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FINITE ENERGY GLOBAL SOLUTIONS TO A TWO-FLUID MODEL ARISING IN SUPERFLUIDITY

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Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

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Abstract

In this paper we study a hydrodynamic system describing two interacting fluids, which can be seen as a toy model to start an investigation on the so called two-fluid models arising in superfluidity and Bose-Einstein condensates at finite temperatures. We show global existence of finite energy weak solutions, by using a fractional step argument which combines the study of a Cauchy problem for an appropriate nonlinear Schrödinger equation and the polar factorisation techinque. The convergence of the sequence of approximate solutions is then proved by using the dispersive properties of the nonlinear Schrödinger equation.

1. Introduction

In this paper we consider a class of hydrodynamic systems describing a two-fluid model. Such models arise in various physical phenomena, e.g. superfluidity [14], or Bose-Einstein at finite temperatures [11, 28].

In the Landau-Khalatnikov two-fluid formulation the system describes a dilute Bose condensed gas at temperature lower than the critical condensation temperature, but not very close to absolute zero, so that the gas has a condensate and non-condensate fraction. It turns out that the condensate part is described by a frictionless, quantum fluid, whereas the non-condensate part is considered as a viscous classical fluid. Hereafter we will

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refer to the condensate part as the superfluid and to the non-condensate one as the normal fluid. The two fluids interact by exchanging mass and momentum. If ρ_s, v_s denote the superfluid mass density and velocity field, respectively, and ρ_n, v_n the analogous quantities for the normal fluid, then the class of two-fluid models we are interested in can be written in the following way, in its most general formulation[1],

$$\begin{cases} \partial_t\rho_s + \operatorname{div}(\rho_s v_s) = -\Gamma_{12} \\ \partial_t(\rho_s v_s) + \operatorname{div}(\rho_s v_s \otimes v_s) + \nabla P_s(\rho_s) + \rho_s\nabla V_{ext} = \dfrac{1}{2}\rho_s\nabla\left(\dfrac{\Delta\sqrt{\rho_s}}{\sqrt{\rho_s}}\right) - Q_{12} \\ \partial_t\rho_n + \operatorname{div}(\rho_n v_n) = -\Gamma_{21} \\ \partial_t(\rho_n v_n) + \operatorname{div}(\rho_n v_n \otimes v_n) \\ \qquad + \nabla P_n(\rho_n) + \rho_n\nabla V_{ext} = \operatorname{div}\left(2\eta\left(\mathrm{D}v_n - \dfrac{1}{3}\mathbf{1}\,\mathrm{trace}\,\mathrm{D}v_n\right)\right) - Q_{21}. \end{cases} \tag{1.1}$$

where P_s and P_n are the self-consistent pressure terms for the superfluid and the normal fluid, respectively, V_{ext} is a given external potential, η is the viscosity in the equation for the normal fluid, and in general it could depend on the unkown variables. $\mathrm{D}v_n$ denotes the symmetric part of the gradient of v_n. Furthermore $\Gamma_{ij} = \Gamma_{ij}(\rho_s, \rho_n, v_n, v_s)$ are the terms accounting for the mass exchange, $Q_{ij} = Q_{ij}(\rho_s, \rho_n, v_s, v_n)$ for the momentum exchange, and they are nonlinear operators, depending on the unknown variables. Various models of this type were derived in the physics literature, see for example [21, 28, 11, 20] and references therein. Here in this paper we consider a toy model in this class of systems. A more general and physically senseful class of two-fluid models will be the subject of future investigations. The main assumptions we make in the model presented here are as follows. First of all we assume the two gas populations don't interact by exchanging mass, i.e. we fix $\Gamma_{12} = \Gamma_{21} = 0$. Furthermore $Q_{21} = 0$, that is we consider a system where the dynamics of the normal fluid is not influenced by the superfluid part. On the other hand the superfluid interacts with the normal fluid through the collision term Q_{12}, which we will assume to be linear in the two velocity fields v_s, v_n. This means we want $Q_{12} = \frac{1}{\tau}\rho_s(v_s - v_n)$, where

[1] Many models consider also a dynamical equation for the energy density, or the entropy. However this is beyond the scope of our study and will be the subject of future investigations

$\tau > 0$ is the average collision time between particles of the two populations. However, to keep some physical features of the superfluid we will assume

$$Q_{12} = \frac{1}{\tau}\rho_s(v_s - \mathbb{Q}v_n), \tag{1.2}$$

where $\mathbb{Q} = -(-\Delta)^{-1}\nabla\mathrm{div}$ is the Helmholtz projection operator. Indeed, we want the superfluid velocity to be irrotational (in $\rho_s\, dx$ almost everywhere): this is one of the main properties for superfluids, strictly related to the quantization of vortices. Thus we fix (1.2) to be the collision operator in the equation for the current density, so to have a toy model which nevertheless conserves the main physical features of general two-fluid models. Moreover, we will assume the viscosity coefficient to be constant. This assumption is physically reasonable in this context and it will not be further discussed here. Finally we assume that there is no external potential, $V_{ext} = 0$ and that the self-consistent pressure terms are given by the usual $\gamma-$law,

$$P_s(\rho_s) = \frac{\gamma_s - 1}{\gamma_s}\rho^{\gamma_s}, \qquad P_n(\rho_n) = \frac{\gamma_n - 1}{\gamma_n}\rho^{\gamma_n}, \tag{1.3}$$

where $1 \le \gamma_s < 3, \gamma_n > \frac{3}{2}$. Those two final assumptions are only for the simplification of the exposition of the result, as it is of no difficulty to extend the study to a class of more general potentials and pressure terms than the ones we are considering here.

We want to study the Cauchy problem for the system above in the space of energy, namely we deal with minimal regularity solutions such that the physical quantities, like the total mass and the total energy, are well defined at any time along the flow of solutions. Our goal is to prove the existence of global in time, finite energy weak solutions to the system (1.1).

Since the early days of quantum mechanics it is well-known [18] that the Quantum Hydrodynamics (QHD) system

$$\begin{cases} \partial_t\rho + \mathrm{div}J = 0 \\ \partial_t J + \mathrm{div}\left(\dfrac{J\otimes J}{\rho}\right) + \rho\nabla V + \nabla P(\rho) = \dfrac{1}{2}\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \end{cases} \tag{1.4}$$

is strictly related to the nonlinear Schrödinger equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V\psi + f'(|\psi|^2)\psi, \tag{1.5}$$

where f is such that $\rho f''(\rho) = P'(\rho)$. This can be shown by using the so called WKB ansatz, i.e. by assuming the wave function ψ can be expressed in terms of its amplitude $\sqrt{\rho}$ and its phase S, $\psi = \sqrt{\rho}e^{iS}$. In this way we may formally see that, if ψ solves (1.5), then the pair (ρ, J), where $J := \rho\nabla S$, solves the QHD system (1.4). Unfortunately this analogy is valid only in the region where $\rho > 0$. Indeed in the nodal region $\{\rho = 0\}$ the phase is not well-defined. As the authors proved in [1], [2], it is possible to overcome the WKB ansatz in order to define a finite energy weak solution to (1.4), given a wave function ψ, solution to the NLS (1.5). This is made rigorous by a polar factorization techinque, which avoids the definition of the (superfluid) velocity field in the vacuum regions. On the other hand, the presence of dissipative terms in the QHD system, as in (1.1) destroys somehow the analogy with nonlinear Schrödinger equations. Thus we need to proceed by constructing a sequence of approximate solutions, through a fractional step argument.

