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EXISTENCE AND STABILITY OF TIME-PERIODIC SOLUTIONS TO THE DRIFT-DIFFUSION MODEL FOR SEMICONDUCTORS

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Dedicated to Professor Tai-Ping Liu on his 70th birthday

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Abstract

We study the existence and the asymptotic stability of time-periodic solutions to the drift-diffusion model for semiconductors. If alternating-current voltage is applied to PN-junction diodes, a time-periodic current flow is observed. The main purpose of the present paper is mathematical analysis on this periodic flow. We construct a time-periodic solution by utilizing the Galerkin method. The solution is unique in a neighborhood of a thermal equilibrium, and it is globally stable. Proofs of the uniqueness and the stability are based on the energy method employing an energy form.

1. Introduction

PN-junction diodes are widely utilized as a rectifier. Joining P-type and N-type semiconductors yields this diode. If applied voltage is negative, a current flows through this diode. On the contrary, if positive voltage is applied, the current is almost zero. Hence, this diode allows a flow of electricity in one direction but not in the opposite direction. This property is called rectification and converts alternating current (AC) into direct-current (DC).

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As a first step of mathematical understanding of this conversion process, we investigate the existence and the time-asymptotic stability of time-periodic solutions to the drift-diffusion model for semiconductors. The drift-diffusion model consists of two continuity equations for the density of electrons and holes, adopting constitutive current relations, coupled with the Poisson equation for the electrostatic potential (for details, see [6, 7, 8, 11, 12]):

$$n_t = (n_x - nv_x)_x - R(n,p), \tag{1.1a}$$

$$p_t = (p_x + pv_x)_x - R(n,p), \tag{1.1b}$$

$$\varepsilon v_{xx} = n - p - D, \quad (t,x) \in I \times \Omega, \tag{1.1c}$$

where $\Omega = (0,1)$ and $I \subset \mathbb{R}$ is an interval. The unknown functions n, p and v denote the electron density, the hole density and the electrostatic potential, respectively. The recombination-generation term R accounts for instantaneous generation or annihilation of electron-hole pairs. In most application, it is given by the Shockley-Read-Hall form

$$R(n,p) := \nu \frac{np-1}{n+p+2},$$

where ν is some positive constant. The positive constant ε is the scaled Debye length. The doping profile D denotes the density of ionized impurities in the semiconductors and determines the performance of the devices. It is supposed that D is a bounded measurable function. We prescribe the boundary data

$$n(t,0) = n_l, \quad p(t,0) = p_l, \quad v(t,0) = 0, \tag{1.2a}$$

$$n(t,1) = n_r, \quad p(t,1) = p_r, \quad v(t,1) = \phi(t) \tag{1.2b}$$

and the initial data

$$(n,p)(0,x) = (n_0,p_0)(x), \tag{1.3a}$$

$$n_0(x), p_0(x) \geq 0 \ \text{ a.e. } \ x \in (0,1). \tag{1.3b}$$

The positive constants n_l, p_l, n_r and p_r in (1.2) are supposed to be in thermal equilibrium

$$n_l p_l = 1, \quad n_r p_r = 1.$$

We do not assume the Ohmic contacts

$$n_l - p_l - D(0) = 0, \quad n_r - p_r - D(1) = 0,$$

although it is adopted to uniquely determine the boundary data n_l, p_l, n_r and p_r in the numerical simulation of the devices. Moreover, since we consider the situation that alternating-current voltage is applied to the devices, the function $\phi \in C(\mathbb{R})$ in (1.2b) is assumed to be periodic with period $T_* > 0$. Namely,

$$\phi(t + T_*) = \phi(t) \quad \text{for any } t \in \mathbb{R}.$$

Let us mention some known results. The drift-diffusion model was derived by Roosbroeck in [12]. Mock in [9] first studied the existence of the stationary solution in a multi-dimensional bounded domain. In his another paper [10], the asymptotic stability of the stationary solution was also discussed. In these two results, he treated the Neumann boundary condition which does not allow any electron and hole flow through the boundary. Gajewski and Gröger in [1] showed the unique existence and the stability for general cases. More precisely, they adopted the Dirichlet-Neumann mixed boundary condition which covers the case that a current permeates the boundary. However, they investigated only a special stationary solution (N, P, V) in thermal equilibrium, that is,

$$NP = 1, \quad (\log N - V)_x = (\log P + V)_x = 0. \tag{1.4}$$

Here the second equality means that the electron and the hole currents are zero. Under the general condition which admits stationary solutions in non-thermal equilibrium, Fang and Ito showed that time global solutions converge to an absorbing set as t tends to infinity in [3, 4, 5]. The relation between the absorbing set and the stationary solution was not made clear.

Time-periodic solutions are closely related to rectification of PN-junction diodes. However, its study is quite limited in the existing literature. Seidman in [13] constructed the time-periodic solutions with adopting the boundary condition different from that in the device simulation. On the other hand, the uniqueness and the stability have been quite open problems. The difficulty to study these problem arises in the fact that time-periodic solutions are usually in non-thermal equilibrium state. Indeed, even the stationary problem, which is a special case of the time-periodic problem, is investigated only in the thermal equilibrium state.

The main purpose of this paper is to show the unique existence and the global stability of the time-periodic solution. This also covers the situation that the stationary current flows in semiconductors if letting $\phi(t)$ be a constant. This situation was not treated in [1]. Before stating our main results, we give a definition of solutions to (1.1).

Definition 1.1. We say that (n, p, v) is a solution to the problem (1.1) and (1.2) if (n, p, v) satisfies (1.1) and (1.2) with the conditions (i)–(iii):

(i) $n, p \in C(I; H^1(\Omega)) \cap L^2_{loc}(\bar{I}; H^2(\Omega)) \cap H^1_{loc}(\bar{I}; L^2(\Omega))$.
(ii) $v \in C(I; H^2(\Omega))$.
(iii) $n(t,x), p(t,x) \geq 0 \quad \text{for any } (t,x) \in I \times \Omega$.

If (n_*, p_*, v_*) is a solution with $I = \mathbb{R}$ and additionally satisfies the following condition, we call (n_*, p_*, v_*) a time-periodic solution with period T_*.

(iv) $(n_*, p_*, v_*)(t + T_*, x) = (n_*, p_*, v_*)(t, x) \quad \text{for any } (t,x) \in \mathbb{R} \times \Omega$.

If a solution (n, p, v) with $I = (0, T)$ satisfies $(n, p)(t, \cdot) \to (n_0, p_0)$ in $L^2(\Omega) \times L^2(\Omega)$ as $t \downarrow 0$, (n, p, v) is said to be a solution to the initial-boundary value problem (1.1)–(1.3).

The existence of time-periodic solutions is established without any restriction of $\phi(t)$. On the other hand, the uniqueness is proved under the assumption that $\max_{t \in [0, T_*]} |\phi(t) - \phi_r|$ is small enough, where

$$\phi_r := V_b(1),\ V_b(x) := \log \frac{N_b(x)}{n_l},\ N_b(x) := n_l(1-x) + n_r x,\ P_b(x) := \frac{1}{N_b(x)} \tag{1.5}$$

(there notations are often used throughout this paper). This smallness assumption implies that the time-periodic solution is in a neighborhood of the stationary solution which is in thermal equilibrium. Note that Gajewski and Gröger in [1] constructed this stationary solution.

Proposition 1.2 ([1]). *The stationary problem to* (1.1) *and* (1.2) *with* $\phi(t) = \phi_r$ *has a unique solution* $(N, P, V) \in H^2(\Omega)$ *satisfying* $N, P > 0$. *Moreover, it satisfies* (1.4).

The unique existence result of time-periodic solutions is summarized in the next theorem.

Theorem 1.3. *The problem* (1.1) *and* (1.2) *has a time-periodic solution. Moreover, there exists* $\delta > 0$ *depending only on* ν, ε, n_l, n_r, p_l, p_r, T_*, ϕ_r *and* $|D|_\infty$ *such that if* $\max_{t\in[0,T_*]} |\phi(t) - \phi_r| < \delta$, *the time-periodic solution to* (1.1) *and* (1.2) *is unique.*

Next let us consider the initial-boundary value problem (1.1)–(1.3). We prove not only the global solvability but also universal a priori estimates.

