Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **10** (2015), No. 3, pp. 311-347

THE VLASOV-POISSON-LANDAU SYSTEM WITH A UNIFORM IONIC BACKGROUND AND ALGEBRAIC DECAY INITIAL PERTURBATION

YUANJIE LEI[1,a], LING WAN[2,b] AND HUIJIANG ZHAO[3,c]

Dedicated to Professor Tai-Ping Liu on the occasion of his 70th birthday

[1]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
[a]E-mail: leiyuanjie@whu.edu.cn
[2]Computational Science Hubei Key Laboratory, Wuhan University, Wuhan 430072, China.
[b]E-mail: ling.wan@whu.edu.cn
[3]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China.
[c]E-mail: hhjjzhao@hotmail.com

Abstract

In the absence of magnetic effects, the dynamics of two-species charged dilute particles (e.g., electrons and ions) interacting with their self-consistent electrostatic field as well as their grazing collisions is described by the two-species Vlasov-Poisson-Landau system, while the one-species Vlasov-Poisson-Landau system models the time evolution of dilute charged particles consisting of electrons interacting through its binary grazing collisions under the influence of the self-consistent internally generated electrostatic forces with a fixed ionic background. To construct global smooth solutions of the two-species Vlasov-Poisson-Landau system near Maxwellians, a time-velocity weighted energy method is developed by Guo in [Guo Y., *J. Amer. Math. Soc.* **25** (2012), 759–812] which yields a satisfactory well-posedness theory for the two-species Vlasov-Poisson-Landau system with algebraic decay initial perturbation in the perturbative context. It is worth emphasizing that such a time-velocity weighted energy method relies heavily on the fact that the potential of the electrostatic field decays sufficiently fast. The main purpose of this paper is to show that, for the one-species Vlasov-Poisson-Landau system, although the temporal decay of the electric potential is worse than that of the two-species Vlasov-Poisson-Landau system, the method developed in [Guo Y., *J. Amer. Math. Soc.* **25** (2012), 759–812] can still be adapted provided that the initial perturbation satisfies the neutral condition.

Received March 19, 2015 and in revised form May 18, 2015.

AMS Subject Classification: 35Q83 (35A01; 35A02; 35B40).

Key words and phrases: One-species Vlasov-Poisson-Landau system, global solutions near Maxwellians, algebraic decay initial perturbation, neutral condition.

1. Introduction and Main Results

In the absence of magnetic effects, the dynamics of two-species charged dilute particles (e.g., electrons and ions) interacting with their self-consistent electrostatic field as well as their grazing collisions is described by the two-species Vlasov-Poisson-Landau (called VPL in the sequel for simplicity of presentation) system (cf. [9, 15, 18]):

$$\begin{aligned} &\partial_t F_+ + v\cdot\nabla_x F_+ + E\cdot\nabla_v F_+ = Q(F_+,F_+)+Q(F_+,F_-),\\ &\partial_t F_- + v\cdot\nabla_x F_- - E\cdot\nabla_v F_- = Q(F_-,F_+)+Q(F_-,F_-). \end{aligned} \tag{1.1}$$

Here $F_\pm(t,x,v)\geq 0$ are the number density functions for the ions (+) and electrons (−), respectively, at time $t\geq 0$, position $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. The collision between charged particles is given by

$$\begin{aligned} &Q\left(G_\pm,G_\mp\right)\\ &= \nabla_v\cdot\int_{\mathbb{R}^3_v}\Phi(v-v')\Big\{G_\pm(v')\nabla_v G_\mp(v)-\nabla_{v'}G_\pm(v')G_\mp(v)\Big\}dv'\\ &= \sum_{i,j=1}^3\partial_i\int_{\mathbb{R}^3_v}\Phi^{ij}(v-v')\Big\{G_\pm(v')\partial_j G_\mp(v)-\partial_j G_\pm(v')G_\mp(v)\Big\}dv', \end{aligned} \tag{1.2}$$

where $\partial_i=\partial_{v_i}$ and $\Phi(v)=\big(\Phi^{ij}(v)\big)_{3\times 3}$ is the famous Landau (Fokker-Planck) kernel (cf. [1, 2, 3, 6, 7, 9, 10, 11, 13, 15, 16, 18]):

$$\Phi^{ij}(v)=\left(\delta_{ij}-\frac{v_iv_j}{|v|^2}\right)|v|^{\gamma+2},\quad \gamma\geq -3 \tag{1.3}$$

and the case of $\gamma=-3$ corresponds to the Coulomb potential (cf. [2, 17]).

The self-consistent electric field $E(t,x)=-\nabla_x\phi$ and the electric potential $\phi(t,x)$ will then satisfy the Poisson equation

$$\nabla_x\cdot E=-\triangle_x\phi=\int_{\mathbb{R}^3_v}(F_+-F_-)\,dv,\qquad \lim_{|x|\to+\infty}\phi(t,x)=0. \tag{1.4}$$

Here, without loss of generality, all the physical constants such as the magnitudes of charge and mass of the two-species charged particles and all the generic constants such as 4π in the Poisson equation (1.4), etc. evolved are normalized to be unit for notational simplicity throughout this manuscript.

In physical situations the ion mass is usually much larger than the electron mass so that the electrons move much faster than the ions. Thus, the ions are often described by a fixed ion background $n_b(x)$ and only the electrons move. For such a case, the dynamics of the electrons interacting with its self-consistent electrostatic field as well as its grazing collisions with a fixed background of ions can be described by the following one-species VPL system

$$\partial_t F + v\cdot\nabla_x F + \nabla_x\phi\cdot\nabla_v F = Q(F,F), \tag{1.5}$$

$$\Delta_x\phi = \int_{\mathbb{R}^3_v} F\,dv - n_b(x), \qquad \lim_{|x|\to+\infty}\phi(t,x)=0. \tag{1.6}$$

Here the unknown $F=F(t,x,v)\geq 0$ is the density distribution function of electrons located at $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ with velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$ at time $t\geq 0$. The potential function $\phi=\phi(t,x)$ generating the self-consistent electric field $E(t,x)\equiv-\nabla_x\phi(t,x)$ in (1.5) is coupled with $F(t,x,v)$ through the Poisson equation (1.6) where $n_b(x)$ is the density of the ionic background. Throughout this manuscript, $n_b(x)$ is assumed to be a positive constant, which means that the density of the ionic background is spatially uniform, and without loss of generality, we can further assume that $n_b(x)=1$.

This paper is concerned with the global solvability of the one-species VPL system (1.5), (1.6) around the following normalized global Maxwellian

$$\mu=(2\pi)^{-\frac{3}{2}}\exp\left(-\frac{|v|^2}{2}\right)$$

in the whole space $\mathbb{R}^+_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v$ with prescribed initial data

$$F(0,x,v)=F_0(x,v). \tag{1.7}$$

For this purpose, we define the perturbation $f(t,x,v)$ by

$$F(t,x,v)=\mu+\mu^{\frac{1}{2}}f(t,x,v),$$

then, such a Cauchy problem (1.5), (1.6), (1.7) is reformulated as

$$\begin{aligned}
&\partial_t f + v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f - \frac{1}{2}v\cdot\nabla_x\phi f - \nabla_x\phi\cdot v\mu^{\frac12} + \mathbf{L}f = \mathbf{\Gamma}(f,f),\\
&\triangle_x\phi = \int_{\mathbb{R}^3_v}\mu^{\frac12} f\,dv, \qquad \lim_{|x|\to+\infty}\phi(t,x)=0, \qquad\qquad (1.8)\\
&f(0,x,v) = f_0(x,v) = \mu^{-\frac12}\left(F_0(x,v)-\mu(v)\right).
\end{aligned}$$

Here the linearized Landau collision operator $\mathbf{L}f$ and the nonlinear collision term $\mathbf{\Gamma}(f,f)$ are defined by

$$\mathbf{L}f \ = \ -\mu^{-1/2}\left\{Q\left(\mu,\mu^{\frac12}f\right)+Q\left(\mu^{\frac12}f,\mu\right)\right\}$$

and

$$\mathbf{\Gamma}(f,f) \ = \ \mu^{-\frac12}Q\left(\mu^{\frac12}f,\mu^{\frac12}f\right),$$

respectively. Recalling (1.8), we can rewrite $\phi(t,x)$ in terms of $f(t,x,v)$ as

$$\phi(t,x) = -\frac{1}{4\pi|x|} *_x \int_{\mathbb{R}^3_v}\mu^{\frac12}(v)f(t,x,v)\,dv.$$

Here $*_x$ denotes the convolution with respect to the x variable.

For the linearized Landau collision operator $\mathbf{L}$, it is well known [7] that it is non-negative and the null space $\mathcal{N}$ of $\mathbf{L}$ is given by

$$\mathcal{N} = \operatorname{Span}\left\{\mu^{\frac12},\ v_i\mu^{\frac12}(1\le i\le 3), \left(|v|^2-3\right)\mu^{\frac12}\right\}.$$

If we define $\mathbf{P}$ as the orthogonal projection from $L^2(\mathbb{R}^3_v)$ to $\mathcal{N}$, then for any given function $f(t,x,v)\in L^2(\mathbb{R}^3_v)$, one has

$$\begin{aligned}
\mathbf{P}f \ &= \ a(t,x)\mu^{\frac12} + b(t,x)\cdot v\mu^{\frac12} + c(t,x)\left(|v|^2-3\right)\mu^{\frac12},\\
a \ &= \ \int_{\mathbb{R}^3_v}\mu^{\frac12} f\,dv = \int_{\mathbb{R}^3_v}\mu^{\frac12}\mathbf{P}f dv \equiv \mathbf{P}_0 f,\\
b_i \ &= \ \int_{\mathbb{R}^3_v} v_i\mu^{\frac12} f\,dv = \int_{\mathbb{R}^3_v} v_i\mu^{\frac12}\mathbf{P}f\,dv, \quad i=1,2,3,\\
c \ &= \ \frac16\int_{\mathbb{R}^3_v}\left(|v|^2-3\right)\mu^{\frac12} f dv = \frac16\int_{\mathbb{R}^3_v}\left(|v|^2-3\right)\mu^{\frac12}\mathbf{P}f dv.
\end{aligned}$$

Therefore, we have the following macro-micro decomposition with respect to the given global Maxwellian μ

$$f(t,x,v) = \mathbf{P}f(t,x,v) + \{\mathbf{I}-\mathbf{P}\}f(t,x,v). \tag{1.9}$$

Here $\mathbf{I}$ denotes the identity operator. $\mathbf{P}f$ and $\{\mathbf{I}-\mathbf{P}\}f$ are called the macroscopic and the microscopic component of $f(t,x,v)$, respectively. For the corresponding macro-microscopic decomposition for the Boltzmann equation, see [8, 14].

Before going on, we first list some basic notations used throughout this paper:

- C denotes some positive constant (generally large) and λ, ϵ, κ, and δ stand for some positive constant (generally small). Note that all these constants may take different values in different places;
- $A \lesssim B$ means that there is a generic constant $C > 0$ such that $A \leq CB$. $A \sim B$ means $A \lesssim B$ and $B \lesssim A$. $A \gtrsim B$ can be defined similarly;
- The multi-indices $\alpha = [\alpha_1, \alpha_2, \alpha_3]$ and $\beta = [\beta_1, \beta_2, \beta_3]$ will be used to record spatial and velocity derivatives, respectively. And $\partial^\alpha_\beta = \partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}$. Similarly, the notation ∂^α will be used when $\beta = 0$ and likewise for ∂_β. The length of α is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$. $\alpha' \leq \alpha$ means that no component of α' is greater than the corresponding component of α, and $\alpha' < \alpha$ means that $\alpha' \leq \alpha$ and $|\alpha'| < |\alpha|$. And it is convenient to write
$$\left|\nabla^k_x f\right| \equiv \sqrt{\sum_{|\alpha|=k} |\partial^\alpha f|^2};$$
- $\langle\cdot,\cdot\rangle$ is used to denote the L^2_v inner product in $\mathbb{R}^3_v$, with the L^2 norm $|\cdot|_{L^2_v}$, while $(\cdot,\cdot)$ denotes the L^2 inner product either in $\mathbb{R}^3_x \times \mathbb{R}^3_v$ or in $\mathbb{R}^3_x$ with the L^2 norm $\|\cdot\|$;
- For $p \geq 1, q \geq 1$, we also define the mixed velocity-space Lebesgue space $L^p_v L^q_x = L^p(\mathbb{R}^3_v; L^q(\mathbb{R}^3_x))$ with the norm
$$\|f\|_{L^p_v L^q_x} = \left(\int_{\mathbb{R}^3_v}\left(\int_{\mathbb{R}^3_x}|f(x,v)|^q dx\right)^{\frac{p}{q}} dv\right)^{\frac{1}{p}}$$
for $f = f(x,v) \in L^p_v L^q_x$ and when $p = 2$, we will use Z_q to denote $L^2_v L^q_x$. For $p \geq 1$, $q \geq 1$, $\ell \in \mathbb{Z}^+$, $L^p_x L^q_v$, $L^p_v H^\ell_x$, etc. can be defined similarly;