Since we are going to work in the same framework as [1, 2], we rewrite the system we are going to study in the following way:

$$\begin{cases} \partial_t\rho_1 + \mathrm{div}J_1 = 0 \\ \partial_t J_1 + \mathrm{div}\left(\frac{J_1\otimes J_1}{\rho_1}\right) + \nabla P_1(\rho_1) = \frac{1}{2}\rho_1\nabla\left(\frac{\Delta\sqrt{\rho_1}}{\sqrt{\rho_1}}\right) - \frac{1}{\tau}\left(J_1 - \rho_1\mathbb{Q}v_2\right) \\ \partial_t\rho_2 + \mathrm{div}(\rho_2 v_2) = 0 \\ \partial_t(\rho_2 v_2) + \mathrm{div}(\rho_2 v_2\otimes v_2) + \nabla P_2(\rho_2) = \eta\Delta v_2 + \frac{1}{3}\eta\nabla\mathrm{div}v_2, \end{cases} \tag{1.6}$$

where the superscripts 1 and 2 stand for the quantities associated to the superfluid and the normal fluid, respectively. That is, here the superfluid is described in terms of its mass and current densities, and we do not deal with the superfluid velocity. More precisely, we will show it is possible to study the superfluid part in terms of the square root of the superfluid mass density and of a vector field which represents the quantity $J_1/\sqrt{\rho_1}$, see the definition below. Furthermore, without loss of generality we fix $\tau = 1$ in (1.2), since at present we are not dealing with singular limits for the system (1.6).

We prescribe initial data for (1.6) as

$$\rho_1(0) = \rho_{1,0}, J_1(0) = J_{1,0}, \quad \rho_2(0) = \rho_{2,0}, (\rho_2 v_2)(0) = J_{2,0} \quad \text{in } \mathbf{R}^3, \tag{1.7}$$

where $J_2 = \rho_2 v_2$, and we will assume they satisfy the following assumptions

$$\begin{cases} \exists\, \psi_0 \in H^1(\mathbf{R}^3) \text{ such that } \rho_{0,1} = |\psi_0|^2, J_{0,1} = \mathrm{Im}(\bar{\psi}_0 \nabla \psi_0), \\ \rho_{2,0} \geq 0 \text{ a. e. in } \mathbf{R}^3, \quad \rho_{2,0} \in L^1(\mathbf{R}^3) \cap L^{\gamma_2}(\mathbf{R}^3), \\ J_{2,0} \in L^{\frac{2\gamma_2}{\gamma_2+1}}(\mathbf{R}^3), \quad J_{2,0} = 0 \text{ a.e. in } \{\rho_{2,0} = 0\} \\ \dfrac{|J_{2,0}|^2}{\rho_{2,0}} \in L^1(\mathbf{R}^3), \quad \dfrac{|J_{2,0}|^2}{\rho_{2,0}} = 0 \text{ on } \{\rho_{2,0} = 0\}. \end{cases} \tag{1.8}$$

The total mass of the system

$$\int_{\mathbf{R}^3} \rho_1(t,x) + \rho_2(t,x)\, dx$$

is conserved along the flow of solutions to (1.6). Moreover, each species total mass $\int \rho_1$ and $\int \rho_2$ is conserved. For the total mass energy we have

$$E(t) = E_1(t) + E_2(t),$$

where

$$\begin{cases} E_1(t) = \displaystyle\int_{\mathbf{R}^3} \frac{1}{2}|\nabla\sqrt{\rho_1}|^2 + \frac{1}{2}|\Lambda_1|^2 + \frac{1}{\gamma_1}\rho_1^{\gamma_1}\, dx, \\ E_2(t) = \displaystyle\int_{\mathbf{R}^3} \frac{1}{2}\rho_2|v_2|^2 + \frac{1}{\gamma_2}\rho_2^{\gamma_2}\, dx, \end{cases} \tag{1.9}$$

and formally we have

$$\begin{aligned} E(t) + \int_0^t \int |\Lambda_1|^2\, dxdt' - \int_0^t \int J_1 \cdot \mathbb{Q} v_2\, dxdt' \\ + \eta \int_0^t \int |\nabla v_2|^2 + \frac{1}{3}|\mathrm{div} v_2|^2\, dxdt' = E(0). \end{aligned}$$

Definition 1. Let $0 < T < \infty$. we say the quadruple $(\rho_1, J_1, \rho_2, J_2)$ is a *finite energy weak solution* for the system (1.6) with initial data (1.7) on the space-time strip $[0,T] \times \mathbf{R}^3$ if

- there exist two functions $\sqrt{\rho_1} \in L^2(0,T; H^1(\mathbf{R}^3)) \cap \mathcal{C}([0,T]; H^1_{loc}(\mathbf{R}^3))$, $\Lambda_1 \in L^2(0,T; L^2(\mathbf{R}^3)) \cap \mathcal{C}([0,T]; L^2_{loc}(\mathbf{R}^3))$ such that $\rho_1 := (\sqrt{\rho_1})^2$ and $J_1 := \sqrt{\rho_1}\Lambda_1$;
- $\rho_2 \in L^\infty(0,T; L^{\gamma_2}(\mathbf{R}^3)) \cap \mathcal{C}([0,T]; L^p(\mathbf{R}^3))$, for $1 \leq p < \gamma_2$, $\rho_2 \geq 0$ a. e.;

- $\nabla v_2 \in L^2(0,T;L^2(\mathbf{R}^3))$, $\rho_2|v_2|^2 \in L^\infty(0,T;L^1(\mathbf{R}^3))$, $\rho_2 v_2 \in \mathcal{C}([0,T]; L^{\frac{2\gamma_2}{\gamma_2+1}}(\mathbf{R}^3))$;
- $\forall\, \eta_1 \in \mathcal{C}_0^\infty([0,T)\times\mathbf{R}^3)$,

$$\int_0^T\int_{\mathbf{R}^3}\rho_1\partial_t\eta + J_1\cdot\nabla\eta\,dxdt + \int_{\mathbf{R}^3}\rho_{1,0}(x)\eta(0,x)\,dx = 0;$$

- $\forall\zeta \in \mathcal{C}_0^\infty([0,T)\times\mathbf{R}^3;\mathbf{R}^3)$,

$$\begin{aligned}\int_0^T\int_{\mathbf{R}^3} & J_1\cdot\partial_t\zeta + \Lambda_1\otimes\Lambda_1 : \nabla\zeta + P_1(\rho_1)\mathrm{div}\zeta + \nabla\sqrt{\rho_1}\otimes\nabla\sqrt{\rho_1} : \nabla\zeta \\ & - \frac{1}{4}\rho_1\Delta\mathrm{div}\zeta + (J_1 - \rho_1\mathbb{Q}v_2)\cdot\zeta\,dxdt + \int_{\mathbf{R}^3} J_{1,0}(x)\cdot\zeta(0,X)\,dx = 0;\end{aligned}$$

- the Navier-Stokes equation for (ρ_2, v_2) in (1.6) holds in $\mathcal{D}'([0,T]\times\mathbf{R}^3)$;
- (generalized irrotationality condition) for almost every $t \in (0,T)$

$$\nabla\wedge J_1 = 2\nabla\sqrt{\rho_1}\wedge\Lambda_1, \tag{1.10}$$

  holds in the sense of distributions.

We say that $(\rho_1, J_1, \rho_2, J_2)$ in a *global in time* finite energy weak solution to (1.6) with initial data (1.7) if the above conditions hold for any $0 < T < \infty$.

Remark 1. Let us consider (1.10) more closely. If we have a sufficiently smooth solution to (1.6), so that we may write $J_1 = \rho_1 v_1$, where v_1 is the superfluid velocity, then it is straightforward to check that (1.10) is equivalent to

$$\rho_1\nabla\wedge v_1 = 0,$$

i.e. the superfluid velocity is irrotational almost everywhere $\rho_1\,dx$. The condition (1.10) is naturally satisfied by the solutions constructed in this paper. On the other hand, as we already noticed before, condition (1.10) has a very strong physical interpretation, being related to the existence of quantized vortices in a superfluid, see for example [3], [25] and references therein.

Condition (1.10) in particular implies that the solutions we deal with are more general than the ones given through a WKB analysis. Indeed for such solutions one has $v_1 = \nabla S$, where S is the phase of the wave function.

Hence, the velocity field is everywhere irrotational, ruling out completely the presence of quantized vortices.

We also remark that formula (1.2) is consistent with the generalized irrotationality condition: if we don't consider the normal velocity projected with $\mathbb{Q}$ in the collision operator Q_{12}, then in general we could not expect a solution which satisfies condition (1.10).

