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GLOBAL SOLUTIONS TO 3D ISENTROPIC COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH FREE BOUNDARY

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Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

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Abstract

In this paper, we consider the free boundary value problem of 3D isentropic compressible Navier-Stokes equations with the stress free boundary condition where the compressible viscous flow of finite mass expands into infinite vacuum. The density changes continuously (or discontinuously) across the interfaces separating the fluid and vacuum. For the spherically symmetric initial data with finite energy, we prove the global existence of spherically symmetric weak solutions. Furthermore, we investigate the expanding rate of the domain occupied by the fluid.

1. Introduction

The Navier-Stokes equations are the basic equations of fluid mechanics, describing the motion of viscous fluid. For isentropic case, the Navier-Stokes equations of compressible viscous fluid in $\mathbb{R}^n$ can be written as

$$\begin{cases} \rho_t + \mathrm{div}(\rho\mathbf{u}) = 0, \\ (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \mathrm{div}(2\xi D(\mathbf{u})) + \nabla(\lambda\mathrm{div}\mathbf{u}), \end{cases} \tag{1.1}$$

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where $\rho(\mathbf{x},t), \mathbf{u}(\mathbf{x},t) = (u_1(\mathbf{x}),\ldots,u_n(\mathbf{x}))$ and $P(\rho) = \rho^\gamma\,(\gamma > 1)$ denote the density, velocity and pressure respectively.

$$D(\mathbf{u}) = \frac{\nabla\mathbf{u} + \nabla^t\mathbf{u}}{2}$$

is the viscous stress tensor. ξ and λ are the Lamé viscosity coefficients satisfying

$$\xi > 0,\ 2\xi + n\lambda \geq 0. \tag{1.2}$$

There are huge literatures on initial boundary value problems and the initial value problems for the compressible Navier-Stokes equations. In one dimension, the problem has been studied extensively, refer to [17, 9, 19] and the references therein. In the multi-dimensional case, the problem is more complicated and is far from being completed. The global existence of smooth solutions to Cauchy problem and initial boundary value problem for (1.1) was first proved by Matsumura and Nishida for smooth initial data with small perturbation [21, 22, 23] and later by Hoff [11] for discontinuous initial data. Danchin [2] obtained existence and uniqueness of global strong solution for initial small perturbation in Besov space. The global existence of renormalized solutions to (1.1) in $\mathbb{R}^n (n = 2,3)$ for arbitrarily large initial data with finite total energy and the vacuum possibly contained was made by Lions [19] for $\gamma \geq \frac{3n}{n+2}$ and by Feireisl, Novotný and Petzeltoý [5] for $\gamma > \frac{n}{2}$. If the initial data is spherically symmetric, Jiang and Zhang [14] obtained a global spherically symmetric weak solution provided $\gamma > 1$. Recently, the classical solution with small initial total energy and possibly vacuum is shown to exist globally in time in $\mathbb{R}^3$ by Huang-Li-Xin [8].

The free boundary problems for the compressible Navier-Stokes equations (1.1) which involves the influence of the vacuum state on the existence and dynamics of global solutions to (1.1) has attracted lots of research interests and been studied with rather abundant results for general initial data and variant boundary conditions imposed on the free surface, refer to [1, 6, 15, 20, 24, 25, 32, 34, 35, 36, 37] and the references therein. The free boundary problems have been studied extensively in one dimension, in particular, global existence of the weak solutions to the free boundary problem was investigated for one boundary fixed and the other connected to vacuum in [24]. Similar results were obtained for the equations of spherically symmetric motion of viscous fluid in [25]. A further understanding of the regularity and the behavior of solutions near the interfaces between the flow and the vacuum was given by Luo-Xin-Yang in [20]. For the free boundary problem in multi-dimension, there are also many important works concerned with the well-posedness and asymptotic behaviors of solutions for either barotropic [6], [30, 35] or heat-conductive fluid [1, 36]. In particular, the local classical solutions to the free boundary value problem for (1.1) is shown in the case that across the free surface stress tensor is balanced by a constant exterior pressure and/or the surface tension for either barotropic flow [30, 34, 37] or heat-conductive flow [28, 36]. The global existence of strong solutions close to the equilibrium state is established in the case that across the free surface the stress tensor is balanced by exterior pressure [34], surface tension[31] or both surface tension and constant exterior pressure [35] in three-dimension. Global solutions to the free boundary value problem for compressible heat-conductive flow are constructed for spherically symmetric initial data of large oscillation between a static solid core and a free boundary connected to a surrounding vacuum state in [1]. Global existence of a spherically symmetric weak solution to the multi-dimensional free boundary value problem with density-dependent viscosity coefficients for arbitrary large data was shown by Guo-Li-Xin [6] subject to the stress free boundary condition and positive flow density near/at the free boundary.

In this paper, we investigate the free boundary value problem (FBVP) for the compressible viscous flow with the stress-free boundary condition and the zero flow density across the free boundary. For spherically symmetric initial data with finite total energy, we prove the global existence of spherically symmetric weak solutions to the FBVP problem for (1.1), establish the local regularity of solution and the positivity of flow density, and obtain the expanding rate of the domain occupied by the fluid (refer to Theorems 2.1−2.2).

The rest part of the paper is arranged as follows. In Sect. 2 we state our main results. In Sect. 3, we construct the global approximate solutions. In Sect. 4, the key uniform estimates are established and the global existence of spherically symmetric solution is proved. In Sect. 5, the expanding rate of the domain is made.

2. Main Results

Consider a spherically symmetric solution $(\rho, \mathbf{u})$ to (1.1) in $\mathbb{R}^3$ so that

$$\rho(\mathbf{x}, t) = \rho(r, t), \;\; \mathbf{u} = u(r, t)\frac{\mathbf{x}}{r}, \;\; r = |\mathbf{x}|,$$

and (1.1) are changed to

$$\begin{cases} \rho_t + (\rho u)_r + \frac{2\rho u}{r} = 0, \\ (\rho u)_t + (\rho u^2 + \rho^\gamma)_r + \frac{2\rho u^2}{r} = (\lambda + 2\xi)(u_r + \frac{2u}{r})_r, \end{cases} \tag{2.1}$$

for $(r, t) \in \Omega_T$ and

$$\Omega_T = \{(r, t)|0 \leq r \leq a(t), \;\; 0 \leq t \leq T\}.$$

The initial data is taken as

$$(\rho, \rho u)(r, 0) = (\rho_1, m_0)(r) := (\rho_1, \rho_1 u_0)(r), \;\; r \in (0, a_0). \tag{2.2}$$

At the center of symmetry we impose the Dirichlet boundary condition

$$u(0, t) = 0, \tag{2.3}$$

and across the free surface $\partial\Omega_t$ which moves in the radial direction along the particle path $r = a(t)$, the vacuum state appears and the stress-free boundary condition holds

$$F(a(t), t) = 0, \quad \rho(a(t), t) = 0, \quad t \geq 0, \tag{2.4}$$

where $a'(t) = u(a(t), t), \;\; t > 0$, $a(0) = a_0 > 0$ and the stress (effective viscous flux) F is defined by

$$F =: \rho^\gamma - (\lambda + 2\xi)\mathrm{div}\mathbf{u} = \rho^\gamma - (\lambda + 2\xi)u_r - (\lambda + 2\xi)\frac{2u}{r}. \tag{2.5}$$

Before state the main result, let us give the definition of weak solution below.

Definition 2.1. Let $n = 3$. $(\rho, \mathbf{u}, a)$ with $\rho \geq 0$ a.e. is said to be a weak solution to the free boundary value problem (1.1)$-$(2.4) on $\Omega_t \times [0, T]$, provided that it holds that

$$\begin{aligned} \rho &\in L^\infty(0,T;L^1(\Omega_t)\cap L^\gamma(\Omega_t)),\ \sqrt{\rho}\mathbf{u}\in L^\infty(0,T;L^2(\Omega_t)),\\ \nabla\mathbf{u} &\in L^2(0,T;L^2(\Omega_t)),\ a(t)\in H^1([0,T]), \end{aligned} \tag{2.6}$$

and the equations are satisfied in the sense of distribution. Namely, it holds for any $t_2 > t_1 \geq 0$ and any $\phi \in C^1([0,T]\times\bar{\Omega}_t)$ that

$$\int_{\Omega_t}\rho\phi\mathrm{d}\mathbf{x}|_{t_1}^{t_2} = \int_{t_1}^{t_2}\int_{\Omega_t}(\rho\phi_t+\rho\mathbf{u}\cdot\nabla\phi)\mathrm{d}\mathbf{x}\mathrm{d}t, \tag{2.7}$$

and for $\psi = (\psi_1,\psi_2,\psi_3) \in C^1([0,T]\times\bar{\Omega}_t)$ satisfying $\psi(\mathbf{x},T) = 0$ and $\psi(\mathbf{x},t) = 0$ on $\partial\Omega_t$ that

$$\begin{aligned} &\int_{\Omega_t}\mathbf{m_0}\cdot\psi(\mathbf{x},0)\mathrm{d}\mathbf{x} + \int_0^T\int_{\Omega_t}[\rho\mathbf{u}\cdot\partial_t\psi+\rho\mathbf{u}\otimes\mathbf{u}:\nabla\psi]\mathrm{d}\mathbf{x}\mathrm{d}t\\ &+\int_0^T\int_{\Omega_t}\rho^\gamma\mathrm{div}\psi\mathrm{d}\mathbf{x}\mathrm{d}t+\int_0^T\int_{\Omega_t}(2\xi D(\mathbf{u}):\nabla\psi+\lambda\mathrm{div}\mathbf{u}\mathrm{div}\psi)\mathrm{d}\mathbf{x}\mathrm{d}t = 0, \end{aligned} \tag{2.8}$$

where $\Omega_t = \{(r,t)|0\leq r\leq a(t)\}\times\{t\}$. And (2.4) is satisfied in the sense of trace.

Notations: Throughout this paper, $C > 0$ and $c > 0$ denote the generic positive constants, $C_{\varepsilon,\delta} > 0$ denotes a generic constant which may depend on the sub-index ε and δ, and $C_T > 0$ denotes a generic constant depending on T.

Theorem 2.1. (Global exsitence) *Let $n = 3$ and $T > 0$. Assume that the spherically symmetric initial data* (2.2) *satisfies the regularity and compatibility conditions*

$$0\leq\rho_1 \in L^1(\Omega_0)\cap L^\infty(\Omega_0),\ (\rho_1)^k\in H^1(\Omega_0),\ \mathbf{u}_0\in H^1(\Omega_0), \tag{2.9}$$

$$\rho_1(r) > 0,\ r \in (0,a_0),\ \rho_1(a_0) = 0,\ u_{0r}(a_0)+\frac{2u_0(a_0)}{a_0} = 0, \tag{2.10}$$

where k is the constant satisfying $0 < k \leq \gamma-\frac{1}{2}$. Then, there exists a global spherically symmetric weak solution $(\rho,\mathbf{u})$ to (1.1) *with free boundary $|\mathbf{x}| = a(t)$ given by*

$$(\rho,\mathbf{u},a)(\mathbf{x},t) = (\rho(r,t),u(r,t)\frac{|\mathbf{x}|}{r},a(t)),\ r = |\mathbf{x}|,$$

for $t\in(0,T]$ *in the sense of Definition* 2.1*, where* $(\rho(r,t),u(r,t),a(t))$ *is a solution to FBVP* (2.1) $-$(2.4) *and satisfies* $\rho(r,t)\geq 0$ *a.e. and*

$$c_0\leq a(t)\leq C_T,\ t\in[0,T];\ \|a\|_{H^1(0,T)}\leq C_T, \tag{2.11}$$

$$\begin{aligned}&\int_{\Omega_t}(\frac{1}{2}\rho|\mathbf{u}|^2+\frac{1}{\gamma-1}\rho^\gamma)\mathrm{d}\mathbf{x}+\xi\int_0^T\int_{\Omega_t}|\nabla\mathbf{u}|^2\mathrm{d}\mathbf{x}\mathrm{d}t\\ &\leq\int_{\Omega_0}(\frac{1}{2}\rho_1|\mathbf{u}_0|^2+\frac{1}{\gamma-1}\rho_1^\gamma)\mathrm{d}\mathbf{x},\end{aligned} \tag{2.12}$$

where $c_0>0$*,* $C_T>0$ *are two constants. Furthermore, the following transport properties and regularities hold for the global weak solution* (ρ,u,a)*:*

(i) (Transport Property) *For any* $r_i\in(0,a_0]$*, there exits a positive constant* $C_{x_i,T}>0$ *so that*

$$\rho_1(r_i)e^{-C_{x_i,T}/(\lambda+2\xi)}\leq\rho(r_{x_i}(t),t)\leq\rho_1(r_i)e^{C_{x_i,T}/(\lambda+2\xi)},\ t\in[0,T], \tag{2.13}$$

$$cx_0^{\overline{\frac{\gamma}{3(\gamma-1)}}}\leq r_{x_0}(t)\leq a(t),\ t\in[0,T], \tag{2.14}$$

$$c(x_2-x_1)^{\overline{\frac{\gamma}{\gamma-1}}}\leq r_{x_2}^3(t)-r_{x_1}^3(t),\ t\in[0,T], \tag{2.15}$$

where $r=r_{x_i}(t)$*,* $i=0,1,2,$ *is the particle path defined by* $\frac{\mathrm{d}r_{x_i}(t)}{\mathrm{d}t}=u(r_{x_i}(t),t)$ *with* $r_{x_i}(0)=r_i\in(0,a_0]$ *and* $x_i=1-\int_{r_i}^{a_0}\rho_1r^2\mathrm{d}r$.

(ii) (Interior regularity) *If the initial velocity also satisfies* $u_0\in H^2([r_0^-,r_b^+])$ *for any* $0<r_0^-<r_0<r_b<r_b^+\leq a_0$*. Then, the following interior regularities hold*

$$\begin{cases}(\rho,u)\in C([r_{x_0}(t),r_{x_b}(t)]\times[0,T]),\\ \rho\in L^\infty(0,T;H^1([r_{x_0}(t),r_{x_b}(t)])),u\in L^\infty(0,T;H^2([r_{x_0}(t),r_{x_b}(t)])),\\ \rho_t\in L^\infty(0,T;L^2([r_{x_0}(t),r_{x_b}(t)]))\cap L^2(0,T;H^1([r_{x_0}(t),r_{x_b}(t)])),\\ u_t\in L^\infty(0,T;L^2([r_{x_0}(t),r_{x_b}(t)]))\cap L^2(0,T;H^1([r_{x_0}(t),r_{x_b}(t)]))\end{cases} \tag{2.16}$$

where $r=r_{x_0}(t)$ *is the particle path defined as the above and* $r=r_{x_b}(t)$ *is the particle path with* $r_{x_b}(0)=r_b$ *and* $x_b=1-\int_{r_b}^{a_0}\rho_1r^2\mathrm{d}r$.