Proposition 1.4. *For any* $(n_0, p_0) \in L^2(\Omega)\times L^2(\Omega)$, *the problem* (1.1)–(1.3) *admits a unique global solution* (n, p, v). *Moreover, the solution satisfies*

$$\limsup_{t\to\infty} (\|n(t)\|_1 + \|p(t)\|_1 + \|v(t)\|_1) \le C, \tag{1.6}$$

$$\limsup_{t\to\infty} \left(|n(t)^{-1}|_\infty + |p(t)^{-1}|_\infty\right) \le C, \tag{1.7}$$

where C *is some positive constant depending only on* ν, ε, n_l, n_r, p_l, p_r, $\max_{t\in[0,T_*]} |\phi(t)|$ *and* $|D|_\infty$.

Finally we give a global stability theorem for the time-periodic solution under the smallness condition on $\max_{t\in[0,T_*]} |\phi(t) - \phi_r|$.

Theorem 1.5. *There exists* $\delta > 0$ *depending only on* ν, ε, n_l, n_r, p_l, p_r, T_*, ϕ_r *and* $|D|_\infty$ *such that if* $\max_{t\in[0,T_*]} |\phi(t)-\phi_r| < \delta$, *then every global solution* (n, p, v) *to the problem* (1.1)–(1.3) *converges to the time-periodic solution* (n_*, p_*, v_*) *exponentially fast in* $L^\infty \times L^\infty \times W^{2,\infty}$ *as* t *goes to infinity.*

Outline of the paper. This paper is organized as follows. Section 2 establishes the existence and the uniqueness of time-periodic solutions. To show the existence, we first find a time-periodic solution to a modified problem by the Galerkin method, and then verify that the solution satisfies the original problem. The uniqueness is proved by employing an energy form. In Section 3, we discuss the global stability of the time-periodic solution. The key to showing the stability is to derive universal bounds for global solutions. Such bounds are obtained by an energy method.

Notation. For $1 \le p \le \infty$, $L^p(\Omega)$ denotes the Lebesgue space equipped with the norm $|\cdot|_p$. For a nonnegative integer $l \ge 0$, $H^l(\Omega)$ denotes the l-th order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. Moreover, for a nonnegative integer $l \ge 0$,

$W^{l,\infty}(\Omega)$ denotes the l-th order Sobolev space in the L^∞ sense. For $a \in \mathbb{R}$ and $k > 0$, we define

$$a_+ := \max\{a, 0\}, \quad a_- := \min\{a, 0\}, \quad a^k := \min\{a_+, k\}.$$

Lastly c and C denote generic positive constants depending only on ν, ε, n_l, n_r, p_l, p_r, T_*, $\max_{t\in[0,T_*]} |\phi(t)|$ and $|D|_\infty$. We denote a generic positive constant depending additionally on other parameters α, β, $\cdots$ by $C[\alpha, \beta, \cdots]$.

2. Construction of a Time-Periodic Solution

2.1. Existence

This subsection is devoted to the construction of a time-periodic solution to the problem (1.1) and (1.2). We begin with seeking a time-periodic solution (n, p, v) to the modified problem

$$n_t = n_{xx} - (n^k v_x)_x - R_k(n, p), \tag{2.1a}$$

$$p_t = p_{xx} + (p^k v_x)_x - R_k(n, p), \tag{2.1b}$$

$$\varepsilon v_{xx} = n - p - D, \quad (t, x) \in \mathbb{R} \times \Omega \tag{2.1c}$$

with (1.2), where $n_+ = \max\{n, 0\}$, $n^k = \min\{n_+, k\}$ for a positive constant k and

$$R_k(n, p) := \nu \frac{(n_+ p_+)^{k^2} - 1}{n_+ + p_+ + 2}.$$

After constructing the solution to (2.1), we show $n^k = n$ and $p^k = p$ by deriving a priori bounds independent of k. More precisely, we prove the next two propositions.

Proposition 2.1. *For any $k > \max\{n_l, p_l, n_r, p_r\}$, the problem* (2.1) *and* (1.2) *has a time-periodic solution (n, p, v) satisfying the conditions* (i)–(iv) *in Definition 1.1.*

Proposition 2.2. *There exists $K > 0$ depending only on ν, ε, n_l, n_r, p_l, p_r, T_*, $\max_{t\in[0,T_*]} |\phi(t)|$ and $|D|_\infty$ such that if $k > K$, then every time-periodic solution (n, p, v) to* (2.1) *and* (1.2) *satisfies $n^k = n$ and $p^k = p$.*

To prove Proposition 2.1, we reduce the problem (2.1) with (1.2) to the problem for $(\tilde{n}, \tilde{p}) = (n - N_b, p - P_b)$:

$$\tilde{n}_t = \tilde{n}_{xx} - \{(\tilde{n}+N_b)^k v_x\}_x - R_k(\tilde{n}+N_b, \tilde{p}+P_b) + N_{bxx}, \tag{2.2a}$$

$$\tilde{p}_t = \tilde{p}_{xx} + \{(\tilde{p}+P_b)^k v_x\}_x - R_k(\tilde{n}+N_b, \tilde{p}+P_b) + P_{bxx}, \tag{2.2b}$$

$$\begin{aligned} v = \Phi[\tilde{n},\tilde{p}] :=& \frac{1}{\varepsilon}\int_0^x\int_0^y (\tilde{n}-\tilde{p})(t,z) + (N_b - P_b - D)(z)\, dzdy \\ & - \frac{x}{\varepsilon}\int_0^1\int_0^y (\tilde{n}-\tilde{p})(t,z) + (N_b - P_b - D)(z)\, dzdy + \phi(t)x, \end{aligned} \tag{2.2c}$$

$$\tilde{n}(t,0) = \tilde{n}(t,1) = \tilde{p}(t,0) = \tilde{p}(t,1) = 0, \tag{2.2d}$$

where N_b and P_b are defined in (1.5) and Φ is a solution operator of the Poisson equation. The Galerkin method establishes the existence of a time-periodic solution to the problem (2.2) (for the details of the Galerkin method, see [15]). We take a complete orthonormal system $\{w_i(x) := \sqrt{2}\sin i\pi x\}_{i=1}^{\infty}$ in $L^2(\Omega)$ and then contract an approximate solution ($\tilde{n}_j(t,x) := \sum_{i=1}^{j} a_i(t)w_i(x)$, $\tilde{p}_j(t,x) := \sum_{i=1}^{j} b_i(t)w_i(x)$) by solving the system of nonlinear ordinary differential equations for $(a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_j)$:

$$\begin{aligned} (\tilde{n}_{jt}, w_i) &= (\tilde{n}_{jxx}, w_i) - (\{(\tilde{n}_j+N_b)^k \Phi[\tilde{n}_j,\tilde{p}_j]_x\}_x + R_k(\tilde{n}_j+N_b, \tilde{p}_j+P_b) \\ &\quad -N_{bxx}, w_i), \end{aligned} \tag{2.3a}$$

$$\begin{aligned} (\tilde{p}_{jt}, w_i) &= (\tilde{p}_{jxx}, w_i) + (\{(\tilde{p}_j+P_b)^k \Phi[\tilde{n}_j,\tilde{p}_j]_x\}_x - R_k(\tilde{n}_j+N_b, \tilde{p}_j+P_b) \\ &\quad +P_{bxx}, w_i), \end{aligned} \tag{2.3b}$$

$$(a_i, b_i)(t+T_*) = (a_i, b_i)(t) \tag{2.3c}$$

for $i = 1, 2, \ldots, j$, where $(\cdot,\cdot)$ denotes the standard L^2 inner product.

Lemma 2.3. *The problem* (2.3) *admits a solution* $(a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_j) \in (C^1(\mathbb{R}))^{2j}$.

