get stable solutions of the system is by a fixed point argument applied to an equivalent system of integral equations

$$\begin{aligned}
 a_k^-(y, \gamma) &= e^{\mu_k^-(V_\infty, \gamma)y} a_{k0}^- + \int_0^y e^{\mu_k^-(V_\infty, \gamma)(y-s)} O(\|a(s)\|^2) ds, \quad k = 1, \dots, N, \\
 a_k^+(y, \gamma) &= e^{\mu_k^+(V_\infty, \gamma)y} a_{k0}^+ + \int_0^y e^{\mu_k^+(V_\infty, \gamma)(y-s)} O(\|a(s)\|^2) ds, \quad k = 1, \dots, p, \quad (7.27) \\
 a_k^+(y, \gamma) &= - \int_y^\infty e^{\mu_k^+(V_\infty, \gamma)(y-s)} O(\|a(s)\|^2) ds, \quad k = p + 1, \dots, N,
 \end{aligned}$$

with  $a_{k0}^-, k = 1, 2, \dots, n$  and  $a_{k0}^+, k = 1, 2, \dots, p$  are constants close to 0.

We note that for stable solutions, we do not prescribe those components of the initial data in the direction of eigenvectors corresponding to positive eigenvalues and impose that these components are in the stable invariant manifold. This means

$$a_k^+(0, \gamma) = - \int_0^\infty e^{-\mu_k^+(V_\infty, \gamma)s} O(\|a(s)\|^2) ds, \quad k = p + 1, \dots, N \quad (7.28)$$

If we denote the right hand side of (7.28) by  $G_k(a_1^-(0), \dots, a_n^-(0), a_1^+(0), \dots, a_p^+(0)), k = p + 1, \dots, N$ , then  $G = (G_{p+1}, \dots, G_N)$  together with partial derivatives of the  $G$  in each of its arguments vanish at 0.

Now let us come to our problem of determining the data  $V_\infty$  for which the system (7.4) has a solution, or equivalently the problem (7.22) and (7.26) has a solution. Above analysis shows that the right hand side of  $G = (G_{p+1}, \dots, G_N)$  depends only on

$$\begin{aligned}
 a_k^-(0, \gamma) &= -\alpha_k(V_\infty, \gamma) l_k(V_\infty) \cdot [+ \mu_k^+(V_\infty, \gamma)(u_B - V_\infty) + W(0)], \quad k = 1, \dots, N \\
 a_k^+(0, \gamma) &= \alpha_k(V_\infty, \gamma) l_k(V_\infty) \cdot [- \mu_k^-(V_\infty, \gamma)(u_B - V_\infty) + W(0)], \quad k = 1, \dots, p
 \end{aligned}$$

but not on

$$a_k^+(0, \gamma) = \alpha_k(V_\infty, \gamma)l_k(V_\infty) \cdot [-\mu_k^-(V_\infty, \gamma)(u_B - V_\infty) + W(0)], \quad k = p+1, \dots, N$$

Let  $F = (F_{p+1}, \dots, F_N)$  with

$$F_k(u_B, V_\infty, \gamma) = \alpha_k(V_\infty, \gamma)l_k(V_\infty) \cdot [-\mu_k^-(V_\infty, \gamma)(u_B - V_\infty) + W(0)], \\ k = p + 1, \dots, N$$

Considering  $G$  as a function of  $(u_B, V_\infty, \gamma)$ , the solvability condition becomes

$$F(u_B, V_\infty, \gamma) - G(u_B, V_\infty, \gamma) = 0. \tag{7.29}$$

Now writing  $W(0) = W(\gamma, u_B, V_\infty, \gamma)$  and using the fact  $W(\gamma, u_B, u_B, \gamma) = 0$ , we get

$$D_{V_\infty} F_k(u_B, u_B, \gamma) = [-\mu_k^-(u_B, \gamma) - D_{V_\infty} W(\gamma, u_B, u_B, \gamma)]\alpha_k(u_B, \gamma)l_k(u_B), \\ k = p + 1, \dots, N$$

and

$$D_{V_\infty} G_k(u_B, u_B, \gamma) = 0.$$

Since the function

$$\alpha_k(V_\infty)Det[-\mu_k^-(u_B, \gamma)Id - D_{V_\infty} W(\gamma, u_B, u_B)] \tag{7.30}$$

is an analytic function of  $\gamma$ , its zeros are isolated and so we find  $\gamma_0 > 0$  such that for  $0 < \gamma < \gamma_0$  this determinant is not zero. Thus the rank of  $D_{V_\infty}(F - G)(u_B, u_B, \gamma) = N - p$ . So in the  $V$ -space the relation (7.29) defines a  $p$  dimensional manifold containing  $u_B$  and if  $V_\infty$  is in this manifold we have a solution for the problem (7.4) satisfying  $u(0) = u_B$  and  $V(\infty) = V_\infty$  and the tangent space at  $u_B$  is spanned by  $r_j(u_B), j = 1, \dots, p$ .  $\square$

**Remark 7.5.** For any stable solution  $(a_k^+(y, \gamma), a_k^-(y, \gamma))$  of (7.22), the components  $a_k^-(y, \gamma)$  goes to zero uniformly for  $y \geq \delta$  for all  $\delta > 0$  since  $\mu_k^-(V_\infty, \gamma) \approx \frac{-1}{\gamma}$  as  $\gamma \approx 0$ . Taking limit in the expression (7.24), we get,

$$a_k^+(y) = l_k(V_\infty) \cdot (V(y) - V_\infty), \quad k = 1, 2, \dots, N,$$

which is exactly the expressions for coefficient of the  $r_k(V_\infty)$  in the expansion of  $V(y)$  for the case  $\gamma = 0$ , which was analyzed in [8]. In fact the corresponding expression appearing in (7.30) is just one.

We summarize our convergence analysis by the following theorem.

**Theorem 7.6** (The Riemann problem for systems with diffusion and dispersion). *Assume that the system (5.1) is strictly hyperbolic with  $p$  negative eigenvalues and  $N - p$  positive eigenvalues (we would like to recall that here  $x$  denotes the space variable, and not the self-similar variable).*

- (Existence) *There exist  $\delta, C > 0$  with the following property. Given  $u_B, u_I \in B(u_*, \delta)$  there exists a weak solution  $u(x, t)$  of (5.1) which is self-similar and of bounded total variation. The solution  $u$  satisfies with initial condition  $u(x, 0) = u_I$  and a weak form of boundary condition  $u(0+, t) \in \Phi_\gamma(u_B)$  the set of boundary values given by (7.3). Further the solution satisfies the Lax entropy condition  $\lambda_k(u(x+, t)) \leq s \leq \lambda_k(u(x-, t))$ , where  $s$  is the speed of the discontinuity.*
- (Local structure of admissible set) *The set  $\Phi_\gamma(u_B)$  defined in (7.3) contains the point  $u_B$  and, locally near  $u_B$ , is a manifold with dimension  $p$  whose tangent space at  $u_B$  is spanned by the eigenvectors  $r_j(u_B)$ ,  $j = 1, 2, \dots, p$ .*

The above facts were already established during our analysis, except the assertion that the solution satisfies the Lax shock inequality, which follows by the same analysis as for the scalar case in Section 3.

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