(iii) (Boundary regularity) *It holds near the free boundary* $r=a(t)$ *that*

$$\begin{aligned}&\|(\rho^k,u)(t)\|_{H^1(\Omega_\eta)}+\|F(t)\|_{L^2(\Omega_\eta)}+\|\sqrt{\rho}\dot{u}\|_{L^2(0,T;L^2(\Omega_\eta))}\\ &+\|F\|_{L^2(0,T;H^1(\Omega_\eta))}+\|u\|_{L^2(0,T;H^2(\Omega_\eta))}+\|a\|_{H^1([0,T])}\leq C_{T,\delta_0},\end{aligned} \tag{2.17}$$

with $\delta_0 =: \|\rho_1\|_{L^\infty([0,a_0])} + \|u_0\|_{H^1([0,a_0])} + \|\rho_1^k\|_{H^1([0,a_0])}$ *and* $\Omega_\eta = (a(t)-\eta, a(t))$ *for some small constant* $\eta > 0$. *In addition, if the initial data* (ρ_1, u_0) *satisfy* $u_0 \in H^2([a_0-\eta, a_0])$, $\rho_1^{-\frac{1}{2}}\partial_r^2 u_0 \in L^2([a_0-\eta, a_0])$ *and compatibility condition, then*

$$\begin{aligned}&\|\sqrt{\rho}\dot{u}(t)\|_{L^2(\Omega_\eta)} + \|u(t)\|_{H^2(\Omega_\eta)} + \|\rho^{-\frac{1}{2}}\partial_r^2 u\|_{L^2(\Omega_\eta)}\\ &+\|F(t)\|_{H^1(\Omega_\eta)} + \|a\|_{H^2([0,T])} \le C_{T,\delta_1},\end{aligned} \tag{2.18}$$

with $\dot{u} = u_t + uu_r$ *and* $\delta_1 = \|\rho_1\|_{L^\infty([0,a_0])} + \|u_0\|_{H^1([0,a_0])}$
$+\|\rho_1^{-\frac{1}{2}}\partial_r^2 u_0\|_{L^2([a_0-\eta,a_0])} + \|\rho_1^k\|_{H^1([0,a_0])}$.

Then, we have the time expanding of the flow domain into the vacuum.

Theorem 2.2 (Long Time Expanding Rate). *Let* $n = 3$, $T > 0$, $\gamma > 1$, *and* (ρ, u, a) *be any global solution to the FBVP* (2.1)−(2.4) *in the sense of Definition* 2.1 *for* $t \in [0, T]$ *with* $F = \rho^\gamma - (\lambda + 2\xi)u_r - (\lambda + 2\xi)\frac{2u}{r} \in L^2(0, T; H^1(\Omega_\eta))$ *and* $\Omega_\eta = (a(t) - \eta, a(t))$ *for some small constant* $\eta > 0$. *Then, for any* $\gamma > 1$ *and* $t > 0$, *it holds*

$$a_1(t) = \max_{s\in[0,t]} a(s) \ge \begin{cases} C(1+t)^{\frac{\gamma-1}{\gamma}}, & 1 < \gamma < \frac{4}{3},\\ C(1+t)^{\frac{1-\nu}{3\gamma}}, & \gamma = \frac{4}{3},\\ C(1+t)^{\frac{1}{3\gamma}}, & \gamma > \frac{4}{3},\end{cases} \tag{2.19}$$

where $\nu > 0$ *is a constant small enough; C is a constant independent of time. In particular, for* $\gamma \ge \frac{4}{3}$, *the expanding rate is more precise as follows:*

$$a(t) \ge \begin{cases} C(1+t)^{\frac{1}{3\gamma}}, & \gamma = \frac{4}{3},\\ C(1+t)^{\frac{\gamma-1}{\gamma}}, & \frac{4}{3} < \gamma < \frac{5}{3},\\ C(1+t)^{\frac{1}{3\gamma}}, & \gamma \ge \frac{5}{3}.\end{cases} \tag{2.20}$$

3. Global Existence of Approximate FBVP Problem

3.1. Approximate FBVP problem

Consider the modified FBVP for Eq. (2.1) with the following initial data and boundary condition for any fixed $\varepsilon > 0$:

$$(\rho, u)(r, 0) = (\rho_0, u_0)(r),\ \rho_0(r) > 0, \quad \varepsilon \le r \le a_0, \tag{3.1}$$

$$u(\varepsilon,t) \;=\; 0,\; (\rho^\gamma-(\lambda+2\xi)u_r-(\lambda+2\xi)\frac{2u}{r})(a(t),t)=0, \quad t>0, \tag{3.2}$$

where $a'(t)=u((a(t),t),\ t>0$ and $a(0)=a_0$. Without the loss of generality, one can assume that the initial data is smooth enough and consistent with the boundary value (3.2) to the higher order.

It is convenient to investigate the approximate FBVP problem (2.1), (3.1)−(3.2) in Lagrangian coordinate. For simplicity we assume that $\int_\varepsilon^{a_0}\rho_0 r^2\mathrm{d}r=1$, which implies

$$\int_\varepsilon^{a(t)}\rho r^2\mathrm{d}r=\int_\varepsilon^{a_0}\rho_0 r^2\mathrm{d}r=1.$$

For $(r,t)\in\Omega_T^\varepsilon=\{(r,t)|\varepsilon\le r\le a(t),\ 0\le t\le T\}$, define the Lagrangian coordinates transform

$$x(r,t)=\int_\varepsilon^r\rho y^2\mathrm{d}y=1-\int_r^{a(t)}\rho y^2\mathrm{d}y,\ \tau=t, \tag{3.3}$$

which translates the domain Ω_T^ε into $[0,T]\times[0,1]$ and satisfies

$$\frac{\partial x}{\partial r}\;=\;\rho r^2,\quad \frac{\partial x}{\partial t}=-\rho u r^2,\quad \frac{\partial\tau}{\partial r}=0,\quad \frac{\partial\tau}{\partial t}=1, \tag{3.4}$$

and

$$r^3(x,\tau)\;=\;\varepsilon^3+3\int_0^x\frac{1}{\rho}(y,\tau)\mathrm{d}y=a^3(t)-3\int_0^x\frac{1}{\rho}(y,\tau)\mathrm{d}y,\ \frac{\partial r}{\partial\tau}=u. \tag{3.5}$$

The free boundary problem (2.1) and (3.1)−(3.2) is changed to

$$\begin{cases}\rho_\tau+\rho^2(ur^2)_x=0,\\ u_\tau+r^2(\rho^\gamma-(\lambda+2\xi)\rho(ur^2)_x)_x=0,\end{cases} \tag{3.6}$$

for $(x,\tau)\in[0,1]\times[0,T]$, with the initial data and boundary conditions given by

$$\begin{aligned}&(\rho,u)(x,0)=(\rho_0,u_0)(x),\\ &u(0,\tau)=0,\ (\rho^\gamma-(\lambda+2\xi)\rho r^2u_x-(\lambda+2\xi)\frac{2u}{r})(1,\tau)=0,\end{aligned} \tag{3.7}$$

where $r=r(x,\tau)$ is defined by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}r(x,\tau)=u(x,\tau),\ x\in[0,1],\ \tau\in[0,T], \tag{3.8}$$

and the fixed boundary $x = 1$ corresponds to the free boundary $a(\tau) = r(1,\tau)$ determined by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}a(\tau) = u(1,\tau),\ \tau \in [0,T];\ a(0) = a_0. \tag{3.9}$$

The main result for the FBVP (2.1) and (3.1)−(3.2) is stated as follows.

Proposition 3.1. *Let $T > 0, \gamma > 1$ and $\varepsilon > 0$ be fixed. Assume that the initial data (ρ_0, u_0) satisfies*

$$\inf_{x\in[0,1]} \rho_0(x) \geq \rho_*^\varepsilon > 0,\ \rho_0^q \in H^1([0,1]),\ u_0 \in H^1[0,1]. \tag{3.10}$$

Then, there exists a unique global strong solution (ρ, u, a) of the FBVP problem (3.6)−(3.9), which satisfies

$$\begin{cases} cx_0^{\frac{\gamma}{3(\gamma-1)}} \leq r(x_0,\tau) \leq a(\tau),\ c_0 \leq a(\tau) \leq C_T,\ (x_0,\tau)\in[0,1]\times[0,T], \\ c(x_1-x_2)^{\frac{\gamma}{\gamma-1}} \leq r^3(x_2,\tau) - r^3(x_1,\tau),\ 0\leq x_1 < x_2 \leq 1,\ \tau\in[0,T], \\ 0 < c_{\varepsilon,T} \leq \rho(x,\tau) \leq C_{\varepsilon,T},\ \forall (x,\tau)\in[0,1]\times[0,T], \end{cases} \tag{3.11}$$

where $r = r(x_i,\tau), i = 0,1,2,$ is the particle path defined by (3.8) with $r(x_i, 0) = r_i \in [\varepsilon, a_0]$ and $x_i = 1 - \int_{r_i}^{a_0} \rho_0 r^2 \mathrm{d}r$, and

$$\|(\rho,u)(\tau)\|_{H^1}^2 + \|(\rho_\tau, F)(\tau)\|_{L^2}^2 + \int_0^T (\|\rho_{x\tau}(\tau)\|_{L^2}^2 + \|F(\tau)\|_{H^1}^2 + \|u(\tau)\|_{H^2}^2)\mathrm{d}\tau$$
$$+ \int_0^T (|a(\tau)|^2 + |a'(\tau)|^2)\mathrm{d}\tau \leq C_{\varepsilon,T,\delta_0}, \tag{3.12}$$

with $C_{\varepsilon,T,\delta_0}$ a constant depending on ε, T, E_0, ρ_*^ε and δ_0 with $\delta_0 =: \|\rho_0^q\|_{H^1([0,1])} + \|\rho_0\|_{L^\infty([0,1])} + \|u_0\|_{H^1([0,1])}$. Furthermore, if $u_0 \in H^2([0,1])$, then it holds

$$\|u(\tau)\|_{H^2}^2 + \|u_\tau(\tau)\|_{L^2}^2 + \|F(\tau)\|_{H^1}^2 + \int_0^T (\|u_\tau(\tau)\|_{H^1}^2 + \|F(\tau)\|_{H^2}^2)\mathrm{d}\tau$$
$$+ \int_0^T (|a(\tau)|^2 + |a'(\tau)|^2 + |a''(\tau)|^2)\mathrm{d}\tau \leq C_{\varepsilon,T,\delta_1}, \tag{3.13}$$

with $\delta_1 =: \|\rho_0^q\|_{H^1([0,1])} + \|\rho_0\|_{L^\infty([0,1])} + \|u_0\|_{H^2([0,1])}$.

3.2. The a-priori estimates

Lemma 3.1. *Let $T>0,\gamma>1$ and (ρ,u,a) be any regular solution to the FBVP* (2.1), (3.1)−(3.2) *under the assumption of Proposition* 3.1. *Then,*

$$\int_{\varepsilon}^{a(t)}\rho r^2\mathrm{d}r=\int_{\varepsilon}^{a_0}\rho_0 r^2\mathrm{d}r, \tag{3.14}$$

$$\int_{\varepsilon}^{a(t)}(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^{\gamma})r^2\mathrm{d}r+(\lambda+2\xi)\int_0^T\int_{\varepsilon}^{a(s)}(u_r^2+\frac{2u^2}{r^2})r^2\mathrm{d}r\mathrm{d}s\le E_0, \tag{3.15}$$

$$a(t)\in H^1([0,T]),\ c_0\le a(t)\le C_T,\ t\in(0,T), \tag{3.16}$$

where $E_0:=\int_{\varepsilon}^{a_0}(\frac{1}{2}\rho_0u_0^2+\frac{1}{\gamma-1}\rho_0^{\gamma})r^2\mathrm{d}r$, c_0 *and* C_T *are positive constants.*

Proof. First, multiplying $(2.1)_1$ by r^2 and integrating the resulted equation over $(\varepsilon,a(t))$ lead to

$$\int_{\varepsilon}^{a(t)}\rho_t r^2\mathrm{d}r+\int_{\varepsilon}^{a(t)}(\rho u r^2)_r=0.$$

Integrating by part and using boundary condition give

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varepsilon}^{a(t)}\rho r^2\mathrm{d}r-\rho(a(t),t)u(a(t),t)a^2(t)+\rho(a(t),t)u(a(t),t)a^2(t)=0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varepsilon}^{a(t)}\rho r^2\mathrm{d}r=0,$$

hence

$$\int_{\varepsilon}^{a(t)}\rho r^2\mathrm{d}r=\int_{\varepsilon}^{a_0}\rho_0 r^2\mathrm{d}r.$$

Next, multiplying $(2.1)_2$ by ur^2 and integrating the resulting equality over $(\varepsilon,a(t))$, we obtain after integrating by part and using (3.2) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\varepsilon}^{a(t)}(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^{\gamma})r^2\mathrm{d}r+(\lambda+2\xi)\int_{\varepsilon}^{a(t)}(u_r^2+\frac{2u^2}{r^2})r^2\mathrm{d}r\mathrm{d}r$$
$$+2(\lambda+2\xi)(a'(t))^2a(t)=0,$$

which gives

$$\int_{\varepsilon}^{a(t)}(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^{\gamma})r^2\mathrm{d}r+(\lambda+2\xi)\int_0^T\int_{\varepsilon}^{a(s)}(u_r^2+\frac{2u^2}{r^2})r^2\mathrm{d}r\mathrm{d}s$$
$$+2(\lambda+2\xi)(a'(t))^2a(t)=E_0,$$

for any $t\in(0,T)$. So, (3.15) holds.

