

CRITICAL EXPONENTS FOR THE CAUCHY PROBLEM TO THE SYSTEM OF WAVE EQUATIONS WITH TIME OR SPACE DEPENDENT DAMPING

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This paper is dedicated to the 70th birthday of Professor Tai-Ping Liu.

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Abstract

We consider the Cauchy problem for the system of weakly coupled semilinear wave equations with space or time dependent damping. Our aim is to determine the critical exponents, when the coupled semilinear terms are polynomial orders and the damping is effective. In fact, the critical exponents are completely determined in the cases of only space dependent damping and only time dependent damping. But, in the case of both space and time dependent damping it remains open, in which the blow-up result is not obtained even for the scalar damped wave equation.

1. Introduction

In this paper we consider the Cauchy problem for 2×2 weakly coupled system of wave equations with time or space dependent damping

$$\begin{cases} u_{tt} - \Delta u + b(t, x)u_t = |v|^p, \\ v_{tt} - \Delta v + b(t, x)v_t = |u|^q, & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t, v, v_t)(0, x) = \varepsilon(u_0, u_1, v_0, v_1)(x), & x \in \mathbf{R}^N, \end{cases} \quad (1.1)$$

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where $p, q > 1, \varepsilon > 0$ and

$$b = b(t, x) = \begin{cases} b_1(x) & := \langle x \rangle^{-\alpha} & (0 \leq \alpha < 1) \\ b_2(t) & := (t + 1)^{-\beta} & (-1 < \beta < 1) \\ b_3(t, x) & := \langle x \rangle^{-\alpha}(t + 1)^{-\beta} & (\alpha, \beta \geq 0, \alpha + \beta < 1). \end{cases} \tag{1.2}$$

with $\langle x \rangle = (1 + |x|^2)^{1/2}$. Our main interest is to obtain the critical exponents for (1.1). When $b(t, x)$ is a positive constant (set $b(t, x) = 1$ without loss of generality), (1.1) is reduced to

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^p, \\ v_{tt} - \Delta v + v_t = |u|^q, & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t, v, v_t)(0, x) = \varepsilon(u_0, u_1, v_0, v_1)(x), & x \in \mathbf{R}^N, \end{cases} \tag{1.3}$$

and the critical exponents are given by

$$\Lambda = \frac{N}{2}, \quad \Lambda := \max(\Lambda_1, \Lambda_2) := \max\left(\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right), \tag{1.4}$$

in the sense that, if $\Lambda < \frac{N}{2}$ (supercritical exponents), then (1.3) has a unique global-in-time solution for sufficiently small data, while, if $\Lambda \geq \frac{N}{2}$ (critical and subcritical exponents), then the solution to (1.3) blows up in a finite time for suitable data.

For the scalar damped wave equations

$$\begin{cases} u_{tt} - \Delta u + b(t, x)u_t = |u|^\rho, & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1), & x \in \mathbf{R}^N, \end{cases} \tag{1.5}$$

with $1 < \rho < \frac{N+2}{[N-2]_+} = \infty$ ($N = 1, 2$) and $= \frac{N+2}{N-2}$ ($N \geq 3$), there are many literatures to this topics. Refer [3, 4, 5, 6, 10, 11, 12, 15, 16, 19, 20, 31, 36] for $b = 1$, [7, 8, 21] for $b = b_1(x)$, [14, 22, 24, 32, 34, 35] for $b = b_2(t)$, [13, 33] for $b = b_3(t, x)$, and references therein.

Let us consider (1.5) from the viewpoint of the diffusion phenomena of solutions of the damped wave equation. The solution of damped wave equation has been recognized to behave like that of the corresponding diffusive equation. In [20] it is shown that the solution $V(t, x)$ to

$$V_{tt} - \Delta V + V_t = 0, \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N \quad (N = 3) \tag{1.6}$$

with the data $(V, V_t)(0, x) = (0, g)(x)$, $x \in \mathbf{R}^3$, has the form

$$\begin{aligned} V(t, x) &= e^{-\frac{t}{2}} \cdot \frac{t}{4\pi} \int_{|\omega|=1} g(x + t\omega) d\omega + \int_{|z|\leq t} \frac{e^{-\frac{t}{2}} I_1(\frac{1}{2}\sqrt{t^2 - |z|^2})}{8\pi t \sqrt{t^2 - |z|^2}} g(x + z) dz \\ &=: e^{-\frac{t}{2}} \cdot [W_3(t)g](x) + [J_{30}(t)g](x), \end{aligned} \tag{1.7}$$