Proof. We define a map $L : (\alpha_1, \alpha_2, \ldots, \alpha_j, \beta_1, \beta_2, \ldots, \beta_j) \mapsto (a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_j)$ over the Banach space $(C^0_{per}(\mathbb{R}))^{2j}$, where $C^0_{per}(\mathbb{R}) := \{f \in C^0(\mathbb{R}); f(t+T_*) = f(t) \text{ for } t \in \mathbb{R}\}$, by solving the following linear problem for $(a_1, a_2, \ldots, a_j, b_1, b_2, \ldots, b_j)$:

$$\begin{aligned} (\tilde{n}_{jt}, w_i) &= (\tilde{n}_{jxx}, w_i) - (\{(n_j+N_b)^k \Phi[n_j,p_j]_x\}_x + R_k(n_j+N_b, p_j+P_b) \\ &\quad -N_{bxx}, w_i), \end{aligned} \tag{2.4a}$$

$$\begin{aligned} (\tilde{p}_{jt}, w_i) &= (\tilde{p}_{jxx}, w_i) + (\{(p_j+P_b)^k \Phi[n_j,p_j]_x\}_x - R_k(n_j+N_b, p_j+P_b) \\ &\quad +P_{bxx}, w_i), \end{aligned} \tag{2.4b}$$

$$(a_i, b_i)(t + T_*) = (a_i, b_i)(t) \tag{2.4c}$$

for $i = 1, 2, \ldots, j$, where

$$n_j(t,x) := \sum_{i=1}^{j} \alpha_i(t) w_i(x), \qquad p_j(t,x) := \sum_{i=1}^{j} \beta_i(t) w_i(x).$$

The solvability of (2.4) follows from the standard theory of ordinary differential equations. It is also straightforward to check that L is continuous and compact over $(C^0_{per}(\mathbb{R}))^{2j}$. Hence, to construct a solution of the nonlinear problem (2.3) by the Leary-Schauder fixed point theorem (Theorem 11.3 in [2]), it is sufficient to show the boundedness

$$\max_{t\in[0,T_*]} (\|\tilde{n}_j(t)\|^2 + \|\tilde{p}_j(t)\|^2) \le M \tag{2.5}$$

for any solution $(\tilde{n}_j(t,x), \tilde{p}_j(t,x))$ of

$$\begin{aligned} (\tilde{n}_{jt}, w_i) &= (\tilde{n}_{jxx}, w_i) - \lambda(\{(\tilde{n}_j + N_b)^k \Phi[\tilde{n}_j, \tilde{p}_j]_x\}_x + R_k(\tilde{n}_j + N_b, \tilde{p}_j + P_b) \\ &\quad - N_{bxx}, w_i), \qquad (2.6a) \\ (\tilde{p}_{jt}, w_i) &= (\tilde{p}_{jxx}, w_i) + \lambda(\{(\tilde{p}_j + P_b)^k \Phi[\tilde{n}_j, \tilde{p}_j]_x\}_x - R_k(\tilde{n}_j + N_b, \tilde{p}_j + P_b) \\ &\quad + P_{bxx}, w_i) \qquad (2.6b) \end{aligned}$$

with (2.4c) for $i = 1, 2, \ldots, j$ and $\lambda \in [0,1]$, where M is some positive constant independent of λ and j.

Multiply (2.6a) by $a_i(t)$ and (2.6b) by $b_i(t)$, respectively. By summing up these resulting equalities for $i = 1, 2, \ldots, j$ and applying integration by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 (\tilde{n}_j)^2 + (\tilde{p}_j)^2\, dx + \int_0^1 (\tilde{n}_{jx})^2 + (\tilde{p}_{jx})^2 dx \\ &= -\lambda \int_0^1 G[\tilde{n}_j + N_b, \tilde{p}_j + P_b]\Phi_{xx}\, dx + \frac{1}{2}\lambda \left(n_r^2 - p_r^2\right) \Phi_x(t,1) \\ &\quad -\frac{1}{2}\lambda \left(n_l^2 - p_l^2\right) \Phi_x(t,0) - \lambda \int_0^1 \left\{(\tilde{n}_j + N_b)^k N_{bx} - (\tilde{p}_j + P_b)^k P_{bx}\right\} \Phi_x \\ &\quad + R_k(\tilde{n}_j + \tilde{p}_j) - N_{bxx}\tilde{n}_j - P_{bxx}\tilde{p}_j\, dx, \qquad (2.7) \end{aligned}$$

where

$$G[n,p] := n^k \left(n - \frac{1}{2}n^k\right) - p^k \left(p - \frac{1}{2}p^k\right).$$

We estimate the right-hand side of (2.7) separately. By (2.2c), the first term is rewritten as

$$
\begin{aligned}
\text{(1st term)} &= -\frac{\lambda}{\varepsilon}\int_0^1 G[\tilde{n}_j+N_b,\tilde{p}_j+P_b](\tilde{n}_j+N_b-\tilde{p}_j-P_b) \\
&\quad - G[\tilde{n}_j+N_b,\tilde{p}_j+P_b]D\,dx \\
&\le \frac{\lambda}{\varepsilon}\int_0^1 G[\tilde{n}_j+N_b,\tilde{p}_j+P_b]D\,dx \\
&\le C[k](1+\|\tilde{n}_{jx}\|+\|\tilde{p}_{jx}\|),
\end{aligned}
$$

where we have used the nonnegativity of $G[n,p](n-p)$ in deriving the first inequality and the Schwarz and the Poincaré inequalities in deriving the last inequality. $C[k]$ is some positive constant depending on k but independent of λ and j. One can also handle the other terms as

$$
\text{(other terms)} \le C[k](1+\|\tilde{n}_{jx}\|+\|\tilde{p}_{jx}\|)
$$

by utilizing $|\Phi[\tilde{n}_j,\tilde{p}_j]|_\infty \le C[k](1+\|\tilde{n}_j\|+\|\tilde{p}_j\|)$. Substituting these estimations into (2.7) and applying the Schwarz inequality give

$$
\frac{d}{dt}\left(\|\tilde{n}_j\|^2+\|\tilde{p}_j\|^2\right)+\|\tilde{n}_{jx}\|^2+\|\tilde{p}_{jx}\|^2 \le C[k]. \tag{2.8}
$$

Integrate this inequality over $[0,T_*]$ and apply the mean value theorem for integration to see that there exists $t_* \in [0,T_*]$ such that $\|\tilde{n}_{jx}(t_*)\|^2 + \|\tilde{p}_{jx}(t_*)\|^2 \le C[k]$. Moreover, integrating (2.8) over $[t_*, t_*+T_*]$ again and utilizing the periodicity and the Poincaré inequality, we conclude

$$
\max_{t\in[0,T_*]}\left(\|\tilde{n}_j(t)\|^2+\|\tilde{p}_j(t)\|^2\right)+\int_0^{T_*}\|\tilde{n}_{jx}(t)\|^2+\|\tilde{p}_{jx}(t)\|^2\,dt \le C[k]. \tag{2.9}
$$

Hence, the desired boundedness (2.5) holds. This ensures the existence of a fixed point of the mappings L. Moreover, it is obvious that this fixed point is a solution $(a_1,a_2,\ldots,a_j,b_1,b_2,\ldots,b_j) \in (C^1(\mathbb{R}))^{2j}$ of (2.3). □

Since the construction of the approximate solution has been complete, we are now at a position to prove Proposition 2.1.

Proof of Proposition 2.1. It is seen from Lemma 2.3 and its proof that the approximate solution $(\tilde{n}_j,\tilde{p}_j) \in C^1(\mathbb{R}; H_0^1(\Omega)\cap H^2(\Omega))$ satisfies (2.9). We derive the estimate of the higher order derivatives. Multiply (2.3a) by

$a_{it}(t) - (i\pi)^2 a_i(t)$ and (2.3b) by $b_{it}(t) - (i\pi)^2 b_i(t)$, respectively. Sum up them for $i = 1, 2, \dots, j$ and integrate by parts. Note that $w_{ixx} = -(i\pi)^2 w_i$. The result is

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 (\tilde{n}_{xj})^2 + (\tilde{p}_{xj})^2\,dx + \int_0^1 (\tilde{n}_{jxx})^2 + (\tilde{p}_{jxx})^2 + (\tilde{n}_{jt})^2 + (\tilde{p}_{jt})^2\,dx \\
&= \int_0^1 (-\{(\tilde{n}_j + N_b)^k \Phi_x\}_x - R_k + N_{bxx})(\tilde{n}_{jt} - \tilde{n}_{jxx})\,dx \\
&\quad + \int_0^1 (\{(\tilde{p}_j + P_b)^k \Phi_x\}_x - R_k + P_{bxx})(\tilde{p}_{jt} - \tilde{p}_{jxx})\,dx.
\end{aligned}$$

In a similar way to the derivation of (2.9), this equality yields

$$\begin{aligned}
&\max_{t\in[0,T_*]} \left(\|\tilde{n}_{xj}(t)\|^2 + \|\tilde{p}_{xj}(t)\|^2\right) + \int_0^{T_*} \|\tilde{n}_{jt}(t)\|^2 + \|\tilde{p}_{jt}(t)\|^2 + \|\tilde{n}_{jxx}(t)\|^2 \\
&\qquad + \|\tilde{p}_{jxx}(t)\|^2\,dt \le C[k],
\end{aligned} \tag{2.10}$$

where $C[k]$ is some positive constant depending on k but independent of λ and j. As this derivation is easier than that of (2.9), we omit the details.