Finally, since

$$\begin{aligned}\int_{\varepsilon}^{a_0}\rho_0r^2\mathrm{d}r &= \int_{\varepsilon}^{a(t)}\rho r^2\mathrm{d}r\le\Big(\int_{\varepsilon}^{a(t)}\rho^{\gamma}r^2\mathrm{d}r\Big)^{\frac{1}{\gamma}}\Big(\int_{\varepsilon}^{a(t)}r^2\mathrm{d}r\Big)^{1-\frac{1}{\gamma}}\\ &\le (\gamma-1)^{\frac{1}{\gamma}}3^{\frac{1}{\gamma}-1}E_0^{\frac{1}{\gamma}}a(t)^{3(1-\frac{1}{\gamma})},\end{aligned}$$

we get

$$a(t)\ge 3^{\frac{1}{3}}(\gamma-1)^{-\frac{1}{3(\gamma-1)}}E_0^{-\frac{1}{3(\gamma-1)}}\Big(\int_{\varepsilon}^{a_0}\rho_0r^2\mathrm{d}r\Big)^{\frac{\gamma}{3(\gamma-1)}}.$$

$$\begin{aligned}&2(\lambda+2\xi)3^{\frac{1}{3}}(\gamma-1)^{-\frac{1}{3(\gamma-1)}}E_0^{-\frac{1}{3(\gamma-1)}}\Big(\int_{\varepsilon}^{a_0}\rho_0r^2\mathrm{d}r\Big)^{\frac{\gamma}{3(\gamma-1)}}\int_0^t(a'(s))^2\mathrm{d}s\\ &\le\ 2(\lambda+2\xi)\int_0^t(a'(s))^2a(s)\mathrm{d}s\le E_0.\end{aligned}$$

Therefore, it is easily deduced that

$$|a(t)|=|a_0+\int_0^t(a'(s))\mathrm{d}s|\le a_0+(\int_0^t(a'(s))^2\mathrm{d}s)^{\frac{1}{2}}t^{\frac{1}{2}}\le C+Ct^{\frac{1}{2}}\le C_T.$$

The proof is completed. □

From Lemma 3.1, we get Lemma 3.2 below.

Lemma 3.2. *Let $\gamma>1,\ T>0$ and (ρ,u,a) be any regular solution to the FBVP* (3.6)-(3.9) *for $\tau\in[0,T]$ under the assumption of Proposition* 3.1*. Then,*

$$\int_0^1(\frac{1}{2}u^2+\frac{1}{\gamma-1}\rho^{\gamma-1})\mathrm{d}x+(\lambda+2\xi)\int_0^{\tau}\int_0^1\{\rho r^4|u_x|^2+\frac{2u^2}{\rho r^2}\}\mathrm{d}x\mathrm{d}s$$
$$+2(\lambda+2\xi)\int_0^{\tau}(a'(s))^2a(s)\mathrm{d}s\le E_0, \tag{3.17}$$

$$a(\tau)\in H^1([0,T]),\ c\le a(\tau)\le C_T, \tag{3.18}$$

where $E_0 = \int_0^1 (\frac{1}{2}u_0^2 + \frac{1}{\gamma-1}\rho_0^{\gamma-1})\mathrm{d}x.$

Use the same method as the proof of Lemma 3.4 in [6], we get the lemma below.

Lemma 3.3. *Let* $\gamma > 1,\ T > 0$ *and* (ρ, u, a) *be any regular solution to the FBVP* (3.6)−(3.9) *for* $\tau \in [0, T]$ *under the assumption of Proposition* 3.1. *Then*

$$E_0^{-\frac{1}{3(\gamma-1)}} x^{\frac{\gamma}{3(\gamma-1)}} \le r(x,\tau) \le a(\tau),\ \ (x,\tau) \in [0,1] \times [0,T] \tag{3.19}$$

$$E_0^{-\frac{1}{\gamma-1}} (x_2 - x_1)^{\frac{\gamma}{\gamma-1}} \le r^3(x_2,\tau) - r^3(x_1,\tau),\ \ 0 \le x_1 < x_2 \le 1,\ \ \tau \in [0,T]. \tag{3.20}$$

Lemma 3.4. *Let* $\gamma > 1,\ T > 0$ *and* (ρ, u, a) *be any regular solution to the FBVP* (3.6)−(3.9) *for* $\tau \in [0, T]$ *under the assumption of Proposition* 3.1. *Then*

$$\begin{aligned} \rho_0(r(0))e^{-C_{x,T}/(\lambda+2\xi)} \le \rho(r(t),t) \le \rho_0(r(0))e^{C_{x,T}/(\lambda+2\xi)}, \\ \forall (r(t),t) \in [\varepsilon, a(t)] \times [0,T], \end{aligned} \tag{3.21}$$

where $r(t)$ *is particle path defined as* (3.26) *and* x *is defined as* (3.28) *and* $C_{x,T} := 4E_0^{\frac{3\gamma+1}{6(\gamma-1)}} x^{-\frac{2\gamma}{3(\gamma-1)}} + 4TE_0^{\frac{\gamma}{\gamma-1}} x^{-\frac{\gamma}{\gamma-1}}$ *is a constant independent of the* ρ_*^ε.

Proof. Define

$$\tilde{\xi} = \int_{a(t)}^r \rho u \mathrm{d}y, \quad \eta = \rho u^2(r,t) - \rho u^2(a(t),t) + \int_{a(t)}^r \frac{2\rho u^2}{y}\mathrm{d}y, \qquad r \in [\varepsilon, a(t)].$$

A direct calculation, together with (2.1) and (2.4) gives rise to

$$\begin{aligned} \tilde{\xi}_t + \eta + F &= \int_{a(t)}^r (\rho u)_t \mathrm{d}y + \rho u^2(r,t) - 2\rho u^2(a(t),t) + \int_{a(t)}^r \frac{2\rho u^2}{y}\mathrm{d}y \\ &\quad + (\rho^\gamma - (\lambda+2\xi)u_r - (\lambda+2\xi)\frac{2u}{r})(r,t) \\ &= \int_r^{a(t)} (\rho u^2 + \rho^\gamma - (\lambda+2\xi)u_y - (\lambda+2\xi)\frac{2u}{y})_y \mathrm{d}y - 2\rho u^2(a(t),t) \\ &\quad + (\rho u^2 + \rho^\gamma - (\lambda+2\xi)u_r - (\lambda+2\xi)\frac{2u}{r})(r,t) \\ &= -\rho u^2(a(t),t). \end{aligned} \tag{3.22}$$

Rewrite $(2.1)_1$ as

$$\rho_t + \rho_r u + \rho(u_r + \frac{2u}{r}) = 0,$$

which, together with $(2.1)_2$, yields

$$((\lambda + 2\xi)\ln\rho)_t + ((\lambda + 2\xi)\ln\rho)_r u + \rho^\gamma - F = 0. \tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\begin{aligned}(\tilde{\xi} + (\lambda + 2\xi)\ln\rho)_t + (\tilde{\xi} + (\lambda + 2\xi)\ln\rho)_r u &= -\rho u^2(a(t),t) - \eta - \rho^\gamma + u\xi_r \\ &= -\rho^\gamma + \int_r^{a(t)} \frac{2\rho u^2}{y}\mathrm{d}y,\end{aligned}$$

Thus, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\xi} + (\lambda + 2\xi)\ln\rho) + \rho^\gamma = \int_r^{a(t)} \frac{2\rho u^2}{y}\mathrm{d}y, \tag{3.24}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + u\partial_r$. Integrating (3.24) with respect to time t shows

$$\begin{aligned}&(\tilde{\xi} + (\lambda + 2\xi)\ln\rho)(r(t),t) + \int_0^t \rho^\gamma(r(s),s)\mathrm{d}s \\ =&(\tilde{\xi} + (\lambda + 2\xi)\ln\rho)(r(0),0) + \int_0^T\int_{r(s)}^{a(s)} \frac{2\rho u^2}{y}\mathrm{d}y\mathrm{d}s,\end{aligned} \tag{3.25}$$

where $r(t)$ is particle path defined as

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}r(t) = u(r(t),t), \\ r(0) = r, \ r \in [\varepsilon, a_0]. \end{cases} \tag{3.26}$$

Therefore, it holds that

$$\begin{aligned}&(\lambda + 2\xi)\ln\frac{\rho(r(t),t)}{\rho_0(r(0))} + \int_0^t \rho^\gamma(r(s),s)\mathrm{d}s \\ =& \int_{r(t)}^{a(t)} \rho u\mathrm{d}y + \int_0^T\int_{r(s)}^{a(s)} \frac{2\rho u^2}{y}\mathrm{d}y\mathrm{d}s - \int_{r(0)}^{a_0} \rho_0 u_0\mathrm{d}y.\end{aligned} \tag{3.27}$$

Let

$$x = \int_\varepsilon^{r(t)} \rho y^2\mathrm{d}y = 1 - \int_{r(t)}^{a(t)} \rho y^2\mathrm{d}y = 1 - \int_{r(0)}^{a_0} \rho_0 y^2\mathrm{d}y. \tag{3.28}$$

It follows from Hölder's inequality, Lemma 3.2−3.3 and (3.19) that

$$\int_{r(t)}^{a(t)}\rho u\mathrm{d}y\leq\frac{1}{r^2(x,t)}\Big(\int_{\varepsilon}^{a(t)}\rho u^2y^2\mathrm{d}y\Big)^{\frac{1}{2}}\Big(\int_{\varepsilon}^{a(t)}\rho y^2\mathrm{d}y\Big)^{\frac{1}{2}}\leq 2E_0^{\frac{2}{3(\gamma-1)}}x^{-\frac{2\gamma}{3(\gamma-1)}}E_0^{\frac{1}{2}}, \tag{3.29}$$

and,

$$\int_0^T\int_{r(s)}^{a(s)}\frac{2\rho u^2}{y}\mathrm{d}y\mathrm{d}s\leq\int_0^T\frac{1}{r^3(x,s)}\int_{\varepsilon}^{a(s)}2\rho u^2y^2\mathrm{d}y\mathrm{d}s\leq 4TE_0^{\frac{1}{\gamma-1}}x^{-\frac{\gamma}{\gamma-1}}E_0. \tag{3.30}$$

It holds that for the initial data

$$\int_r^{a_0}\rho_0u_0\mathrm{d}y\leq\frac{1}{r^2}\Big(\int_{\varepsilon}^{a_0}\rho_0u_0^2y^2\mathrm{d}y\Big)^{\frac{1}{2}}\Big(\int_{\varepsilon}^{a_0}\rho_0y^2\mathrm{d}y\Big)^{\frac{1}{2}}\leq 2E_0^{\frac{2}{3(\gamma-1)}}x^{-\frac{2\gamma}{3(\gamma-1)}}E_0^{\frac{1}{2}}. \tag{3.31}$$

From the above estimates, we have

$$\begin{aligned}\rho_0(r(0))e^{-C_{x,T}/(\lambda+2\xi)}\leq\rho(r(t),t)\leq\rho_0(r(0))e^{C_{x,T}/(\lambda+2\xi)},&\\ \forall(r(t),t)\in[\varepsilon,a(t)]\times[0,T],&\end{aligned}$$

where $C_{x,T}:=4E_0^{\frac{3\gamma+1}{6(\gamma-1)}}x^{-\frac{2\gamma}{3(\gamma-1)}}+4TE_0^{\frac{\gamma}{\gamma-1}}x^{-\frac{\gamma}{\gamma-1}}$.

Lemma 3.5. *Let $\gamma>1,\ T>0$ and (ρ,u,a) be any regular solution to the FBVP (3.6)−(3.9) for $\tau\in[0,T]$ under the assumption of Proposition* 3.1*. Then*

$$\int_0^1[r^{-2}u+(\lambda+2\xi)(\ln\rho)_x]^2\mathrm{d}x+\int_0^{\tau}\int_0^1\rho^{\gamma-2}(\rho_x)^2\mathrm{d}x\mathrm{d}s\leq C_{\varepsilon,T,\delta_0},\ \tau\in[0,T], \tag{3.32}$$

where $C_{\varepsilon,T,\delta_0}$ is a constant depending on $\varepsilon,\ T$ and initial data.

Proof. $(3.6)_1$ can be rewritten as

$$(\ln\rho)_\tau+\rho(r^2u)_x=0. \tag{3.33}$$

Differentiating (3.33) with respect to x and substituting the resulted equation into $(3.6)_2$, we have

$$[r^{-2}u+(\lambda+2\xi)(\ln\rho)_x]_\tau+(\rho^\gamma)_x+\frac{2u^2}{r^3}=0, \tag{3.34}$$

since $r^{-2}u_\tau = (r^{-2}u)_\tau + \frac{2u^2}{r^3}$. Multiplying (3.34) by $r^{-2}u + (\lambda+2\xi)(\ln\rho)_x$, integrating the resulted equation over $[0,1]\times[0,\tau]$, we obtain after integrating by parts that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [r^{-2}u + (\lambda+2\xi)(\ln\rho)_x]^2\mathrm{d}x\\ &\quad+\int_0^1 [(\rho^\gamma)_x + \frac{2u^2}{r^3}][r^{-2}u + (\lambda+2\xi)(\ln\rho)_x]\mathrm{d}x = 0,\end{aligned} \tag{3.35}$$

which yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [r^{-2}u + (\lambda+2\xi)(\ln\rho)_x]^2\mathrm{d}x + \gamma(\lambda+2\xi)\int_0^1 \rho^{\gamma-2}(\rho_x)^2\mathrm{d}x\\ &= -\int_0^1 \gamma\rho^{\gamma-1}(\rho_x)r^2\mathrm{d}x - \int_0^1 \frac{2u^3}{r^5}\mathrm{d}x - \int_0^1 (\ln\rho)_x\frac{2u^2}{r^3}\mathrm{d}x\\ &\le \delta\int_0^1 \rho^{\gamma-2}(\rho_x)^2\mathrm{d}x + C(\delta)\int_0^1 \rho^\gamma r^4\mathrm{d}x\\ &\quad+C\int_0^1 [r^{-2}u + (\lambda+2\xi)(\ln\rho)_x]^2\mathrm{d}x + C\int_0^1 \frac{u^4}{r^6}\mathrm{d}x + C\int_0^1 [r^{-2}u]^2\mathrm{d}x,\end{aligned} \tag{3.36}$$

where $\delta\in(0,1)$ is small enough.