- For some time and velocity dependent weight function $w(t,v)\gtrsim 1$, as in [7], we define the the weighted norms $|\cdot|_{\sigma,w}$ and $\|\cdot\|_{\sigma,w}$ as

$$
\begin{aligned}
|f|_{\sigma,w} \sim\ & \left|w(t,v)\langle v\rangle^{\frac{\gamma+2}{2}} f\right|_{L^2_v}+\left|w(t,v)\langle v\rangle^{\frac{\gamma}{2}}\nabla_v f\cdot\frac{v}{|v|}\right|_{L^2_v}\\
&+\left|w(t,v)\langle v\rangle^{\frac{\gamma+2}{2}}\nabla_v f\times\frac{v}{|v|}\right|_{L^2_v}
\end{aligned}
\tag{1.10}
$$

and $\|f\|_{\sigma,w}=\big\||f|_{\sigma,w}\big\|$. Moreover, $|f|_\sigma\equiv|f|_{\sigma,1}$ and $\|f\|_\sigma\equiv\|f\|_{\sigma,1}$;
- For $p\in[1,+\infty]$ and some time and velocity dependent weight function $w(t,v)\gtrsim 1$, $L^p(\mathbb{R}^3_v)$ denotes the usual Lebesgue space in $\mathbb{R}^3_v$ with the usual norm $|\cdot|_{L^p_v}$, $L^p_w(\mathbb{R}^3_v)$ stands for the weighted Lebesgue space in $\mathbb{R}^3_v$ with norm $|f|_{p,w}=|wf|_{L^p_v}$ for $f(v)\in L^p_w(\mathbb{R}^3_v)$. Similarly, $L^p(\mathbb{R}^3_v\times\mathbb{R}^3_x)$ (or $L^p(\mathbb{R}^3_x)$) in $\mathbb{R}^3_v\times\mathbb{R}^3_x$ (or $\mathbb{R}^3_x$) and $L^p_w(\mathbb{R}^3_v\times\mathbb{R}^3_x)$ (or $L^p_w(\mathbb{R}^3_x)$) in $\mathbb{R}^3_v\times\mathbb{R}^3_x$ (or $\mathbb{R}^3_x$) can be defined similarly with the corresponding norms $\|\cdot\|_{L^p_{x,v}}$ (or $\|\cdot\|_{L^p_x}$) and $\|\cdot\|_{p,w}$, respectively. It is easy to see that $|\cdot|_{2,1}=|\cdot|_{L^2_v}$, $\|\cdot\|_{2,1}=\|\cdot\|_{L^2_{x,v}}=\|\cdot\|$, and $\|\cdot\|_{2,1}=\|\cdot\|_{L^2_x}=\|\cdot\|$.

With the above notations in hand, it is well known, cf. [7], that the linear operator $\mathbf{L}\geq 0$ is locally coercive in the sense that

$$
\langle \mathbf{L}f,f\rangle\gtrsim|\{\mathbf{I}-\mathbf{P}\}f|^2_\sigma. \tag{1.11}
$$

As pointed out in [9], the main difficulty for the construction of solutions, either local or global, to the Cauchy problem of both the two-species VPL system (1.1), (1.4) and the one-species VPL system (1.5), (1.6) for the case of $-3\leq\gamma<-2$ is due to the degeneration of coercive estimate (1.11) at large velocity for the linearized Landau collision operator $\mathbf{L}$ and the velocity-growth of the nonlinear term $-\frac{1}{2}v\cdot\nabla_x\phi f$ with the first order velocity-growth rate induced by the electrostatic force. To overcome such a difficulty, a time-velocity weighted energy method is introduced in [9] for the two-species VPL system (1.1), (1.4) in a periodic box to capture the dissipation for controlling the velocity growth in the nonlinear term for $-3\leq\gamma<-1$ and to overcome the large-velocity degeneracy in the energy dissipation for $-3\leq\gamma<-2$. The main ideas developed in [9] can be outlined as in the following:

- An exponential weight of electric potential $e^{\mp\phi}$ is introduced to cancel the growth of the velocity in the nonlinear term $\mp\frac{1}{2}\nabla_x\phi\cdot vf_\pm$, which is mainly due to the following identity

$$e^{\mp\phi}f_\pm\left(\mp\frac{1}{2}\nabla_x\phi\cdot vf_\pm+v\cdot\nabla_x f_\pm\right)=\frac{1}{2}v\cdot\left(e^{\mp\phi}f_\pm^2\right). \tag{1.12}$$

Here $f_\pm(t,x,v)=\mu(v)^{-\frac{1}{2}}\left(F_\pm(t,x,v)-\mu(v)\right)$;

- A velocity weight

$$w_\ell(\alpha,\beta)(v)=\langle v\rangle^{-(\gamma+1)(\ell-|\alpha|-|\beta|)},\quad \langle v\rangle=\sqrt{1+|v|^2} \tag{1.13}$$

is designed to capture the weak velocity diffusion in the linearized Landau kernel $\mathbf{L}$ for the case of $-3\leq\gamma<-2$;

- A temporal decay of the electric potential $\phi(t,x)$ is obtained to close the energy estimate.

We note, however, that the fact

$$\|\partial_t\phi(t)\|_{L^\infty(\mathbb{R}^3_x)}\in L^1\left(\mathbb{R}^+\right)$$

plays an essential role in the analysis in [9], cf. [15] and [18] for the corresponding results in the whole space. But for the Cauchy problem of the one-species VPL system (1.8), the temporal decay analysis for the solution operator of the corresponding linearized system in [6] tells us that even if the initial perturbation $f_0(x,v)$ is assumed to satisfy the neutral condition

$$\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v}\mu^{\frac{1}{2}}(v)f_0(x,v)dvdx=0, \tag{1.14}$$

one can only deduce that $\partial_t\phi(t,x)$ decays at most like

$$\|\partial_t\phi(t)\|_{L^\infty(\mathbb{R}^3_x)}\lesssim(1+t)^{-1}$$

and consequently, the arguments developed in [9, 15, 18], which have been proved to be effective for the construction of global solutions to the Cauchy problem of the two-species VPL system (1.1), (1.4), can not be adopted directly to deal with the one-species VPL system (1.5), (1.6).

To study the global solvability of the Cauchy problem (1.8) of the one-species VPL system, another time-velocity energy method is introduced in [6] and [12] which is first used in [4, 5] to deal with the Vlasov-Poisson-Boltzmann system and is based on the following weight function $\widetilde{w}_{\ell-|\beta|}(t,v)$

$$\widetilde{w}_{\ell-|\beta|}(t,v)=\langle v\rangle^{-(\gamma+2)(\ell-|\beta|)}\exp\left(\frac{q\langle v\rangle^2}{(1+t)^{\vartheta}}\right),\quad 0<q\ll 1,\quad 0<\vartheta\leq\frac{1}{4}. \tag{1.15}$$

By combining the dissipation term

$$\left\|\widetilde{w}_{\ell-|\beta|}(t,v)\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\sigma} \tag{1.16}$$

which is the weighted variant of the coercive estimate (1.11) of the linearized Landau collision operator $\mathbf{L}$ with respect to the weight function $\widetilde{w}_{\ell-|\beta|}(t,v)$ and the extra dissipation term

$$(1+t)^{-1-\vartheta}\left\|\langle v\rangle\widetilde{w}_{\ell-|\beta|}(t,v)\partial^{\alpha}_{\beta}\{\mathbf{I}-\mathbf{P}\}f\right\|^2 \tag{1.17}$$

induced by the exponential factor $e^{\frac{q\langle v\rangle^2}{(1+t)^{\vartheta}}}$ of the weight function $\widetilde{w}_{\ell-|\beta|}(t,v)$, a somewhat satisfactory well-posedness theory is obtained for the Cauchy problem (1.8) for the whole range of $\gamma\geq-3$ and for any small initial perturbation $f_0(x,v)$ which is not necessarily assumed to satisfy the neutral condition (1.14), but at the price that the initial perturbation $f_0(x,v)$ decays exponentially for large $|v|$.

Thus a natural question is: *Does similar global solvability result hold for the Cauchy problem* (1.8) *of the one-species VPL system with algebraic decay initial perturbation?* Or in other words, *Can the approach developed in [9, 15, 18] still be adapted in the one-species case?*

The main purpose of this paper is devoted to the above problem and we will show that it is indeed the case if the initial perturbation $f_0(x,v)$ satisfies the neutral condition (1.14). Before stating such a result, for some integer $N\geq 1$ and some constant $\ell\geq N$, we define the energy functional $\mathcal{E}_{N,\ell}(t)$ and the corresponding dissipation rate functional $\mathcal{D}_{N,\ell}(t)$ of a given $f(t,x,v)$ by

$$\mathcal{E}_{N,\ell}(t)\sim\sum_{|\alpha|+|\beta|\leq N}\left\|w_{\ell}(\alpha,\beta)\partial^{\alpha}_{\beta}f\right\|^2+\|\nabla_x\phi\|^2_{H^N} \tag{1.18}$$

and

$$\mathcal{D}_{N,\ell}(t) = \|a\|^2 + \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha(a,b,c)\|^2 + \sum_{|\alpha|+|\beta|\leq N} \left\|w_\ell(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\sigma . \tag{1.19}$$

Now our main result can be stated as follows.

Theorem 1.1. *Assume that*

- $-3\leq\gamma<-2$, $N\geq 3$, $0<\epsilon_0\leq\frac{5}{4}$;
- *The parameters* $l_j (j=0,1,2,3)$ *are chosen such that*

$$l_0>\frac{3}{2},\quad l_1\geq\frac{l_0}{2}+N,\quad l_2\geq l_1+\frac{3\gamma+4}{2(\gamma+1)},\quad l_3\geq l_2+\frac{1}{2};$$

- *The initial perturbation* $f_0(x,v)$ *is assumed to satisfy* $F_0(x,v)=\mu+\sqrt{\mu}f_0(x,v)\geq 0$ *and the neutral condition* (1.14);
- $Y_0=\sum_{|\alpha|+|\beta|\leq N}\left\|\langle v\rangle^{-(\gamma+1)(l_3-|\alpha|+|\beta|)}\partial^\alpha_\beta f_0\right\|+\|\nabla_x\phi_0\|_{H^N}+\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}f_0\right\|_{Z_1}$ $+\|(1+|x|)a_0\|_{L^1_x}$ *is assumed to be sufficiently small.*

Then the Cauchy problem (1.8) *of the one-species VPL system exists a unique global solution* $f(t,x,v)$ *satisfying* $F(t,x,v)=\mu+\sqrt{\mu}f(t,x,v)\geq 0$ *and the following temporal decay estimates:*

$$\begin{aligned} \|(f(t),\nabla_x\phi(t))\|^2 &\lesssim Y_0^2(1+t)^{-\frac{3}{2}},\\ \|\nabla_x(f(t),\nabla_x\phi(t))\|^2 &\lesssim Y_0^2(1+t)^{-\frac{5}{2}}, \end{aligned}\tag{1.20}$$

and

$$\mathcal{E}_{N,l_1}(t)\lesssim Y_0^2(1+t)^{-\frac{3}{2}}. \tag{1.21}$$

Remark 1.1. Several remarks concerning Theorem 1.1 are listed below:

- Although only the case of $-3\leq\gamma<-2$ is treated in Theorem 1.1, the case of $\gamma\geq-2$ has been studied in [12] and it is shown in Theorem 1.1 in [12] that similar result holds even without the neutral condition (1.14);
- For the case of $-3\leq\gamma<-2$, it would be of some interest to see whether similar result holds or not if the initial perturbation $f_0(x,v)$ is not assumed to satisfy the neutral condition (1.14). Such a problem is under our current study.

Before concluding this section, we point out the key technical points in the proof of Theorem 1.1. First of all, similar to that of [9], we use the velocity weight $w_\ell(\alpha,\beta)$ defined in (1.13) and an exponential weight of the electric potential $e^{-\phi}$ to cancel the velocity growth induced by the self-consistent electrostatic field $\nabla_x\phi$. As a consequence, the corresponding weighted energy estimate based on these two weight functions $w_{l_3}(\alpha,\beta)$ and $e^{-\phi}$ will result in a Lyapunov type differential inequality for some energy functional $\mathcal{E}_{N,l_3}(t)$ like

$$\frac{d}{dt}\mathcal{E}_{N,l_3}(t)+\mathcal{D}_{N,l_3}(t)\lesssim\|\partial_t\phi(t)\|_{L^\infty_x}\mathcal{E}_{N,l_3}(t). \tag{1.22}$$

Since, unlike the case of two-species VPL system (1.1), (1.4) studied in [9, 15, 18], the temporal decay of the electric potential ϕ for the one-species VPL system (1.5), (1.6) is worse such that $\|\partial_t\phi(t)\|_{L^\infty(\mathbb{R}^3_x)}$ does not belong to $L^1(\mathbb{R}^+)$ any longer, the argument in [9, 15, 18] can not be adopted directly to deal with the term $\|\partial_t\phi(t)\|_{L^\infty_x}\mathcal{E}_{N,l_3}(t)$ appeared in the right hand side of (1.22).