As it is clear from the system itself, the dynamics of the normal fluid is not affected at all by the superfluid, whereas the latter interacts with the former through the collision term. Thus we may solve separately the dynamics for the normal fluid and consider it as given in the equation for the superfluid current density.

The classical fluid is described by the Navier-Stokes system for a compressible fluid, with a constant viscosity coefficient:

$$\begin{cases} \partial_t\rho_2 + \mathrm{div}(\rho_2 v_2) = 0 \\ \partial_t(\rho_2 v_2) + \mathrm{div}(\rho_2 v_2 \otimes v_2) + \nabla P_2(\rho_2) = \eta\Delta v_2 + \dfrac{1}{3}\eta\nabla\mathrm{div} v_2 \\ \rho_2(0) = \rho_{2,0},\ (\rho_2 v_2)(0) = J_{2,0}, \end{cases} \tag{1.11}$$

with $P_2(\rho_2) = \frac{\gamma_2-1}{\gamma_2}\rho^{\gamma_2}$, $\gamma_2 > \frac{3}{2}$, and $\rho_{2,0}, J_{2,0}$ which satisfy the assumptions in (1.8).

The compressible Navier-Stokes system is well studied, see for example [17, 9, 23] and references therein. Here we only state a global existence result for weak solutions to (1.11) which will be useful to our analysis. It is proved in [8] in a more general case, namely for our purposes we set $\lambda = -\frac{2}{3}\mu$ and $G = 0$ in the main theorem there.

Theorem 1. *Under the assumptions in* (1.8) *for* $\rho_{2,0}, J_{2,0}$*, there exists a finite energy weak solution to the Cauchy problem* (1.11) *in* $[0,T]$*, for any* $T > 0$*. Furthermore, the solution satisfies*

- $\rho_2 \in L^\infty([0,T]; L^1 \cap L^{\gamma_2}(\mathbf{R}^3))$, $\rho_2 \geq 0$;
- $v_2 \in L^2([0,T]; \dot{H}^1(\mathbf{R}^3))$;
- $$E_2(t) + \eta\int_0^t \int |\nabla v_2|^2 + \frac{1}{3}|\mathrm{div} v_2|^2\, dxdt' \leq E_2(0),$$

where the energy $E_2(t)$ *is defined as in* (1.9).

Once we solve the Cauchy problem for the normal fluid part of the gas, we may regard it as a source term appearing in the equation for the superfluid current density. That is, we need to solve the following Cauchy problem

$$\begin{cases} \partial_t \rho_1 + \operatorname{div} J_1 = 0 \\ \partial_t J_1 + \operatorname{div}(\Lambda_1 \otimes \Lambda_1) + \nabla P_1(\rho_1) \\ \qquad = \frac{1}{4}\nabla\Delta\rho_1 - \operatorname{div}(\nabla\sqrt{\rho_1} \otimes \nabla\sqrt{\rho_1}) - (J_1 - \rho_1 \mathbb{Q} v_2) \\ \rho_1(0) = \rho_{1,0},\ J_1(0) = J_{1,0}, \end{cases} \tag{1.12}$$

in accordance with the Definition 1. Here the initial data are given as in (1.8): $\rho_{1,0} = |\psi_0|^2$, $J_{1,0} = \operatorname{Im}(\bar{\psi}_0 \nabla \psi_0)$, for some $\psi_0 \in H^1(\mathbf{R}^3)$.

In analogy with the WKB analysis (see also [1, 2]), we see that (1.12) is formally equivalent to the following NLS equation

$$\begin{cases} i\partial_t \psi = -\frac{1}{2}\Delta\psi + \tilde{V}\psi + W\psi + |\psi|^{2(\gamma_1 - 1)}\psi \\ \psi(0) = \psi_0, \end{cases} \tag{1.13}$$

where $W = \frac{1}{2i}\log\left(\psi/\bar{\psi}\right)$ and $\tilde{V} = (-\Delta)^{-1}\operatorname{div} v_2$, v_2 being the normal fluid velocity, solution to the Navier-Stokes system (1.11), see Theorem 1. The self-consistent potential W is ill-posed and to the best of our knowledge there is no satisfactory theory for the Cauchy problem for NLS equations with such potentials. For this reason we will avoid studying (1.13) and we will overcome this difficulty by using a fractional step argument, in order to construct a sequence of approximate solutions for the hydrodynamic problem (1.12). That is, (1.13) cannot be studied in the framework we are considering, but we will show the existence of a finite energy weak solution to the hydrodynamic system related to (1.13), namely (1.12).

On the other hand, we can study the term $(-\Delta)^{-1}\operatorname{div} v_2$ in the framework of NLS equations. For this purpose we actually need a further assumption on v_2, namely we will assume that

$$v_2 \in L^\infty([0,T]; \dot{H}^1(\mathbf{R}^3)), \tag{1.14}$$

for any $0 < T < \infty$. By this assumption and Theorem 1 we know then

$$v_2 \in L^2([0,T]; \dot{H}^1(\mathbf{R}^3)) \cap L^\infty([0,T]; \dot{H}^1(\mathbf{R}^3)). \tag{1.15}$$

By means of this information, we are going to give a precise meaning to $(-\Delta)^{-1}\mathrm{div}v_2$ in the NLS equation. Indeed, by (1.15) we only know that $\nabla\tilde{V} = -\mathbb{Q}v_2 \in (L^2 \cap L^\infty)_t L^6_x$. We are going to use a result on a note by Ortner, Süli [19]. Here we only state the result we are going to use for our analysis, for a more general statement we refer the reader to Theorem 2.2 in [19].

Theorem 2. *Let $\tilde{V}$ be such that $\nabla\tilde{V} \in L^\infty([0,T]; L^6(\mathbf{R}^3))$. Then there exist two functions $V_\infty \in L^\infty_{loc}([0,T] \times \mathbf{R}^3)$, $V_p \in L^\infty([0,T]; W^{1,6}(\mathbf{R}^3))$, such that*

(i) $V_\infty(t) \in \mathcal{C}^\infty(\mathbf{R}^3)$, *for a.a.* $t \in [0,T]$;

(ii) $\tilde{V} = V_\infty + V_p$;

(iii) $\|V_p\|_{L^\infty_t W^{1,6}_x} \leq C\|\nabla\tilde{V}\|_{L^\infty_t L^6_x}$;

(iv) $\|\nabla V_\infty\|_{L^\infty_t L^6_x} + \|\nabla V_\infty\|_{L^\infty_{t,x}} \leq C\|\nabla\tilde{V}\|_{L^\infty_t L^6_x}$;

(v) $|V_\infty(t,x)| \leq C|x|^{5/6}\|\nabla\tilde{V}\|_{L^\infty_t L^6_x}$ *for a.a.* $t \in [0,T]$;

(vi) $\|\partial^\alpha V_\infty\|_{L^\infty_t L^6_x} \leq C\|\nabla\tilde{V}\|_{L^\infty_t L^6_x}$, *for any* $\alpha \in \mathbf{N}^3$, $|\alpha| \geq 1$.

Remark 2. The result stated in Theorem 2 is slightly different from Theorem 2.2 in [19], namely in [19] the result is for time independent functions and property (vi) is not stated. However, those modification can be easily inferred from the proof in [19].