Thus, by (3.36), (3.17), (3.19) and (3.21), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 [r^{-2}u + (\lambda+2\xi)(\ln\rho)_x]^2\mathrm{d}x + \int_0^1 \rho^{\gamma-2}(\rho_x)^2\mathrm{d}x\\ &\quad\le C\int_0^1 [r^{-2}u + (\lambda+2\xi)(\ln\rho)_x]^2\mathrm{d}x\\ &\qquad+C_{\varepsilon,T}\Big(\int_0^1 u^2\mathrm{d}x + \int_0^1 u^4\mathrm{d}x\Big) + C_{\varepsilon,T}.\end{aligned} \tag{3.37}$$

It follows from Sobolev embedding and Lemma 3.2$-$(3.5) that

$$\int_0^\tau\int_0^1 u^2\mathrm{d}x\mathrm{d}s \le C, \tag{3.38}$$

$$\begin{aligned}\int_0^\tau\int_0^1 u^4\mathrm{d}x\mathrm{d}s &\le \int_0^\tau \Big(\|u\|^2_{L^\infty[0,1]}\int_{x_0}^1 u^2\mathrm{d}x\Big)\mathrm{d}s\\ &\le \int_0^\tau \Big(\int_0^1 |u|\mathrm{d}x + \int_0^1 |u_x|\mathrm{d}x\Big)^2\int_0^1 u^2\mathrm{d}x\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^\tau \Big(\int_0^1 u^2\mathrm{d}x+\int_0^1 u_x^2\mathrm{d}x\Big)\int_0^1 u^2\mathrm{d}x\mathrm{d}s\\
&\leq \sup_{0\leq s\leq\tau}\int_0^1 u^2\mathrm{d}x\Big\{\int_0^\tau\int_0^1 u^2\mathrm{d}x\mathrm{d}s+\int_0^\tau\int_0^1 u_x^2\mathrm{d}x\mathrm{d}s\Big\}\\
&\leq C_{\varepsilon,T}. \qquad (3.39)
\end{aligned}
$$

By Gronwall inequality, it holds that

$$
\int_0^1 [r^{-2}u+(\lambda+2\xi)(\ln\rho)_x]^2\mathrm{d}x+\int_0^\tau\int_0^1 \rho^{\gamma-2}(\rho_x)^2\mathrm{d}x\mathrm{d}s\leq C_{\varepsilon,T,\delta_0},\ \tau\in[0,T].
$$

Finally, we have the following higher order regularity estimates.

Lemma 3.6. *Let $\gamma>1,\ T>0$ and (ρ,u,a) be any regular solution to the FBVP (3.6)–(3.9) for $\tau\in[0,T]$ under the assumption of Proposition* 3.1. *Then*

$$
\begin{aligned}
&\int_0^1 (u_x^2+\rho_x^2+\rho_\tau^2+F^2)\mathrm{d}x+\int_0^T\int_0^1 (u_\tau^2+u_{xx}^2+\rho_{x\tau}^2+F_x^2)\mathrm{d}x\mathrm{d}\tau\\
&\qquad+\int_0^T (|a(\tau)|^2+|a'(\tau)|^2)\mathrm{d}\tau\leq C_{\varepsilon,T,\delta_0}, \qquad (3.40)
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 (u_{xx}^2+F_x^2+u_\tau^2)\mathrm{d}x+\int_0^T\int_0^1 (u_{\tau x}^2+F_{xx}^2)\mathrm{d}x\mathrm{d}\tau+\int_0^T u_\tau^2(1,\tau)\mathrm{d}\tau\\
&\qquad+\int_0^T (|a(\tau)|^2+|a'(\tau)|^2+|a''(\tau)|^2)\mathrm{d}\tau\leq C_{\varepsilon,T,\delta_1}, \qquad (3.41)
\end{aligned}
$$

with $C_{\varepsilon,T,\delta_0}$ and $C_{\varepsilon,T,\delta_1}$ being constants the same as Proposition 3.1.

Proof. Taking the inner production of $(3.6)_2$ with $\rho^{-1}u_\tau$ over $[0,1]\times[0,\tau]$ and making use of the boundary condition (3.7)–(3.9), we can obtain after a direction computation that

$$
\begin{aligned}
&\frac{\lambda+2\xi}{2}\int_0^1 r^2u_x^2\mathrm{d}x+\int_0^\tau\int_0^1 \rho^{-1}r^{-2}u_s^2\mathrm{d}x\mathrm{d}s+(\lambda+2\xi)\rho^{-1}r^{-1}u^2(1,\tau)\\
&=\ (\lambda+2\xi)\int_0^\tau\int_0^1 ruu_x^2\mathrm{d}x\mathrm{d}s-\int_0^\tau\int_0^1 (\rho^{\gamma-1})_s u_x\mathrm{d}x\mathrm{d}s\\
&\quad-\int_0^\tau\int_0^1 (\rho^\gamma-(\lambda+2\xi)\rho r^2u_x)\frac{\rho_x}{\rho^2}u_s\mathrm{d}x\mathrm{d}s\\
&\quad+(\lambda+2\xi)\int_0^\tau\int_0^1 \frac{2u_xu_\tau}{\rho r}\mathrm{d}x\mathrm{d}s-(\lambda+2\xi)\int_0^\tau\int_0^1 \frac{2uu_s}{\rho^2r^4}\mathrm{d}x\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1 \rho^{\gamma-1}u_x\mathrm{d}x+\int_0^1 \rho^{\gamma-1}u_x\Big|_{\tau=0}\mathrm{d}x \\
&+\frac{\lambda+2\xi}{2}\int_{x_0}^1 r^2u_x^2\Big|_{\tau=0}\mathrm{d}x+(\lambda+2\xi)\rho^{-1}r^{-1}u^2(1,0) \\
&+\int_0^\tau (\rho^{-1}r^{-1})_s u^2(1,s)\mathrm{d}s. \qquad (3.42)
\end{aligned}
$$

Using the Lemma 3.2 and the following facts

$$
C_{\varepsilon,T}\le a(\tau)\le C_T,\ 0<c_{\varepsilon,T}\le \rho(x,\tau)\le C_{\varepsilon,T},\ \forall (x,\tau)\in[0,1]\times[0,T], \quad (3.43)
$$

we have

$$
\begin{aligned}
&\int_0^1 r^2u_x^2\mathrm{d}x+\int_0^\tau\int_0^1 \rho^{-1}r^{-2}u_s^2\mathrm{d}x\mathrm{d}s+\rho^{-1}r^{-1}u^2(1,\tau) \\
&\le\ C_{\varepsilon,T}\int_0^\tau\Big(1+\int_0^1 u_x^2\mathrm{d}x\Big)\int_0^1 r^2u_x^2\mathrm{d}x\mathrm{d}s \\
&\quad+\int_0^\tau \sup_{x\in[0,1]}|u_x|^2\int_0^1|\rho_x|^2\mathrm{d}x\mathrm{d}s+C_{\varepsilon,T}, \qquad (3.44)
\end{aligned}
$$

which together with Lemma 3.5 and Gronwall's inequality yields

$$
\int_0^1 r^2u_x^2\mathrm{d}x+\int_0^\tau\int_0^1 \rho^{-1}r^{-2}u_s^2\mathrm{d}x\mathrm{d}s\le C_{\varepsilon,\delta_0,T}, \qquad (3.45)
$$

where we have used the fact that

$$
\begin{aligned}
\int_0^T \sup_{x\in[0,1]}|u_x|^2\mathrm{d}\tau\ &\le\ C_{\varepsilon,T}\int_0^T\sup_{x\in[0,1]}|\rho(ur^2)_x|\mathrm{d}\tau+C_{\varepsilon,T} \\
&\le\ C_{\varepsilon,T}\int_0^T\int_0^1 u_\tau^2\mathrm{d}x\mathrm{d}\tau+C_{\varepsilon,T}\le C_{\varepsilon,T}. \qquad (3.46)
\end{aligned}
$$

It follows from $(3.6)_2$, (3.32), (3.43) and (3.45) that

$$
\begin{aligned}
\int_0^T\int_0^1 u_{xx}^2\mathrm{d}x\mathrm{d}\tau\ &\le\ C_{\varepsilon,T}\int_0^T\int_0^1(u_\tau^2+u_x^2+\rho_x^2)\mathrm{d}x\mathrm{d}\tau \\
&\quad+C_{\varepsilon,T}\sup_{\tau\in[0,T]}\|\rho_x\|^2_{L^2[0,1]}\int_0^T\sup_{x\in[0,1]}|u_x|^2\mathrm{d}\tau\le C_{\varepsilon,\delta_0,T}. \qquad (3.47)
\end{aligned}
$$

The combination of (3.6), Lemmas 3.2–3.5, and (3.45)–(3.47) leads to

(3.40).

Next, differentiating $(3.6)_2$ with respect to τ gives

$$r^{-2}u_{\tau\tau}-2r^{-3}uu_\tau+(\rho^\gamma-(\lambda+2\xi)\rho r^2u_x-(\lambda+2\xi)\frac{2u}{r})_{x\tau}=0. \tag{3.48}$$

Taking the inner product of (3.48) with u_τ over $[0,1]$ and using (3.7)$-$(3.9), one may get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1 r^{-2}u_\tau^2\mathrm{d}x+\frac{\lambda+2\xi}{2}\int_0^1\rho r^2u_{\tau x}^2\mathrm{d}x+(\lambda+2\xi)\frac{u_\tau^2(1,\tau)}{a(\tau)}\\ &\le\ C_{\varepsilon,T}\int_0^1 r^{-2}u_\tau^2\mathrm{d}x+C_{\varepsilon,T}\int_0^1(u_{xx}^2+u_\tau^2)\mathrm{d}x+C_{\varepsilon,T}.\end{aligned} \tag{3.49}$$

Applying Gronwall's inequality to (3.49) and using (3.40) shows

$$\int_0^1 r^{-2}u_\tau^2\mathrm{d}x+\int_0^T\int_0^1\rho r^2u_{\tau x}^2\mathrm{d}x\mathrm{d}\tau+\int_0^T u_\tau^2(1,\tau)\mathrm{d}\tau\le C_{\varepsilon,\delta_1,T}. \tag{3.50}$$

Furthermore, it follows from $(3.6)_2$, (3.50) that

$$\int_0^1(u_{xx}^2+F_x^2)\mathrm{d}x+\int_0^T\int_0^1 F_{xx}^2\mathrm{d}x\mathrm{d}\tau\le C_{\varepsilon,\delta_1,T}. \tag{3.51}$$

The combination of (3.6), Lemma 3.2$-$3.5, (3.50)$-$(3.51) leads to (3.41).

The Proof of Proposition 3.1. At this stage of argument, ε is fixed and positive, so that the FBVP (3.6) is essentially an one-dimensional problem. Under the assumptions of Propositions 3.1, one can apply the standard argument to obtain the short time existence of the unique regular solution (ρ,u,a). By the a-prior estimates established in Lemma 3.1$-$3.6 and a continuity argument, we can continue the local solution globally in time and obtain the global regular solution to the FBVP satisfying (3.11)$-$(3.13).

4. Uniform Estimates and the Proof of Theorem 2.1

In this section, we will obtain the uniform estimates of the approximate solutions established in Proposition 3.1. We establish uniform estimates containing symmetry center in Eulerian coordinate and uniform estimates away from symmetry center in Lagrangian coordinate. First we begin with the uniform estimates around symmetry center.

Lemma 4.1. *Under the same assumptions as Lemma 3.1, there exists a positive constant $C_{0,T} > 0$ depending on E_0 and T, but independent of ε, such that*

$$\int_0^T \int_\varepsilon^{a(t)} \rho^{2\gamma}(r,t) r^{12} \mathrm{d}r\mathrm{d}t \le C_{0,T}. \tag{4.1}$$

Proof. We multiply $(2.1)_2$ by $\varphi(r)(= r^3)$ and integrate over $(r, a(t))$ $(r \in [\varepsilon, a(t)])$ to obtain

$$\begin{aligned}
&\Big(\rho u^2 + \rho^\gamma - (\lambda+2\xi)u_r - (\lambda+2\xi)\frac{2u}{r}\Big)\varphi \\
&= \partial_t \int_r^{a(t)} \rho u \varphi \mathrm{d}y - \int_r^{a(t)} (\rho u^2 + \rho^\gamma - (\lambda+2\xi)u_y \\
&\quad -(\lambda+2\xi)\frac{2u}{y})\varphi_y \mathrm{d}y + \int_r^{a(t)} \frac{2\rho u^2}{y}\varphi \mathrm{d}y,
\end{aligned} \tag{4.2}$$

which yields

$$\begin{aligned}
\rho^{2\gamma}\varphi &= (\lambda+2\xi)\rho^\gamma(u_r + \frac{2u}{r})\varphi - \rho^{1+\gamma}u^2\varphi \\
&\quad +\rho^\gamma \partial_t \int_r^{a(t)} \rho u \varphi \mathrm{d}y - \rho^\gamma \int_r^{a(t)} (\rho u^2 + \rho^\gamma))\varphi_y \mathrm{d}y \\
&\quad +(\lambda+2\xi)\rho^\gamma \int_r^{a(t)} (u_y + \frac{2u}{y})\varphi_y \mathrm{d}y + \rho^\gamma \int_r^{a(t)} \frac{2\rho u^2}{y}\varphi \mathrm{d}y.
\end{aligned} \tag{4.3}$$

It follows from $(2.1)_1$ that ρ^γ satisfies

$$(\rho^\gamma)_t + (\rho^\gamma u)_r + \frac{2\gamma\rho^\gamma u}{r} = (1-\gamma)\rho^\gamma u_r. \tag{4.4}$$

Then, one has

$$\begin{aligned}
\rho^\gamma \partial_t \int_r^{a(t)} \rho u \varphi \mathrm{d}y &= \partial_t\Big(\rho^\gamma \int_r^{a(t)} \rho u \varphi \mathrm{d}y\Big) - \partial_t \rho^\gamma \int_r^{a(t)} \rho u \varphi \mathrm{d}y \\
&= \partial_t\Big(\rho^\gamma \int_r^{a(t)} \rho u \varphi \mathrm{d}y\Big) + \partial_r\Big(\rho^\gamma u \int_r^{a(t)} \varphi \rho u \mathrm{d}y\Big) \\
&\quad +\Big\{\frac{2\gamma\rho^\gamma u}{r} + (\gamma-1)\rho^\gamma u_r\Big\} \int_r^{a(t)} \rho u \varphi \mathrm{d}y + \varphi \rho^{1+\gamma} u^2.
\end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.3), we get

$$\rho^{2\gamma}\varphi = (\lambda+2\xi)\rho^\gamma(u_r+\frac{2u}{r})\varphi + \partial_t\Big(\rho^\gamma\int_r^{a(t)}\rho u\varphi \mathrm{d}y\Big) + \partial_r\Big(\rho^\gamma u\int_r^{a(t)}\rho u\varphi \mathrm{d}y\Big)$$
$$+\Big\{\frac{2\gamma\rho^\gamma u}{r}+(\gamma-1)\rho^\gamma u_r\Big\}\int_r^{a(t)}\rho u\varphi \mathrm{d}y - \rho^\gamma\int_r^{a(t)}(\rho u^2+\rho^\gamma))\varphi_y \mathrm{d}y$$
$$+(\lambda+2\xi)\rho^\gamma\int_r^{a(t)}(u_y+\frac{2u}{y})\varphi_y \mathrm{d}y + \rho^\gamma\int_r^{a(t)}\frac{2\rho u^2}{y}\varphi \mathrm{d}y. \tag{4.6}$$