To deal with such a term, we have by multiplying (1.22) by $(1+t)^{-\epsilon_0}$ for some constant $\epsilon_0>0$ that

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_{N,l_3}(t)\right\}+\epsilon_0(1+t)^{-1-\epsilon_0}\mathcal{E}_{N,l_3}(t)+(1+t)^{-\epsilon_0}\mathcal{D}_{N,l_3}(t)\\ &\lesssim(1+t)^{-\epsilon_0}\|\partial_t\phi(t)\|_{L^\infty_x}\mathcal{E}_{N,l_3}(t).\end{aligned} \tag{1.23}$$

The estimate (1.23) tells us that if the electric potential ϕ decays suitably such that

$$\|\partial_t\phi(t)\|_{L^\infty(\mathbb{R}^3_x)}\le\epsilon(1+t)^{-1} \tag{1.24}$$

holds for some sufficiently small positive constant $\epsilon>0$, then we can use the new dissipation term $(1+t)^{-1-\epsilon_0}\mathcal{E}_{N,l_3}(t)$, i.e. the second term in the left hand side of the differential inequality (1.23), to absorb the term $(1+t)^{-\epsilon_0}\|\partial_t\phi(t)\|_{L^\infty_x}\mathcal{E}_{N,l_3}(t)$ appeared in the right hand side of (1.23).

Now the problem is how to get the temporal decay of the electric potential, especially the temporal decay estimate on $\|\partial_t\phi(t)\|_{L^\infty_x}$. To this end, by combining the temporal decay analysis on the solution operator of the corresponding linearized VPL system with the Duhamel principle as in [6], we can indeed deduce the temporal decay estimate (1.24) together with the following temporal decay estimate on $\|\nabla_x\phi(t)\|_{L^\infty(\mathbb{R}^3_x)}$

$$\|\nabla_x\phi(t)\|_{L^\infty(\mathbb{R}^3_x)} \leq \epsilon(1+t)^{-\frac{5}{4}} \tag{1.25}$$

under the assumption that

$$\mathcal{E}_{N,l_1}(t) \lesssim \epsilon(1+t)^{-\frac{3}{2}}. \tag{1.26}$$

To verify the temporal decay estimate (1.26) on $\mathcal{E}_{N,l_1}(t)$, on the one hand, one has for any $\ell \geq N \geq 2$ that

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t) + \mathcal{D}_{N,\ell}(t) \leq 0, \tag{1.27}$$

where we need the smallness of $\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)$ which is caused by the velocity growth comes from the nonlinear term $\nabla_x\phi\cdot vf$ (For details, see the estimates on $I_1,\ I_2$, and I_4 in the proof of Lemma 3.2). On the other hand, the analysis on the temporal decay estimate (1.26) on the energy functional $\mathcal{E}_{N,l_1}(t)$ in Lemma 3.3 asks that (1.27) also holds for $\ell = l_1 + 2l^*$ with $l^* = \frac{\gamma+2}{2(\gamma+1)}$. Consequently, the smallness of $\mathcal{E}_{N,l_1+2l^*+\frac{\gamma}{2(\gamma+1)}}(t)$ needs to be justified.

To guarantee the smallness of $\mathcal{E}_{N,l_1+2l^*+\frac{\gamma}{2(\gamma+1)}}(t)$, by performing the weighted energy estimates with respect to the weight $w_\ell(\alpha,\beta)$ only and by re-examining the terms related to the nonlinear term $\nabla_x\phi\cdot vf$, one can deduce the following Lyapunov type differential inequality for some energy functional $\mathcal{E}_{N,\ell}(t)$

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t) + \mathcal{D}_{N,\ell}(t) \lesssim \|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,\ell+\frac{1}{2}}(t) \tag{1.28}$$

for any $\ell \geq N$.

Thus, if we replace ℓ in (1.28) by $l_2 \geq l_1+2l^*+\frac{\gamma}{2(\gamma+1)}$ and take $l_3 \geq l_2+\frac{1}{2}$ in (1.23), respectively, we can then close the whole analysis by combining (1.23) with $\ell = l_3$, (1.28) with $\ell = l_2$, and the estimate (1.25) provided that ϵ_0 is suitably chosen such that $\epsilon_0 \in (0,1]$.

The rest of this paper is arranged as follows. In section 2, we will first give some weighted estimates on the linearized Landau collision operator $\mathbf{L}$

and the nonlinear collision term $\mathbf{\Gamma}$. For our later use, some weighted estimates on the nonlinear term related to the electric potential $\phi(t,x)$ will also be given in this section. Section 3 is first devoted to deducing some a priori estimates on the local solutions to the Cauchy problem (1.8) constructed by repeating the argument used in [9, 15, 18] and then to proving our main result Theorem 1.1 by the continuation argument in the usual way.

2. Preliminaries

In this section, we first collect several fundamental results to be used frequently later. The first lemma is devoted to the estimates on the linearized Landau operator $\mathbf{L}$ and the nonlinear term $\mathbf{\Gamma}$, whose proofs can be found in [18].

Lemma 2.1 (cf. [18]). *Assume $-3\leq\gamma<-2$ and let $w=w_\ell(\alpha,\beta)(v)$ be the weight function defined in* (1.13)*, then we can get that*

(i) *There exist $\kappa>0$ and $C_\kappa>0$ such that*

$$\left\langle w_\ell^2(\alpha,0)(v)\partial^\alpha\mathbf{L}f,\partial^\alpha f\right\rangle\geq\kappa|f|^2_{\sigma,w_\ell(\alpha,0)}-C_\kappa|f|^2_\sigma. \tag{2.1}$$

Let $|\beta|>0$, for $\eta>0$ small enough there exists $C_\eta>0$ such that

$$\begin{aligned}\left\langle w_\ell^2(\alpha,\beta)(v)\partial^\alpha_\beta\mathbf{L}f,\partial^\alpha_\beta f\right\rangle \;\geq\; &\kappa\left|\partial^\alpha_\beta f\right|^2_{\sigma,w_\ell(\alpha,\beta)}-\eta\sum_{|\beta'|=|\beta|}\left|\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right|^2_{\sigma,w_\ell(\alpha,\beta)}\\ &-C_\eta\sum_{|\beta'|<|\beta|}\left|\partial^\alpha_{\beta'}f\right|^2_{\sigma,w_\ell(\alpha,\beta)}.\end{aligned} \tag{2.2}$$

(ii) *It follows that*

$$\begin{aligned}&\left\langle w_\ell^2(\alpha,\beta)(v)\partial^\alpha_\beta\mathbf{\Gamma}(g_1,g_2),\partial^\alpha_\beta g_3\right\rangle\\ &\lesssim\sum_{\substack{\alpha_1\leq\alpha\\ \bar\beta\leq\beta_1\leq\beta}}\left|\mu^\delta\partial^{\alpha_1}_{\bar\beta}g_1\right|_{L^2_v}\left|\partial^{\alpha-\alpha_1}_{\beta-\beta_1}g_2\right|_{\sigma,w_\ell(\alpha,\beta)}\left(\left|\partial^\alpha_\beta g_3\right|_{\sigma,w_\ell(\alpha,\beta)}+\ell\left|\partial^\alpha_\beta g_3\right|_{2,\frac{w_\ell(\alpha,\beta)}{\langle v\rangle^{-\frac{\gamma}{2}}}}\right).\end{aligned} \tag{2.3}$$

Here $\delta>0$ is a sufficiently small universal constant. In particular, we have

$$\langle\mathbf{\Gamma}(g_1,g_2),g_3\rangle\lesssim\left|\mu^\delta g_1\right|_{L^2_v}|g_2|_\sigma\,|g_3|_\sigma. \tag{2.4}$$

Next we turn to deduce some weighted estimates on the nonlinear terms. Our first lemma is concerned with the term $v\cdot\nabla_x f$ which can be stated as in the following lemma:

Lemma 2.2. *Assume $-3\le\gamma<-2$ and take $N\ge 2$ and $\ell\ge N$, then it follows that*

$$(v\cdot\nabla_x\phi f,f)\lesssim\frac{d}{dt}\int_{\mathbb{R}^3_x}|b|^2(a+2c)dx+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t). \tag{2.5}$$

Furthermore, we also have the following weighted estimates with respect to the weight function $w_\ell(\alpha,\beta)$:

$$\sum_{1\le|\alpha|\le N}\left(v\cdot\nabla_x\phi\partial^\alpha f,w_\ell^2(\alpha,0)\partial^\alpha f\right)\lesssim\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)+\varepsilon\mathcal{D}_{N,\ell}(t), \tag{2.6}$$

$$\begin{aligned}\sum_{|\alpha|+|\beta|\le N}&\left(v\cdot\nabla_x\phi\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f),w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\\ &\lesssim\ \mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)+\varepsilon\mathcal{D}_{N,\ell}(t),\end{aligned} \tag{2.7}$$

$$\sum_{\substack{1\le|\alpha|\le N\\ \alpha_1\neq0}}\left(v\cdot\nabla_x\partial^{\alpha_1}\phi\partial^{\alpha-\alpha_1}f,w_\ell^2(\alpha,0)\partial^\alpha f\right)\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t), \tag{2.8}$$

and

$$\begin{aligned}\sum_{\substack{|\alpha|+|\beta|\le N,\\ \alpha_1\neq0,\alpha_1\le\alpha,\beta_1\le\beta}}&\left(\partial_{\beta_1}v\cdot\nabla_x\partial^{\alpha_1}\phi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f,w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\\ &\lesssim\ \mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).\end{aligned} \tag{2.9}$$

Proof. For (2.5), as in Lemma 3.2 of [6], one has

$$\begin{aligned}(v\cdot\nabla_x\phi f,f)\ &\lesssim\ \frac{d}{dt}\int_{\mathbb{R}^3_x}|b|^2(a+2c)dx+\left\|\nabla_x^2\phi\right\|_{H^1}\left\|\langle v\rangle^{\frac12}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\\ &\quad+\Big\{\|(a,b,c)\|_{H^2}+\|\nabla_x\phi\|_{H^1}+\|\nabla_x\phi\|\,\|\nabla_x b\|\Big\}\\ &\quad\times\left\{\|\nabla_x(a,b,c)\|^2+\left\|\langle v\rangle^{\frac{\gamma+2}{2}}\{\mathbf{I}-\mathbf{P}\}f\right\|^2\right\}\\ &\lesssim\ \frac{d}{dt}\int_{\mathbb{R}^3_x}|b|^2(a+2c)dx+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t),\end{aligned} \tag{2.10}$$

where we have used the fact that $\ell\ge N$ and $N\ge 2$.

As for (2.6), one has from Sobolev's inequality that

$$
\begin{aligned}
&\sum_{1\le|\alpha|\le N}\big(v\cdot\nabla_x\phi\partial^\alpha f, w_\ell^2(\alpha,0)\partial^\alpha f\big)\\
&\lesssim \sum_{1\le|\alpha|\le N}\|\nabla_x\phi\|_{L^\infty_x}\left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{-\frac{\gamma}{2}}\right\|\left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{\frac{\gamma+2}{2}}\right\|\\
&\lesssim \sum_{1\le|\alpha|\le N}\left\|\nabla_x^2\phi\right\|^2_{H^1_x}\left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{-\frac{\gamma}{2}}\right\|^2+\varepsilon\|w_\ell(\alpha,0)\partial^\alpha f\|^2_\sigma\\
&\lesssim \mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).
\end{aligned}
$$

This proves (2.6) and (2.7) can be proved by the same way as (2.6).

Now for (2.8), we have

$$
\begin{aligned}
&\sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}}\big(v\cdot\nabla_x\partial^{\alpha_1}\phi\partial^{\alpha-\alpha_1}f, w_\ell^2(\alpha,0)\partial^\alpha f\big)\\
&\lesssim \sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}}\|\nabla_x\partial^{\alpha_1}\phi\|_{L^\infty_x}\left\|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}\mathbf{P}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|\left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{\frac{\gamma+2}{2}}\right\|\\
&+\sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}}\|\nabla_x\partial^{\alpha_1}\phi\|_{L^\infty_x}\left\|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\|\left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{\frac{\gamma+2}{2}}\right\|\\
&\lesssim \sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}}\left\|\partial^{\alpha_1}\nabla^2\phi\right\|^2_{H^1_x}\Big\{\left\|\mu^\delta\partial^{\alpha-\alpha_1}f\right\|^2\\
&+\left\|w_\ell(\alpha-\alpha_1,0)\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma+2}{2}}\right\|^2\Big\}+\varepsilon\sum_{1\le|\alpha|\le N}\|w_\ell(\alpha,0)\partial^\alpha f\|^2_\sigma\\
&\lesssim \mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).
\end{aligned}
$$

Here we use the fact that

$$
w_\ell(\alpha,0)\langle v\rangle^{-\frac{\gamma}{2}}=w_\ell(\alpha-\alpha_1,0)\langle v\rangle^{|\alpha_1|(\gamma+1)-\frac{\gamma}{2}}\le w_\ell(\alpha-\alpha_1,0)\langle v\rangle^{\frac{\gamma+2}{2}},\quad 1\le|\alpha_1|\le|\alpha|.
$$

In the similar way, we can also obtain

$$
\sum_{\substack{1\le|\alpha|\le N\\2\le|\alpha_1|\le N}}\big(v\cdot\nabla_x\partial^{\alpha_1}\phi\partial^{\alpha-\alpha_1}f, w_\ell^2(\alpha,0)\partial^\alpha f\big)\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).
$$

Collecting the above two estimates gives (2.8).