Given the above Theorem we may then study the following Cauchy problem

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + V_\infty\psi + V_p\psi + |\psi|^{2(\gamma_1-1)}\psi \\ \psi(0) = \psi_0. \end{cases} \tag{1.16}$$

In particular, let us notice that we have a NLS equation with a smooth potential V_∞, growing at infinity and whose derivatives are uniformly bounded for a.e. time. For this reason we need to study the above Cauchy problem in the space

$$\Sigma(\mathbf{R}^3) = \{\psi \in H^1(\mathbf{R}^3) \ : \ |\cdot|\psi \in L^2(\mathbf{R}^3)\}.$$

We will show that (1.16) is globally well-posed in $\Sigma(\mathbf{R}^3)$. Then, by using this well-posedness result and the polar factorisation technique we will set up a fractional step argument in order to construct a sequence of approximate solutions for the system (1.12). The main result we are going to prove in this paper is the following

Theorem 3. *Let* (ρ_2, v_2) *be a finite energy weak solution to the Navier-Stokes system as in Theorem* 1 *and let us assume* v_2 *further satisfies* (1.14). *For any* $\psi_0 \in \Sigma(\mathbf{R}^3)$ *let us define* $\rho_{1,0} = |\psi_0|^2, J_{1,0} = \mathrm{Im}(\bar{\psi}_0 \nabla \psi_0)$. *Then for any* $0 < T < \infty$ *there exists a finite energy weak solution for the QHD system* (1.12). *Furthermore, the energy satisfies*

$$E_1(t) + c \int_0^t \int |\Lambda(t', x)|^2 \, dx dt' \leq C \left(E_1(0) + \|\rho_{1,0}\|_{L^1} \|v_2\|^4_{L^2_t \dot{H}^1_x([0,T]\times\mathbf{R}^3)} \right),$$

for $t \in [0, T]$.

Corollary 1. *Let us assume* $\rho_{1,0}, J_{1,0}, \rho_{2,0}, J_{2,0}$ *satisfy conditions* (1.8) *and let us furthermore assume that* ψ_0 *in* (1.8) *satisfies* $\psi_0 \in \Sigma(\mathbf{R}^3)$ *and that the normal velocity* v_2 *satisfies* $v_2 \in L^\infty([0,T]; \dot{H}^1(\mathbf{R}^3))$, *for any* $0 < T < \infty$. *Then there exists a global in time finite energy weak solution to the two-fluid model* (1.6).

The paper is organised as follows. In Section 2 we are going to study the Cauchy problem (1.16) in the space of energy $\Sigma(\mathbf{R}^3)$, in Section 3 we will recall the polar factorisation technique and we will give an intermediate result for our study of the system (1.12). In Section 4 we will set up the fractional step argument to define the sequence of approximate solutions for (1.12) and then Section 5 we will prove some a priori estimates for this sequence, showing the convergence to a finite energy weak solution to (1.12).

2. The Cauchy Problem for the NLS Equation

In this Section we are going to study the following Cauchy problem

$$\begin{cases} i\partial_t \psi = -\frac{1}{2}\Delta\psi + V_\infty \psi + V_p \psi + |\psi|^{2(\gamma_1 - 1)}\psi \\ \psi(0) = \psi_0, \end{cases} \tag{2.1}$$

where $\tilde{V} = V_\infty + V_p$ is such that $\nabla \tilde{V} = (-\Delta)^{-1} \mathrm{div} v_2 \in L^\infty_t L^6_x([0,T] \times \mathbf{R}^3)$ and is well defined by Theorem 2. In particular we have that V_∞ is a potential which for almost every time is smooth, growing at infinity and whose derivatives are uniformly bounded. As we already remarked this requires we study the Cauchy problem (1.16) in the space of energy

$$\Sigma(\mathbf{R}^3) = \{\psi \in H^1(\mathbf{R}^3) \ : \ |\cdot|\psi \in L^2(\mathbf{R}^3)\}.$$

The energy we associate to problem (1.16) is the following[2]

$$E_1(t) = \int \frac{1}{2}|\nabla\psi|^2 + \frac{1}{\gamma_1}|\psi|^{2\gamma_1}.$$

Along the flow of solutions to (1.16) we have

$$\frac{d}{dt}E_1(t) = -\int J_1 \cdot \mathbb{Q}v_2 \, dx,$$

where $J_1 = \mathrm{Im}(\bar{\psi}\nabla\psi)$. From the bound $v_2 \in L_t^\infty \dot{H}_x^1$ and the conservation of mass for (1.16) we have the following Gronwall-type inequality for the energy

$$E_1(t) \leq e^{Ct}E_1(0). \tag{2.2}$$

Proposition 1. *For any $\psi_0 \in \Sigma(\mathbf{R}^3)$ there exists a unique global solution $\psi \in \mathcal{C}(\mathbf{R};\Sigma(\mathbf{R}^3))$ such that*

$\psi, \nabla\psi, |\cdot|\psi \in L^q([0,T];L^r(\mathbf{R}^3)), \forall\, (q,r)$ *Strichartz admissible pairs*, $0<T<\infty$.

Moreover the solution depends continuously on the initial data and the energy satisfies (2.2).

To study the Cauchy problem (1.16) we first define the semigroup associated to the linear equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V_\infty\psi. \tag{2.3}$$

Let $U(t,s)f$ be the solution to the above linear equation with initial datum $\psi(s) = f$. By [10] and Theorem 2 we know it is well defined and moreover

$$\|U(t,s)f\|_{L^\infty} \lesssim |t|^{-\frac{3}{2}}\|f\|_{L^1}, \quad \text{for } |t| < \delta.$$

[2]There is no ambiguity in denoting the energy E_1 as the one in (1.9) as in Section 3 we shall indeed prove they are the same object.

From [13] we may infer the Strichartz estimates for the semigroup $U(t,s)$, see also [5]. For this purpose we recall that the pair of exponents (q,r) is called *admissible* is $2 \le q \le \infty$, $2 \le r \le 6$ and $\frac{1}{q} = \frac{3}{2}\left(\frac{1}{r} - \frac{1}{2}\right)$. For a more detailed introduction to Strichartz estimates we refer to [13] and the references therein.

Proposition 2. *For any $(q,r), (\tilde{q}, \tilde{r})$ arbitrary admissible pairs we have*

$$\begin{aligned} \|U(t,0)f\|_{L^q_t L^r_x(I\times\mathbf{R}^3)} &\le C(|I|)\|f\|_{L^2} \\ \|\int_0^t U(t,s)F(s)\,ds\|_{L^q_t L^r_x(I\times\mathbf{R}^3)} &\le C(|I|)\|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(I\times\mathbf{R}^3)}. \end{aligned}$$

We may now use the Strichartz estimates for the semigroup $U(t,s)$ to prove the existence of local in time solutions by a fixed point argument. Fixed $\psi_0 \in \Sigma(\mathbf{R}^3)$, we define the space

$$\begin{aligned} E = &\{\psi \in L^\infty([0,T];\Sigma(\mathbf{R}^3)) \; : \; \psi, \nabla\psi, |\cdot|\psi \in L^q_t L^r_x([0,T]\times\mathbf{R}^3), \; \forall\,(q,r) \\ &\quad \text{admissible}, \; \|\psi\|_{L^q_t L^r_x} + \|\nabla\psi\|_{L^q_t L^r_x} + \||\cdot|\psi\|_{L^q_t L^r_x} \le M\}, \end{aligned}$$

where M, T will be chosen depending on $\|\psi_0\|_{\Sigma(\mathbf{R}^3)}$. It is standard now to prove there exists a unique solution $\psi \in \mathcal{C}([0,T_{max});\Sigma(\mathbf{R}^3))$ to

$$\psi(t) = U(t,0)\psi_0 - i\int_0^t U(t,s)\left(V_p\psi + |\psi|^{2(\gamma_1-1)}\psi\right)(s)\,ds,$$

see for example [5], [6] and references therein. Furthermore we also have the following blow-up alternative for (2.1),

$$T_{max} < \infty \Leftrightarrow \lim_{t\to T_{max}} \|\nabla\psi(t)\|_{L^2} = \infty.$$

By combining this with the energy inequality (2.2) we then infer that the solution is global in time, indeed.

3. Polar Factorisation and an Intermediate Result

In this Section we review the polar factorisation technique introduced in [1, 2], establishing its main properties. By using this, we will then provide an intermediate result for our analysis of system (1.12).