By virtue of the boundedness (2.9) and (2.10), there exists a subsequence, still denoted by $\{(\tilde{n}_j, \tilde{p}_j)\}_{j=1}^{\infty}$, and $\tilde{n}, \tilde{p} \in C([0,T_*]; L^2) \cap L^2(0, T_*; H_0^1 \cap H^2) \cap H^1(0, T_*; L^2)$ such that

$$\begin{array}{lllll}
\tilde{n}_j, \tilde{p}_j & \to \tilde{n}, \tilde{p} & \text{in} & C([0,T_*]; L^2(\Omega)) & \text{strongly}, \\
\tilde{n}_j, \tilde{p}_j & \rightharpoonup \tilde{n}, \tilde{p} & \text{in} & L^2(0,T_*; H_0^1(\Omega) \cap H^2(\Omega)) & \text{weakly}, \\
\tilde{n}_{jt}, \tilde{p}_{jt} & \rightharpoonup \tilde{n}_t, \tilde{p}_t & \text{in} & L^2(0,T_*; L^2(\Omega)) & \text{weakly}.
\end{array}$$

Notice that $(\tilde{n}, \tilde{p})(0) = (\tilde{n}, \tilde{p})(T_*)$ holds (in L^2) thanks to $(\tilde{n}_j, \tilde{p}_j)(0) = (\tilde{n}_j, \tilde{p}_j)(T_*)$. Hence, extending the domain of $\tilde{n}$ and $\tilde{p}$ from $[0, T_*]$ to $\mathbb{R}$ periodically, we see that $(\tilde{n}, \tilde{p})$ is a desired time-periodic solution to (2.2). The standard theory of parabolic equations ensures the regularity $\tilde{n}, \tilde{p} \in C([0,T_*]; H^1)$. Consequently, it is immediately seen that $(n, p, v) = (\tilde{n} + N_b, \tilde{p} + P_b, \Phi[\tilde{n}, \tilde{p}])$ is a time-periodic solution to (2.1) satisfying the conditions (i), (ii) and (iv) in Definition 1.1.

Lastly, we show that the constructed time-periodic solution satisfies the condition (iii) in Definition 1.1. Multiply (2.1a) by $n_- := \min\{n, 0\}$ and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (n_-)^2\,dx + \int_0^1 \{(n_-)_x\}^2\,dx = \int_0^1 n^k v_x (n_-)_x\,dx - \int_0^1 R_k(n,p) n_-\,dx.$$

Notice that the first term of the right-hand side is zero and the second term is nonpositive. Thus, integrating the above equality over $[0,T_*]$ yields $\int_0^{T_*}\|(n_-)_x(t)\|^2\,dt \le 0$ owing to the periodicity. This together with the Poincaré inequality gives $n_- = 0$, which means $n \ge 0$. In the same way, we have $p \ge 0$ from (2.1b). $\square$

Next we show Lemmas 2.4 and 2.5 which immediately lead to Proposition 2.2 owing to the Morrey inequality. In the proofs, we utilize

$$\|u\| \le \|u_x\| \le \|u_{xx}\| \quad \text{for any } u \in H_0^1(\Omega)\cap H^2(\Omega). \tag{2.11}$$

Here and hereafter, we need the arguments using the mollifier with respect to the time variable t due to the insufficiency of the regularity of solutions and ϕ. However, we omit these arguments since they are standard.

Lemma 2.4. *Let (n,p,v) be a time-periodic solution to (2.1) and (1.2) satisfying the conditions (i)−(iv) in Definition 1.1. For any $k > \max\{n_l, p_l, n_r, p_r\}$, it holds that*

$$\max_{t\in[0,T_*]}(\|n(t)\|^2+\|p(t)\|^2+\|v_x(t)\|^2)+\int_0^{T_*}\|(n_x(t)\|^2+\|p_x(t)\|^2+\|v_{xx}(t)\|^2dt\le C, \tag{2.12}$$

where C is some positive constant independent of k.

Proof. Differentiating (2.1c) with respect to t and utilizing the equations (2.1a) and (2.1b) give

$$\varepsilon v_{xxt} = n_t - p_t = n_{xx} - p_{xx} - \{(n^k+p^k)v_x\}_x.$$

Multiply this equation by $-(v-\phi x)$ and integrate it by parts over Ω to obtain

$$\begin{aligned}&\frac{\varepsilon}{2}\frac{d}{dt}\int_0^1 \{(v-\phi x)_x\}^2\,dx\\ &= \int_0^1 -(n-N_b-p+P_b)(v-\phi x)_{xx} + (N_b-P_b)_x(v-\phi x)_x\end{aligned}$$

$$
\begin{aligned}
& -(n^k+p^k)\{(v-\phi x)_x\}^2 - \phi(n^k+p^k)(v-\phi x)_x\,dx \\
= \;& -\varepsilon\int_0^1 (v_{xx})^2 + (n^k+p^k)(v_x)^2\,dx + I_1, \qquad (2.13)\\
I_1 := \;& \int_0^1 (N_b-P_b-D)v_{xx} + (N_b-P_b)_x(v-\phi x)_x + \phi(n^k+p^k)v_x\,dx,
\end{aligned}
$$

where we have used (2.1c) in deriving the last equality of (2.13). By the Schwarz inequality and (2.11), I_1 is estimated as

$$
|I_1| \le \mu(\|v_{xx}\|^2 + \|\sqrt{n^k}v_x\|^2 + \|\sqrt{n^k}v_x\|^2 + \|n\|^2 + \|p\|^2) + C[\mu], \qquad (2.14)
$$

where μ is a positive constant to be determined later and $C[\mu]$ is some constant depending on μ but independent of k.

Multiply (2.1a) by $n-N_b$ and (2.1b) by $p-P_b$, respectively. Then adding both results and integrating by parts over Ω lead to

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int_0^1 (n-N_b)^2 + (p-P_b)^2\,dx + \int_0^1 (n_x)^2 + (p_x)^2\,dx \\
= \;& \int_0^1 n_xN_{bx} - \{(n^k)_xv_x + n^kv_{xx}\}n + (n^kv_x)_xN_b \\
& +p_xP_{bx} + \{(p^k)_xv_x + p^kv_{xx}\}p - (p^kv_x)_xP_b \\
& -R_k(n,p)\{(n-N_b)+(p-P_b)\}\,dx \\
= \;& -\int_0^1 G[n,p]v_{xx}dx - \int_0^1 R_k(n,p)\{(n-N_b)+(p-P_b)\}dx + I_2, \qquad (2.15)\\
I_2 := \;& \int_0^1 n_xN_{bx} + p_xP_{bx} - (n^kN_{bx} - p^kP_{bx})v_x\,dx + \frac{1}{2}\left(n_r^2 - p_r^2\right)v_x(t,1) \\
& -\frac{1}{2}\left(n_l^2 - p_l^2\right)v_x(t,0),
\end{aligned}
$$

where $G[n,p]$ is given in (2.7). Let us estimate each term of the right-hand side of (2.15). Substituting (2.1c) into the first term and using $2G[n,p](n-p) \ge (n^k+p^k)(n-p)^2$ and $|G[n,p]| \le 2(n^k+p^k)|n-p|$, we have

$$
\begin{aligned}
& -\int_0^1 G[n,p]v_{xx}dx = -\frac{1}{\varepsilon}\int_0^1 G[n,p](n-p)\,dx - \frac{1}{\varepsilon}\int_0^1 G[n,p]D\,dx \\
\le \;& -\frac{1}{4\varepsilon}\int_0^1 (n^k+p^k)(n-p)^2\,dx + \mu(\|n\|^2 + \|p\|^2) + C[\mu]. \qquad (2.16)
\end{aligned}
$$