Multiplying (4.6) by φ^3 and integrating over $[\varepsilon, a(t)]\times[0,T]$ lead to

$$\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4 \mathrm{d}r\mathrm{d}t = \int_0^T\int_\varepsilon^{a(t)}\{R.H.S\ of\ (4.6)\}\varphi^3 \mathrm{d}r\mathrm{d}t = \sum_{i=1}^5 I_i. \tag{4.7}$$

The right hand side terms of (4.7) can be estimated as follows:

$$\begin{aligned}|I_1| &= |(\lambda+2\xi)\int_0^T\int_\varepsilon^{a(t)}\rho^\gamma(u_r+\frac{2u}{r})\varphi^4 \mathrm{d}r\mathrm{d}t| \\ &\le \delta\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4 \mathrm{d}r\mathrm{d}t + C\delta^{-1}\int_0^T(u_r^2+\frac{4u^2}{r^2})\varphi^4 \mathrm{d}r\mathrm{d}t \\ &\le \delta\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4 \mathrm{d}r\mathrm{d}t + \delta^{-1}C_{0,T}.\end{aligned}$$

It is easy to see that by (3.14) and (3.15):

$$\Big|\int_r^{a(t)}\rho u\varphi \mathrm{d}y\Big| \le \frac{1}{2}\int_r^{a(t)}(\rho u^2+\rho)\varphi \mathrm{d}y \le C_{0,T}, \tag{4.8}$$

hence,

$$\begin{aligned}|I_2| &= \Big|\int_0^T\int_\varepsilon^{a(t)}\varphi^3\partial_t(\rho^\gamma\int_r^{a(t)}\rho u\varphi \mathrm{d}y)\mathrm{d}r\mathrm{d}t\Big| \\ &\le C\sup_{0\le t\le T}\Big|\int_\varepsilon^{a(t)}\rho^\gamma\varphi^3\Big(\int_r^{a(t)}\rho u\varphi \mathrm{d}y\Big)\mathrm{d}r\mathrm{d}t\Big| \\ &\le C_{0,T}\sup_{0\le t\le T}\int_\varepsilon^{a(t)}\rho^\gamma\varphi^4 \mathrm{d}r \le C_{0,T}, \\ |I_3| &= \Big|\int_0^T\int_\varepsilon^{a(t)}\partial_r(\rho^\gamma u\int_r^{a(t)}\varphi\rho u \mathrm{d}y)\varphi^3 \mathrm{d}r\mathrm{d}t\Big|\end{aligned}$$

$$
\begin{aligned}
&= \Big|\int_0^T\int_\varepsilon^{a(t)} 3\rho^\gamma u\varphi^2\varphi_r\Big(\int_r^{a(t)}\varphi\rho u\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}t\Big| \\
&\le \delta\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4\mathrm{d}r\mathrm{d}t+\delta^{-1}C_{0,T}\int_0^T\int_\varepsilon^{a(t)}u^2\varphi_r^2\mathrm{d}r\mathrm{d}t \\
&\le \delta\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4\mathrm{d}r\mathrm{d}t+\delta^{-1}C_{0,T},
\end{aligned}
$$

$$
\begin{aligned}
|I_4| &= \Big|\int_0^T\int_\varepsilon^{a(t)}\varphi^3\Big\{\frac{2\gamma\rho^\gamma u}{r}+(\gamma-1)\rho^\gamma u_r\Big\}\Big(\int_r^{a(t)}\rho u\varphi\mathrm{d}y\Big)\mathrm{d}r\mathrm{d}t\Big| \\
&\le C_{0,T}\int_0^T\int_\varepsilon^{a(t)}\varphi^3\Big|\frac{2\gamma\rho^\gamma u}{r}+(\gamma-1)\rho^\gamma u_r\Big|\mathrm{d}r\mathrm{d}t \\
&\le \delta\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4\mathrm{d}r\mathrm{d}t+\delta^{-1}C_{0,T}\int_0^T\int_\varepsilon^{a(t)}(\frac{u^2}{r^2}+u_r^2)\varphi^2\mathrm{d}r\mathrm{d}t \\
&\le \delta\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4\mathrm{d}r\mathrm{d}t+\delta^{-1}C_{0,T}.
\end{aligned}
$$

By (3.15) and (3.16), we get

$$
\Big|\int_r^{a(t)}(\rho u^2+\rho^\gamma))\varphi_y\mathrm{d}y\Big| \le CE_0,\ \Big|\int_r^{a(t)}\frac{2\rho u^2}{y}\varphi\mathrm{d}y\Big|\le CE_0, \tag{4.9}
$$

$$
\Big|\int_r^{a(t)}(u_y+\frac{2u}{y})\varphi_y\mathrm{d}y\Big| \le a(t)+\int_\varepsilon^{a(t)}(u_y^2+\frac{4u^2}{y^2})\varphi_y^2\mathrm{d}y\le C_{0,T}, \tag{4.10}
$$

which gives

$$
\begin{aligned}
|I_5| &= \Big|\int_0^T\int_\varepsilon^{a(t)}\rho^\gamma\varphi^3\Big\{\int_r^{a(t)}(\rho u^2+\rho^\gamma)\varphi_y\mathrm{d}y \\
&\qquad +(\lambda+2\xi)\int_r^{a(t)}(u_y+\frac{2u}{y})\varphi_y\mathrm{d}y+\int_r^{a(t)}\frac{2\rho u^2}{y}\varphi\mathrm{d}y\Big\}\mathrm{d}r\mathrm{d}t\Big| \\
&\le C_{0,T}\int_0^T\int_\varepsilon^{a(t)}\rho^\gamma\varphi^3\mathrm{d}r\mathrm{d}t\le C_{0,T}.
\end{aligned}
$$

We finally get

$$
\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4\mathrm{d}r\mathrm{d}t\le 3\delta\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4\mathrm{d}r\mathrm{d}t+\delta^{-1}C_{0,T},
$$

Choosing $\delta=\frac{1}{6}$, we obtain

$$\int_0^T\int_\varepsilon^{a(t)}\rho^{2\gamma}\varphi^4\mathrm{d}r\mathrm{d}t\le C_{0,T}.$$

Then, we establish the uniform estimates of solutions away from symmetry center.

Lemma 4.2. *Let $T>0$ and $\gamma>1$, (ρ,u,a) be the solution to FBVP* (2.1) *with* (3.1)*)–*(3.2) *for $(r,t)\in[\varepsilon,a(t)]\times[0,T]$ constructed in Proposition* 3.1. *Assume further that*

$$\rho_0\in L^\infty([r_0,a_0]),\ u_0\in H^1[r_0,a_0]. \tag{4.11}$$

where $0<r_0<a_0$. Then

$$\int_0^T\int_{x_1}^1(u_\tau^2+F_x^2)\mathrm{d}x\mathrm{d}\tau+\int_{x_1}^1\frac{F^2}{\rho}\mathrm{d}x+\int_{x_1}^1(\rho r^4|u_x|^2+\frac{2u^2}{\rho r^2}+|u_x|)\mathrm{d}x\le C_{x_0,\bar\delta_0}, \tag{4.12}$$

where $C_{x_0,\bar\delta_0}$ is a positive constant just depending on T, x_1, $x_0(0<x_0<x_1<1)$, E_0, $\|\rho\|_{L^\infty[r_0,a_0]}$ and $\|u_0\|_{H^1[r_0,a_0]}$.

Proof. Multiplying $(3.6)_2$ with $u_\tau\phi$ and integrating the resulted equation over$[0,1]$, then we have

$$\int_0^1u_\tau^2\phi\mathrm{d}x+\int_0^1F_x(ur^2)_\tau\phi\mathrm{d}x-\int_0^1F_x2ru^2\phi\mathrm{d}x=0, \tag{4.13}$$

where $\phi=\chi^2(x)$ and $\chi\in C^\infty([0,1])$ satisfies $0\le\chi(x)\le1,\ \chi(x)=1$ for $x\in[x_1,1](0<x_0<x_1<1)$, $\chi(x)=0$ for $x\in[0,x_0]$ and $|\chi'|\le\frac{2}{x_1-x_0}$. After integrating by part, it holds that

$$\int_0^1u_\tau^2\phi\mathrm{d}x+\int_0^1F(\frac{F-\rho^\gamma}{(\lambda+2\xi)\rho})_\tau\phi\mathrm{d}x-\int_0^1F(ur^2)_\tau\phi'\mathrm{d}x+\int_0^1r^{-2}u_\tau2ru^2\phi\mathrm{d}x=0, \tag{4.14}$$

which implies

$$\begin{aligned}&\int_0^1u_\tau^2\phi\mathrm{d}x+\frac{1}{2(\lambda+2\xi)}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1\frac{F^2}{\rho}\phi\mathrm{d}x\\&=\frac{1}{2(\lambda+2\xi)}\int_0^1F\frac{F}{\sqrt\rho}\frac{\rho_\tau}{\rho^{\frac32}}\phi\mathrm{d}x+\frac{\gamma-1}{(\lambda+2\xi)}\int_0^1F\rho^{\gamma-1}\rho_\tau\phi\mathrm{d}x\end{aligned}$$

$$-2\int_0^1 r^{-1}u_\tau u^2\phi \mathrm{d}x+\int_0^1 Fu_\tau r^2\phi' \mathrm{d}x+2\int_0^1 Fu^2r\phi'\mathrm{d}x=\sum_{i=1}^5 I_i, \quad (4.15)$$

where we have used the fact

$$-(ur^2)_x=\frac{F-\rho^\gamma}{(\lambda+2\xi)\rho},\ \ F_x=-r^{-2}u_\tau.$$

Using Lemma 3.1−3.3, Hölder's inequality and (3.21), we can estimate each term in the following way:

$$\begin{cases} |I_1|\le \delta\|\sqrt{\phi}F\|^2_{L^\infty[0,1]}+\delta^{-1}C(\int_{x_0}^1\rho^{-3}|\rho_\tau|^2\mathrm{d}x)\int_0^1\frac{F^2}{\rho}\phi\mathrm{d}x,\\ |I_2|\le C\int_{x_0}^1F^2\mathrm{d}x+C_{x_0}\int_{x_0}^1\rho^{-3}|\rho_\tau|^2,\\ |I_3|\le \delta\int_0^1u_\tau^2\phi\mathrm{d}x+\delta^{-1}C_{x_0}\|u\phi\|^2_{L^\infty[0,1]}\int_0^1u^2\mathrm{d}x,\\ |I_4|\le\delta\int_0^1u_\tau^2\phi\mathrm{d}x+\delta^{-1}C_{x_0}\int_{x_0}^1F^2\mathrm{d}x,\\ |I_5|\le C\int_{x_0}^1F^2\mathrm{d}x+C_{x_0}\|u\phi\|^2_{L^\infty[0,1]}\int_0^1u^2\mathrm{d}x.\end{cases} \quad (4.16)$$

where $\delta\in(0,1)$ is a small positive constant and C_{x_0} is a constant depending on $x_0,\ E_0,\ T$ and $\|\rho\|_{L^\infty[r_0,a_0]}$. Since

$$\sqrt{\phi}F=-\int_x^1(\sqrt{\phi}F)_y\mathrm{d}y=\int_x^1u_\tau r^{-2}\sqrt{\phi}\mathrm{d}y-\int_x^1(\sqrt{\phi})'F\mathrm{d}y, \quad (4.17)$$

we obtain

$$\|\sqrt{\phi}F\|^2_{L^\infty[0,1]}\le C_{x_0}\int_0^1u_\tau^2\phi\mathrm{d}x+C_{x_0}\int_{x_0}^1F^2\mathrm{d}x. \quad (4.18)$$

Substituting (4.16) and (4.18) into (4.15), we obtain

$$\begin{aligned}&\int_0^1u_\tau^2\phi\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1\frac{F^2}{\rho}\phi\mathrm{d}x\\ &\le\ C_{x_0}(\int_{x_0}^1\rho^{-3}|\rho_\tau|^2\mathrm{d}x)\int_0^1\frac{F^2}{\rho}\phi\mathrm{d}x+C_{x_0}\int_{x_0}^1(F^2+\rho^{-3}|\rho_\tau|^2)\mathrm{d}x\\ &\quad+C_{x_0}\|u\phi\|^2_{L^\infty[0,1]},\end{aligned} \quad (4.19)$$

By $(3.6)_1$, Lemma 3.1−3.3 and (3.21), we get

$$\int_0^T\int_{x_0}^1 F^2\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_0^1 \rho^{-3}|\rho_\tau|^2\mathrm{d}x\mathrm{d}\tau\le C_{x_0}. \tag{4.20}$$

$$\begin{aligned}
&\int_0^T \|u\phi\|^2_{L^\infty[0,1]}\mathrm{d}\tau\le C\int_0^T\Big(\int_0^1|u|\mathrm{d}x+\int_0^1|u_x|\phi\mathrm{d}x\Big)^2\mathrm{d}\tau\\
\le&\int_0^T\int_0^1 u^2\mathrm{d}x\mathrm{d}\tau+C\int_0^T\int_0^1\rho r^4|u_x|^2\phi\mathrm{d}x\mathrm{d}\tau+C_{x_0}\int_0^T\int_0^1\frac{1}{\rho r^2}\phi\mathrm{d}x\mathrm{d}\tau\\
\le&\ C\int_0^T\int_0^1\rho r^4|u_x|^2\phi\mathrm{d}x\mathrm{d}\tau+C_{x_0}\le C_{x_0}.
\end{aligned}\tag{4.21}$$

It follows (4.19)−(4.21) and Gronwall inequality that

$$\int_0^T\int_0^1 u_\tau^2\phi\mathrm{d}x\mathrm{d}\tau+\int_0^1\frac{F^2}{\rho}\phi\mathrm{d}x\le C_{x_0,\bar\delta_0}. \tag{4.22}$$

Notice that

$$\begin{aligned}
&\int_0^1\frac{((\lambda+2\xi)\rho(r^2u)_x)^2}{\rho}\phi\mathrm{d}x\\
=&\ (\lambda+2\xi)^2\int_0^1(\rho r^4|u_x|^2+\frac{2u^2}{\rho r^2})\phi\mathrm{d}x+2(\lambda+2\xi)^2u^2(1,\tau)a(\tau)\\
&-2(\lambda+2\xi)^2\int_0^1 u^2\phi'\mathrm{d}x,
\end{aligned}\tag{4.23}$$

which, together with (4.22), gives

$$\begin{aligned}
&\int_0^1(\rho r^4|u_x|^2+\frac{2u^2}{\rho r^2})\phi\mathrm{d}x+u^2(1,\tau)a(\tau)\\
\le&\ C\int_0^1\frac{((\lambda+2\xi)\rho(r^2u)_x)^2}{\rho}\phi\mathrm{d}x+C\int_0^1u^2\mathrm{d}x\\
\le&\ C\int_0^1\frac{F^2}{\rho}\phi\mathrm{d}x+\int_0^1\rho^{2\gamma-1}\phi\mathrm{d}x+C\int_0^1u^2\mathrm{d}x\\
\le&\ C\int_0^1\frac{F^2}{\rho}\phi\mathrm{d}x+C_{x_0}\\
\le&\ C_{x_0,\bar\delta_0}.
\end{aligned}\tag{4.24}$$

Therefore, there is

$$\int_0^1 |u_x|\phi \mathrm{d}x \leq C\int_0^1 \rho r^4|u_x|^2\phi \mathrm{d}x + C_{x_0}\int_0^1 \frac{1}{\rho}\phi \mathrm{d}x \leq C_{x_0,\bar{\delta}_0}. \tag{4.25}$$

Lemma 4.3. *Let $T>0$ and $\gamma>1$, (ρ,u,a) be the solution to FBVP (2.1) and (3.1)–(3.2) for $(r,t)\in[\varepsilon,a(t)]\times[0,T]$ constructed in Proposition 3.1. If the initial data ρ_0 satisfies $(\rho_0^q)_x\in L^2[x_1,1]$, then*

$$\int_{x_1}^1 |(\rho^q)_x|^2 \mathrm{d}x \leq C_{x_1,\delta_0}, \tag{4.26}$$

where $\frac{1}{2}<q=k+\frac{1}{2}\leq\gamma$ and the constant C_{x_1,δ_0} depends on T, x_1, $x_0(0<x_0<x_1<1)$, E_0 and δ_0 with $\delta_0=\|u_0\|_{H^1[r_0,a_0]}+\|\rho\|_{L^\infty[r_0,a_0]}+\|(\rho_0^q)_x\|_{L^2[x_1,1]}$.