The main advantage of the polar factorization is that vacuum regions are allowed in the theory. More precisely, we write the wave function ψ in terms of its amplitude $\sqrt{\rho} := |\psi|$ and its unitary factor ϕ, namely a function taking its values in the unitary disk of the complex plane, such that $\psi = \sqrt{\rho}\phi$. In the WKB setting the polar factor would be $\phi = e^{iS/\hbar}$, however this equality holds only in the case of a smooth, nowhere vanishing, wave function. The idea of polar factorization is similar in spirit to the one used by Brenier in [4] to decompose a vector-valued function by means of a gradient of a convex function and a measure preserving map. Our case is much simpler than [4] and it can be studied directly. Given any function $\psi \in H^1(\mathbf{R}^3)$ we define the set

$$P(\psi) := \{\phi \in L^\infty(\mathbf{R}^3) \ : \ \|\phi\|_{L^\infty} \leq 1, \psi = \sqrt{\rho}\phi \ \text{ a.e. in } \mathbf{R}^3\},$$

where $\sqrt{\rho} := |\psi|$. For any polar factor $\phi \in P(\psi)$, we have $|\phi| = 1$ $\sqrt{\rho}\, dx$ a.e. in $\mathbf{R}^3$ and ϕ is uniquely defined $\sqrt{\rho}\, dx$ a.e. in $\mathbf{R}^3$.

The next Lemma uses the polar factor to define the hydrodynamic quantities in terms of the underlying wave function, in the framework of finite energy states. It shows then how this structure is stable in $H^1(\mathbf{R}^3)$ in a sense which will be specified below. Moreover we see that any current density originated from a wave function in $H^1(\mathbf{R}^3)$ satisfies the generalized irrotationality condition.

Lemma 1. *Let $\psi \in H^1(\mathbf{R}^3)$, $\sqrt{\rho} := |\psi|$ its amplitude and let $\phi \in P(\psi)$ be a polar factor associated to ψ. Then $\sqrt{\rho} \in H^1(\mathbf{R}^3)$ and we have $\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\phi}\nabla\psi)$. Moreover, if we define $\Lambda := \mathrm{Im}(\bar{\phi}\nabla\psi)$, then $\Lambda \in L^2(\mathbf{R}^3)$ and the following identity holds*

$$\mathrm{Re}(\nabla\bar{\psi} \otimes \nabla\psi) = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda, \quad \text{a.e. in } \mathbf{R}^3. \tag{3.1}$$

Furthermore, if $\{\psi_n\} \subset H^1(\mathbf{R}^3)$ is a strongly converging sequence in H^1, say $\psi_n \to \psi$, then we have

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad \text{in } L^2(\mathbf{R}^3),$$

where $\sqrt{\rho_n} := |\psi_n|, \Lambda_n := \mathrm{Im}(\bar{\phi}_n\nabla\psi_n)$, ϕ_n being a unitary factor for ψ_n. Finally the current density

$$J := \mathrm{Im}(\bar{\psi}\nabla\psi) = \sqrt{\rho}\Lambda,$$

satisfies

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda, \quad \text{a.e. in } \mathbf{R}^3.$$

Proof. For the proof, we refer to [1, 2] □

In view of Lemma 1, we may now prove an intermediate result for our analysis of the system (1.12).

Proposition 3. *let v_2 be given by Theorem 1 and which satisfies* (1.14). *Let $\psi_0 \in \Sigma(\mathbf{R}^3)$ and let $\psi \in \mathcal{C}(\mathbf{R}; \Sigma(\mathbf{R}^3))$ be the solution to* (2.1) *with initial datum $\psi(0) = \psi_0$. Then $(\sqrt{\rho}, \Lambda)$, defined by $\sqrt{\rho} = |\psi|$, $\Lambda = \mathrm{Im}(\bar{\phi}\nabla\psi)$, ϕ being a polar factor for ψ, determine a finite energy weak solution to the following hydrodynamic system*

$$\begin{cases} \partial_t \rho + \mathrm{div} J = 0 \\ \partial_t J + \mathrm{div}(\Lambda \otimes \Lambda) + \nabla P_1(\rho) = \dfrac{1}{4}\nabla\Delta\rho - \mathrm{div}(\nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho}) - \rho\mathbb{Q}v_2, \end{cases} \tag{3.2}$$

with initial data $\rho(0) = |\psi_0|^2, J(0) = \mathrm{Im}(\bar{\psi}_0\nabla\psi_0)$. Moreover, the energy

$$E_1(t) = \int \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2 + \frac{1}{\gamma_1}\rho^{\gamma_1}\, dx$$

satisfies the Gronwall-type inequality

$$E_1(t) \le e^{Ct} E_1(0).$$

The last Proposition provides the rigorous correspondence between solutions to the NLS equation (1.16) in the space of energy and finite energy weak solutions to (3.2). It generalises the WKB approach, which is valid only for regular enough solutions and only where the wave function does not vanish.

Proof. Let $\psi \in \mathcal{C}(\mathbf{R}; \Sigma(\mathbf{R}^3))$ be the solution to (1.16) with initial datum $\psi(0) = \psi_0$. By defining the mass density $\rho = |\psi|^2$ it is straightforward to see that is satisfies the continuity equation

$$\partial_t \rho + \mathrm{div} J = 0,$$

where the current density is given by $J = \mathrm{Im}(\bar{\psi}\nabla\psi)$. By differentiating J with respect to time we find, after some calculations,

$$\partial_t J + \mathrm{div}(\mathrm{Re}(\nabla\bar{\psi} \otimes \nabla\psi)) + \rho\nabla(V_\infty + V_p) + \nabla P_1(\rho) = \frac{1}{4}\nabla\Delta\rho.$$

Now remember that $\tilde{V} = V_\infty + V_p$ and that $\nabla\tilde{V} = (-\Delta)^{-1}\nabla\mathrm{div}v_2 = -\mathbb{Q}v_2$. Furthermore by the polar decomposition Lemma we have

$$\mathrm{Re}(\nabla\bar{\psi} \otimes \nabla\psi) = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda,$$

where $\Lambda = \mathrm{Im}(\bar{\phi}\nabla\psi)$, ϕ being a polar factor for ψ.

The above argument is rigorously justified for regular enough ψ. If we want to consider an arbitrary $\psi \in \mathcal{C}(\mathbf{R}; \Sigma(\mathbf{R}^3))$ it suffices to use a density argument, the propagation of regularity for solutions to the NLS equation (2.1) and the stability in H^1 of the polar factorisation. This proves the Proposition. $\square$

We conclude the Section by stating a Lemma which is a direct consequence of the polar factorisation Lemma 1, but it will be useful as a building block for our fractional step and the derivation of a priori estimates for the approximate solutions.

Lemma 2. *Let $\psi \in H^1(\mathbf{R}^3)$ and let $\tau > 0$ be an arbitrary (small) parameter. Then there exists $\tilde{\psi} \in H^1(\mathbf{R}^3)$ such that*

$$\begin{aligned} \sqrt{\tilde{\rho}} =& \sqrt{\rho} \\ \tilde{\Lambda} =& (1-\tau)\Lambda \\ \nabla\tilde{\psi} =& \nabla\psi - i\tau\hat{\phi}\Lambda + R^\tau, \\ \tilde{\psi} =& \psi + r^\tau \end{aligned}$$

where $\sqrt{\rho}, \Lambda, \sqrt{\tilde{\rho}}, \tilde{\Lambda}$ are the hydrodynamic quantities associated to $\psi, \tilde{\psi}$, respectively, and we have

$$\begin{aligned} \|\hat{\phi}\|_{L^\infty} \leq& 1, \\ \|R^\tau\|_{L^2} \lesssim& \tau\|\nabla\psi\|_{L^2}, \\ \|r^\tau\|_{L^2} \lesssim& \tau\|\psi\|_{L^2}. \end{aligned}$$