(2.9) follows by employing the same argument used to deduce (2.8). Thus the proof of Lemma 2.2 is complete. □

Remark 2.1. For the estimates (2.5)-(2.9), only the estimates (2.6) and (2.7) can lead to the increase of the order of the weight with respect to v. It is worth to emphasizing that if we perform the corresponding energy type estimates by using the weights $w_\ell(\alpha,\beta)(v)$ and $e^{-\phi}$ simultaneously, we do not need to deal with these two terms since, due to

$$\begin{aligned}
&e^{-\phi}w_\ell^2(\alpha,0)\partial^\alpha f\times\left(-\frac{1}{2}v\cdot\nabla_x\phi\partial^\alpha f+v\cdot\nabla_x\partial^\alpha f\right)\\
&=\frac{w_\ell^2(\alpha,0)}{2}v\cdot\nabla_x\left(e^{-\phi}\left|\partial^\alpha f\right|^2\right),\\
&e^{-\phi}w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\times\left(-\frac{1}{2}v\cdot\nabla_x\phi\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f+v\cdot\nabla_x\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\\
&=\frac{w_\ell^2(\alpha,\beta)}{2}v\cdot\nabla_x\left(e^{-\phi}\left|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right|^2\right),
\end{aligned}$$

they will vanish after integration with respect to v and x over $\mathbb{R}^3_x\times\mathbb{R}^3_v$.

The next lemma is concerned with $\nabla_x\phi\cdot\nabla_v f$, which can be stated as follows:

Lemma 2.3. *Let $N\geq 2$, $\ell\geq N$, we have the estimates on the nonlinear term $\nabla_x\phi\cdot\nabla_v f$ as follows:*

$$\sum_{1\leq|\alpha|\leq N}\left(\partial^\alpha(\nabla_x\phi\cdot\nabla_v f),w_\ell^2(\alpha,0)\partial^\alpha f\right)\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t)\quad(2.11)$$

and

$$\begin{aligned}
&\sum_{|\alpha|+|\beta|\leq N}\left(\partial^\alpha_\beta(\nabla_x\phi\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f),w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\\
&\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).
\end{aligned}\quad(2.12)$$

Proof. We only need to prove (2.11) since (2.12) can be proved similarly. For this purpose, one can get first that

$$\sum_{1\leq|\alpha|\leq N}\left(\nabla_x\phi\cdot\nabla_v\partial^\alpha f,w_\ell^2(\alpha,0)\partial^\alpha f\right)$$

$$
\begin{aligned}
&= \sum_{1\le|\alpha|\le N} \frac{1}{2}\left(\nabla_x\phi\cdot\nabla_v(\partial^\alpha f)^2, w_\ell^2(\alpha,0)\right)\\
&\lesssim \sum_{1\le|\alpha|\le N} \int_{\mathbb{R}^3_x} |\nabla_x\phi| \left|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{-\frac12}\right|^2_{L^2_v} dx\\
&\lesssim \sum_{1\le|\alpha|\le N} \|\nabla_x\phi\|_{L^\infty_x} \left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{-\frac12}\right\|^2\\
&\lesssim \sum_{1\le|\alpha|\le N} \left\|\nabla_x^2\phi\right\|^2_{H^1_x} \left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{-\frac12}\right\|^2 + \varepsilon\left\|w_\ell(\alpha,0)\partial^\alpha f\right\|^2_\sigma\\
&\lesssim \mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}} \left(\nabla_x\partial^{\alpha_1}\phi\cdot\partial^{\alpha-\alpha_1}\nabla_v f, w_\ell^2(\alpha,0)\partial^\alpha f\right)\\
&= -\sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}} \left(\nabla_x\partial^{\alpha_1}\phi\cdot\partial^{\alpha-\alpha_1} f, \nabla_v(w_\ell^2(\alpha,0)\partial^\alpha f)\right)\\
&\lesssim \sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}} \|\nabla_x\partial^{\alpha_1}\phi\|_{L^\infty_x} \left\|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}\mathbf{P}f\langle v\rangle^{-\frac{\gamma}{2}}\right\| \left\|\nabla_v(w_\ell(\alpha,0)\partial^\alpha f)\langle v\rangle^{\frac{\gamma}{2}}\right\|\\
&\quad+\sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}} \|\nabla_x\partial^{\alpha_1}\phi\|_{L^\infty_x} \left\|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac{\gamma}{2}}\right\| \left\|\nabla_v(w_\ell(\alpha,0)\partial^\alpha f)\langle v\rangle^{\frac{\gamma}{2}}\right\|\\
&\quad+\sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}} \|\nabla_x\partial^{\alpha_1}\phi\|_{L^\infty_x} \left\|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}\mathbf{P}f\langle v\rangle^{-\frac12}\right\| \left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{-\frac12}\right\|\\
&\quad+\sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}} \|\nabla_x\partial^{\alpha_1}\phi\|_{L^\infty_x} \left\|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{-\frac12}\right\| \left\|w_\ell(\alpha,0)\partial^\alpha f\langle v\rangle^{-\frac12}\right\|\\
&\lesssim \sum_{\substack{1\le|\alpha|\le N\\|\alpha_1|=1}} \left\|\partial^{\alpha_1}\nabla^2\phi\right\|^2_{H^1_x}\left\{\left\|\mu^\delta\partial^{\alpha-\alpha_1}f\right\|^2+\left\|w_\ell(\alpha-\alpha_1,0)\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma+2}{2}}\right\|^2\right\}\\
&\quad+\varepsilon\sum_{1\le|\alpha|\le N}\left\|w_\ell(\alpha,0)\partial^\alpha f\right\|^2_\sigma\\
&\lesssim \mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t),
\end{aligned}
$$

where we have used again the fact that

$$
w_\ell(\alpha,0)\langle v\rangle^{-\frac{\gamma}{2}} = w_\ell(\alpha-\alpha_1,0)\langle v\rangle^{|\alpha_1|(\gamma+1)-\frac{\gamma}{2}} \le w_\ell(\alpha-\alpha_1,0)\langle v\rangle^{\frac{\gamma+2}{2}},\ 1\le|\alpha_1|\le|\alpha|.
$$

Similarly, one can also conclude that

$$\sum_{\substack{1\leq|\alpha|\leq N\\ 2\leq|\alpha_1|\leq N}}\left(v\cdot\nabla_x\partial^{\alpha_1}\phi\cdot\partial^{\alpha-\alpha_1}\nabla_v f, w_\ell^2(\alpha,0)\partial^\alpha f\right)\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).$$

Collecting the above three estimates yields (2.11). Thus we have completed the proof of Lemma 2.3. □

For the estimates on $\mathbf{\Gamma}(f,f)$ with respect to the weight function $w_\ell(\alpha,\beta)$, from (2.4) and (2.3) in Lemma 2.1, we can obtain that

Lemma 2.4. *Let $N\geq 2$, $\ell\geq N$, we have the following estimates on the nonlinear term $\mathbf{\Gamma}(f,f)$ with respect to the weight function $w_\ell(\alpha,\beta)$*

$$\left(\mathbf{\Gamma}(f,f), w_\ell^2(0,0)\{\mathbf{I}-\mathbf{P}\}f\right)\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t),\tag{2.13}$$

$$\sum_{1\leq|\alpha|\leq N}\left(\partial^\alpha\mathbf{\Gamma}(f,f), w_\ell^2(\alpha,0)\partial^\alpha f\right)\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t),\tag{2.14}$$

and

$$\sum_{\substack{|\alpha|+|\beta|\leq N\\ |\beta|\geq 1}}\left(\partial^\alpha_\beta\mathbf{\Gamma}(f,f), w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).\tag{2.15}$$

Proof. We only prove (2.14) in the following since the proofs of (2.13) and (2.15) are similar. It follows from (2.3) of Lemma 2.1 that

$$\begin{aligned}
&\sum_{1\leq|\alpha|\leq N}\left(\partial^\alpha\mathbf{\Gamma}(f,f), w_\ell^2(\alpha,0)\partial^\alpha f\right)\\
\lesssim&\sum_{\substack{1\leq|\alpha|\leq N,\\ \alpha_1\leq\alpha}}\int_{\mathbb{R}^3_x}\left|\mu^\delta\partial^{\alpha_1}f\right|_{L^2_v}\left|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}f\right|_{L^2_\sigma}\left|w_\ell(\alpha,0)\partial^\alpha f\right|_{L^2_\sigma}dx\\
\lesssim&\sum_{\substack{1\leq|\alpha|\leq N,\\ \alpha_1=0}}\int_{\mathbb{R}^3_x}\left|\mu^\delta f\right|_{L^2_v}\left|w_\ell(\alpha,0)\partial^\alpha f\right|^2_{L^2_\sigma}dx\\
&+\sum_{\substack{1\leq|\alpha|\leq N,\\ 1\leq|\alpha_1|\leq N-1}}\int_{\mathbb{R}^3_x}\left|\mu^\delta\partial^{\alpha_1}f\right|_{L^2_v}\left|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1}f\right|_{L^2_\sigma}\left|w_\ell(\alpha,0)\partial^\alpha f\right|_{L^2_\sigma}dx\\
&+\sum_{\substack{1\leq|\alpha|\leq N,\\ \alpha_1=\alpha,|\alpha|\geq 2}}\int_{\mathbb{R}^3_x}\left|\mu^\delta\partial^{\alpha}f\right|_{L^2_v}\left|w_\ell(\alpha,0)f\right|_{L^2_\sigma}\left|w_\ell(\alpha,0)\partial^\alpha f\right|_{L^2_\sigma}dx
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{\substack{1\le|\alpha|\le N,\\ \alpha_1=0}} \left\|\mu^\delta f\right\|_{L^2_vL^\infty_x} \|w_\ell(\alpha,0)\partial^\alpha f\|^2_\sigma \\
&\quad + \sum_{\substack{1\le|\alpha|\le N,\\ 1\le|\alpha_1|\le N-1}} \left\|\mu^\delta\partial^{\alpha_1} f\right\|_{L^2_vL^3_x} \|w_\ell(\alpha,0)\partial^{\alpha-\alpha_1} f\|_{L^2_\sigma L^6_x} \|w_\ell(\alpha,0)\partial^\alpha f\|_\sigma \\
&\quad + \sum_{\substack{1\le|\alpha|\le N,\\ \alpha_1=\alpha,|\alpha|\ge 2}} \left\|\mu^\delta\partial^{\alpha} f\right\| \|w_\ell(\alpha,0) f\|_{L^2_\sigma L^\infty_x} \|w_\ell(\alpha,0)\partial^\alpha f\|_\sigma \\
&\lesssim \mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t) + \varepsilon\mathcal{D}_{N,\ell}(t).
\end{aligned}
$$

This proves (2.13) and the proof of Lemma 2.4 is complete. □

For later use, we also need the time-decay property of the linearized VPL system

$$
\partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot v\mu^{1/2} + \mathbf{L}f = 0, \quad \Delta_x\phi = \int_{\mathbb{R}^3_v} \mu^{\frac12} f dv, \quad \lim_{|x|\to\infty}\phi(t,x) = 0 \tag{2.16}
$$

with the prescribed initial data $f(0,x,v) = f_0(x,v)$. We use e^{tB} to denote the evolution operator. The following lemma is concerned with the linearized VPL system (2.16).

Lemma 2.5. *(cf. [6]) Set $\kappa = \kappa(v) := \langle v\rangle^{-\frac{\gamma+1}{2}}$. Let $-3 \le \gamma < -2$, $l \ge 0$, $l_0 > \frac32$, $\alpha \ge 0$, $m = |\alpha|$, and assume*

$$
\int_{\mathbb{R}^3_x} a_0 dx \equiv \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v} \mu(v)^{\frac12} f_0(x,v) dv dx = 0, \quad \int_{\mathbb{R}^3_x} (1+|x|)|a_0| dx < \infty, \tag{2.17}
$$

and

$$
\left\|\kappa^{l+l_0} f_0\right\|_{Z_1} + \left\|\kappa^{l+l_0}\partial^\alpha f_0\right\| < \infty. \tag{2.18}
$$

Then, the evolution operator e^{tB} satisfies

$$
\begin{aligned}
&\left\|\kappa^l\partial^\alpha e^{tB} f_0\right\| + \left\|\partial^\alpha\nabla_x\triangle_x^{-1}\mathbf{P_0} e^{tB} f_0\right\| \\
&\lesssim (1+t)^{-\sigma_m}\left(\left\|\kappa^{l+l_0} f_0\right\|_{Z_1} + \left\|\kappa^{l+l_0}\partial^\alpha f_0\right\| + \|(1+|x|)a_0\|_{L^1_x}\right)
\end{aligned}
\tag{2.19}
$$

for any $t \ge 0$, where

$$
\sigma_m = \frac34 + \frac m2.
$$

Proof. The only difference between this lemma and Lemma 3.5 in [6] is that we use $\kappa(v) := \langle v\rangle^{-\frac{\gamma+1}{2}}$ to replace $\mu(v) = \langle v\rangle^{-\frac{\gamma+2}{2}}$ used there. Since the modification is straightforward, we thus omit the details for brevity. □

The next lemma is concerned with the estimates on $\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\mathbf{\Gamma}(f,f)\right\|_{Z_1}$ and $\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\mathbf{\Gamma}(f,f)\right\|$ for later use.