where $I_\nu(y)$ is the modified Bessel function of order ν and $W_3(t)g$ is the Kirchhoff formula of the solution to the 3-dimensional wave equation. Moreover, $J_{30}(t)g$ satisfies the following estimates

$$\begin{aligned} \|J_{30}(t)g\|_{L^p} &\leq C(t + 1)^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p})} \|g\|_{L^q} \quad (t \geq 0), \\ \|(J_{30}(t) - e^{t\Delta})g\|_{L^p} &\leq Ct^{-\frac{3}{2}(\frac{1}{q} - \frac{1}{p}) - 1} \|g\|_{L^q} \quad (t > 0) \end{aligned}$$

for $1 \leq q \leq p \leq \infty$ (see [15] for $N = 1$, [5] for $N = 2$, and [28] in case of general dimension). In the results, the solution of (1.6) behaves like that of the corresponding diffusive equation, whose property is called the diffusion phenomena. Note that $V(t, x)$ still has the wave property from the first term in the right hand side of (1.7) though this decays exponentially. Thus we can expect that the critical exponent $\rho_c(N)$ for (1.5) is the same as that for the corresponding diffusive equation

$$-\Delta\phi + b(t, x)\phi_t = |\phi|^\rho, \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N \tag{1.8}$$

if the damping term is effective. Note that, if it is not effective, then the solution has the wave property. (See Mochizuki [18], Wirth [34, 35].) When $b(t, x) = 1$, the damping term $+b(t, x)u_t$ is effective and the critical exponent $\rho_c(N)$ for (1.5) is the same as the critical exponent

$$\rho_F(N) = 1 + \frac{2}{N} \tag{1.9}$$

for (1.8), which is called the Fujita exponent, named after his pioneering work [2]. The critical exponent for (1.5) is known in the references listed above as follows:

- (i) $b = b_1(x) = \langle x \rangle^{-\alpha}$ ($0 \leq \alpha < 1$) $\Rightarrow \rho_c(N, \alpha) = 1 + \frac{2}{N-\alpha}$,
- (ii) $b = b_2(t) = (t + 1)^{-\beta}$ ($-1 < \beta < 1$) $\Rightarrow \rho_c(N, \beta) = 1 + \frac{2}{N} (= \rho_F(N))$.

When $b = b_3(t, x)$, the critical exponent $\rho_c(N, \alpha, \beta)$ is not known. Though there exists a unique global-in-time solution for small data if $\rho > 1 + \frac{2}{N-\alpha}$, the blow-up result is not yet known. Our conjecture is

$$\begin{aligned} \text{(iii)} \quad & b = b_3(t, x) = \langle x \rangle^{-\alpha} (t+1)^{-\beta} \quad (\alpha, \beta \geq 0, \alpha + \beta < 1) \\ & \Rightarrow \rho_c(N, \alpha, \beta) = 1 + \frac{2}{N-\alpha}. \end{aligned}$$

We shall apply these considerations on the scalar damped wave equations to the system (1.1). In fact, the system of diffusive equations

$$\begin{cases} -\Delta\phi + \phi_t = |\psi|^p, \\ -\Delta\psi + \psi_t = |\phi|^q, \end{cases} \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N, \quad (1.10)$$

corresponding to (1.3), was considered in [1] and (1.4) was shown to be critical. After [1], the critical exponents for (1.3) have been investigated to be the same as for (1.4). Refer Sun and Wang [29], Narazaki [19], Nishihara [23], Nishihara and Wakasugi [25], Takeda [30], Ogawa and Takeda [26, 27] etc.

Our aim in this paper is to show that the critical exponents are given by

$$\begin{aligned} \text{(I)} \quad & b = b_1(x) = \langle x \rangle^{-\alpha} \quad (0 \leq \alpha < 1) \Rightarrow \Lambda = \frac{N-\alpha}{2}, \\ \text{(II)} \quad & b = b_2(t) = (t+1)^{-\beta} \quad (-1 < \beta < 1) \Rightarrow \Lambda = \frac{N}{2}, \end{aligned}$$

where Λ is given in (1.4). Note that in Case (II) the critical exponents are independent of β . In Case (III) of $b = b_3(t, x)$ the critical exponents are not obtained as same as the scalar case. But, we conjecture that $\Lambda = \frac{N-\alpha}{2}$ gives the critical exponents. In fact, we show that there exists a unique global-in-time solution for suitably small data when $\Lambda < \frac{N-\alpha}{2}$, which also covers the small data global existence of solutions in the supercritical exponent cases in both Cases (I) and (II). While the blow-up result is not known in Case (III).