Proof. Let $\psi \in H^1(\mathbf{R}^3)$ and let us consider a sequence of smooth compactly supported functions which converge to ψ in $H^1(\mathbf{R}^3)$, $\{\psi_n\} \subset \mathcal{C}_0^\infty(\mathbf{R}^3)$, $\|\psi_n - \psi\|_{H^1} \to 0$, as $n \to \infty$. Then for such smooth functions we may write

$$\psi_n(x) = \sqrt{\rho_n}(x)e^{i\theta_n(x)},$$

where $\theta_n : \mathrm{supp}\psi_n \to [0, 2\pi]$ is a piecewise smooth function. For any $n \in \mathbf{N}$ we define

$$\tilde{\psi}_n(x) = \sqrt{\rho_n}(x)e^{i(1-\tau)\theta_n(x)}.$$

Then clearly we have

$$\begin{aligned}\sqrt{\tilde{\rho}_n}(x) =& \sqrt{\rho_n}(x)\\ \tilde{\Lambda}_n(x) =& (1-\tau)\Lambda_n(x),\end{aligned}$$

and

$$\begin{aligned}\nabla\tilde{\psi}_n(x) =& e^{-i\tau\theta_n(x)}\nabla\psi_n(x) - i\tau\tilde{\psi}_n(x)\nabla\theta_n(x)\\ =& \nabla\psi_n(x) - i\tau e^{i(1-\tau)\theta_n(x)}\Lambda_n(x) + \left(e^{-i\tau\theta_n(x)} - 1\right)\nabla\psi_n(x).\end{aligned}$$

For the last term we have

$$e^{-i\tau\theta_n(x)} - 1 = \int_0^t \frac{d}{ds}e^{-is\theta_n(x)}\,ds = -i\theta_n(x)\int_0^\tau e^{-is\theta_n(x)}\,ds$$

and by the properties of θ_n we have

$$\left|e^{-i\tau\theta_n(x)} - 1\right| \leq 2\pi\tau.$$

Now let us recall the polar factorisation Lemma, from $\|\psi_n - \psi\|_{H^1} \to 0$ we infer that $\|\Lambda_n - \Lambda\|_{L^2} \to 0$. This implies that $\{\tilde{\psi}_n\}$ is uniformly bounded in $H^1(\mathbf{R}^3)$, hence $\tilde{\psi}_n \rightharpoonup \tilde{\psi}$ in $H^1(\mathbf{R}^3)$ for some $\tilde{\psi} \in H^1(\mathbf{R}^3)$. Moreover we have

$$\nabla\tilde{\psi} = \nabla\psi - i\tau\hat{\phi}\Lambda + R_\tau,$$

where $\hat{\phi}, R^\tau$ are the weak limits for $e^{i(1-\tau)\theta_n}, (e^{i(1-\tau)\theta_n} - 1)$, respectively, whence $\|\hat{\phi}\|_{L^\infty} \leq 1$ and $\|R^\tau\|_{L^2} \lesssim \tau\|\nabla\psi\|_{L^2}$. In the same way we have

$$\tilde{\psi}_n(x) = \psi_n(x) - i\tau\theta_n(x)\int_0^\tau e^{-is\theta_n(x)}\,ds\psi_n(x).$$

By passing to the limit we obtain the analogue inequality for $\tilde{\psi}$. The Lemma is thus proved. □

The last Proposition 3 is an intermediate step for our problem (1.12), since it shows the existence of finite energy solutions for the hydrodynamic system (1.12) without the collision term $-J$ in the current density. As we already pointed out we are not able to treat this term at the level of wave function dynamics. Thus to overcome this difficulty we need to use a fractional step argument which allows us to define a sequence of approximate solutions. Then by the compactness properties of approximate solutions we will show that they converge to a finite energy weak solution to (1.12). In the next section we will set up the fractional step argument and derive some properties for the approximate solutions.

4. The Fractional Step

After having recalled the polar factorisation technique and the correspondence between finite energy solutions to (2.1) and finite energy weak solutions to (3.2), we now turn our attention to the QHD system (1.12). In this Section we are going to set up the fractional step argument we need in order to define a sequence of approximate solutions. We will split the evolution into two parts: the first one which will be solved by means of the NLS equation (2.1) and the second one which will take into account the collision term $-J$ in the superfluid current density. To implement this idea we proceed as follows: having fixed a small time step $\tau > 0$, we divide the positive time semi-axis into subintervals $[k\tau, (k+1)\tau)$, $k = 0, 1, \dots$. Then on each subinterval we solve the QHD system (3.2) and at the end of those subintervals we update the quantities in order to take into account the collision term.

More precisely, let $\psi_0 \in \Sigma(\mathbf{R}^3)$, we then consider the solution ψ to (2.1) with initial datum $\psi(0) = \psi_0$ in the spacetime slab $[0, \tau) \times \mathbf{R}^3$. By Proposition 3 we know that $(\sqrt{\rho}, \Lambda) = (|\psi|, \mathrm{Im}(\bar{\phi}\nabla\psi))$ determine a weak solution to (3.2) in $[0, \tau) \times \mathbf{R}^3$. Now we need to update the quantities in order to take into account the collisions, i.e. we want

$$\begin{cases} \rho(\tau+) = \rho(\tau-) \\ J(\tau+) = (1-\tau)J(\tau-). \end{cases} \tag{4.1}$$

On the other hand, when we start again in $[\tau, 2\tau)\times\mathbf{R}^3$ we would then need an initial datum for (2.1), thus we need to translate (4.1) at a wave function level. For this purpose we are going to use Lemma 2.

Thus, assume we already constructed our approximate solution in the space-times slab $[(k-1)\tau, k\tau)\times\mathbf{R}^3$. We then apply Lemma 2 with $\psi = \psi(k\tau-)$ and we shall define $\psi(k\tau+)=\tilde{\psi}$. In this way by the statement of the Lemma, we have that the hydrodynamic quantities satisfy (4.1), and we can start again with (2.1) on the spacetime slab $[k\tau,(k+1)\tau)\times\mathbf{R}^3$ by considering $\psi(k\tau+)$ as initial datum.

In this way, for any fixed $\tau>0$ small enough we define ψ^τ on the whole $[0,\infty)\times\mathbf{R}^3$. We may now define our aproximate solutions through the hydrodynamic quantities $\sqrt{\rho^\tau}=|\psi^\tau|, \Lambda^\tau=\mathrm{Im}(\bar{\phi}^\tau\nabla\psi^\tau)$.

First of all, we show that $(\sqrt{\rho^\tau},\Lambda^\tau)$ is indeed a sequence of approximate solutions for (1.12).

Lemma 3. *Let $(\sqrt{\rho^\tau},\Lambda^\tau)$ be given by the construction above, then for any $\eta,\zeta\in\mathcal{C}_0^\infty([0,\infty)\times\mathbf{R}^3)$ we have*

$$
\begin{aligned}
&\int_{\mathbf{R}^3}\rho_{1,0}(x)\eta(0,x)\,dx+\int_0^\infty\int_{\mathbf{R}^3}\rho^\tau\partial_t\eta+J^\tau\cdot\nabla\eta\,dxdt=0\\
&\int_{\mathbf{R}^3}J_{1,0}(x)\cdot\zeta(0,x)\,dx+\int_0^\infty\int_{\mathbf{R}^3}J^\tau\cdot\partial_t\zeta+(\Lambda^\tau\otimes\Lambda^\tau+\nabla\sqrt{\rho^\tau}\otimes\nabla\sqrt{\rho^\tau}):\nabla\zeta\\
&\qquad\qquad+P_1(\rho^\tau)\mathrm{div}\zeta\quad-\frac{1}{4}\rho^\tau\Delta\mathrm{div}\zeta\\
&\qquad\qquad-(J^\tau-\rho^\tau\mathbb{Q}v_2)\cdot\zeta\,dxdt=o(1),\ \textit{as}\ \tau\to0,
\end{aligned}
\tag{4.2}
$$

where $\rho_{1,0}=|\psi_0|^2, J_{1,0}=\mathrm{Im}(\bar{\psi}_0\nabla\psi_0)$

The proof is straightforward and a simple consequence of the construction above and it will be left to the reader. As a consequence from this Lemma we obtain a consistence property for the approximate solutions.