Moreover, from the fact that $-C \le R_k(a,b) \le C(1+a+b)$ for any $a,b \ge 0$, one can handle the second term as

$$
\begin{aligned}
&-\int_0^1 R_k(n,p)\{(n-N_b)+(p-P_b)\}dx \\
&\quad = -\int_0^1 R_k(n,p)\{(n-N_b)_+ + (p-P_b)_+\}dx \\
&\qquad -\int_0^1 R_k(n,p)\{(n-N_b)_- + (p-P_b)_-\}dx \\
&\quad \le C\int_0^1 \{(n-N_b)_+ + (p-P_b)_+\}dx \\
&\qquad - C\int_0^1 (1+n+p)\{(n-N_b)_- + (p-P_b)_-\}dx \\
&\quad \le \mu(\|n\|^2+\|p\|^2)+C[\mu]. \qquad (2.17)
\end{aligned}
$$

For the estimation of I_2, we use

$$\|n\|+\|p\| \;\le\; C(1+\|n_x\|+\|p_x\|), \qquad (2.18)$$

$$|v_x|_\infty \;\le\; C(1+\|n\|+\|p\|). \qquad (2.19)$$

Note that the first inequality follows from (2.11) and the second inequality is derived from the solution formula $v=\Phi[n-N_b,p-P_b]$. From these we have

$$|I_2| \le \mu(\|n_x\|^2+\|p_x\|^2)+C[\mu]. \qquad (2.20)$$

Add (2.13) and (2.15) and substitute (2.14), (2.16), (2.17) and (2.20). Then, taking μ appropriately and using (2.18), we conclude

$$\frac{d}{dt}\left(\|n-N_b\|^2+\|p-P_b\|^2+\|(v-\phi x)_x\|^2\right)+\frac{1}{2}\left(\|n_x\|^2+\|p_x\|^2+\|v_{xx}\|^2\right)\le C, \qquad (2.21)$$

where C is some positive constant independent of k. Integrate this inequality over $[0,T_*]$ and apply the mean value theorem for integration to see that there exists $t_* \in [0,T_*]$ such that $\|n_x(t_*)\|^2+\|p_x(t_*)\|^2+\|v_{xx}(t_*)\|^2 \le C$. In addition, integrate (2.21) over $[t_*,t_*+T_*]$ again and utilize (2.11), (2.18) and the periodicity to obtain (2.12). The proof is complete. □

Lemma 2.5. *Under the same assumptions as in Lemma* 2.4*, it holds that*

$$\max_{t\in[0,T_*]}(\|n_x(t)\|^2+\|p_x(t)\|^2)+\int_0^{T_*}\|n_{xx}(t)\|+\|p_{xx}(t)\|^2\,dt\le C. \tag{2.22}$$

Proof. By multiplying (2.1a) by $-n_{xx}$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_0^1(n_x)^2dx&=-\int_0^1(n_{xx})^2dx+\int_0^1\{(n^k)_xv_x+n^kv_{xx}+R(n,p)\}n_{xx}dx\\&\le-\frac{1}{2}\|n_{xx}\|^2+|v_x|_\infty^2\|n_x\|^2+\|v_{xx}\|^2|n|_\infty^2\\&\quad+C(\|n\|^2+\|p\|^2+1).\end{aligned} \tag{2.23}$$

The Morrey inequality, (2.1c), (2.18) and (2.19) yield

$$\begin{aligned}|v_x|_\infty^2\|n_x\|^2+\|v_{xx}\|^2|n|_\infty^2&\le C(\|n\|^2+\|p\|^2+1)(\|n_x\|^2+1)\\&\le\frac{1}{4}\|n_{xx}\|^2+C(\|n\|^4+\|p\|^4+1).\end{aligned} \tag{2.24}$$

Substituting (2.24) into (2.23) gives $(d/dt)\|n_x\|^2+\|n_{xx}\|^2/2\le C$. In a similar way, we have $(d/dt)\|p_x\|^2+\|p_{xx}\|^2/2\le C$. Then we obtain (2.22) in the same way as the derivation of (2.12) from (2.21). $\square$

2.2. A priori bounds

In this subsection, we discuss several estimates of a time-periodic solution (n_*,p_*,v_*) to (1.1) and (1.2). More precisely, we show a priori bounds in Lemma 2.6 and estimates of the difference between the time-periodic solution and the stationary solution (N,P,V) to (1.1) and (1.2) with $\phi(t)=\phi_r$ in Lemma 2.7. These estimates are utilized in the proof of the uniqueness.

Lemma 2.6. *Every time-periodic solution* (n_*,p_*,v_*) *to* (1.1) *and* (1.2) *satisfies*

$$\max_{t\in[0,T_*]}(\|n_*(t)\|_1^2+\|p_*(t)\|_1^2+\|v_{*x}(t)\|^2)+\int_0^{T_*}\|n_{*x}(t)\|_1^2+\|p_{*x}(t)\|_1^2dt\le C, \tag{2.25}$$

$$\max_{t\in[0,T_*]}(|n_*(t)|_\infty+|p_*(t)|_\infty)\le C, \tag{2.26}$$

$$\max_{t\in[0,T_*]}(|n_*(t)^{-1}|_\infty + |p_*(t)^{-1}|_\infty) \le C, \tag{2.27}$$

where C is some positive constant depending only on ν, ε, n_l, n_r, p_l, p_r, T_*, $\max_{t\in[0,T_*]}|\phi(t)|$ and $|D|_\infty$.

Proof. By the same manner as in the proof of Lemmas 2.4 and 2.5, we obtain (2.25). The Morrey inequality together with (2.25) leads to (2.26). To derive (2.27), let us take $m_0 > 0$ sufficiently small so that

$$m_0 \le \min\{n_l, n_r, p_l, p_r\}, \quad \frac{\nu}{2(C_1+1)} - m_0 - \frac{m_0}{\varepsilon}(2C_1 + \|D\|_\infty) - \nu\frac{m_0C_1}{2} \ge 0,$$

where C_1 is a positive constant verifying (2.26) with $C = C_1$, and put $m(t) := m_0(1-e^{-(t-t_0)})$ for $t_0 \in \mathbb{R}$. Multiplying (1.1a) by $(n_* - m)_-$, integrating the resulting equality by parts and utilizing the fact that $0 \le n_* \le m_0$ if $(n_* - m)_- < 0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \{(n_*-m)_-\}^2\,dx + \int_0^1 [\{(n_*-m)_-\}_x]^2\,dx \\
&= \int_0^1 \left\{-m_0e^{-(t-t_0)} - \frac{1}{2\varepsilon}(n_*-p_*-D)(n_*+m) - \nu\frac{n_*p_*-1}{n_*+p_*+2}\right\}(n_*-m)_-dx \\
&\le \left\{\frac{\nu}{2(C_1+1)} - m_0 - \frac{m_0}{\varepsilon}(2C_1+\|D\|_\infty) - \nu\frac{m_0C_1}{2}\right\}\int_0^1 (n_*-m)_-dx \\
&\le 0.
\end{aligned}$$

This together with the nonnegativity of n_* yields $\|(n_*-m_0)_-(t)\|^2 = 0$ for any $t > t_0$. Therefore

$$\limsup_{t\to\infty}|n_*(t)^{-1}|_\infty \le \lim_{t\to\infty} m(t)^{-1} = m_0^{-1}.$$

Hence, $n_* \ge m_0$ holds owing to the periodicity. The same argument is valid for (1.1b) and thus we conclude (2.27) . $\square$

To estimate the difference between the time-periodic solution (n_*, p_*, v_*) and the stationary solution (N, P, V), we employ an energy form

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}[n_1, p_1, v_1, n_2, p_2, v_2] \\
&:= \int_0^\varphi \log\left(1+\frac{y}{n_2}\right)dy + \int_0^\psi \log\left(1+\frac{y}{p_2}\right)dy + \frac{\varepsilon}{2}(\eta_x - \eta|_{x=1})^2,
\end{aligned}$$

where

$$\varphi = \varphi[n_1,n_2] := n_1-n_2,\ \psi=\psi[p_1,p_2] := p_1-p_2,\ \eta=\eta[v_1,v_2] := v_1-v_2.$$