We now define the solution to (1.1) and state our results. Some notations can be referred at the end of this section.

The function $(u, v) \in [C([0, T]; H^1) \cap C^1([0, T]; L^2)]^2$ for $T > 0$ is said to be a weak solution to (1.1) on $[0, T]$ if, for any $\psi \in C_0^\infty([0, T] \times \mathbf{R}^N)$,

$$\begin{aligned} & \int_{[0, T] \times \mathbf{R}^N} \{u(\psi_{tt} - \Delta\psi - (b\psi)_t)(t, x) - |v|^p\psi(t, x)\} dx dt \\ &= \varepsilon \int_{\mathbf{R}^N} \{(b(0, x)u_0(x) + u_1(x))\psi(0, x) - u_0(x)\psi_t(0, x)\} dx, \\ & \int_{[0, T] \times \mathbf{R}^N} \{v(\psi_{tt} - \Delta\psi - (b\psi)_t)(t, x) - |u|^q\psi(t, x)\} dx dt \\ &= \varepsilon \int_{\mathbf{R}^N} \{(b(0, x)v_0(x) + v_1(x))\psi(0, x) - v_0(x)\psi_t(0, x)\} dx. \end{aligned}$$

The local existence of the solution is obtained in a standard way (cf. [13, 33]).

Proposition 1.1. *Assume $b = b_3(t, x) = \langle x \rangle^{-\alpha}(t + 1)^{-\beta}$ ($\alpha, \beta \geq 0, \alpha + \beta < 1$), $1 < p, q \leq \frac{N}{[N-2]_+}$ and $(u_0, u_1, v_0, v_1) \in [H^1 \times L^2]^2$ with compact supports. Then there exists a unique weak solution $(u, v) \in [C([0, T]; H^1) \cap C^1([0, T]; L^2)]^2$ to (1.1) for some $T > 0$.*

Our main results are following two theorems.

Theorem 1.1 (Global-in-time solution). *Under the assumptions in Proposition 1.1, if $\varepsilon > 0$ is suitably small, then there exists a unique weak solution $(u, v) \in [C([0, \infty); H^1) \cap C^1([0, \infty); L^2)]^2$ to (1.1) when $\Lambda < \frac{N-\alpha}{2}$.*

Theorem 1.2 (Blow-up of solutions). *Under the assumptions in Proposition 1.1, let $(u, v) \in [C([0, T]; H^1) \cap C^1([0, T]; L^2)]^2$ ($0 < T < \infty$) be the solution to (1.1). Then the following two assertions hold.*

(I) (Damping of space dependent coefficient) *When $b = b_1(x) = \langle x \rangle^{-\alpha}$ ($0 \leq \alpha < 1$) and $\Lambda \geq \frac{N-\alpha}{2}$, if the data satisfy*

$$\int_{\mathbf{R}^N} (\langle x \rangle^{-\alpha} u_0(x) + u_1(x)) dx > 0, \quad \int_{\mathbf{R}^N} (\langle x \rangle^{-\alpha} v_0(x) + v_1(x)) dx > 0,$$

then the solution (u, v) does not exist time-globally.

(II) (Damping of time dependent coefficient) *When $b = b_2(t) = (t + 1)^{-\beta}$ ($-1 < \beta < 1$) and $\Lambda \geq \frac{N}{2}$, if the data satisfy*

$$\int_{\mathbf{R}^N} (u_0(x) + g(0)u_1(x)) dx > 0, \quad \int_{\mathbf{R}^N} (v_0(x) + g(0)v_1(x)) dx > 0$$

with $g(0) = \int_0^\infty \exp(-\int_0^\tau b_2(s) ds) d\tau$, then the solution (u, v) does not exist time-globally.