Lemma 4. *Let $(\sqrt{\rho^\tau},\Lambda^\tau)$ be the sequence of approximate solutions constructed above. If we have*

$$
\begin{aligned}
\sqrt{\rho^\tau}\to&\sqrt{\rho},\quad \textit{in}\ L^2([0,T];H^1_{loc}(\mathbf{R}^3))\\
\Lambda^\tau\to&\Lambda,\quad \textit{in}\ L^2([0,T];L^2_{loc}(\mathbf{R}^3)),
\end{aligned}
$$

for some $\sqrt{\rho} \in L^2([0,T]; H^1_{loc}(\mathbf{R}^3))$, $\Lambda \in L^2([0,T]; L^2_{loc}(\mathbf{R}^3))$, *then* $(\sqrt{\rho}, \Lambda)$ *is a finite energy weak solution to* (1.12).

It thus remains to prove that the sequence of approximate solutions indeed converge strongly in the spaces specified in the Lemma above, that is we need to derive some compactness properties for the approximate solutions.

5. Compactness for the Sequence of Approximate Solutions

In this Section we are going to derive some a priori estimates for the sequence of approximate solutions we constructed in the previous Section. More precisely we are going to show some compactness estimates for the sequence $\{\psi^\tau\}_{\tau>0}$, which will then yield the boundedness for $(\sqrt{\rho^\tau}, \Lambda^\tau)$ in the right spaces. The first estimates we are going to prove is a uniform estimate for $\{\psi^\tau\}$ in the space of energy.

Formally for a solution to (1.12) we would have

$$E_1(t) - \int_0^t \int |\Lambda|^2 \, dx dt' + \int_0^t \int J \cdot \mathbb{Q} v_2 \, dx dt' = E_1(0).$$

Now recall that by Theorem 1 we have $v_2 \in L^2_t \dot{H}^1_x([0,T] \times \mathbf{R}^3)$ for any $0 < T < \infty$. Then by using Hölder's inequality and Sobolev embedding we have

$$\begin{aligned}
\int_0^t \int J \cdot \mathbb{Q} v_2 \, dx dt' \leq & C \|\sqrt{\rho}\|_{L^\infty_t L^3_x} \|\Lambda\|_{L^2_{t,x}} \|v_2\|_{L^2_t \dot{H}^1_x} \\
\leq & C(\varepsilon) \|\sqrt{\rho}\|_{L^\infty_t L^2_x} \|\nabla \sqrt{\rho}\|_{L^\infty_t L^2_x} \|v_2\|^2_{L^2_t \dot{H}^1_x} + \varepsilon \|\Lambda\|^2_{L^2_{t,x}} \\
\leq & C(\varepsilon) \|\rho_{1,0}\|_{L^1} \|v_2\|^4_{L^2_t \dot{H}^1_x} + \varepsilon \|\nabla \sqrt{\rho}\|^2_{L^\infty_t L^2} + \varepsilon \|\Lambda\|^2_{L^2_{t,x}}.
\end{aligned}$$

By resuming we have

$$E_1(t) - (1-\varepsilon) \int_0^t \int |\Lambda|^2 dx dt' \leq E_1(0) + \varepsilon \|\nabla \sqrt{\rho}\|^2_{L^\infty_t L^2} + C(\varepsilon) \|\rho_{1,0}\|_{L^1} \|v_2\|^4_{L^2_t \dot{H}^1_x}.$$

Now, let us take the supremum in time on the left hand side. For $\varepsilon > 0$ small enough we then get

$$\sup_t \left(E_1(t) - c \int_0^t \int |\Lambda|^2 \, dx dt' \right) \lesssim E_1(0) + \|\rho_{1,0}\|_{L^1} \|v_2\|^4_{L^2_t \dot{H}^1_x([0,t] \times \mathbf{R}^3)},$$

which in particular implies the following inequality

$$E_1(t) - c\int_0^t\int |\Lambda|^2\,dxdt' \lesssim E_1(0) + \|\rho_{1,0}\|_{L^1}\|v_2\|^4_{L^2_t\dot{H}^1_x([0,T]\times\mathbf{R}^3)}, \tag{5.1}$$

for $t\in[0,T]$. Our aim is to give a discrete version of (5.1) which holds for approximate solutions

Lemma 5. *Let $\tau>0$ be sufficiently small. Then*

$$E_1^\tau(t) + c\tau\sum_{k=1}^{[t/\tau]}\int |\Lambda^\tau(k\tau-)|^2\,dx \lesssim E_1(0) + \|\rho_{1,0}\|_{L^1}\|v_2\|^4_{L^2_t\dot{H}^1_x} \tag{5.2}$$

Proof. First of all we see that because of the updating procedure in the previous Section, we have

$$E_1^\tau(k\tau+) - E_1^\tau(k\tau-) = -\left(\tau - \frac{\tau^2}{2}\right)\int |\Lambda^\tau(k\tau-)|^2\,dx.$$

Hence we have

$$E_1^\tau(t) = -\left(\tau-\frac{\tau^2}{2}\right)\sum_{k=1}^{[t/\tau]}\int |\Lambda^\tau(k\tau-)|^2\,dx + \int_0^t\int J^\tau\cdot\mathbb{Q}v_2\,dxdt' + E_1(0).$$

Now we argue as before to estimate

$$\int_0^t\int J^\tau\cdot\mathbb{Q}v_2\,dxdt' \le C(\varepsilon)\|\rho_{1,0}\|_{L^1}\|v_2\|^4_{L^2_t\dot{H}^1_x} + \varepsilon\|\nabla\sqrt{\rho^\tau}\|^2_{L^\infty_t L^2_x} + \varepsilon\|\Lambda^\tau\|^2_{L^2_tL^2_x}.$$

For $\tau>0$ sufficiently small and by the continuity of $\|\Lambda^\tau(t)\|_{L^2}$ on each of the subintervals $[k\tau,(k+1)\tau)$ we have

$$\|\Lambda^\tau\|_{L^2_{t,x}} = \tau\sum_{k=1}^N\int |\Lambda^\tau|^2\,dx + \int_{N\tau}^t\int |\Lambda^\tau|^2\,dxdt' + o(1),$$

as $\tau\to 0$, where $N=[t/\tau]$. By resuming we have

$$\begin{aligned}E_1^\tau(t) &+ \tau\left(1-\frac{\tau}{2}-\varepsilon\right)\sum_{k=1}^{[t/\tau]}\int|\Lambda^\tau(k\tau-)|^2\,dx\\ &\le C(\varepsilon)\|\rho_{1,0}\|_{L^1}\|v_2\|^4_{L^2_t\dot{H}^1_x} + \varepsilon\left(\|\nabla\sqrt{\rho^\tau}\|_{L^\infty_tL^2_x} + \|\Lambda^\tau\|_{L^\infty_tL^2_x}\right) + E_1(0).\end{aligned}$$

Now we use the same trick as before to incorporate $\|\nabla\sqrt{\rho^\tau}\|^2_{L^\infty_t L^2}+\|\Lambda^\tau\|^2_{L^\infty_t L^2}$ in the left hand side and get

$$\sup_t\left(E_1^\tau(t)+c\tau\sum_{k=1}^{[t/\tau]}\int|\Lambda^\tau(k\tau-)|^2\,dx\right)\lesssim E_1(0)+\|\rho_{1,0}\|_{L^1}\|v_2\|^4_{L^2_t\dot{H}^1_x},$$

which proves the desired bound. □

The Lemma above implies the sequence of approximate solutions $\{\psi^\tau\}_{\tau>0}$ is uniformly bounded in $L^\infty_t H^1_x([0,T]\times\mathbf{R}^3)$ for any $0<T<\infty$. We can thus infer there exists a weak$-\star$ limit for the sequence in $L^\infty_t H^1_x$,

$$\psi^\tau\rightharpoonup\psi,\quad \star-L^\infty_t H^1_x([0,T]\times\mathbf{R}^3).$$

Unfortunately the energy estimate is not sufficient to pass to the limit the quadratic terms in (4.2). We thus need to derive further compactness properties for $\{\psi^\tau\}$. For this purpose we are going to fully exploit the dispersive properties inherited by the nonlinear Schrödinger equation we used in the fractional step. Recall the semigroup $U(t,s)$ defined by (2.3). We are going to write a formula for ψ^τ in terms of this linear evolution operator.