It is seen that the energy form $\mathcal{E}$ for any two solutions (n_1,p_1,v_1) and (n_2,p_2,v_2) to (1.1) satisfies

$$\mathcal{E}_t = -\mathcal{D}+\mathcal{J}+\mathcal{K}+\mathcal{L}+\mathcal{M}+\mathcal{B}_x, \tag{2.28}$$

where $\mathcal{D},\mathcal{J},\dots,\mathcal{B}$ are defined by

$$\begin{aligned}
\mathcal{D} &= \mathcal{D}[n_1,p_1,v_1,n_2,p_2,v_2] \\
&:= n_1\left\{\left(\log\frac{n_1}{n_2}-\eta\right)_x\right\}^2 + p_1\left\{\left(\log\frac{p_1}{p_2}+\eta\right)_x\right\}^2, \\
\mathcal{J} &= \mathcal{J}[n_1,p_1,v_1,n_2,p_2,v_2] \\
&:= -\left\{n_1\left(\log\frac{n_1}{n_2}-\eta\right)_x - p_1\left(\log\frac{p_1}{p_2}+\eta\right)_x\right\}(\eta|_{x=1}), \\
\mathcal{K} &= \mathcal{K}[n_1,p_1,n_2,p_2] \\
&:= -(R(n_1,p_1)-R(n_2,p_2))(\log n_1p_1-\log n_2p_2), \\
\mathcal{L} &= \mathcal{L}[n_1,p_1,n_2,p_2] \\
&:= R(n_2,p_2)\left(\int_0^{\varphi}\frac{y}{n_2(n_2+y)}dy+\int_0^{\psi}\frac{y}{p_2(p_2+y)}dy\right), \\
\mathcal{M} &= \mathcal{M}[n_1,p_1,v_1,n_2,p_2,v_2] \\
&:= \{\varphi(\log n_2-v_2)_x-\psi(\log p_2+v_2)_x\}\{\eta_x-(\eta|_{x=1})\}, \\
\mathcal{B} &= \mathcal{B}[n_1,p_1,v_1,n_2,p_2,v_2] \\
&:= n_1\left(\log\frac{n_1}{n_2}-\eta\right)_x\left(\log\frac{n_1}{n_2}-\{\eta-(\eta|_{x=1})x\}\right) \\
&\quad +p_1\left(\log\frac{p_1}{p_2}+\eta\right)_x\left(\log\frac{p_1}{p_2}+\{\eta-(\eta|_{x=1})x\}\right) \\
&\quad +\{n_1(\log n_2-v_2)_x\}\log\frac{n_1}{n_2}+\{n_2(\log n_2-v_2)_x\}\left(1-\frac{n_1}{n_2}\right) \\
&\quad +\{p_1(\log p_2+v_2)_x\}\log\frac{p_1}{p_2}+\{p_2(\log p_2+v_2)_x\}\left(1-\frac{p_1}{p_2}\right) \\
&\quad -\{\varphi(\log n_2-v_2)_x-\psi(\log p_2+v_2)_x\}\{\eta-(\eta|_{x=1})x\} \\
&\quad +\varepsilon\{\eta-(\eta|_{x=1})x\}\{\eta-(\eta|_{x=1})x\}_{xt}.
\end{aligned}$$

Moreover, we utilize the elementary inequalities

$$\frac{b^2}{2(a+b_+)} \leq \int_0^b \log\left(1+\frac{y}{a}\right) dy \leq b \log\left(1+\frac{b}{a}\right) \quad \text{for any } a>0, b>-a \tag{2.29}$$

and the equations for (φ, ψ, η)

$$\varphi_t = \varphi_{xx} - (n_1\eta_x + \varphi v_{2x})_x - R(n_1,p_1) + R(n_2,p_2), \tag{2.30a}$$

$$\psi_t = \psi_{xx} + (p_1\eta_x + \psi v_{2x})_x - R(n_1,p_1) + R(n_2,p_2), \tag{2.30b}$$

$$\varepsilon\{\eta - (\eta|_{x=1})x\}_{xx} = \varphi - \psi. \tag{2.30c}$$

Lemma 2.7. *Let (N,P,V) be the stationary solution to* (1.1) *and* (1.2) *with $\phi(t)=\phi_r$. Then every time-periodic solution (n_*,p_*,v_*) to* (1.1) *and* (1.2) *satisfies*

$$\max_{t\in[0,T_*]}(\|(n_*-N)(t)\|_1^2 + \|(p_*-P)(t)\|_1^2 + \|(v_{*x}-V_x)(t)\|^2) \\ \leq C \max_{t\in[0,T_*]}|\phi(t)-\phi_r|^2, \tag{2.31}$$

$$|n_*p_*-1|_\infty + \|(\log n_* - v_*)_x\| + \|(\log p_* + v_*)_x\| \\ \leq C \max_{t\in[0,T_*]}|\phi(t)-\phi_r|, \tag{2.32}$$

where C is some positive constant depending only on ν, ε, n_l, n_r, p_l, p_r, T_*, $\max_{t\in[0,T_*]}|\phi(t)|$ and $|D|_\infty$.

Proof. Since (2.32) follows from (1.4) and (2.31), it is sufficient to show (2.31). Substitute $(n_1,p_1,v_1,n_2,p_2,v_2)=(n_*,p_*,v_*,N,P,V)$ into φ, ψ, η, $\mathcal{E}$, $\mathcal{D}$, $\ldots$, $\mathcal{B}$ and write them by φ_e, ψ_e, η_e, $\mathcal{E}_e$, $\mathcal{D}_e$, $\ldots$, $\mathcal{B}_e$, respectively. Integrate (2.28) over Ω and use (1.2) and (1.4). The result is

$$\frac{d}{dt}\int_0^1 \mathcal{E}_e\,dx + \int_0^1 \mathcal{D}_e\,dx = \int_0^1 \mathcal{J}_e\,dx + \int_0^1 \mathcal{K}_e\,dx. \tag{2.33}$$

First we estimate $\int_0^1 \mathcal{E}_e\,dx$ from below. Owing to (2.11), (2.26) and (2.29), it holds that

$$\int_0^1 \mathcal{E}_e\,dx \geq C(\|\varphi_e\|^2 + \|\psi_e\|^2 + \|\eta_e - (\eta_e|_{x=1})x\|_1^2). \tag{2.34}$$

For the estimation of $\int_0^1 \mathcal{E}_e\,dx$ from above, we use the inequality

$$\varepsilon\|\eta_{ex} - (\eta_e|_{x=1})\|^2 = \int_0^1 (-\varphi_e + \psi_e)\eta_e\,dx + (\eta_e|_{x=1}) \int_0^1 x(\varphi_e - \psi_e)\,dx,$$

which follows from (2.30c). This together with (2.11) and (2.29) leads to

$$\begin{aligned}\int_0^1 \mathcal{E}_e\,dx \leq & \int_0^1 \varphi_e \left(\log \frac{n_*}{N} - \eta_e\right) + \psi_e \left(\log \frac{p_*}{P} + \eta_e\right)\,dx \\ & + (\eta_e|_{x=1}) \int_0^1 x(\varphi_e - \psi_e)\,dx \\ \leq & \mu \left(\|\varphi_e\|^2 + \|\psi_e\|^2\right) + C[\mu] \left(\int_0^1 \mathcal{D}_e\,dx + (\eta_e|_{x=1})^2\right),\end{aligned} \tag{2.35}$$

where $\mu > 0$ is an arbitrary constant. Combining (2.34) and (2.35) and taking μ appropriately, we deduce

$$\int_0^1 \mathcal{E}_e\,dx \leq C \left(\int_0^1 \mathcal{D}_e\,dx + (\eta_e|_{x=1})^2\right). \tag{2.36}$$

Let us estimate the right-hand side of (2.33). By the Schwarz inequality and (2.25), the first term is handled as

$$\int_0^1 \mathcal{J}_e\,dx \leq \mu \int_0^1 \mathcal{D}_e\,dx + C[\mu](\eta_e|_{x=1})^2. \tag{2.37}$$

Moreover, $\int_0^1 \mathcal{K}_e\,dx \geq 0$ follows from (1.4) and $(n_*p_* - 1)\log n_*p_* \geq 0$. Thus the second term is negligible. Substituting (2.37) into (2.33), letting μ sufficiently small and then using (2.36) and $\eta_e|_{x=1} = \phi(t) - \phi_r$, we conclude

$$\frac{d}{dt}\int_0^1 \mathcal{E}_e\,dx + c\int_0^1 \mathcal{E}_e\,dx \leq C \max_{t\in[0,T_*]} |\phi(t) - \phi_r|^2.$$

Hence, by the same calculations as the derivation of (2.12) from (2.21) with the aid of (2.34), we have

$$\max_{t\in[0,T_*]} (\|\varphi_e(t)\|^2 + \|\psi_e(t)\|^2 + \|\eta_{ex}(t)\|^2) \leq C \max_{t\in[0,T_*]} |\phi(t) - \phi_r|^2.$$

Let us derive the estimate of the first order derivatives of φ_e and ψ_e.