Remark 1.1. Here we derive how to get $\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}$ in (1.4), which will play an important role in the proof of Theorem 1.1. Let $P = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$, where p, q are the exponents of semilinear terms. Then $P - I = \begin{pmatrix} -1 & p \\ q & -1 \end{pmatrix}$ is regular and

$$(P - I)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{1 - pq} \begin{pmatrix} -1 & -p \\ -q & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}. \quad (1.11)$$

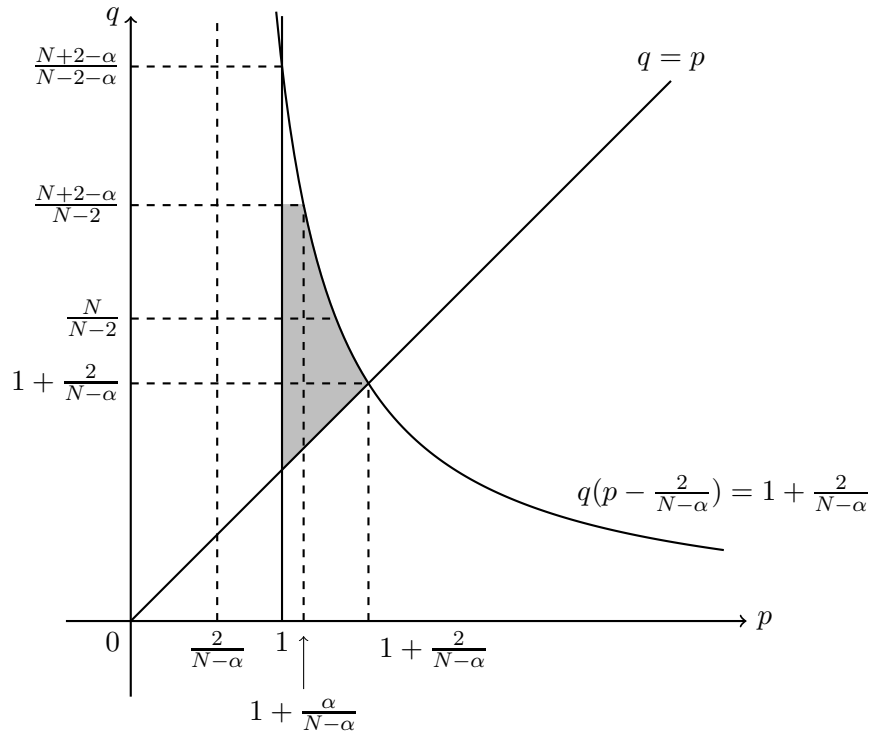
Remark 1.2. In Theorem 1.2, if there exists a solution $(u, v) \in [C([0, T]; H^1) \cap C^1([0, T]; L^2)]^2$ without the restriction on p, q , then the assertion (II) is still true for any $p, q > 1$. However, when $\alpha > 0$, for the assertion (I), we need the condition

$$p, q \leq \frac{N + 2 - \alpha}{[N - 2]_+} \quad (1.12)$$

in addition to $\Lambda \geq \frac{N - \alpha}{2}$. In other words, when $N = 1, 2$, this additional restriction (1.12) is not necessary since any $p, q (1 < p, q < \infty)$ are admitted. When $N \geq 3$, for $q \geq p$ without loss of generality, $\Lambda \geq \frac{N - \alpha}{2}$ is equivalent to

$$q(p - \frac{2}{N - \alpha}) \leq 1 + \frac{2}{N - \alpha}.$$

So, for $(p, q) \in \{(p, q) | 1 < p < 1 + \frac{\alpha}{N - \alpha}, q > \frac{N + 2 - \alpha}{N - 2}, q(p - \frac{2}{N - \alpha}) \leq 1 + \frac{2}{N - \alpha}\}$, we don't know whether the blow-up result holds or not. See Figure.



Our plan of this paper is as follows. In Section 2 we prove Theorem 1.1. Theorem 1.2 will be shown in Section 3. In the final section we discuss further problems.

We finish up this section by introducing some notations. By $L^p = L^p(\mathbf{R}^N)$, $H^1 = H^1(\mathbf{R}^N)$, denote the usual Lebesgue space and Sobolev space, respectively, equipped with the norms $\|f\|_{L^p} = (\int_{\mathbf{R}^N} |f(x)|^p dx)^{1/p}$ ($1 \leq p < \infty$), $\|f\|_{L^\infty} = \text{ess. sup}_{x \in \mathbf{R}^N} |f(x)|$, and $\|f\|_{H^1} = (\sum_{|\alpha| \leq 1} \|D^\alpha f\|_{L^2}^2)^{1/2}$. For an interval I and Banach space X , define $C^m(I; X)$ as the space of m -times continuously differentiable map from I to X . Also, by C or C_i ($i = 1, 2, \dots$) we indicate the positive constant which may change from line to line. We write $C_i(a, b, c, \dots)$ when we emphasize the dependence on a, b, c, \dots

2. Existence of Global-in-time Solution in the Supercritical Exponents

Since Proposition 1.1 is shown in a standard way, in order to prove Theorem 1.1, it suffices to derive the a priori estimates.