Lemma 2.6. *Assume $-3 \leq \gamma < -2$ and $N \geq 3$, it holds that*

$$\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\mathbf{\Gamma}(f,f)\right\|_{Z_1} + \left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_x\mathbf{\Gamma}(f,f)\right\| \lesssim \mathcal{E}_{N,\frac{l_0}{2}+N}(t). \tag{2.20}$$

Proof. By the definition of $\mathbf{\Gamma}(f,f)$, one has

$$\begin{aligned}
&\mathbf{\Gamma}(f,f)\\
&= \sum_{i,j=1}^{3} \mu^{-\frac{1}{2}}\partial_i \int_{\mathbb{R}^3_v} \Phi^{ij}(v-v')\Big\{\mu^{\frac{1}{2}}(v')f(v')\partial_j\left(\mu^{\frac{1}{2}}(v)f(v)\right)\\
&\quad -\partial_j\left(\mu^{\frac{1}{2}}(v')f(v')\right)\mu^{\frac{1}{2}}(v)f(v)\Big\}dv'\\
&= \sum_{i,j=1}^{3}\Phi^{ij} *_v \partial_i\left(\mu^{\frac{1}{2}}f\right)(v)\left(\partial_j f - \frac{1}{2}v_j f\right)(v) - \sum_{i,j=1}^{3}\Phi^{ij} *_v \partial_{ij}\left(\mu^{\frac{1}{2}}f\right)(v)f(v)\\
&\quad +\sum_{i,j=1}^{3}\Phi^{ij} *_v \left(\mu^{\frac{1}{2}}f\right)(v)\left(\partial_{ij}f - \frac{1}{2}v_i\partial_j f - \frac{1}{2}v_j\partial_i f + \frac{1}{4}v_iv_jf - \frac{1}{2}\delta_{ij}f\right)(v)\\
&\quad -\sum_{i,j=1}^{3}\Phi^{ij} *_v \partial_j\left(\mu^{\frac{1}{2}}f\right)(v)\left(\partial_i f - \frac{1}{2}v_i f\right)(v).
\end{aligned}\tag{2.21}$$

Here $*_v$ denotes the convolution with respect to the v variable.

Therefore, one can deduce that

$$\begin{aligned}
&\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\mathbf{\Gamma}(f,f)\right\|_{Z_1}\\
&\lesssim \sum_{i,j=1}^{3}\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\Big\{\Phi^{ij} *_v \partial_i\left(\mu^{\frac{1}{2}}f\right)(v)\left(\partial_j f - \frac{1}{2}v_j f\right)(v)\right.\\
&\quad -\Phi^{ij} *_v \partial_{ij}\left(\mu^{\frac{1}{2}}f\right)(v)f(v)\\
&\quad +\Phi^{ij} *_v \left(\mu^{\frac{1}{2}}f\right)(v)\left(\partial_{ij}f - \frac{1}{2}v_i\partial_j f - \frac{1}{2}v_j\partial_i f + \frac{1}{4}v_iv_jf - \frac{1}{2}\delta_{ij}f\right)(v)
\end{aligned}$$

$$-\Phi^{ij} *_v \partial_j \left(\mu^{\frac{1}{2}} f\right)(v) \left(\partial_i f - \frac{1}{2} v_i f\right)(v) \Big\} \Big\|_{Z_1} \tag{2.22}$$

and

$$\begin{aligned}
&\left\| \langle v\rangle^{-\frac{\gamma+1}{2} l_0} \nabla_x \mathbf{\Gamma}(f,f) \right\| \\
\lesssim\ & \sum_{i,j=1}^{3} \Big\| \langle v\rangle^{-\frac{\gamma+1}{2} l_0} \nabla_x \Big\{ \Phi^{ij} *_v \partial_i \left(\mu^{\frac{1}{2}} f\right)(v) \left(\partial_j f - \frac{1}{2} v_j f\right)(v) \\
& -\Phi^{ij} *_v \partial_{ij} \left(\mu^{\frac{1}{2}} f\right)(v) f(v) \\
& +\Phi^{ij} *_v \left(\mu^{\frac{1}{2}} f\right)(v) \left(\partial_{ij} f - \frac{1}{2} v_i \partial_j f - \frac{1}{2} v_j \partial_i f + \frac{1}{4} v_i v_j f - \frac{1}{2} \delta_{ij} f\right)(v) \\
& -\Phi^{ij} *_v \partial_j \left(\mu^{\frac{1}{2}} f\right)(v) \left(\partial_i f - \frac{1}{2} v_i f\right)(v) \Big\} \Big\|.
\end{aligned} \tag{2.23}$$

Now we estimate the following two terms

$$K_1 = \left\| \langle v\rangle^{-\frac{\gamma+1}{2} l_0} \Phi^{ij} *_v \left(\mu^{\frac{1}{2}} f\right)(v) \left(\partial_{ij} f\right)(v) \right\|_{Z_1}$$

and

$$K_2 = \left\| \langle v\rangle^{-\frac{\gamma+1}{2} l_0} \Phi^{ij} *_v \left(\mu^{\frac{1}{2}} f\right)(v) \left(\nabla_x \partial_{ij} f\right)(v) \right\|$$

which are typical terms in the right hand sides of (2.22) and (2.23).

To this end, noticing that for each sufficiently small $\delta' > 0$

$$\left| \Phi^{ij} *_v \left(\mu^{\frac{1}{2}} f\right)(v) \right| \le \left(\left|\Phi^{ij}\right|^2 *_v \left(\mu^{1-\delta'}\right) \right) \left|\mu^{\delta'} f\right|_{L^2_v} \lesssim \langle v\rangle^{\gamma+2} \left|\mu^{\delta'} f\right|_{L^2_v},$$

which follows from the Cauchy-Schwarz inequality, we can get that

$$\begin{aligned}
K_1 &\lesssim \left\| \left| \Phi^{ij} *_v \left(\mu^{\frac{1}{2}} f\right)(v) \langle v\rangle^{-\frac{\gamma+1}{2} l_0} \partial_{ij} f \right|_{L^2_v} \right\|_{L^1_x} \\
&\lesssim \left\| \left|\mu^{\delta'} f\right|_{L^2_v} \left| \langle v\rangle^{\gamma+2} \langle v\rangle^{-\frac{\gamma+1}{2} l_0} \partial_{ij} f \right|_{L^2_v} \right\|_{L^1_x} \\
&\lesssim \left\|\mu^{\delta'} f\right\| \left\| \langle v\rangle^{\gamma+2-\frac{\gamma+1}{2} l_0} \partial_{ij} f \right\| \\
&\lesssim \mathcal{E}_{2,\frac{l_0}{2}+2}(t) \\
&\lesssim \mathcal{E}_{N,\frac{l_0}{2}+N}(t)
\end{aligned} \tag{2.24}$$

and

$$K_2 \lesssim \left\| \left| \Phi^{ij} *_v \left(\mu^{\frac{1}{2}} f\right)(v) \langle v\rangle^{-\frac{\gamma+1}{2} l_0} \nabla_x \partial_{ij} f \right|_{L^2_v} \right\|$$

$$
\begin{aligned}
&\lesssim \left\| \left|\mu^{\delta'} f\right|_{L^2_v} \left|\langle v\rangle^{\gamma+2}\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_x\partial_{ij}f\right|_{L^2_v}\right\| \\
&\lesssim \left\|\mu^{\delta'} f\right\| \left\|\langle v\rangle^{\gamma+2-\frac{\gamma+1}{2}l_0}\nabla_x\partial_{ij}f\right\| \\
&\lesssim \mathcal{E}_{3,\frac{l_0}{2}+3}(t) \\
&\lesssim \mathcal{E}_{N,\frac{l_0}{2}+N}(t) \qquad (2.25)
\end{aligned}
$$

if $N \geq 3$. It is worth emphasizing that it was here that we need to ask that $N \geq 3$.

It is direct to verify that for all other terms in the right hand sides of (2.22) and (2.23), the same estimate still holds and hence it proves (2.20). This completes the proof of Lemma 2.6. □

The next lemma is concerned with the macro dissipation. For this purpose, by applying the macro-micro decomposition (1.9) introduced in [8], for any $f(t,x,v)$, define moment functions $A_{mj}(f)$ and $B_j(f)$, $1 \leq m,j \leq 3$, by

$$A_{mj}(f) = \int_{\mathbb{R}^3_v} (v_m v_j - 1)\,\mu^{1/2} f dv, \quad B_j(f) = \frac{1}{10}\int_{\mathbb{R}^3_v} \left(|v|^2 - 5\right) v_j \mu^{1/2} f dv.$$

Then, one can derive form (1.8) a fluid-type system of equations

$$
\begin{cases}
\partial_t a + \nabla_x\cdot b = 0, \\
\partial_t b + \nabla_x(a+2c) + \nabla_x A(\{\mathbf{I}-\mathbf{P}\}f) - \nabla_x\phi = \nabla_x\phi a, \\
\partial_t c + \frac{1}{3}\nabla_x\cdot b + \frac{5}{3}\nabla_x\cdot B(\{\mathbf{I}-\mathbf{P}\}f) = \frac{1}{3}\nabla_x\phi\cdot b, \\
\triangle_x\phi = a
\end{cases}
\tag{2.26}
$$

and

$$
\begin{cases}
\partial_t A_{mj}(\{\mathbf{I}-\mathbf{P}\}f) + \partial_{x_m} b_j + \partial_{x_j} b_m - \frac{2}{3}\delta_{mj}\nabla_x\cdot B(\{\mathbf{I}-\mathbf{P}\}f) \\
= A_{mj}(r+G) - \frac{2}{3}\delta_{mj}\nabla_x\phi\cdot b, \\
\partial_t B_j(\{\mathbf{I}-\mathbf{P}\}f) + \partial_{x_j} c = B_j(r+G)
\end{cases}
\tag{2.27}
$$

with

$$r = -v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f - \mathbf{L}\{\mathbf{I}-\mathbf{P}\}f,\ G = \mathbf{\Gamma}(f,f) + \frac{1}{2}v\cdot\nabla_x\phi f - \nabla_x\phi\cdot\nabla_v f, \tag{2.28}$$

where r is a linear term related only to the microscopic component $\{\mathbf{I}-\mathbf{P}\}f$

and G is a quadratic nonlinear term.

Using (2.26) and (2.27), one has

Lemma 2.7. *There is a temporal interactive functional* $\mathcal{E}_N^{int}(t)$ *such that*

$$\left|\mathcal{E}_N^{int}(t)\right| \lesssim \sum_{|\alpha|\leq N} \|\partial^\alpha f\|^2 \tag{2.29}$$

and

$$\frac{d}{dt}\mathcal{E}_N^{int}(t)+\mathcal{D}_N^{int}(t) \lesssim \sum_{|\alpha|\leq N} \|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,0}(t) \tag{2.30}$$

hold for any $0\leq t<T$, *where*

$$\mathcal{D}_N^{int}(t)\sim\|a\|^2+\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha(a,b,c)\|^2.$$

Since the proof of Lemma 2.7 is similar to that of Lemma 5.1 in [5], we thus omit the details for brevity.

3. The Proofs of Our Main Results

This section is devoted to proving our main results based on the continuation argument. Local existence for the Cauchy problem (1.8) in certain weighted Sobolev space is now well-understood, cf. [9], thus the proof will consist in deducing the global a priori estimate in the same weighted Sobolev space in which the local solution is constructed.

To make the presentation clear, we divide the rest of this section into two subsections. The first one focuses on deducing some a priori estimates on $f(t,x,v)$.

3.1. Some a priori estimates

This subsection is devoted to deducing some a priori estimates on $f(t,x,v)$. For this purpose, let $f(t,x,v)$ be the unique solution to the Cauchy problem (1.8) defined on some time interval $0\leq t<T$ constructed in [9].

To this end, set

$$X(t) = \sup_{0\le s\le t}\Big\{(1+s)^{\frac{3}{2}}\left\|(f,\nabla_x\phi)(s)\right\|^2+(1+s)^{\frac{5}{2}}\left\|\nabla_x(f,\nabla_x\phi)(s)\right\|^2 +(1+s)^{\frac{3}{2}}\mathcal{E}_{N,l_1}(s)+\mathcal{E}_{N,l_2}(s)+(1+s)^{-\epsilon_0}\mathcal{E}_{N,l_3}(s)\Big\}, \tag{3.1}$$

where the precise ranges of the parameters N and $l_j(j=1,2,3)$ will be specified later, we now try to deduce certain a priori estimates on $f(t,x,v)$ and $\phi(t,x)$ in terms of $X(t)$ and Y_0 in this subsection.

Our first result is to deduce certain basic temporal decay estimates on $f(t,x,v)$ and $\phi(t,x)$. For result in this direction, we have from Lemma 2.4 and Duhamel's principle that

Lemma 3.1. *Assume $-3\le\gamma<-2$ and let $N\ge 3$, $l_0>\frac{3}{2}$, and $l_1\ge\frac{l_0}{2}+N$. Then it holds that*

$$\sup_{0\le s\le t}\Big\{(1+s)^{\frac{3}{2}}\left\|(\nabla_x\phi,f)(s)\right\|^2+(1+s)^{\frac{5}{2}}\left\|\left(\nabla_x^2\phi,\nabla_x f\right)(s)\right\|\Big\}\lesssim Y_0^2+X^2(t) \tag{3.2}$$

for any $0\le t<T$.