Lemma 6. *We have*

$$\begin{aligned}\psi^\tau(t) &= U(t,0)\psi_0-i\int_0^t U(t,s)\left(V_p\psi^\tau+|\psi^\tau|^{2(\gamma_1-1)}\psi^\tau\right)(s)\,ds\\ &\quad+\sum_{k=1}^{[t/\tau]}U(t,k\tau)r_k^\tau\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}\nabla\psi^\tau(t) &= U(t,0)\nabla\psi_0-i\int_0^t U(t,s)\nabla\left(V_p\psi^\tau+|\psi^\tau|^{2(\gamma_1-1)}\psi^\tau\right)(s)\,ds\\ &\quad-i\int_0^t U(t,s)\left(\nabla V_\infty\psi^\tau\right)(s)\,ds\\ &\quad+\sum_{k=1}^{[t/\tau]}U(t,k\tau)\left[-i\tau\hat{\phi}_k^\tau\Lambda^\tau(k\tau-)+R_k^\tau\right].\end{aligned}\tag{5.4}$$

Proof. We will prove (5.4), formula (5.3) can be proved in a similar way. Let $t\in[N\tau(N+1)\tau)$, then since ψ^τ is a solution to (2.1) on $[N\tau,(N+1)\tau)$, we have

$$\nabla\psi^\tau(t) = U(t,N\tau)\nabla\psi^\tau(N\tau+)-i\int_{N\tau}^t U(t,s)\nabla\left(V_p\psi^\tau+|\psi^\tau|^{2(\gamma_1-1)}\psi^\tau\right)(s)ds$$
$$-i\int_{N\tau}^t U(t,s)\left(\nabla V_\infty\psi^\tau\right)(s)ds.$$

The last term in the formula is due to the commutator between the Schrödinger operator $-\frac{1}{2}\Delta+V_\infty$ and the gradient. By Lemma 2 we have

$$\nabla\psi^\tau(k\tau+)=\nabla\psi^\tau(k\tau-)-i\tau\hat{\phi}_k^\tau\Lambda^\tau(k\tau-)+R_k^\tau$$

for any k. By plugging it into the formula for $\nabla\psi^\tau$ and by interating the argument we then find (5.4). $\square$

Now we may use Strichartz estimates on (5.3) and (5.4) to obtain a priori bounds for $\{\psi^\tau\}$ in the Strichartz spaces.

Proposition 4. *For any* $0<T<\infty$ *and for any admissible pair* (q,r) *we have*

$$\|\nabla\psi^\tau\|_{L_t^qW_x^{1,r}([0,T]\times\mathbf{R}^3)}\leq C(\|\psi_0\|_\Sigma,T,\|v_2\|_{L_t^2\dot{H}_x^1}),\tag{5.5}$$

uniformly in $\tau>0$.

The proof of the above Proposition is similar to the proof of existence of local solutions for NLS equations. Let us define the Strichartz norm

$$\|f\|_{S^1([0,T])}=\sup_{(q,r)}\|f\|_{L_t^qW_x^{1,r}([0,T]\times\mathbf{R}^3)},$$

where the sup is taken over all admissible pairs (q,r). Let us first consider a small time interval $[0,T_1]$, by using formulas (5.3), (5.4) and Strichartz estimates we obtain

$$\|\psi^\tau\|_{S^1([0,T_1])}\lesssim\|\psi_0\|_\Sigma+T^\alpha\Big(\|V_p\|_{L_t^\infty W_x^{1,6}}\|\psi^\tau\|_{S^1}+\|\psi^\tau\|_{S^1}^{2\gamma_1-1}$$
$$+\|\nabla V_\infty\|_{L_t^\infty L_x^6}\|\psi^\tau\|_{S^1}+\|\psi^\tau\|_{S^1}\Big),$$

for some positive $\alpha>0$. If T_1 is sufficiently small depending on $\|\psi_0\|_\Sigma$, $\|V_p\|_{L_t^\infty W_x^{1,6}}$, $\|\nabla V_\infty\|_{L_t^\infty L_x^6}$, then we have

$$\|\psi^\tau\|_{S^1([0,T_1])}\lesssim\|\psi_0\|_\Sigma.$$

Now we may divide any bounded time interval $[0,T]$ into many small subintervals over we can argue as before. The details are left to the reader and we refer to [1] for a similar calculation.

Now we can use the above Strichartz estimates to obtain further regularity properties for the sequence of approximate solutions. Indeed, it is a common feature of dispersive equations that solutions are more regular than their initial data, see for example [12], [7]. Here we will use a result by Yajima [26] which shows that $U(t,s)$ has the same local smoothing property which holds for the free Schrödinger propagator, at the price of having a constant depending on the length of the considered time interval. The results in [26] hold for a more general class of Hamiltonian, anyway here we will state the result only for the case of our interest.

Theorem 4. *For any* $0 < T < \infty$

$$\|U(t,0)f\|_{L^2([0,T];H^{1/2}_{loc}(\mathbf{R}^3))} + \|\int_0^t U(t,s)F(s)\,ds\|_{L^2([0,T];H^{1/2}_{loc}(\mathbf{R}^3))} \leq C(T)\left(\|f\|_{L^2} + \|F\|_{L^1_tL^2}\right).$$

Now we can use the above Theorem, combined with the Strichartz bounds (5.5) and formulas (5.3), (5.4) to infer that

$$\|\psi^\tau\|_{L^2([0,T];H^{\frac{3}{2}}_{loc}(\mathbf{R}^3))} \leq C(T, \|\psi_0\|_\Sigma, \|v_2\|_{L^2_t\dot{H}^1_x}),$$

for any $0 < T < \infty$. This inequality is proved in the same way as the Strichartz bounds (5.5), i.e. by using a bootstrap argument. Finally we have the sufficient compactness for the approximate solutions to yield the convergence to a finite energy weak solution to (1.12). More precisely, we may prove by an Aubin-Lions type lemma that, up to passing to subsequences, $\{\psi^\tau\}$ converges strongly. We will use the following result by Rakotoson and Temam [24].

Theorem 5. *Let* $(V, \|\cdot\|_V)$, $(H; \|\cdot\|_H)$ *be two separable Hilbert spaces. Assume that* $V \subset H$ *with a compact and dense embedding. Consider a sequence* $\{u^\varepsilon\}$, *converging weakly to a function* u *in* $L^2([0,T];V)$, $T < \infty$. *Then* u^ε *converges strongly to* u *in* $L^2([0,T];H)$, *if and only if*

1. $u^\varepsilon(t)$ *converges to* $u(t)$ *weakly in* H *for a.e.* t*;*

2. $\lim_{|E|\to 0, E\subset[0,T]} \sup_{\varepsilon>0} \int_E \|u^\varepsilon(t)\|_H^2\,dt = 0$.

We can apply now this Theorem to our sequence by choosing $V = H^{3/2}_{loc}(\mathbf{R}^3)$, $H = H^1_{loc}(\mathbf{R}^3)$. Indeed by (5.2) we know that $\psi^\tau(t)$ converges weakly to $\psi(t)$ for almost every $t \in [0,T]$, so that the first condition is satisfied. On the other hand, the second one is a consequence of the bound $\|\psi^\tau\|_{L^2([0,T];H^{3/2}_{loc}(\mathbf{R}^3)}$.

Resuming, we may prove that, up to passing to subsequences

$$\psi^\tau \to \psi, \quad \text{in } L^2([0,T];H^1_{loc}(\mathbf{R}^3)),$$

for any $0 < T < \infty$. This in particular implies that

$$\begin{aligned} \sqrt{\rho^\tau} \to \sqrt{\rho} &\quad \text{in } L^2([0,T];H^1_{loc}(\mathbf{R}^3)) \\ \Lambda^\tau \to \Lambda &\quad \text{in } L^2([0,T];L^2_{loc}(\mathbf{R}^3)). \end{aligned}$$

By Lemma 4 we then infer that $(\sqrt{\rho}, \Lambda)$ is a finite energy weak solution to the QHD system (1.12) and thus Theorem 3 is proved.

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