Proof. Multiplying $(3.6)_1$ by $q\rho^{q-1}$ and differentiating the resulted equation with respect to x, we obtain

$$\rho^q_{x\tau} + q(\rho^{q+1}(ur^2)_x)_x = 0. \tag{4.27}$$

Multiplying (4.27) by $(\rho^q)_x$ and integrating over $[x_1,1]$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{x_1}^1 |(\rho^q)_x|^2\mathrm{d}x\\ &= -q\int_{x_1}^1 \rho(ur^2)_x|(\rho^q)_x|^2\mathrm{d}x + \frac{q}{\lambda+2\xi}\int_{x_1}^1 \rho^q(\rho^q)_x(r^{-2}u_\tau + (\rho^\gamma)_x)\mathrm{d}x\end{aligned} \tag{4.28}$$

from which we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{x_1}^1 |(\rho^q)_x|^2\mathrm{d}x\\ &\leq C_{x_1}(\|F\|_{L^\infty[x_1,1]}+1)\int_{x_1}^1 |(\rho^q)_x|^2\mathrm{d}x + C_{x_1}\int_{x_1}^1 \rho^{2q}u_\tau^2\mathrm{d}x + C\int_{x_1}^1 \rho^\gamma|(\rho^q)_x|^2\mathrm{d}x\\ &\leq C_{x_1}\Big(\int_{x_1}^1 |u_\tau|^2\mathrm{d}x + 1\Big)\int_{x_1}^1 |(\rho^q)_x|^2\mathrm{d}x + C_{x_1}\int_{x_1}^1 |u_\tau|^2\mathrm{d}x,\end{aligned} \tag{4.29}$$

where C_{x_1} is a constant depending on x_1, E_0, T and $\|\rho\|_{L^\infty[r_0,a_0]}$.
By Gronwall inequality and Lemma 4.2, it holds that

$$\int_{x_1}^1 |(\rho^q)_x|^2\mathrm{d}x \leq C_{x_1,\delta_0}. \tag{4.30}$$

Lemma 4.4. *Let $T>0$ and $\gamma>1$, (ρ,u,a) be the solution to FBVP* (2.1) *and* (3.1)–(3.2) *for $(r,t)\in[\varepsilon,a(t)]\times[0,T]$ constructed in Proposition* 3.1. *Assume further $\rho_0^{-\frac{1}{2}}\partial_r^2u_0(r)\in L^2[r_1,1]$, namely $(\rho_0r^2u_{0x}(x))_x\in L^2[x_1,1]$, then*

$$\int_{x_2}^1 u_\tau^2\mathrm{d}x+\int_0^T\int_{x_2}^1\frac{F_\tau^2}{\rho}\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_{x_2}^1(\rho r^4u_{x\tau}^2+\frac{2u_\tau^2}{\rho r^2})\mathrm{d}x\mathrm{d}\tau+2\int_0^T|a''(\tau)|^2\mathrm{d}\tau \\ \le C_{x_1,\delta_1}, \tag{4.31}$$

and

$$\int_{x_2}^1|(\rho(ur^2)_x)_x|^2\mathrm{d}x\le C_{x_1,\delta_1}, \tag{4.32}$$

where C_{x_1,δ_1} depends on x_2, C_{x_1,δ_0} and $\|\rho_0r^2u_{0x}\|_{L^2[x_1,1]}$ with $\delta_1=\|u_0\|_{H^2[r_0,a_0]}+\|\rho\|_{L^\infty[r_0,a_0]}+\|(\rho_0^q)_x\|_{L^2[x_1,1]}+\|\rho_0r^2u_{0x}\|_{L^2[x_1,1]}$.

Proof. Differentiating $(3.6)_2$ with respect to τ, we obtain

$$u_{\tau\tau}+r^2F_{x\tau}+2ruF_x=0. \tag{4.33}$$

Choose a smooth function $\psi=\zeta^2(x)$ where $\zeta\in C^\infty([0,1])$ satisfies $0\le\zeta(x)\le1$, $\zeta(x)=1$ for $x\in[x_2,1](0<x_0<x_1<x_2<1)$, $\zeta(x)=0$ for $x\in[0,x_1]$ and $|\zeta'|\le\frac{2}{x_2-x_1}$.
Taking inner product of (4.33) with $u_\tau\psi$ and integrating by part, it holds that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^1u_\tau^2\psi\mathrm{d}x+\frac{1}{(\lambda+2\xi)}\int_0^1\frac{F_\tau^2}{\rho}\psi\mathrm{d}x\\
&=\frac{1}{(\lambda+2\xi)}\int_0^1\frac{FF_\tau}{\rho^2}\rho_\tau\psi\mathrm{d}x+\int_0^1F_\tau u_\tau r^2\psi'\mathrm{d}x\mathrm{d}\tau-\frac{4}{(\lambda+2\xi)}\int_0^1F_\tau\rho^\gamma\frac{u}{\rho r}\psi\mathrm{d}x\\
&\quad+\frac{\gamma-1}{(\lambda+2\xi)^2}\int_0^1F_\tau F\rho^{\gamma-1}\psi\mathrm{d}x-\frac{\gamma-1}{(\lambda+2\xi)^2}\int_0^1F_\tau\rho^{2\gamma-1}\psi\mathrm{d}x\\
&\quad+\frac{4}{(\lambda+2\xi)}\int_0^1F_\tau F\frac{u}{\rho r}\psi\mathrm{d}x+6\int_0^1F_\tau\frac{u^2}{\rho r^2}\psi\mathrm{d}x+2\int_0^1u_\tau^2ur^3\psi\mathrm{d}x\\
&\le\delta\int_0^1\frac{F_\tau^2}{\rho}\psi\mathrm{d}x+C\|u\|_{L^\infty([x_1,1]}\int_0^1u_\tau^2\psi\mathrm{d}x\\
&\quad+\delta^{-1}C\|F\|^2_{L^\infty([x_1,1]}\int_{x_1}^1(\rho^{-3}|\rho_\tau|^2+\frac{u^2}{\rho r^2})\mathrm{d}x\\
&\quad+\delta^{-1}C(\|\rho\|^{2\gamma}_{L^\infty([x_1,1]}+\|u\|^2_{L^\infty([x_1,1])})\int_{x_1}\frac{u^2}{\rho r^2}\mathrm{d}x+\delta^{-1}C\|\rho\|_{L^\infty([x_1,1]}\int_{x_1}^1u_\tau^2\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\delta^{-1}C\|\rho\|^{3\gamma}_{L^\infty([x_1,1]}\int_0^1\rho^{\gamma-1}\psi\mathrm{d}x+\delta^{-1}C\|\rho\|^{2\gamma}_{L^\infty([x_1,1]}\int_0^1F^2\psi\mathrm{d}x\\
&\le\delta\int_0^1\frac{F_\tau^2}{\rho}\psi\mathrm{d}x+C\|u\|_{L^\infty([x_1,1]}\int_0^1u_\tau^2\psi\mathrm{d}x\\
&\quad+\delta^{-1}C_{x_1}\Big(\|u\|^2_{L^\infty([x_1,1]}+\int_{x_1}^1u_\tau^2\mathrm{d}x+1\Big),
\end{aligned}\tag{4.34}
$$

since

$$
\begin{aligned}
\int_{x_0}^1\rho^{-3}|\rho_\tau|^2\mathrm{d}x &\le \int_0^1\Big(\rho r^4|u_x|^2+\frac{2u^2}{\rho r^2}\Big)\phi\mathrm{d}x\le C, &(4.35)\\
|F(y,\tau)| &= \Big|-\int_y^1F_x\mathrm{d}x\Big|=\Big|\int_y^1r^{-2}u_\tau\mathrm{d}x\Big|\\
&\le \frac{1}{r^2(x_1,\tau)}\Big(\int_{x_1}^1u_\tau^2\mathrm{d}x\Big)^{\frac12},\ y\in[x_1,1], &(4.36)
\end{aligned}
$$

where C_{x_1} is a constant depending on $x_1,\ E_0,\ T$ and $\|\rho\|_{L^\infty[r_0,a_0]}$. Choosing a small $\delta\in(0,1)$ and using Gronwall inequality, we get

$$
\int_0^1u_\tau^2\psi\mathrm{d}x+\int_0^T\int_0^1\frac{F_\tau^2}{\rho}\psi\mathrm{d}x\mathrm{d}\tau\le C_{x_1,\delta_1}.\tag{4.37}
$$

Furthermore, we have

$$
\begin{aligned}
&\int_0^T\int_0^1(\rho r^4u_{x\tau}^2+\frac{2u_\tau^2}{\rho r^2})\psi\mathrm{d}x\mathrm{d}\tau+2\int_0^T|a''(\tau)|^2a(\tau)\mathrm{d}\tau\\
&=\int_0^T\int_0^1\frac{(\rho r^2u_{x\tau}+\frac{2u_\tau}{r})^2}{\rho}\psi\mathrm{d}x\mathrm{d}\tau+2\int_0^T\int_0^1u_\tau^2r\psi'\mathrm{d}x\mathrm{d}\tau\\
&\le\int_0^T\int_0^1\frac{|(\rho r^2u_x+\frac{2u^2}{r})_\tau|^2}{\rho}\psi\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_0^1\frac{|\rho_\tau r^2u_x|^2}{\rho}\psi\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_0^T\int_0^1\frac{|\rho ruu_x|^2}{\rho}\psi\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_0^1\frac{u^4}{\rho r^2}\mathrm{d}x\mathrm{d}\tau+2\int_0^T\int_0^1u_\tau^2r|\psi'|\mathrm{d}x\mathrm{d}\tau\\
&\le\int_0^T\int_0^1\frac{|F_\tau|^2}{\rho}\psi\mathrm{d}x\mathrm{d}\tau+\int_0^T\int_0^1\rho^{2\gamma-3}|\rho_\tau|^2\psi\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_0^T\|\rho r^2u_x\|^2_{L^\infty([x_1,1]}\int_{x_0}^1\rho^{-3}|\rho_\tau|^2\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_0^T(\|\rho r^2u_x\|^2_{L^\infty([x_1,1]}+\|u\|^2_{L^\infty([x_1,1]})\int_{x_1}^1\frac{u^2}{\rho r^2}\mathrm{d}x\mathrm{d}\tau+2\int_0^T\int_{x_1}^1u_\tau^2\mathrm{d}x\mathrm{d}\tau
\end{aligned}
$$

$$\leq C_{x_1,\delta_1}. \tag{4.38}$$

By (4.37) and Lemma 4.3, we obtain

$$\begin{aligned}\int_0^1 |(\rho(ur^2)_x)_x|^2\psi \mathrm{d}x \ &\leq \ C\int_0^1 |F_x|^2\psi \mathrm{d}x + C\int_0^1 |(\rho^\gamma)_x|^2\psi \mathrm{d}x \\ &\leq \ C_{x_1}\int_0^1 |u_\tau|^2\psi \mathrm{d}x + C_{x_1}\int_{x_1}^1 |(\rho^\gamma)_x|^2 \mathrm{d}x \\ &\leq \ C_{x_1,\delta_1}.\end{aligned}$$

The proof is completed. □

At last, we obtain the interior estimates below.