Multiplying (2.30a) by $-\varphi_{exx}$ and integrating over Ω give

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 (\varphi_{ex})^2 dx \\
&= -\int_0^1 (\varphi_{exx})^2 dx + \int_0^1 \{(n_*\eta_{ex} + \varphi_e V_x)_x + R(n_*, p_*) - R(N,P)\}\,\varphi_{exx} dx \\
&\le -\frac{1}{2}\|\varphi_{xx}\|^2 + |\eta_{ex}|_\infty^2\|n_{*x}\|^2 + \|\eta_{exx}\|^2|n_*|_\infty^2 + |V_x|_\infty^2\|\varphi_{ex}\|^2 + \|V_{xx}\|^2|\varphi_e|_\infty^2 \\
&\quad + C(\|\varphi_e\|^2 + \|\psi_e\|^2). \qquad (2.38)
\end{aligned}$$

The Morrey inequality, (2.11), (2.25), (2.26) and (2.30c) yield

$$\begin{aligned}
&|\eta_{ex}|_\infty^2\|n_{*x}\|^2 + \|\eta_{exx}\|^2|n_*|_\infty^2 + |V_x|_\infty^2\|\varphi_{ex}\|^2 + \|V_{xx}\|^2|\varphi_e|_\infty^2 \\
&\quad \le C\left(\|\varphi_e\|_1^2 + \|\psi_e\|^2\right) \\
&\quad \le \frac{1}{4}\|\varphi_{exx}\|^2 + C\left(\|\varphi_e\|^2 + \|\psi_e\|^2\right). \qquad (2.39)
\end{aligned}$$

Substituting (2.39) into (2.38) gives $(d/dt)\|\varphi_{ex}\|^2+\|\varphi_{exx}\|^2/2 \le C\max_t|\phi(t) -\phi_r|^2$. Similarly, the inequality $(d/dt)\|\psi_{ex}\|^2 + \|\psi_{exx}\|^2/2 \le C\max_t|\phi(t) - \phi_r|^2$ follows from (2.30b). Therefore we obtain

$$\max_{t\in[0,T_*]}(\|\varphi_{ex}(t)\|^2 + \|\psi_{ex}(t)\|^2) \le C\max_{t\in[0,T_*]}|\phi(t) - \phi_r|^2$$

by the same computations as the derivation of (2.12) from (2.21). The proof is complete. □

2.3. Uniquness

The existence of a time-periodic solution has been established in Subsection 2.1. To complete the proof of Theorem 1.3, it is sufficient to prove the uniqueness under the smallness assumption on $\max_t|\phi(t) - \phi_r|$.

***Proof of Theorem* 1.3.** Let (n_1, p_1, v_1) and (n_2, p_2, v_2) be time-periodic solutions to (1.1) and (1.2) with period $T_* > 0$. Substitute $(n_1, p_1, v_1, n_2, p_2, v_2)$ into φ, ψ, η, $\mathcal{E}$, $\mathcal{D}$, …, $\mathcal{B}$ and write them by φ_*, ψ_*, η_*, $\mathcal{E}_*$, $\mathcal{D}_*$, …, $\mathcal{B}_*$, respectively. Integrate (2.28) over Ω and use (1.2) to obtain

$$\frac{d}{dt}\int_0^1 \mathcal{E}_*\,dx + \int_0^1 \mathcal{D}_*\,dx = \int_0^1 \mathcal{K}_*\,dx + \int_0^1 \mathcal{L}_* + \mathcal{M}_*\,dx. \qquad (2.40)$$

Firstly, we have

$$c(\|\varphi_*\|^2 + \|\psi_*\|^2 + \|\eta_*\|^2) \le \int_0^1 \mathcal{E}_*\,dx \le C\int_0^1 \mathcal{D}_*\,dx \tag{2.41}$$

in a similar way to the derivations of (2.34) and (2.36) with the aid of $\eta_*|_{x=1} = 0$. The term $\int_0^1 \mathcal{K}_*\,dx$ is estimated from above as

$$\begin{aligned}\int_0^1 \mathcal{K}_*\,dx &= -\nu\int_0^1 \frac{n_1p_1 - n_2p_2}{n_1+p_1+2}\log\frac{n_1p_1}{n_2p_2}dx \\ &\quad + \nu\int_0^1 \frac{(n_2p_2-1)(\varphi_*+\psi_*)}{(n_1+p_1+2)(n_2+p_2+2)}\log\frac{n_1p_1}{n_2p_2}dx \\ &\le \frac{\nu}{4}|n_2p_2-1|_\infty \int_0^1 (|\varphi_*|+|\psi_*|)\left|\log\frac{n_1p_1}{n_2p_2}\right|dx \\ &\le C\max_{t\in[0,T_*]}|\phi(t)-\phi_r|\int_0^1 \mathcal{D}_*\,dx, \end{aligned}\tag{2.42}$$

where we have used the nonnegativity $(a-b)(\log a - \log b) \ge 0$ in deriving the first inequality and used (2.26), (2.27), (2.32) and (2.41) in deriving the last inequality. By (2.26) and (2.27), one can handle $\int_0^1 \mathcal{L}_* + \mathcal{M}_*\,dx$ as

$$\begin{aligned}\int_0^1 \mathcal{L}_* + \mathcal{M}_*\,dx \le& C\|R(n_2,p_2)\|_\infty(\|\varphi_*\|^2 + \|\psi_*\|^2) \\ &+ C(\|\varphi_*\|\|(\log n_2 - v_2)_x\| + \|\psi_*\|\|(\log p_2 + v_2)_x\|)|\eta_{*x}|_\infty \\ \le& C\max_{t\in[0,T_*]}|\phi(t)-\phi_r|\int_0^1 \mathcal{D}_*\,dx,\end{aligned}\tag{2.43}$$

where we have used (2.32), (2.41) and the elliptic estimate $\|\eta_*\|_2^2 \le C(\|\varphi_*\|^2 + \|\psi_*\|^2)$ in deriving the last inequality.

Substituting (2.42) and (2.43) into (2.40), making $\max_t|\phi(t)-\phi_r|$ small enough and then utilizing (2.41), we conclude

$$\frac{d}{dt}\int_0^1 \mathcal{E}_*\,dx + c\int_0^1 \mathcal{E}_*\,dx \le 0.$$

Consequently, integrating this inequality over $[0,T_*]$ together with the periodicity and (2.41) yields $(n_1,p_1,v_1) = (n_2,p_2,v_2)$. □

3. Asymptotic Behavior of Solutions

In this section we consider the global stability of the time-periodic solution constructed in the previous section. Throughout this section, the initial value (n_0, p_0) is assumed to belong to $L^2(\Omega) \times L^2(\Omega)$ and a constant $C > 0$ is independent of (n_0, p_0).

3.1. Global solvability and a priori bounds

This subsection is devoted to proving Proposition 1.4 which ensures the global existence and the universal bounds of solutions to the initial-boundary value problem (1.1)–(1.3).

Before proving Proposition 1.4, we show the local solvability.

Lemma 3.1. *The problem* (1.1)–(1.3) *admits a unique solution for some interval* $I = (0, T)$ *with* $T = C^{-1}e^{-C(\|n_0\|^2+\|p_0\|^2)}$.