We first derive the linear estimate on a weak solution $u \in C([0, T]; H^1) \cap C^1([0, T]; L^2)$ to the single damped wave equation (first equation of (1.1))

$$\begin{cases} u_{tt} - \Delta u + b(t, x)u_t = f := |v|^p, & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N \\ (u, u_t)(0, x) = \varepsilon(u_0, u_1)(x), & x \in \mathbf{R}^N, \end{cases} \quad (2.1)$$

where $b(t, x) = b_3(t, x) = \langle x \rangle^{-\alpha}(t+1)^{-\beta}$ ($\alpha, \beta \geq 0, \alpha + \beta < 1$). The derivation of the following lemma is almost same as in Lin, Nishihara and Zhai [13] and Wakasugi [33].

Lemma 2.1. *Define $\psi(t, x) = a_1 \frac{\langle x \rangle^{2-\alpha}}{(1+t)^{1+\beta}}$ with $a_1 = \frac{(1+\beta)}{(2-\alpha)^2(2+\delta)}$ ($\delta > 0$), then it holds that for a suitably small constant $\bar{\varepsilon} > 0$*

$$\begin{aligned} & (t+1)^{l_1-\bar{\varepsilon}} \bar{E}(t) + \int_0^t (\tau+1)^{l_1-\bar{\varepsilon}} \left\{ E(\tau) + \frac{B-l_1}{\tau+1} \int_{\mathbf{R}^N} b(\tau, x) e^{2\psi} u^2 dx \right\} d\tau \\ & \leq C\varepsilon^2 I_1 + C \int_0^t \int_{\mathbf{R}^N} (\tau+1)^{l_1-\bar{\varepsilon}} f e^{2\psi} (|u| + (\tau+1)^{\alpha+\beta} |u_t|) dx d\tau \end{aligned} \quad (2.2)$$

and

$$\begin{aligned} & (t+1)^{l_1+1-\bar{\varepsilon}} E(t) + \int_0^t (\tau+1)^{l_1+1-\bar{\varepsilon}} \int_{\mathbf{R}^N} b(\tau, x) e^{2\psi} u_t^2 dx d\tau \\ & \leq C\varepsilon^2 I_1 + C \int_0^t \int_{\mathbf{R}^N} (\tau+1)^{l_1-\bar{\varepsilon}} f e^{2\psi} (|u| + (\tau+1)|u_t|) dx d\tau \end{aligned} \quad (2.3)$$

for

$$l_1 \leq B := \frac{(1+\beta)(N-\alpha)}{2-\alpha} + \beta, \quad (2.4)$$

where

$$E(t) = \int_{\mathbf{R}^N} e^{2\psi} (u_t^2 + |\nabla u|^2)(t, x) dx, \quad (2.5)$$

$$\bar{E}(t) = E(t) + \int_{\mathbf{R}^N} e^{2\psi} b(t, x) u^2(t, x) dx, \quad (2.6)$$

$$I_1 = I(u_0, u_1) := \int_{\mathbf{R}^N} e^{2\psi(0, x)} (u_1^2 + |\nabla u_0|^2 + \langle x \rangle^{-\alpha} u_0^2) dx. \quad (2.7)$$

Remark 2.3. The estimates (2.2) and (2.3) with $l_1 = B$ are derived in Section 2 in [33] together with the calculations of the right hand side. Here

we have only estimated the left hand side and given the linear estimates.