Proof. By Duhamel's principle, we can write the solution $f(t,x,v)$ to the Cauchy problem (1.8) as

$$f(t)=e^{tB}f_0+\int_0^t e^{(t-s)B}G(s)ds,$$

where $G=\frac{1}{2}v\cdot\nabla_x\phi f-\nabla_x\phi\cdot\nabla_v f+\mathbf{\Gamma}(f,f)$.

Notice that $\mathbf{P}_0G(t)\equiv 0$ for all $t\ge 0$, one has by applying Lemma 2.4 that

$$\begin{aligned}
&\|\nabla_x f(t)\|+\left\|\nabla_x^2\phi(t)\right\|\\
&\lesssim (1+t)^{-\frac{5}{4}}\Big\{\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}f_0\right\|_{Z_1}+\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_x f_0\right\|+\|(1+|x|)a_0\|_{L^1_x}\Big\}\\
&\quad+\int_0^t(1+t-s)^{-\frac{5}{4}}\Big\{\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}G(s)\right\|_{Z_1}+\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_x G(s)\right\|\Big\}ds,
\end{aligned} \tag{3.3}$$

where we recall that $l_0>\frac{3}{2}$ is a constant by Lemma 2.5.

To bound the second term in the right hand side of (3.3), we first estimate the Z_1-norm of the terms containing ϕ as follows

$$
\begin{aligned}
&\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\left\{-\nabla_x\phi\cdot\nabla_v f+\frac{1}{2}v\cdot\nabla_x\phi f\right\}\right\|_{Z_1}\\
\lesssim\ &\left\||\nabla_x\phi|\left\{\left|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_v f\right|_{L^2_v}+\left|\langle v\rangle^{-\frac{\gamma+1}{2}l_0+1}f\right|_{L^2_v}\right\}\right\|_{L^1_x}\\
\lesssim\ &\|\nabla_x\phi\|\left\{\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_v f\right\|+\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0+1}f\right\|\right\}\\
\lesssim\ &\mathcal{E}_{2,\frac{l_0}{2}+2}(t).
\end{aligned}
$$

For the L^2-norm, one can get by applying the similar way that

$$
\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_x\left\{-\nabla_x\phi\cdot\nabla_v f+\frac{1}{2}v\cdot\nabla_x\phi f\right\}\right\|\lesssim\mathcal{E}_{2,\frac{l_0}{2}+2}(t).
$$

Combining the above two estimates and the assumption (H_1) with the estimate (2.20) yield that

$$
\begin{aligned}
\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}G(s)\right\|_{Z_1}+\left\|\langle v\rangle^{-\frac{\gamma+1}{2}l_0}\nabla_x G(s)\right\|&\lesssim\mathcal{E}_{N,\frac{l_0}{2}+N}(s)\\
&\lesssim\mathcal{E}_{N,l_1}(s)\\
&\lesssim(1+s)^{-\frac{3}{2}}\sup_{0\le s\le t}\left\{(1+s)^{\frac{3}{2}}\mathcal{E}_{N,l_1}(s)\right\}\\
&\lesssim(1+s)^{-\frac{3}{2}}X(t),
\end{aligned}
$$

where we have taken $l_1\ge\frac{l_0}{2}+N$ with $N\ge 3$.

Therefore, plugging the above estimate into (3.3) yields that

$$
\sup_{0\le s\le t}\left\{(1+s)^{\frac{5}{2}}\left\|\left(\nabla_x^2\phi,\nabla_x f\right)(s)\right\|^2\right\}\lesssim Y_0^2+X^2(t),
$$

where we used the following inequality

$$
\int_0^t(1+t-s)^{-\frac{5}{4}}(1+s)^{-\frac{3}{2}}ds\lesssim(1+t)^{-\frac{5}{4}}.
$$

In the similar way, we can also obtain that

$$
\sup_{0\le s\le t}\left\{(1+s)^{\frac{3}{2}}\left\|\left(\nabla_x\phi,f\right)(s)\right\|^2\right\}\lesssim Y_0^2+X^2(t).
$$

Thus the proof of Lemma 3.1 is complete. □

The essential assumption imposed in Lemma 3.1 is to assume that

$$(H_1)\qquad \sup_{0\leq t<T}\left\{(1+t)^{\frac{3}{2}}\mathcal{E}_{N,l_1}(t)\right\}\lesssim 1.$$

To verify such an assumption, we need to deduce the temporal decay of the energy functional $\mathcal{E}_{N,l_1}(t)$. For this purpose, we need first to deduce the desired Lyapurov type differential inequality for some suitably constructed energy functional $\mathcal{E}_{N,\ell}(t)$ and the corresponding dissipation rate functional $\mathcal{D}_{N,\ell}(t)$ which is the main content of the following lemma.

Lemma 3.2. *Let $-3\leq\gamma<-2$, $N\geq 2$, and $\ell\geq N$. If we assume further that*

$$(H_2)\qquad \sup_{0\leq t<T}\left\{\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\right\}\leq M$$

for some sufficiently small $M>0$, then there exist an energy functional $\mathcal{E}_{N,\ell}(t)$ and the corresponding dissipation rate functional $\mathcal{D}_{N,\ell}(t)$ which satisfy (1.18) *and* (1.19) *respectively such that*

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t)+\mathcal{D}_{N,\ell}(t)\leq 0 \tag{3.4}$$

holds for any $0\leq t<T$.

Proof. The proof of (3.4) is divided into the following three steps:

Step 1. Applying ∂^α with $|\alpha|\leq N$ to (1.8), then taking the L^2 inner product of the resulting identity with $\partial^\alpha f$, one can get from (1.11), (2.5) and (2.13) that

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\leq N}\left\{\|\partial^\alpha f\|^2+\|\partial^\alpha\nabla_x\phi\|^2-\int_{\mathbb{R}^3_x}|b|^2(a+2c)dx\right\}+\sum_{|\alpha|\leq N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2\\&\lesssim\ \mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).\end{aligned}\tag{3.5}$$

Step 2. This step is concerned with the desired energy estimates with respect to the weight function $w_\ell(\alpha,\beta)(v)$, which is further divided into the following three sub-steps:

Step 2.1. For the weighted estimates on the terms containing only x derivatives, we can get by replacing $\partial^\alpha f$ with $w_\ell^2(\alpha,0)\partial^\alpha f$ in step 1 and by using (2.1) that

$$\begin{aligned}
&\frac{d}{dt}\sum_{1\leq|\alpha|\leq N}\|w_\ell(\alpha,0)\partial^\alpha f\|^2+\sum_{1\leq|\alpha|\leq N}\|w_\ell(\alpha,0)\partial^\alpha f\|_\sigma^2\\
&\lesssim \sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\nabla_x\phi\|^2+\sum_{1\leq|\alpha|\leq N}\|\partial^\alpha f\|_\sigma^2+\sum_{1\leq|\alpha|\leq N}\underbrace{\left|\left(v\cdot\nabla_x\phi\partial^\alpha f, w_\ell^2(\alpha,0)\partial^\alpha f\right)\right|}_{I_1}\\
&\quad+\sum_{1\leq|\alpha|\leq N,\alpha_1\neq 0}\left|\left(v\cdot\nabla_x\partial^{\alpha_1}\phi\partial^{\alpha-\alpha_1}f, w_\ell^2(\alpha,0)\partial^\alpha f\right)\right|\\
&\quad+\sum_{1\leq|\alpha|\leq N}\left|\left(\partial^\alpha(\nabla_x\phi\cdot\nabla_v f), w_\ell^2(\alpha,0)\partial^\alpha f\right)\right|\\
&\quad+\sum_{1\leq|\alpha|\leq N}\left|\left(\partial^\alpha\mathbf{\Gamma}(f,f), w_\ell^2(\alpha,0)\partial^\alpha f\right)\right|.
\end{aligned}$$

From (2.6), I_1 can be dominated by

$$I_1\lesssim \mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).$$

Moreover, with the help of (2.8), (2.11), and (2.14), the last three terms in the right hand side of the above inequality can be bounded by $C\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)$ $+C\varepsilon\mathcal{D}_{N,\ell}(t)$. Thus we arrive at

$$\begin{aligned}
&\frac{d}{dt}\sum_{1\leq|\alpha|\leq N}\|w_\ell(\alpha,0)\partial^\alpha f\|^2+\sum_{1\leq|\alpha|\leq N}\|w_\ell(\alpha,0)\partial^\alpha f\|_\sigma^2\\
&\lesssim \sum_{1\leq|\alpha|\leq N}\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2+\mathcal{D}_{N,0}(t)+\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)\\
&\quad+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).
\end{aligned}\tag{3.6}$$

Step 2.2. Applying $\{\mathbf{I}-\mathbf{P}\}$ to the first equation of (1.8), we get the micro-equation for f:

$$\begin{aligned}
&\partial_t\{\mathbf{I}-\mathbf{P}\}f+v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f+\nabla_x\phi\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f-\frac{1}{2}v\cdot\nabla_x\phi\{\mathbf{I}-\mathbf{P}\}f\\
&\quad+\mathbf{L}\{\mathbf{I}-\mathbf{P}\}f\\
&=\mathbf{\Gamma}(f,f)+\mathbf{P}\Big(v\cdot\nabla_x f+\nabla_x\phi\cdot\nabla_v f-\frac{1}{2}v\cdot\nabla_x\phi f\Big)-v\cdot\nabla_x\mathbf{P}f\\
&\quad-\nabla_x\phi\cdot\nabla_v\mathbf{P}f+\frac{1}{2}v\cdot\nabla_x\phi\mathbf{P}f.
\end{aligned}\tag{3.7}$$

For brevity, we denote the right hand side of (3.7) by

$$\begin{aligned} I_{mac}(t) \;\equiv\; & \mathbf{P}\Big(v\cdot\nabla_x f+\nabla_x\phi\cdot\nabla_v f-\frac{1}{2}v\cdot\nabla_x\phi f\Big)-v\cdot\nabla_x\mathbf{P}f \\ & -\nabla_x\phi\cdot\nabla_v\mathbf{P}f+\frac{1}{2}v\cdot\nabla_x\phi\mathbf{P}f. \end{aligned}$$

Taking the L^2 inner product of (3.7) with $w_\ell^2(0,0)\{\mathbf{I}-\mathbf{P}\}f$ and by using (2.2), (2.13), and (2.12), one obtains

$$\begin{aligned} & \frac{d}{dt}\|w_\ell(0,0)\{\mathbf{I}-\mathbf{P}\}f\|^2+\|w_\ell(0,0)\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2 \\ \lesssim\; & \|\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2+\underbrace{\big|\big(v\cdot\nabla_x\phi\{\mathbf{I}-\mathbf{P}\}f,w_\ell^2(0,0)\{\mathbf{I}-\mathbf{P}\}f\big)\big|}_{I_2} \\ & +\big|\big(\nabla_x\phi\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f,w_\ell^2(0,0)\{\mathbf{I}-\mathbf{P}\}f\big)\big| \\ & +\big|\big(\mathbf{\Gamma}(f,f),w_\ell^2(0,0)\{\mathbf{I}-\mathbf{P}\}f\big)\big|+\big|\big(I_{mac}(t),w_\ell^2(0,0)\{\mathbf{I}-\mathbf{P}\}f\big)\big| \\ \lesssim\; & \mathcal{D}_{N,0}(t)+\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t), \end{aligned} \tag{3.8}$$

where we have used the following estimates

$$I_2\lesssim\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)+\varepsilon\mathcal{D}_{N,\ell}(t)$$

and

$$\big(I_{mac},w_\ell^2(0,0)\{\mathbf{I}-\mathbf{P}\}f\big)\lesssim\Big\|\mu^\delta\{\mathbf{I}-\mathbf{P}\}f\Big\|_{H^1_xL^2_v}^2+\|\nabla_x(a,b,c)\|^2+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,0}(t),$$

which follows from (2.6).