Lemma 4.5. *Let* $T > 0$ *and* $\gamma > 1$, (ρ, u, a) *be the solution to FBVP* (2.1) *and* (3.1)−(3.2) *for* $(r, t) \in [\varepsilon, a(t)] \times [0, T]$ *constructed in Proposition* 3.1. *Assume further there exists* $0 < r_0^- < r_0 < r_b < r_b^+ \leq a_0$ *and a positive constant* ρ_* *such that*

$$\inf_{r\in[r_0^-,r_b^+]} \rho_0(r) \geq \rho_* > 0, \ \ u_0 \in H^2([r_0^-, r_b^+]), \tag{4.39}$$

that is

$$\inf_{x\in[x_0^-,x_b^+]} \rho_0(x) \geq \rho_* > 0, \ \ u_0 \in H^2([x_0^-, x_b^+]), \tag{4.40}$$

then it holds that

$$0 < c_{x_0^-,T} \leq \rho(r,t) \leq \widetilde{C}_{x_0^-,T}, \ \ \forall (r,t) \in [r_{x_0^-}(t), r_{x_b^+}(t)] \times [0,T], \tag{4.41}$$

$$\begin{aligned}&\sup_{\tau\in[0,T]} (\|u_x\|^2_{L^2[x_0,x_b]} + \|\rho_x\|^2_{L^2[x_0,x_b]} + \|\rho_\tau\|^2_{L^2[x_0,x_b]} + \|F\|^2_{L^2[x_0,x_b]}) \\ &+ \int_0^T \|(u_\tau, F_x)(\tau)\|^2_{L^2[x_0,x_b]}\mathrm{d}\tau + \int_0^T \|(u_{xx}, \rho_{x\tau}(\tau)\|^2_{L^2[x_0,x_b]}\mathrm{d}\tau \leq C_1,\end{aligned} \tag{4.42}$$

$$\begin{aligned}&\sup_{\tau\in[0,T]} (\|u_\tau\|^2_{L^2[x_0,x_b]} + \|F_x\|^2_{L^2[x_0,x_b]} + \|u_{xx}\|^2_{L^2[x_0,x_b]}) + \int_0^T \|F_\tau(\tau)\|^2_{L^2[x_0,x_b]}\mathrm{d}\tau \\ &+ \int_0^T \|(u_{\tau x}, F_{xx})(\tau)\|^2_{L^2[x_0,x_b]}\mathrm{d}\tau \leq C_2,\end{aligned} \tag{4.43}$$

where $c_{x_0^-,T}$, $\widetilde{C}_{x_0^-,T}$ *are positive constants depending on* x_0^-, T *and the initial data, and* $r_{x_0^-}(t)$, $r_{x_b^+}(t)$ *are particle paths with* $x_0^- = 1-\int_{r_0^-}^{a_0}\rho_0 r^2\mathrm{d}r$, $x_b^+ = 1-\int_{r_b^+}^{a_0}\rho_0 r^2\mathrm{d}r$ *and* $x_i = 1-\int_{r_i}^{a_0}\rho_0 r^2\mathrm{d}r$ $(i=0,b)$*. The constant* C_1 *depends on* E_0, $c_{x_0^-,T}$, $\widetilde{C}_{x_0^-,T}$, x_b^+, $x_i(i=0,b)$, $\|\rho_0\|_{H^1[x_0^-,r_x^+]}$ *and* $\|u_0\|_{H^1[x_0^-,r_x^+]}$ *and* C_2 *depends on* C_1 *and* $\|u_0\|_{H^2[x_0^-,r_x^+]}$.

Proof. (4.41) can be deduced from (3.21) and (4.39).
Multiplying $(3.6)_2$ with $u_\tau\widetilde{\phi}$ and integrating the resulted equation over $[0,1]$, then we have

$$\int_0^1 u_\tau^2\widetilde{\phi}\mathrm{d}x+\int_0^1 F_x(ur^2)_\tau\widetilde{\phi}\mathrm{d}x-\int_0^1 F_x 2ru^2\widetilde{\phi}\mathrm{d}x=0, \tag{4.44}$$

where $\widetilde{\phi}=\widetilde{\chi}^2(x)$ and $\widetilde{\chi}\in C^\infty([0,1])$ satisfies $0\le\widetilde{\chi}(x)\le 1$, $\widetilde{\chi}(x)=1$ for $x\in[\frac{x_0+x_0^-}{2},\frac{x_b+x_b^+}{2}]$, $\widetilde{\chi}(x)=0$ for $x\in[0,x_0^-]\cup[x_b^+,1]$ and $|\widetilde{\chi}'|\le\frac{4}{x_0-x_0^-}+\frac{4}{x_b^+-x_b}$.
Similarly as the proof of Lemma 4.2, we obtain

$$\int_0^T\int_0^1 u_\tau^2\widetilde{\phi}\mathrm{d}x\mathrm{d}\tau+\int_0^1\frac{F^2}{\rho}\widetilde{\phi}\mathrm{d}x\le C_0, \tag{4.45}$$

and

$$\begin{aligned}&(\lambda+2\xi)^2\int_0^1(\rho r^4|u_x|^2+\frac{2u^2}{\rho r^2})\widetilde{\phi}\mathrm{d}x\\ &=\int_0^1\frac{((\lambda+2\xi)\rho(r^2u)_x)^2}{\rho}\widetilde{\phi}\mathrm{d}x+2(\lambda+2\xi)^2\int_0^1 u^2\widetilde{\phi}'\mathrm{d}x\le C_0,\end{aligned} \tag{4.46}$$

where C_0 depends on E_0, $\widetilde{C}_{x_0^-,T}$ and $\|u_0\|_{H^1[x_0^-,r_x^+]}$.
From (4.45)−(4.46), $(3.6)_2$ and (4.41), we get

$$\begin{aligned}&\sup_{\tau\in[0,T]}\Big(\|u_x\|^2_{L^2[\frac{x_0+x_0^-}{2},\frac{x_b+x_b^+}{2}]}+\|F\|^2_{L^2[\frac{x_0+x_0^-}{2},\frac{x_b+x_b^+}{2}]}\Big)\\ &\quad+\int_0^T\|(u_\tau,F_x)(\tau)\|^2_{L^2[\frac{x_0+x_0^-}{2},\frac{x_b+x_b^+}{2}]}\mathrm{d}\tau\le\widetilde{C}_0,\end{aligned} \tag{4.47}$$

where $\widetilde{C}_0$ depends on C_0 and $c_{x_0^-,T}$. Similarly as the Lemma 4.3, we obtain

$$\sup_{\tau\in[0,T]}\|\rho_x\|^2_{L^2[\frac{x_0+x_0^-}{2},\frac{x_b+x_b^+}{2}]}\le C_1 \tag{4.48}$$

Thus, (4.42) can be deduced from (3.6) and (4.47)−(4.48).

Choose a smooth function $\widetilde{\psi}=\widetilde{\zeta}^2(x)$ and take inner product of (4.33) with $u_\tau\widetilde{\psi}$ where $\widetilde{\zeta}\in C^\infty([0,1])$ satisfies $0\le\widetilde{\zeta}(x)\le 1$, $\widetilde{\zeta}(x)=1$ for $x\in[x_0,x_b]$, $\widetilde{\zeta}(x)=0$ for $x\in[0,\frac{x_0+x_0^-}{2}]\cup[\frac{x_b+x_b^+}{2},1]$ and $|\widetilde{\zeta}'|\le\frac{4}{x_0-x_0^-}+\frac{4}{x_b^+-x_b}$. Then, similarly as Lemma 4.5, there is

$$\int_0^1 u_\tau^2\widetilde{\psi}\mathrm{d}x+\int_0^T\int_0^1\frac{F_\tau^2}{\rho}\widetilde{\psi}\mathrm{d}x\mathrm{d}\tau\le C_2, \tag{4.49}$$

and

$$\begin{aligned}&\int_0^T\int_0^1(\rho r^4u_{x\tau}^2+\frac{2u_\tau^2}{\rho r^2})\widetilde{\psi}\mathrm{d}x\mathrm{d}\tau\\ &=\int_0^T\int_0^1\frac{(\rho r^2u_{x\tau}+\frac{2u_\tau}{r})^2}{\rho}\widetilde{\psi}\mathrm{d}x\mathrm{d}\tau+2\int_0^T\int_0^1u_\tau^2r\widetilde{\psi}'\mathrm{d}x\mathrm{d}\tau\le C_2.\end{aligned} \tag{4.50}$$

The combination of (3.6), (4.41), (4.49)−(4.50) leads to (4.43).

The Proof of Theorem 2.1. We give the proof of Theorem 2.1. Indeed, we can construct global solutions to the approximate FBVP (1.1)−(2.1) with (3.1)−(3.2), establish uniform a-prior estimates based on Lemma 3.1-Lemma 3.3 and Lemma 4.1−Lemma 4.5, show their convergence of the original FBVP problem, and justify the expected properties in Theorem 2.1 for the limiting solution.

We can modify the initial data (ρ_1,u_0) in Theorem 2.1 properly such that the modified initial data $(\rho_0^\varepsilon,u_0^\varepsilon)$ satisfies all assumptions in Proposition 3.1 on $[\varepsilon,a_0]$ and the following properties:

$$\begin{aligned}&\inf_{r\in[\varepsilon,a_0]}\rho_0^\varepsilon(r)>0,\ \inf_{r\in[\varepsilon,a_0]}\rho_0^\varepsilon(r)>\inf_{r\in[\varepsilon,a_0]}\rho_0(r),\\ &\int_\varepsilon^{a_0}\rho_0^\varepsilon r^2\mathrm{d}r=\int_\varepsilon^{a_0}\rho_0r^2\mathrm{d}r,u_0^\varepsilon(\varepsilon)=0,\end{aligned} \tag{4.51}$$

In particular, $((\rho_1^\varepsilon)^k,u_0^\varepsilon)\to(\rho_0^k,u_0)$ strongly in $H^1([0,a_0])$ as $\varepsilon\to0_+$ and $\rho_1^{\varepsilon,k}\to\rho_1(r)$ as $\varepsilon\to0_+$, refer to [6, 14] for construction of such function. By Proposition 3.1, the FBVP (2.1) and (3.1)−(3.2) admits a unique global strong solution $(\rho_\varepsilon,u_\varepsilon,a_\varepsilon)$ on the domain $[\varepsilon,a^\varepsilon(t)]\times[0,T]$ with the initial data $(\rho_0^\varepsilon,u_0^\varepsilon)$. Similarly to those in [6] we can construct the global solution $(\rho^\varepsilon,u^\varepsilon,a^\varepsilon)$ to the FBVP (1.1)−(2.4) by setting $(\rho^\varepsilon,u^\varepsilon,a^\varepsilon)=(\rho_\varepsilon,u_\varepsilon,a_\varepsilon)$ for $(r,t)\in[\varepsilon,a^\varepsilon(t)]\times[0,T]$ and $(\rho^\varepsilon,u^\varepsilon)(r,t)=(\rho_\varepsilon(\varepsilon,t),0)$ for $0\le r\le\varepsilon$ and $t\in[0,T]$.

First, we prove the strong convergence of $(\rho^\varepsilon, u^\varepsilon, a^\varepsilon)$ near the free boundary. It's enough to prove the strong convergence on the domain $[r^\varepsilon_{x_b}, a^\varepsilon] \times [0,T]$, where $r = r^\varepsilon_{x_b}$ is a particle path with $r^\varepsilon_{x_b}(0) = r_b \in (b_0, a_0]$ and $x_b = \int_{r_b}^{a_0} \rho_0 r^2 \mathrm{d}r$ and the initial data satisfies $(\rho_0^k, u_0) \in H^1[b_0, a_0]$. It's convenient to show the strong convergence in Lagrangian coordinate on $[x_b, 1] \times [0,T]$. Indeed, we can show $(\rho^\varepsilon, u^\varepsilon, a^\varepsilon)$ satisfies the uniform estimate established in Lemmas 4.2−4.4 on $[x_b, 1] \times [0,T]$. Thus, by Lions-Aubin's lemma, there is a limiting function (ρ_b, u_b, a) so that up to a subsequence $(\rho^{\varepsilon_j}, u^{\varepsilon_j}, a^{\varepsilon_j})$, it holds that

$$\begin{cases} (\rho^{\varepsilon_j}, u^{\varepsilon_j}) \to (\rho_b, u_b) & \text{strongly in } C([0,T]\times[x_b,1])\times C([0,T]; L^p[x_b,1]), \\ F^{\varepsilon_j} \to F & \text{strongly in } L^2([0,T]\times[x_b,1]), \\ a^{\varepsilon_j} \to a & \text{strongly in } C^\alpha([0,1]), \alpha \in (0, \frac{1}{2}), \end{cases} \tag{4.52}$$

where $r_\tau = u_b$ and $(r^3)_x = \frac{3}{\rho_b}, F = \rho_b^\gamma - (\lambda + 2\xi)\rho_b(r^2 u_b)_x = \rho_b^\gamma - (\lambda + 2\xi)\rho_b r^2 \partial_x u_b - (\lambda + 2\xi)\frac{2u_b}{r}$. In addition, by Lemma 3.4 and the construction that $\rho_0^\varepsilon(r) \to \rho_1(r)$ as $\varepsilon \to 0_+$ we conclude that the boundary condition $\rho_b(a(t), t) = 0$ holds.

Next, we show the convergence of$(\rho^{\varepsilon_j}, u^{\varepsilon_j}, a^{\varepsilon_j})$ on an interior domain $\Omega_{in}^{\varepsilon_j}$ defined by

$$\Omega_{in}^{\varepsilon_j} =: \{(r,t) | 0 \le r < a^{\varepsilon_j}(t),\ 0 \le t \le T\} \cap \{(r,t) | 0 \le r \le a(t),\ 0 \le t \le T\}.$$

Due to the strong convergence (4.52) of velocity and the particle path as $\varepsilon_j \to 0_+$, it holds that for $\varepsilon_j > 0$ small enough

$$\Omega_{in} =: \{(r,t) | 0 \le r \le r_{x_{in}}(t),\ 0 \le t \le T\} \subset\subset \Omega_{in}^{\varepsilon_j}, \tag{4.53}$$

where $r = r_{x_{in}}(t)$ is a particle path defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} r_{x_{in}}(t) = u_b(r_{x_{in}}(t), t),\ r_{x_{in}}(0) = r_{in} \in (r_b, a_0), \tag{4.54}$$

which satisfies that for $x_b < x_{in} = 1 - \int_{r_{in}}^{a_0} \rho_0 r^2 \mathrm{d}r$,

$$0 < c(x_{in} - x_b)^{\frac{\gamma}{\gamma-1}} \le r^3_{x_{in}}(t) - r^3_{x_b}(t),\ t \in [0,T]. \tag{4.55}$$

With help of Lemma 4.1, a proper cut-off function and a similar compactness argument as [14], we can show that there is a limiting function $(\rho_{in}, u_{in})(r,t)$ $((r,t) \in \Omega_{in})$, so that up to a sub-subsequence $(\rho^{\varepsilon_j}, \rho^{\varepsilon_j} u^{\varepsilon_j})$ converge to $(\rho_{in}, \rho_{in} u_{in})$ in the sense that

$$\begin{cases} \rho^{\varepsilon_j} \to \rho_{in} & \text{strongly in } L^p(0,T;\mathcal{L}^p(0,r_{in}(t))),\ \forall\ 1\le p\le 2\gamma \\ \rho^{\varepsilon_j} u^{\varepsilon_j} \rightharpoonup \rho_{in} u_{in} & \text{weakly in } L^\infty(0,T;\mathcal{L}^{\frac{2\gamma}{\gamma+1}}(0,r_{in}(t))), \end{cases} \tag{4.56}$$

and (ρ_{in}, u_{in}) satisfies (1.1)−(2.1) on Ω_{in} in the sense of distribution. As [14], we define $\mathcal{L}^p(\Omega)) := \{f \in L^1_{loc}(\Omega) | \int_\Omega |f(r)|^p r^2 \mathrm{d}r < \infty\}$ with norm $\|\cdot\|_{\mathcal{L}^p(\Omega))} := (\int_\Omega |\cdot|^p r^2 \mathrm{d}r)^{\frac{1}{p}}$. Finally, define

$$(\rho, \rho\mathbf{u}) = \begin{cases} (\rho_b, \rho_b \mathbf{u}_b)(\mathbf{x},t), & r_{x_b}(t) \le |\mathbf{x}| \le a(t),\ t\in[0,T], \\ (\rho_{in}, \rho_{in}\mathbf{u}_{in})(\mathbf{x},t), & 0 \le |\mathbf{x}| \le r_{x_{in}}(t),\ t\in[0,T], \end{cases} \tag{4.57}$$

where $\mathbf{u} = u\frac{\mathbf{x}}{r}$, $\mathbf{u_b} = u_b\frac{\mathbf{x}}{r}$, $\mathbf{u}_{in} = u_{in}\frac{\mathbf{x}}{r}$ and $r = |\mathbf{x}|$. This is well defined and

$$(\rho_b, \rho_b\mathbf{u}_b) = (\rho_{in}, \rho_{in}\mathbf{u}_{in}),\ a.e.\ (r,t) \in [r_{x_b(t)}, r_{x_{in}}(t)] \times [0,T]. \tag{4.58}$$

We can easily deduce that $(\rho, \rho\mathbf{u}, a(t))$ is a weak solution to FBVP (1.1)−(2.4) in the sense of Definition 2.1, and by similar argument to [6] verify that $(\rho, \rho\mathbf{u}, a(t))$ satisfies the properties (2.11)−(2.18) and the free boundary condition with the help of Lemmas 4.2−4.5. The proof of Theorem 2.1 is completed. □

5. Long Time Expanding Rate

In this section, we investigate the large time behavior of any global spherical symmetric weak solutions to FBVP (1.1)−(2.4). Indeed, we can obtain an expanding rate of the domain occupied by the fluid.