Proof. Let $T > 0$ and define a mapping $S : (L^2(0, T; L^2))^2 \to (L^2(0, T; L^2))^2$ as follows. Let $(n, p) \in (L^2(0, T; L^2))^2$ and set $v = \Phi[n - N_b, p - P_b] \in L^2(0, T; H^2)$, where Φ is defined in (2.2c). The Morrey inequality gives

$$\|v_x\|_{L^2(0,T;L^\infty)} \leq C(\|n\|_{L^2(0,T;L^2)} + \|p\|_{L^2(0,T;L^2)}).$$

From this and the fact that $|R(a_+, b_+)| \leq C(|a| + |b| + 1)$, one can show by the Galerkin method that there exists a unique solution $(\tilde{n}, \tilde{p})$ of the linear system

$$\begin{cases} \tilde{n}' = (\tilde{n}_x - v_x\tilde{n})_x - R(n_+, p_+), \\ \tilde{p}' = (\tilde{p}_x + v_x\tilde{p})_x - R(n_+, p_+), \quad \text{a.e. } \ t \in (0, T) \ \text{ in } \ H^{-1} \end{cases}$$

with the conditions

$$(\tilde{n} - N_b, \tilde{p} - P_b) \in (L^2(0, T; H_0^1))^2, \quad (\tilde{n}', \tilde{p}') \in (L^2(0, T; H^{-1}))^2,$$
$$(\tilde{n}, \tilde{p})(0, \cdot) = (n_0, p_0).$$

Furthermore, it is straightforward to derive the estimates

$$\max_{t\in[0,T]} \|\tilde{n}(t)\|^2 + \max_{t\in[0,T]} \|\tilde{p}(t)\|^2$$

$$\leq C(\|n_0\|^2 + \|p_0\|^2 + 1) \exp\left(C \int_0^T \left(\|n(t)\|^2 + \|p(t)\|^2 + 1 \right) dt \right), \tag{3.1}$$

$$\int_0^T \|\tilde{n}(t)\|_1^2 dt + \int_0^T \|\tilde{p}(t)\|_1^2 dt + \int_0^T \|\tilde{n}'(t)\|_{H^{-1}}^2 dt + \int_0^T \|\tilde{p}'(t)\|_{H^{-1}}^2 dt$$
$$\leq C \left(\max_{t\in[0,T]} \|\tilde{n}(t)\|^2 + \max_{t\in[0,T]} \|\tilde{p}(t)\|^2 \right) \int_0^T \left(\|n(t)\|^2 + \|p(t)\|^2 + 1 \right) dt. \tag{3.2}$$

Then the mapping S is defined by $S((n, p)) := (\tilde{n}, \tilde{p})$.

Fixed points of S and solutions of (1.1)–(1.3) are one-to-one correspondence. Indeed, we see from the regularity theory for parabolic equations that every fixed point (n, p) of S satisfies the condition (i) in Definition 1.1. In addition, the fixed point (n, p) also satisfies the condition (iii) since testing n_- and p_- gives

$$\frac{d}{dt}(\|n_-\|^2 + \|p_-\|^2) \leq C(1 + \|n\|^2 + \|p\|^2)(\|n_-\|^2 + \|p_-\|^2).$$

Note that $v := \Phi[n - N_b, p - P_b]$ satisfies the condition (ii).

For $(\tilde{n}_1, \tilde{p}_1) = S((n_1, p_1))$ and $(\tilde{n}_2, \tilde{p}_2) = S((n_2, p_2))$ one can derive

$$\begin{aligned} &\frac{d}{dt}\left(\|\tilde{n}_1 - \tilde{n}_2\|^2 + \|\tilde{p}_1 - \tilde{p}_2\|^2\right) \\ &\leq C(1 + \|n_1\|^2 + \|p_1\|^2)(\|\tilde{n}_1 - \tilde{n}_2\|^2 + \|\tilde{p}_1 - \tilde{p}_2\|^2) \\ &\quad + C(1 + \|\tilde{n}_2\|^2 + \|\tilde{p}_2\|^2)(\|n_1 - n_2\|^2 + \|p_1 - p_2\|^2). \end{aligned}$$

This shows that S is continuous and S has at most one fixed point.

Let us show the existence of a fixed point of S. Put $M := 1 + \|n_0\|^2 + \|p_0\|^2$ and

$$K_T := \left\{ (n, p) \in (L^2(0, T; L^2))^2 \, ; \int_0^T \left(\|n(t)\|^2 + \|p(t)\|^2 \right) dt \leq M \right\}.$$

Since (3.1) yields $\int_0^T \|\tilde{n}(t)\|^2 dt + \int_0^T \|\tilde{p}(t)\|^2 dt \leq CMTe^{C(M+T)}$ for all $(n, p) \in K_T$, S maps from K_T to K_T if $T \leq C^{-1}e^{-CM}$. Moreover, by (3.1), (3.2) and the Aubin-Lions lemma (see [14, Section 8]), we see that the image $S(K_T)$ is precompact in $(L^2(0, T; L^2))^2$. Thus the Schauder fixed point theorem (see, for instance, [2, Corollary 11.2]) shows that S has a fixed point in K_T, provided $T \leq C^{-1}e^{-CM}$. Therefore we obtain the conclusion. □

We are now in a position to prove Proposition 1.4.

***Proof of Proposition* 1.4.** Let (n, p, v) be a solution of (1.1)–(1.3) with $I = (0, T)$. Then we can show by the same computations as in the proof of Lemma 2.4 that

$$\frac{d}{dt}\left\{\|n - N_b\|^2 + \|p - P_b\|^2 + \|(v - \phi x)_x\|^2\right\} + \frac{1}{2}\left\{\|n_x\|^2 + \|p_x\|^2 + \|v_{xx}\|^2\right\} \le C$$

for $0 < t < T$. Using (2.11) and multiplying the above inequality by $e^{t/2}$, we have

$$\|n(t)\| + \|p(t)\| + \|v_x(t)\| \le (\|n(0)\| + \|p(0)\| + \|v_x(0)\|)\, e^{-t/2} + C.$$

From this estimate and Lemma 3.1, we conclude that the problem (1.1)–(1.3) has a unique global solution. Furthermore, the estimates (1.6) and (1.7) can be derived in the same way as in the proofs of Lemmas 2.5 and 2.6. Therefore the assertion follows. □

3.2. Global stability of the time-periodic solution

In this subsection we give the proof of Theorem 1.5.

***Proof of Theorem* 1.5.** Let (n, p, v) be a global solution to (1.1)–(1.3) and (n_*, p_*, v_*) be a time-periodic solution to (1.1) and (1.2). By Lemma 2.6 and Proposition 1.4, we can take $t_0 \ge 0$ such that

$$\sup_{t \ge t_0}(\|n(t)\|_1 + \|p(t)\|_1 + \|n_*(t)\|_1 + \|p_*(t)\|_1) \le C, \tag{3.3}$$

$$\sup_{t \ge t_0}(|n(t)^{-1}|_\infty + |p(t)^{-1}|_\infty + |n_*(t)^{-1}|_\infty + |p_*(t)^{-1}|_\infty) \le C. \tag{3.4}$$

Let $\mathcal{E} = \mathcal{E}[n, p, v, n_*, p_*, v_*]$, $\varphi = \varphi[n, n_*]$, $\psi = \psi[p, p_*]$ and $\eta = \eta[v, v_*]$ be defined as before. By using (3.3) and (3.4) and making the same arguments as in the proof of Theorem 1.3, we find

$$\int_0^1 \mathcal{E}\, dx \ge C(\|\varphi\|^2 + \|\psi\|^2 + \|\eta\|_1^2), \qquad \frac{d}{dt}\int_0^1 \mathcal{E}\, dx + c\int_0^1 \mathcal{E}\, dx \le 0$$

for $t \ge t_0$, provided that $\max_t |\phi(t) - \phi_r|$ is small enough. These inequalities immediately give

$$\|\varphi(t)\| + \|\psi(t)\| + \|\eta(t)\|_1 \le Ce^{-c(t-t_0)}. \tag{3.5}$$

Thanks to (3.3), (3.5) and the Gagliardo-Nirenberg interpolation inequality, φ and ψ decay exponentially fast in L^∞ as $t \to \infty$. From (2.30c), we deduce the decay of η in $W^{2,\infty}$. Thus the proof is complete. $\square$

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