Step 2.3. For the weighted estimates on the mixed $x-v$ derivatives, applying ∂^α_β with $|\alpha|+|\beta|\leq N, |\beta|\geq 1$ to (3.7), multiplying it by $w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f$ and integrating the final results with respect to v and x over $\mathbb{R}^3_x\times\mathbb{R}^3_v$, we obtain

$$\begin{aligned} & \frac{d}{dt}\big\|w_\ell(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|^2+\big\|w_\ell(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_\sigma^2 \\ \lesssim\; & \sum_{|\beta'|<|\beta|}\big\|w_\ell(\alpha,\beta')\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\big\|_\sigma^2+\eta\sum_{|\beta'|=|\beta|}\big\|w_\ell(\alpha,\beta')\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\big\|_\sigma^2 \\ & +\underbrace{\big|\big(\partial^\alpha_\beta(v\cdot\nabla_x\{\mathbf{I}-\mathbf{P}\}f),w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big)\big|}_{I_3} \\ & +\underbrace{\big|\big(v\cdot\nabla_x\phi\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f,w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big)\big|}_{I_4} \end{aligned}$$

$$
\begin{aligned}
&+\underbrace{\sum_{\alpha_1\leq\alpha,\beta_1\leq\beta}\Big(\partial_{\beta_1}v\cdot\partial^{\alpha_1}\phi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f,w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big)}_{I_5}\\
&+\underbrace{\left|\left(\partial^\alpha_\beta(\nabla_x\phi\cdot\nabla_v\{\mathbf{I}-\mathbf{P}\}f),w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_6}\\
&+\underbrace{\left|\left(\partial^\alpha_\beta(\mathbf{\Gamma}(f,f)),w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_7}\\
&+\underbrace{\left|\left(\partial^\alpha_\beta I_{mac}(t),w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\right|}_{I_8}. \qquad (3.9)
\end{aligned}
$$

Now we estimate $I_j (j=3,\cdots,8)$ term by term. Firstly, I_3 can be bounded by

$$
\begin{aligned}
I_3 &= \Big(\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f,w_\ell^2(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\Big)\\
&= \int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_v}\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}fw_\ell(\alpha+e_i,\beta-e_i)\langle v\rangle^{\frac{\gamma+2}{2}}\\
&\qquad\qquad\cdot\partial_{e_i}\partial^\alpha_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}fw_\ell(\alpha,\beta-e_i)\langle v\rangle^{\gamma+1}\langle v\rangle^{-\frac{\gamma+2}{2}}dvdx\\
&\lesssim \varepsilon\left\|w_\ell(\alpha+e_i,\beta-e_i)\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_\sigma^2+\left\|w_\ell(\alpha,\beta-e_i)\partial^\alpha_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_\sigma^2\\
&\lesssim \varepsilon\left\|w_\ell(\alpha+e_i,\beta-e_i)\partial^{\alpha+e_i}_{\beta-e_i}\{\mathbf{I}-\mathbf{P}\}f\right\|_\sigma^2+\sum_{\substack{|\alpha|+|\beta|\leq N\\|\beta'|=|\beta|-1}}\left\|w_\ell(\alpha,\beta')\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|_\sigma^2.
\end{aligned}
$$

With regard to I_4, (2.7) tells us that

$$
I_4\lesssim\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).
$$

From (2.9), (2.12) and (2.15) respectively, one obtains

$$
I_5+I_6+I_7\lesssim\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t).
$$

As for the last term I_8, it is straight to get that

$$
I_8\lesssim\mathcal{D}_{N,0}(t)+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,0}(t).
$$

Therefore, inserting the above estimates on $I_3\sim I_8$ into (3.9), taking summation over $\{|\beta|=m,|\alpha|+|\beta|\leq N\}$ for each given $1\leq m\leq N$, and then taking a proper linear combination of those $N-1$ estimates with properly chosen constants $C_m > 0(1 \leq m \leq N)$, we obtain

$$\frac{d}{dt}\sum_{m=1}^{N}\sum_{\substack{|\alpha|+|\beta|\leq N\\|\beta|=m}}\left\|w_\ell(\alpha,\beta)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|^2+\sum_{m=1}^{N}\sum_{\substack{|\alpha|+|\beta|\leq N\\|\beta|=m}}\left\|w_\ell(\alpha,\beta)\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_\sigma^2$$
$$\lesssim \sum_{|\alpha|\leq N}\|w_\ell(\alpha,0)\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2+\mathcal{D}_{N,0}(t)+\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)\mathcal{D}_{N,0}(t)$$
$$+\mathcal{E}_{N,0}(t)\mathcal{D}_{N,\ell}(t)+\varepsilon\mathcal{D}_{N,\ell}(t). \tag{3.10}$$

Step 3. With the above estimates in hand, if we assume that $\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)$ is sufficiently small, we can finally get by taking a proper linear combination of (2.30), (3.5), (3.6), (3.8), and (3.10) that

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t)+\mathcal{D}_{N,\ell}(t)\leq 0,\quad 0\leq t<T,$$

which completes the proof of Lemma 3.2. □

Based on the above lemma, we can indeed yield the desired temporal decay of the energy functional $\mathcal{E}_{N,l_1}(t)$ in the following lemma

Lemma 3.3. *Let $-3\leq\gamma<-2$ and assume that the initial perturbation $f_0(x,v)$ satisfies the neutral condition*

$$\int_{\mathbb{R}_x^3}a_0(x)dx\equiv\int_{\mathbb{R}_x^3}\int_{\mathbb{R}_v^3}\mu^{\frac12}(v)f_0(x,v)dvdx=0.$$

Fix parameters N, l_1 as stated in Theorem 1.1 and let $l^*=\frac{\gamma+2}{2(\gamma+1)}$. If we assume further that

$$(H_3)\qquad \sup_{0\leq t<T}\left\{\mathcal{E}_{N,l_1+2l^*}(t)\right\}\leq M$$

holds for some sufficiently small $M>0$. Then, one has

$$\sup_{0\leq s\leq t}\left\{(1+s)^{\frac32}\mathcal{E}_{N,l_1}(s)\right\}\lesssim Y_0^2+X^2(t) \tag{3.11}$$

for any $0\leq t<T$.

Proof. Firstly, if we take $\epsilon>0$ small enough, we can get by noticing that

(3.4) holds true when ℓ there is replaced by ℓ_1 that

$$\frac{d}{dt}\mathcal{E}_{N,l_1}(t)+\mathcal{D}_{N,l_1}(t)\leq 0.$$

Multiplying the above inequality by $(1+t)^{\frac{3}{2}+\epsilon}$ gives

$$\frac{d}{dt}\left\{(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_{N,l_1}(t)\right\}+(1+t)^{\frac{3}{2}+\epsilon}\mathcal{D}_{N,l_1}(t)\lesssim(1+t)^{\frac{1}{2}+\epsilon}\mathcal{E}_{N,l_1}(t). \quad (3.12)$$

Secondly, from the assumption (H_3) and the result obtained in Lemma 3.2, we can deduce that the estimate (3.4) also holds true when ℓ there is replaced by l_1+l^* and l_1+2l^*. Thus

$$\frac{d}{dt}\mathcal{E}_{N,l_1+l^*}(t)+\mathcal{D}_{N,l_1+l^*}(t)\ \leq\ 0 \quad (3.13)$$

and

$$\frac{d}{dt}\mathcal{E}_{N,l_1+2l^*}(t)+\mathcal{D}_{N,l_1+2l^*}(t)\ \leq\ 0 \quad (3.14)$$

hold for all $0\leq t<T$.

Multiplying (3.13) by $(1+t)^{\frac{1}{2}+\epsilon}$ gives

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{\frac{1}{2}+\epsilon}\mathcal{E}_{N,l_1+l^*}(t)\right\}+(1+t)^{\frac{1}{2}+\epsilon}\mathcal{D}_{N,l_1+l^*}(t)\leq(1+t)^{-\frac{1}{2}+\epsilon}\mathcal{E}_{N,l_1+l^*}(t)\\ &\leq\ \mathcal{E}_{N,l_1+l^*}(t). \quad (3.15)\end{aligned}$$

Moreover, (1.18) and (1.19) tell us that

$$\mathcal{E}_{N,\ell}(t)\lesssim\mathcal{D}_{N,\ell+l^*}(t)+\|(b,c)(t)\|^2+\|\nabla_x\phi(t)\|^2$$

holds for any given ℓ. Then, by taking the time integrating over $[0,t]$ of (3.12), (3.15), and (3.14), one can get from an appropriate linear combination of the resulting inequalities that

$$\begin{aligned}&(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_{N,l_1}(t)+(1+t)^{\frac{1}{2}+\epsilon}\mathcal{E}_{N,l_1+l^*}(t)+\mathcal{E}_{N,l_1+2l^*}(t)\\ &\lesssim\ \mathcal{E}_{N,l_1+2l^*}(0)+\int_0^t(1+s)^{\frac{1}{2}+\epsilon}\Big\{\|(b,c)(s)\|^2+\|\nabla_x\phi(s)\|^2\Big\}ds.\end{aligned}$$

Applying the estimate (3.2) to the second term in the right hand side of the above inequality, it follows that

$$
\begin{aligned}
&(1+t)^{\frac{3}{2}+\epsilon}\mathcal{E}_{N,l_1}(t)+(1+t)^{\frac{1}{2}+\epsilon}\mathcal{E}_{N,l_1+l^*}(t)+\mathcal{E}_{N,l_1+2l^*}(t)\\
&\lesssim \mathcal{E}_{N,l_1+2l^*}(0)+(1+t)^{\epsilon}\left\{Y_0^2+X^2(t)\right\},
\end{aligned}
$$

which implies further that

$$
\begin{aligned}
&\sup_{0\leq s\leq t}\left\{(1+s)^{\frac{3}{2}}\mathcal{E}_{N,l_1}(s)+(1+s)^{\frac{1}{2}}\mathcal{E}_{N,l_1+l^*}(s)+(1+s)^{-\epsilon}\mathcal{E}_{N,l_1+2l^*}(s)\right\}\\
&\lesssim Y_0^2+X^2(t) \qquad (3.16)
\end{aligned}
$$

holds for any $0\leq t<T$. Thus we conclude the proof of Lemma 3.3. □

From Lemma 3.2 and Lemma 3.3, it is easy to see that to guarantee the temporal decay estimate of the energy functional $\mathcal{E}_{N,l_1}(t)$, we need only to verify the smallness of $\mathcal{E}_{N,l_1+2l^*+\frac{\gamma}{2(\gamma+1)}}(t)\equiv\mathcal{E}_{N,l_1+\frac{3\gamma+4}{2(\gamma+1)}}(t)$. The main purpose of the rest of this subsection is to do so. To this end, we first have the following result

Lemma 3.4. *Let $-3\leq\gamma<-2$, $N\geq 2$, and $\ell\geq N$, if we assume further that*

$$
(H_4) \qquad \sup_{0\leq t<T}\left\{\mathcal{E}_{N,0}(t)\right\}\leq M
$$

holds for some sufficiently small $M>0$, then we can deduce that

$$
\frac{d}{dt}\mathcal{E}_{N,\ell}(t)+\mathcal{D}_{N,\ell}(t)\lesssim\|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,\ell+\frac{1}{2}}(t) \qquad (3.17)
$$

holds for any $0\leq t<T$.

Proof. If one checks the proofs of Lemma 3.2, it is easy to find that for the weight function $w_\ell(\alpha,\beta)(v)$ given by (1.13) with $\ell\geq N$ and $N\geq 2$, if we perform the weighted energy estimate with respect to the weight function $w_\ell(\alpha,\beta)(v)$ as in Lemma 3.2, all the nonlinear terms involved except the terms I_1, I_2, and I_4 are bounded by $(\mathcal{E}_{N,0}(t)+\varepsilon)\,\mathcal{D}_{N,\ell}(t)$ with $\varepsilon>0$ being any sufficiently small positive constant, which can further be absorbed by the corresponding energy dissipation rate functional $\mathcal{D}_{N,\ell}(t)$ if $\mathcal{E}_{N,0}(t)$ is assumed to be sufficiently small for all $0\leq t<T$. Thus to conclude the proof of Lemma 3.2, we only need to bound I_1, I_2, and I_4 suitably. To this end, since

such three terms can be treated by the same way, we just give the estimate on I_4 as follows

$$
\begin{aligned}
I_4 &\lesssim \left|\left(v\cdot\nabla_x\phi\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f, w^2_\ell(\alpha,\beta)\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right)\right| \\
&\lesssim \int_{\mathbb{R}^3_x}|\nabla_x\phi|\left|\langle v\rangle^{\frac12}w_\ell(\alpha,\beta)\{\mathbf{I}-\mathbf{P}\}f\right|^2_{L^2_v}dx \\
&\lesssim \|\nabla_x\phi(t)\|_{L^\infty_x}\left\|\langle v\rangle^{-\frac{\gamma+1}{2}}w_\ell(\alpha,\beta)\{\mathbf{I}-\mathbf{P}\}f\langle v\rangle^{\frac{\gamma+2}{2}}\right\|^2 \\
&\lesssim \|\nabla_x\phi(t)\|_{L^\infty_x}\left\|\langle v\rangle^{-\frac{\gamma+1}{2}}w_\ell(\alpha,\beta)\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\sigma \\
&\lesssim \|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,\ell+\frac12}(t). \qquad (3.18)
\end{aligned}
$$

Here we have used the fact the $-3\leq\gamma<-2$.

With the above observation and the estimates on $I_j(j=1,2,4)$ in hand, the estimate (3.17) follows easily.

It is worth pointing out that although in the proof of (3.17), we do not need the smallness of $\mathcal{E}_{N,\ell+\frac{\gamma}{2(\gamma+1)}}(t)$ as in the proof of Lemma 3.2, we note however, that the estimates on I_1, I_2 and I_4 lead to the new term $\|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,\ell+\frac12}(t)$. This completes the proof of Lemma 3.4. $\square$

In order to absorb the term $\|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,\ell+\frac12}(t)$, which appears in the right hand side of (3.17), we need the following lemma, which plays a key role in closing the time-velocity weighted energy estimates.