Proof. We only sketch the proof following [33]. The details are referred in [33]. Multiplying (2.1) by $e^{2\psi}u_t$ and $e^{2\psi}u$, we have

$$\begin{aligned}
 fe^{2\psi}u_t &= \frac{\partial}{\partial t} \left[\frac{1}{2}e^{2\psi}(u_t^2 + |\nabla u|^2) \right] - \nabla \cdot (e^{2\psi}u_t \nabla u) \\
 &\quad + e^{2\psi} \left(b(t, x) - \frac{|\nabla \psi|^2}{-\psi_t} - \psi_t \right) u_t^2 + \frac{e^{2\psi}}{-\psi_t} |\psi_t \nabla u - u_t \nabla \psi|^2 \\
 &\geq \frac{\partial}{\partial t} \left[\frac{1}{2}e^{2\psi}(u_t^2 + |\nabla u|^2) \right] - \nabla \cdot (e^{2\psi}u_t \nabla u) \\
 &\quad + e^{2\psi} \left\{ \left(\frac{1}{4}b(t, x) - \psi_t \right) u_t^2 + \frac{1}{5}(-\psi_t) |\nabla u|^2 \right\}
 \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
 fe^{2\psi}u &= \frac{\partial}{\partial t} \left[e^{2\psi}(uu_t + \frac{1}{2}b(t, x)u^2) \right] - \nabla \cdot (e^{2\psi}u \nabla u) \\
 &\quad + e^{2\psi} \left\{ |\nabla u|^2 + \left(-\psi_t + \frac{\beta}{2(t+1)} \right) b(t, x)u^2 + 2u \nabla \psi \cdot \nabla u - 2\psi_t uu_t - u_t^2 \right\} \\
 &\geq \frac{\partial}{\partial t} \left[e^{2\psi}(uu_t + \frac{1}{2}b(t, x)u^2) \right] - \nabla \cdot \{ e^{2\psi}(u \nabla u + u^2 \nabla \psi) \} \\
 &\quad + \delta_3 e^{2\psi} |\nabla u|^2 + e^{2\psi} \left(\delta_3 |\nabla \psi|^2 + \frac{\delta}{3}(-\psi_t)b(t, x) + (B - 2\delta_1) \frac{b(t, x)}{2(t+1)} \right) u^2 \\
 &\quad + e^{2\psi}(-2\psi_t uu_t - u_t^2),
 \end{aligned} \tag{2.9}$$

where, by $\delta_i (i = 1, 2, \dots)$, we denote positive constants depending on δ satisfying $\delta_i \rightarrow 0$ as $\delta \rightarrow 0$. Here we have used

$$\begin{aligned}
 -\psi_t &= a_1(1 + \beta) \frac{\langle x \rangle^{2-\alpha}}{(t+1)^{2+\beta}}, \quad \nabla \psi = a_1 \frac{(2-\alpha)\langle x \rangle^{-\alpha} x}{(t+1)^{1+\beta}}, \\
 (-\psi_t)b(t, x) &= a_1(1 + \beta) \frac{\langle x \rangle^{2-2\alpha}}{(t+1)^{2+2\beta}} \geq \frac{(1 + \beta)}{(2-\alpha)^2 a_1} |\nabla \psi|^2 = (2 + \delta) |\nabla \psi|^2, \\
 \Delta \psi &= a_1(2-\alpha)(N-\alpha) \frac{\langle x \rangle^{-\alpha}}{(t+1)^{1+\beta}} + a_1(2-\alpha)\alpha \frac{\langle x \rangle^{-2-\alpha}}{(t+1)^{1+\beta}} \\
 &\geq \frac{1}{2}(B - 2\delta_1) \frac{b(t, x)}{t+1}.
 \end{aligned}$$

Dividing \mathbf{R}^N into $\Omega(t; K, t_0) := \{x; (t+t_0)^2 \geq K + |x|^2\}$ and $\Omega^c(t; K, t_0) = \mathbf{R}^N \setminus \Omega(t; K, t_0)$ for suitable constants $K, t_0 > 0$ and adding (2.8) to (2.9),