The proof of Theorem 2.2. Define an energy functional for a spherically symmetric solution (ρ, u, a) as

$$\begin{aligned} H(t) &= \int_0^{a(t)} (r-(1+t)u)^2 \rho r^2 \mathrm{d}r + \frac{2}{\gamma-1}(1+t)^2 \int_0^{a(t)} \rho^\gamma r^2 \mathrm{d}r \\ &= \int_0^{a(t)} \rho r^4 \mathrm{d}r - 2(1+t)\int_0^{a(t)} \rho u r^3 \mathrm{d}r + (1+t)^2 \int_0^{a(t)} \rho u^2 r^2 \mathrm{d}r \end{aligned}$$

$$+\frac{2}{\gamma-1}(1+t)^2\int_0^{a(t)}\rho^\gamma r^2\mathrm{d}r. \tag{5.1}$$

A direct computation gives

$$\begin{aligned} H'(t) &= \int_0^{a(t)}(\rho_t r^4-2\rho u r^3)\mathrm{d}r+(1+t)^2\int_0^{a(t)}((\rho u^2)_t+\frac{2}{\gamma-1}(\rho^\gamma)_t)r^2\mathrm{d}r \\ &\quad +2(1+t)\int_0^{a(t)}(\rho u^2r^2-(\rho u)_t r^3+\frac{2}{\gamma-1}\rho^\gamma r^2)\mathrm{d}r \\ &\quad +(\rho u r^4-2(1+t)\rho u^2r^3+(1+t)^2\rho u^3r^2+\frac{2}{\gamma-1}(1+t)^2\rho^\gamma u r^2)\Big|_{r=a(t)} \\ &=: I_1+I_2+I_3+I_B. \end{aligned} \tag{5.2}$$

By (2.1) and (2.4), one has

$$\begin{aligned} I_1 &= -\int_0^{a(t)}\big((\rho u r^2)_r r^2-\rho u r^2 2r\big)\mathrm{d}r=-\int_0^{a(t)}(\rho u r^4)_r\mathrm{d}r \\ &= -(\rho u r^4)\Big|_{r=a(t)}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} I_2 &= (1+t)^2\int_0^{a(t)}2u\rho u_t r^2\mathrm{d}r+(1+t)^2\int_0^{a(t)}\rho_t u^2r^2\mathrm{d}r \\ &\quad +(1+t)^2\int_0^{a(t)}\frac{2\gamma}{\gamma-1}\rho^{\gamma-1}\rho_t r^2\mathrm{d}r \\ &= -2(\lambda+2\xi)(1+t)^2\int_0^{a(t)}(u_r+\frac{2u}{r})^2r^2\mathrm{d}r \\ &\quad -(1+t)^2(\rho u^3r^2+\frac{2}{\gamma-1}\rho^\gamma u r^2)\Big|_{r=a(t)}, \\ I_3 &= 2(1+t)\int_0^{a(t)}\Big\{\rho u^2r^2-(\rho u^2r^2)_r r+(\rho^\gamma-(\lambda+2\xi)u_r \\ &\qquad -(\lambda+2\xi)\frac{2u}{r})_r r^3+\frac{2}{\gamma-1}\rho^\gamma r^2\Big\}\mathrm{d}r \\ &= 2(1+t)(\rho u^2r^3)\Big|_{r=a(t)} \\ &\quad +6(1+t)\int_0^{a(t)}\Big\{(\rho^\gamma-(\lambda+2\xi)u_r-(\lambda+2\xi)\frac{2u}{r})r^2+\frac{2}{\gamma-1}\rho^\gamma r^2\Big\}\mathrm{d}r \\ &= \tfrac{2(2-3(\gamma-1))}{\gamma-1}(1+t)\int_0^{a(t)}\rho^\gamma r^2\mathrm{d}r+2(1+t)(3(\lambda+2\xi)ur^2+\rho u^2r^3)\Big|_{r=a(t)}. \end{aligned} \tag{5.4}$$

Substituting the above estimates into (5.2) yields that

$$\begin{aligned} H'(t) &= -2(\lambda+2\xi)(1+t)^2\int_0^{a(t)}(u_r^2+\frac{2u^2}{r^2})r^2\mathrm{d}r \\ &\quad +\frac{2(2-3(\gamma-1))}{\gamma-1}(1+t)\int_0^{a(t)}\rho^\gamma r^2\mathrm{d}r-4(\lambda+2\xi)(1+t)^2u^2(a(t),t)a(t) \\ &\quad +6(\lambda+2\xi)(1+t)u(a(t),t)a^2(t), \qquad (5.5) \\ &\le \frac{2(2-3(\gamma-1))}{\gamma-1}(1+t)\int_0^{a(t)}\rho^\gamma r^2\mathrm{d}r+6(\lambda+2\xi)a^3(t), \qquad (5.6) \end{aligned}$$

where we have used $(1+t)u(a(t),t)a^2(t)\le\frac{1}{2}\{(1+t)^2u^2(a(t),t)a(t)+a^3(t)\}$.

Therefore, we deduce from (5.6) that for $\gamma\ge\frac{5}{3}$

$$H'(t)\le 6(\lambda+2\xi)a^3(t), \tag{5.7}$$

which leads to

$$H(t)\le H(0)+Ca_1^3(t)t\le C(1+t)+C(1+t)a_1^3(t)\le C(1+t)a_1^3(t), \tag{5.8}$$

where

$$a_1(t):=\max_{s\in[0,t]}a(s)\ge c>0. \tag{5.9}$$

Thus, the combination of (5.1) and (5.8) lead to

$$\int_0^{a(t)}\rho^\gamma r^2\mathrm{d}r\le C(1+t)^{-1}a_1^3(t),\ \gamma\ge\frac{5}{3}. \tag{5.10}$$

For $1<\gamma<\frac{5}{3}$, we get from (5.1) and (5.6) that

$$H'(t)\le\frac{2-3(\gamma-1)}{1+t}H(t)+6(\lambda+2\xi)a^3(t), \tag{5.11}$$

from which it follows that for $1<\gamma<\frac{5}{3}$ and $\gamma\ne\frac{4}{3}$

$$H(t)\le C(1+t)^{5-3\gamma}+C(1+t)a_1^3(t), \tag{5.12}$$

and for $\gamma=\frac{4}{3}$

$$H(t)\le C(1+t)a_1^3(t)+C(1+t)\log(1+t)a_1^3(t). \tag{5.13}$$

By (5.9)−(5.10) and (5.12)−(5.13), one has

$$\int_0^{a(t)} \rho^\gamma r^2 \mathrm{d}r \le \begin{cases} C(1+t)^{-1} a_1^3(t), & \gamma > \frac{4}{3}, \\ C(1+t)^{-1}\log(1+t) a_1^3(t), & \gamma = \frac{4}{3}, \\ C(1+t)^{-3(\gamma-1)} a_1^3(t), & \gamma \in (1, \frac{4}{3}). \end{cases} \tag{5.14}$$

Due to

$$\int_0^{a_0} \rho_0 r^2 \mathrm{d}r = \int_0^{a(t)} \rho r^2 \mathrm{d}r \le C a(t)^{\frac{3(\gamma-1)}{\gamma}} \Big(\int_0^{a(t)} \rho^\gamma r^2 \mathrm{d}r \Big)^{\frac{1}{\gamma}}, \tag{5.15}$$

combining (5.14) with (5.15) implies

$$a_1(t) = \max_{s\in[0,t]} a(s) \ge \begin{cases} C(1+t)^{\frac{\gamma-1}{\gamma}}, & \gamma \in (1, \frac{4}{3}), \\ C(1+t)^{\frac{1-\nu}{3\gamma}}, & \gamma = \frac{4}{3}, \\ C(1+t)^{\frac{1}{3\gamma}}, & \gamma > \frac{4}{3}, \end{cases} \tag{5.16}$$

where we have used $(1+t)^\nu \sim \log(1+t)$ for any $\nu > 0$ small enough, and

$$a_1(t) \to +\infty, \ as\ t \to +\infty. \tag{5.17}$$

Next, we show the exact expanding rate of the interface $r = a(t)$. By (5.5), we have

$$H'(t) \le \frac{2(2-3(\gamma-1))}{\gamma-1}(1+t)\int_0^{a(t)} \rho^\gamma r^2 \mathrm{d}r + 2(\lambda+2\xi)(1+t)\frac{\mathrm{d}}{\mathrm{d}t} a^3(t) \tag{5.18}$$

due to the fact $3u(a(t),t)a^2(t) = \frac{\mathrm{d}}{\mathrm{d}t} a^3(t)$. For $\gamma \ge \frac{5}{3}$, it holds

$$\begin{aligned} H'(t) &\le 2(\lambda+2\xi)(1+t)\frac{\mathrm{d}}{\mathrm{d}t} a^3(t) = 2(\lambda+2\xi)\frac{\mathrm{d}}{\mathrm{d}t}[(1+t)a^3(t)] - 2(\lambda+2\xi)a^3(t) \\ &\le 2(\lambda+2\xi)\frac{\mathrm{d}}{\mathrm{d}t}[(1+t)a^3(t)] \end{aligned} \tag{5.19}$$

because of $(\lambda + 2\xi) > 0$ and $a(t) \ge c_0 > 0$. Therefore, we obtain finally

$$H(t) \le C + C(1+t)a^3(t), \tag{5.20}$$

which, together with (5.1), leads to

$$\int_0^{a(t)} \rho^\gamma r^2 \mathrm{d}r \leq C(1+t)^{-1} a^3(t). \tag{5.21}$$

The combination of (5.21) and (5.15) gives rise to

$$a(t) \geq C(1+t)^{\frac{1}{3\gamma}},\ \gamma \geq \frac{5}{3}. \tag{5.22}$$

For $\gamma \leq \frac{5}{3}$, (5.1) and (5.6) lead to

$$H'(t) \leq \frac{2-3(\gamma-1)}{1+t} H(t) + 2(\lambda+2\xi)(1+t)\frac{\mathrm{d}}{\mathrm{d}t} a^3(t). \tag{5.23}$$

If $\gamma = \frac{4}{3}$, by Gronwall inequality, it holds

$$H(t) \leq C(1+t)(H(0) + a^3(t) - a^3(0)) \leq C(1+t)a^3(t),$$

which leads to

$$\int_0^{a(t)} \rho^\gamma r^2 \mathrm{d}r \leq C(1+t)^{-1} a^3(t).$$

Therefore,

$$a(t) \geq C(1+t)^{\frac{1}{3\gamma}},\ \gamma = \frac{4}{3}. \tag{5.24}$$

If $\frac{4}{3} < \gamma < \frac{5}{3}$, the application of Gronwall inequality to (5.23) gives

$$\begin{aligned} H(t) \ &\leq \ C(1+t)^{5-3\gamma}\Big\{H(0) + 2(\lambda+2\xi)(1+t)^{3\gamma-4}a^3(t) - 2(\lambda+2\xi)a^3(0) \\ &\quad -2(\lambda+2\xi)(3\gamma-4)\int_0^t \frac{a^3(s)}{(1+s)^{5-3\gamma}}\mathrm{d}s\Big\} \\ &\leq \ C(1+t)^{5-3\gamma}\big(H(0) + 2(\lambda+2\xi)(1+t)^{3\gamma-4}a^3(t) - 2(\lambda+2\xi)a^3(0)\big) \\ &\leq \ C(1+t)^{5-3\gamma} + C(1+t)a^3(t), \end{aligned}$$

which leads to

$$\begin{aligned} \int_0^{a(t)} \rho^\gamma r^2 \mathrm{d}r \ &\leq \ C(1+t)^{-3(\gamma-1)} + C(1+t)^{-1}a^3(t) \\ &\leq \ C\{(1+t)^{-3(\gamma-1)} + (1+t)^{-1}\}a^3(t). \end{aligned} \tag{5.25}$$

Thus, it follows

$$a(t) \geq C(1+t)^{\frac{\gamma-1}{\gamma}} + C(1+t)^{\frac{1}{3\gamma}} \geq C(1+t)^{\frac{\gamma-1}{\gamma}},\ \frac{4}{3} < \gamma < \frac{5}{3}. \tag{5.26}$$

To conclude from (5.22), (5.24), and (5.26), it holds that

$$a(t) \geq \begin{cases} C(1+t)^{\frac{1}{3\gamma}}, & \gamma = \frac{4}{3}, \\ C(1+t)^{\frac{\gamma-1}{\gamma}}, & \gamma \in (\frac{4}{3}, \frac{5}{3}), \\ C(1+t)^{\frac{1}{3\gamma}}, & \gamma \geq \frac{5}{3}. \end{cases} \tag{5.27}$$

The proof is completed. □

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