Lemma 3.5. *Assume $-3\leq\gamma<-2$, $N\geq 2$, and $\ell\geq N$. If we assume further that*

$$(H_5) \qquad Y_0\leq M$$

and

$$(H_6) \qquad X(t)\leq M,\quad 0\leq t<T$$

hold for some sufficiently small positive constant $M>0$, then it follows that

$$\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_{N,\ell}(t)\right\}+(1+t)^{-1-\epsilon_0}\mathcal{E}_{N,\ell}(t)+(1+t)^{-\epsilon_0}\mathcal{D}_{N,\ell}(t)\leq 0 \quad (3.19)$$

holds for any $0\leq t<T$.

Proof. Firstly, repeating the analysis performed in step 1 and step 2 in the proof of Lemma 3.2, if we introduce the weight function $e^{-\phi}$ as in [9] to the weighted energy estimates done in step 1 and step 2 in the proof of Lemma 3.2, it is easy to see that the terms $I_j (j = 1, 2, 4)$ and as a consequence, the term $\|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,\ell+\frac{1}{2}}(t)$ in the right hand side of (3.17) does not appear. We note, however, that this argument will induce the additional term $\|\partial_t\phi(t)\|_{L^\infty_x}\mathcal{E}_{N,\ell}(t)$. And we can deduce that

$$\frac{d}{dt}\mathcal{E}_{N,\ell}(t) + \mathcal{D}_{N,\ell}(t) \lesssim \|\partial_t\phi(t)\|_{L^\infty_x}\mathcal{E}_{N,\ell}(t). \tag{3.20}$$

Since, for the solution of the Cauchy problem (1.8) of the one-species VPL system, as pointed out in the introduction, the temporal decay of the electric potential $\phi(t, x)$ is worse such that $\|\partial_t\phi(t)\|_{L^\infty_x} \in L^1(\mathbb{R}^+)$ can not hold any more, to control such a term, we can get by multiplying (3.20) by $(1 + t)^{-\epsilon_0}$ for some $\epsilon_0 > 0$ to yield that

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_{N,\ell}(t)\right\} + \epsilon_0(1+t)^{-1-\epsilon_0}\mathcal{E}_{N,\ell}(t) + (1+t)^{-\epsilon_0}\mathcal{D}_{N,\ell}(t)\\ &\lesssim\ (1+t)^{-\epsilon_0}\ \|\partial_t\phi(t)\|_{L^\infty_x}\ \mathcal{E}_{N,\ell}(t).\end{aligned} \tag{3.21}$$

To control the term in the right hand side of (3.21), recalling $(2.26)_1$, it holds that

$$\partial_t\phi = \Delta_x^{-1}\partial_t a = -\Delta_x^{-1}\nabla_x\cdot b,$$

then, one has from the estimate (3.2) obtained in Lemma 3.1 and the Sobolev equality that

$$\begin{aligned}\|\partial_t\phi(t)\|_{L^\infty_x} &\lesssim \|\nabla_x\partial_t\phi(t)\|^{\frac{1}{2}}\left\|\nabla_x^2\partial_t\phi(t)\right\|^{\frac{1}{2}}\\ &\lesssim \|b(t)\|^{\frac{1}{2}}\|\nabla_x b(t)\|^{\frac{1}{2}}\\ &\lesssim \|f(t)\|^{\frac{1}{2}}\|\nabla_x f(t)\|^{\frac{1}{2}}\\ &\lesssim (1+t)^{-1}\Big(X^{\frac{1}{2}}(t) + Y_0^{\frac{1}{2}}\Big).\end{aligned}$$

Thus the term $(1 + t)^{-\epsilon_0}\|\partial_t\phi(t)\|_{L^\infty_x}\mathcal{E}_{N,\ell}(t)$ in the right hand side of (3.21) can be controlled by the second term $(1+t)^{-1-\epsilon_0}\mathcal{E}_{N,\ell}(t)$ in the left hand side of (3.21) if the assumptions (H_5) and (H_6) are satisfied, which gives (3.19). This completes the proof of Lemma 3.5. □

Based on the above lemmas, we are ready to deduce the uniform-in-time boundedness of the energy functional $\mathcal{E}_{N,l_2}(t)$ and the temporal growth rate of the energy functional $\mathcal{E}_{N,l_3}(t)$, which are the main content of the following lemma.

Lemma 3.6. *Under the assumptions listed in Lemma* 3.5 *and let* $l_2 \geq l_1 + \frac{3\gamma+4}{2(\gamma+1)}$, $l_3 \geq l_2 + \frac{1}{2}$, *and* $0 < \epsilon_0 \leq 1$, *then it is follows that*

$$\mathcal{E}_{N,l_2}(t) + (1+t)^{-\epsilon_0}\mathcal{E}_{N,l_3}(t) \lesssim Y_0^2 \tag{3.22}$$

for any $0 \leq t < T$.

Proof. Taking $\ell = l_2 \geq l_1 + \frac{3\gamma+4}{2(\gamma+1)}$ in (3.17) and $\ell = l_3 \geq l_2 + \frac{1}{2}$ in (3.19), respectively, we obtain

$$\frac{d}{dt}\mathcal{E}_{N,l_2}(t) + \mathcal{D}_{N,l_2}(t) \lesssim \|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,l_3}(t) \tag{3.23}$$

and

$$\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_{N,l_3}(t)\right\} + (1+t)^{-1-\epsilon_0}\mathcal{E}_{N,l_3}(t) + (1+t)^{-\epsilon_0}\mathcal{D}_{N,l_3}(t) \lesssim 0. \tag{3.24}$$

On the other hand, the estimate (3.2) obtained in Lemma 3.1 and the estimate (3.11) obtained in Lemma 3.3 together with the Gagliardo-Nirenberg inequality tell us that

$$\begin{aligned}\|\nabla_x\phi(t)\|_{L^\infty_x} &\lesssim \left\|\nabla_x^2\phi(t)\right\|^{\frac{1}{2}}\left\|\nabla_x^3\phi(t)\right\|^{\frac{1}{2}}\\ &\lesssim (1+t)^{-\frac{5}{4}}\left(X^{\frac{1}{2}}(t) + Y_0^{\frac{1}{2}}\right).\end{aligned}$$

Consequently, if we take $0 < \epsilon_0 \leq 1$ in (3.24), the term $\|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,l_3}(t)$ appeared in the right hand side of (3.23) can be bounded by

$$\begin{aligned}\|\nabla_x\phi(t)\|_{L^\infty_x}\mathcal{D}_{N,l_3}(t) &\lesssim (1+t)^{-\frac{5}{4}}\left(X^{\frac{1}{2}}(t) + Y_0^{\frac{1}{2}}\right)\mathcal{D}_{N,l_3}(t)\\ &\lesssim (1+t)^{-\epsilon_0}\left(X^{\frac{1}{2}}(t) + Y_0^{\frac{1}{2}}\right)\mathcal{D}_{N,l_3}(t),\end{aligned}$$

which can further be controlled by the second term in the left hand side of (3.24) successfully provided that $X(t)$ and Y_0 are chosen sufficiently small.

Therefore, combing (3.23) with (3.24) gives

$$\frac{d}{dt}\left\{\mathcal{E}_{N,l_2}(t)+(1+t)^{-\epsilon_0}\mathcal{E}_{N,l_3}(t)\right\}\leq 0, \tag{3.25}$$

which implies that

$$\mathcal{E}_{N,l_2}(t)+(1+t)^{-\epsilon_0}\mathcal{E}_{N,l_3}(t)\lesssim \mathcal{E}_{N,l_3}(0).$$

This is (3.22) and the proof of Lemma 3.6 is complete. □

3.2. The proof of Theorem 1.1

This subsection is devoted to proving our main result Theorem 1.1 by the continuation argument. Since the local solvability result to the Cauchy problem (1.8) in certain weighted Sobolev space is well-established in [9, 15, 18], to extend such a solution $f(t,x,v)$ step by step to a global one, all that we need to do is to deduce some uniform-in-time a priori estimates on $f(t,x,v)$ in the same weighted Sobolev space. To this end, suppose that the local solution $f(t,x,v)$ to the Cauchy problem (1.8) constructed in [9, 15, 18] has been extended to the time interval $[0,T]$ for some $0<T<\infty$ and satisfies the a priori assumption

$$X(t)\leq M,\quad 0\leq t<T \tag{3.26}$$

for some sufficiently small positive constant $M>0$, now we turn to deduce certain a priori estimates on $f(t,x,v)$ such that the a priori assumption (3.26) can be closed. Here the parameters N,l_1,l_2,l_3, and ϵ_0 satisfy the conditions listed in Theorem 1.1.

In fact, if Y_0 is sufficiently small and the a priori assumption (3.26) is assumed to be hold, then the conditions listed in Lemmas 3.1-3.6 are satisfied, thus we can get from the definition of $X(t)$ and the estimates (3.2), (3.11), and (3.22) obtained in Lemmas 3.1-3.6 respectively that

$$X(t)\lesssim Y_0^2+X^2(t),\quad 0\leq t<T, \tag{3.27}$$

from which one can then deduce that

$$X(t)\lesssim Y_0^2 \tag{3.28}$$

holds for all $t \in [0, T]$ if Y_0 is suitably chosen such that

$$Y_0 \leq \delta_1$$

for some positive constant $\delta_1 > 0$.

The (3.28) not only yields the desired unform-in-time estimate on $f(t, x, v)$, but also verifies the a priori assumption (3.26) if Y_0 is chosen to be sufficiently small further such that

$$Y_0 \leq \delta_2$$

for some positive constant $\delta_2 > 0$.

Thus if the initial perturbation $f_0(x, v)$ is assumed to be sufficiently small such that

$$Y_0 \leq \min\{\delta_1, \delta_2\},$$

then the global existence result to the Cauchy problem (1.8) follows from the local existence result obtained in [9, 15, 18] and the standard continuity argument in the usual way. As a byproduct, the temporal decay estimates (1.20) and (1.21) follow directly from the definition of $X(t)$. Thus we have completed the proof of Theorem 1.1.

Acknowledgment

This work was supported by the Fundamental Research Funds for the Central Universities and three grants from the National Natural Science Foundation of China under contracts 10925103, 11271160, and 11261160485, respectively.

References

1. R. Alexandre and C. Villani C., On the Landau approximation in plasma physics, *Ann. Inst. H. Poincare Anal. Non Lineaire*, **21** (2004), No. 1, 61-95.

2. A. A. Arsenev and O. E. Buryak, On the connection between a solution of the Boltzmann equation and a solution of the Landau-Fokker-Planck equation, *Math. USSR. Sbornik*, **69** (1991), No. 2, 465-478.

3. P. Degond and M. Lemou, Dispersion relations for the linearized Fokker-Planck equation, *Arch. Ration. Mech. Anal.*, **138** (1997), 137-167.

4. R.-J. Duan, T. Yang and H.-J. Zhao, The Vlasov-Poisson-Boltzmann system in the whole space: The hard potential case, *J. Differential Equations*, **252** (2012), No.12, 6356-6386.

5. R.-J. Duan, T. Yang and H.-J. Zhao, The Vlasov-Poisson-Boltzmann system for soft potentials. *Mathematical Models and Methods in Applied Sciences*, **23**(2013), No.6, 979-1028.

6. R.-J. Duan, T. Yang and H.-J. Zhao, Global solutions to the Vlasov-Poisson-Landau system, Preprint 2011. See also arXiv:1112.3261.

7. Y. Guo, The Landau equation in a periodic box, *Comm. Math. Phys.*, **231**(2002), 391-434.

8. Y. Guo, The Boltzmann equation in the whole space, *Indiana Univ. Math. J.*, **53**(2004), 1081-1094.

9. Y. Guo, The Vlasov-Poisson-Laudau system in a periodic box, *J. Amer. Math. Soc.*, **25** (2012), 759-812.

10. F. Hilton, Collisional transport in plasma (M. N. Rosenbluth and R. Z. Sagdeev, eds.), Handbook of Plasma Physics, Volume I: Basic Plasma Physics I, North-Holland Publishing Company, 1983, pp. 147.

11. N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics*, McGraw-Hill, 1973.

12. Y.-J. Lei, L.-J. Xiong and H.-J. Zhao, One-species Vlasov-Poisson-Landau system near Maxwellians in the whole space, *Kinetic and Related Models*, **7**(2014), No. 3, 551-590.

13. P.-L. Lions, On Boltzmann and Landau equations. *Phil Trans. R. Soc. Lond. A*, **346** (1994), 191-204.

14. T.-P. Liu, T. Yang and S.-H. Yu, Energy method for the Boltzmann equation. *Physica D*, **188** (2004), 178-192.

15. R. M. Strain and K.-Y. Zhu, The Vlasov-Poisson-Landau System in $\mathbb{R}^3_x$, *Arch. Ration. Mech. Anal.*, **210** (2013), No.2, 615-671.

16. C. Villani, On the Cauchy problem for Landau equation: Sequential stability, global existence, *Adv. Diff. Eq.*, **1** (1996), No. 5, 793-816.

17. C. Villani, A review of mathematical topics in collisional kinetic theory, North-Holland, Amsterdam, Handbook of mathematical fluid dynamics, Vol. I, 2002, 71-305.

18. Y.-J. Wang, Golobal solution and time decay of the Vlasov-Poisson-Landau System in $\mathbb{R}^3_x$, *SIAM J. Math. Anal.*, **44** (2012), No.5, 3281-3323.