we have

$$\begin{aligned} & \frac{d}{dt} [\bar{E}_\psi(t; \Omega) + \bar{E}_\psi(t; \Omega^c)] + H_\psi(t; \Omega) + H_\psi(t; \Omega^c) \\ & \leq C \int_{\mathbf{R}^N} f e^{2\psi} ((t + t_0)^{\alpha+\beta} + \langle x \rangle_K^{\alpha+\beta}) |u_t| + |u| \, dx, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \bar{E}_\psi(t; \Omega) &= \int_{\Omega} e^{2\psi} \left(\frac{(t + t_0)^{\alpha+\beta}}{2} u_t^2 + \nu u u_t + \frac{\nu b(t, x)}{2} u^2 \right. \\ & \quad \left. + \frac{(t + t_0)^{\alpha+\beta}}{2} |\nabla u|^2 \right) dx, \\ \bar{E}_\psi(t; \Omega^c) &= \int_{\Omega^c} e^{2\psi} \left(\frac{\langle x \rangle_K^{\alpha+\beta}}{2} u_t^2 + \hat{\nu} u u_t + \frac{\hat{\nu} b(t, x)}{2} u^2 + \frac{\langle x \rangle_K^{\alpha+\beta}}{2} |\nabla u|^2 \right) dx, \\ H_\psi(t; \Omega) &= c_0 \int_{\Omega} e^{2\psi} (1 + (t + t_0)^{\alpha+\beta} (-\psi_t)) (u_t^2 + |\nabla u|^2) dx \\ & \quad + \nu (B - 2\delta_1) \int_{\Omega} \frac{e^{2\psi} b(t, x)}{2(t + 1)} u^2 dx, \\ H_\psi(t; \Omega^c) &= c_0 \int_{\Omega^c} e^{2\psi} (1 + \langle x \rangle_K^{\alpha+\beta} (-\psi_t)) (u_t^2 + |\nabla u|^2) dx \\ & \quad + \hat{\nu} (B - 2\delta_1) \int_{\Omega^c} \frac{e^{2\psi} b(t, x)}{2(t + 1)} u^2 dx \end{aligned}$$

with $\langle x \rangle_K = \sqrt{K + |x|^2}$, $0 < \nu, \hat{\nu} \ll 1$ and $c_0 > 0$. Noting $\bar{E}_\psi(t; \Omega) + \bar{E}_\psi(t; \Omega^c) \geq C^{-1} \bar{E}(t)$, and multiplying (2.10) by $(t + t_0)^{l_1 - \bar{\varepsilon}}$ ($\bar{\varepsilon} > 2\delta_1$) we have

$$\begin{aligned} & (t + t_0)^{l_1 - \bar{\varepsilon}} \bar{E}(t) + \frac{c_2}{2} \int_0^t (\tau + t_0)^{l_1 - \bar{\varepsilon}} E(\tau) \, d\tau \\ & \quad + \mu (B - 2\delta_1 - l_1 + \bar{\varepsilon}) \int_0^t (\tau + t_0)^{l_1 - 1 - \bar{\varepsilon}} \int_{\mathbf{R}^N} e^{2\psi} b(\tau, x) u^2 \, dx \, d\tau \\ & \leq C(t_0) \varepsilon^2 I_1 + C \int_0^t (\tau + t_0)^{l_1 - \bar{\varepsilon}} \int_{\mathbf{R}^N} f e^{2\psi} ((t + t_0)^{\alpha+\beta} |u_t| + |u|) \, dx \, d\tau, \end{aligned} \tag{2.11}$$

where $\mu = \min\{\nu, \hat{\nu}\}$. Hence we obtain (2.2) by taking the constant $C = C(t_0, K, \nu, \hat{\nu})$. To obtain (2.3), multiply (2.8) integrated over \mathbf{R}^N by $(t + t_0)^{l_1 + 1 - \bar{\varepsilon}}$, we have

$$(t + t_0)^{l_1 + 1 - \bar{\varepsilon}} E(t) + \int_0^t (\tau + t_0)^{l_1 + 1 - \bar{\varepsilon}} \int_{\mathbf{R}^N} e^{2\psi} b(\tau, x) u_t^2 \, dx \, d\tau$$

$$\leq C\varepsilon^2 I_1 + C \int_0^t (\tau + t_0)^{l_1 - \varepsilon} \int_{\mathbf{R}^N} f e^{2\psi} ((\tau + t_0)|u_t| + |u|) dx d\tau, \quad (2.12)$$

since $\int_0^t (\tau + t_0)^{l_1 - \varepsilon} E(\tau) d\tau$ is already estimated in (2.11). Hence (2.3) is obtained by taking $C = C(t_0, K, \nu, \hat{\nu})$ as above and the proof is completed. \square

To treat the right hand side, we prepare the following lemma.

Lemma 2.2.

$$e^{2\psi} |v|^p u_t \leq \frac{1}{2} b(t, x) e^{2\psi} u_t^2 + C(\lambda)(t + 1)^{\beta + \alpha(1 + \beta)/(2 - \alpha)} e^{(2 + \lambda)\psi} |v|^{2p},$$

where $\lambda > 0$ is arbitrarily small.

Proof. By the Schwarz inequality,

$$e^{2\psi} |v|^p u_t \leq \frac{1}{2} b(t, x) e^{2\psi} u_t^2 + \frac{1}{2} b(t, x)^{-1} e^{2\psi} |v|^{2p}.$$

Noting that for $s \geq 0, r \geq 0, \lambda > 0$ it is true that

$$s^r \leq C(\lambda) e^{\lambda s},$$

and

$$\begin{aligned} \langle x \rangle^\alpha e^{2\psi} &= (1 + t)^{\alpha(1 + \beta)/(2 - \alpha)} \left(\frac{\langle x \rangle^{2 - \alpha}}{(1 + t)^{1 + \beta}} \right)^{\alpha/(2 - \alpha)} e^{2\psi} \\ &\leq C(\lambda)(1 + t)^{\alpha(1 + \beta)/(2 - \alpha)} e^{(2 + \lambda)\psi}. \end{aligned}$$

Hence

$$\frac{1}{2} b(t, x)^{-1} e^{2\psi} |v|^{2p} \leq C(\lambda)(1 + t)^{\beta + \alpha(1 + \beta)/(2 - \alpha)} e^{(2 + \lambda)\psi} |v|^{2p}.$$

Therefore we obtain the desired estimate. \square

We now add (2.2) to (2.3), then the term $\int_{\mathbf{R}^N} b(t, x) e^{2\psi} u_t^2 dx$ is absorbed into the second term in (2.3) by Lemma 2.2. Hence by $\alpha + \beta < 1$ we have

$$\begin{aligned} &(t + 1)^{l_1 + 1 - \varepsilon} E(t) + \frac{1}{2} \int_0^t (\tau + 1)^{l_1 + 1 - \varepsilon} \int_{\mathbf{R}^N} e^{2\psi} b(\tau, x) u_t^2 dx dt \\ &\leq C I_1 + C \int_0^t (\tau + 1)^{l_1 + 1 - \varepsilon + \beta + \alpha(1 + \beta)/(2 - \alpha)} \int_{\mathbf{R}^N} |f|^2 e^{2\psi} dx d\tau \\ &\quad + C \int_0^t (\tau + 1)^{l_1 - \varepsilon} \int_{\mathbf{R}^N} e^{2\psi} |f| |u| dx d\tau. \end{aligned} \quad (2.13)$$

We are now back to (1.1). Define the weighted energy of solutions by

$$W_1(t) = (1+t)^{l_1+1-\bar{\epsilon}} \int_{\mathbf{R}^N} e^{2\psi} (u_t^2 + |\nabla u|^2) dx + (1+t)^{l_1-\bar{\epsilon}} \int_{\mathbf{R}^N} e^{2\psi} b(t, x) u^2 dx \quad (2.14)$$

$$W_2(t) = (1+t)^{l_2+1-\bar{\epsilon}} \int_{\mathbf{R}^N} e^{2\psi} (v_t^2 + |\nabla v|^2) dx + (1+t)^{l_2-\bar{\epsilon}} \int_{\mathbf{R}^N} e^{2\psi} b(t, x) v^2 dx. \quad (2.15)$$

and

$$M(t) = \sup_{0 \leq s \leq t} (W_1(s) + W_2(s)),$$

and also $I_1 = I(u_0, u_1)$ in (2.7), $I_2 = I(v_0, v_1)$ with $I_0 = I_1 + I_2$.

By (2.13) and the application of Lemmas 2.1–2.2 to the second equation of (1.1), we easily have the following lemma.

Lemma 2.3. *For $l_j \leq B = \frac{(1+\beta)(N-\alpha)}{2-\alpha} + \beta$ ($j = 1, 2$), it is true that*

$$W_1(t) \leq C\epsilon^2 I_1 + CN_1(t), \quad W_2(t) \leq C\epsilon^2 I_2 + CN_2(t),$$

where

$$N_1(t) = \int_0^t \left[(1+\tau)^{l_1+1-\bar{\epsilon}+\beta+\alpha(1+\beta)/(2-\alpha)} \int_{\mathbf{R}^N} e^{(2+\lambda)\psi} |v|^{2p} dx + (1+\tau)^{l_1-\bar{\epsilon}} \int_{\mathbf{R}^N} e^{2\psi} |v|^p |u| dx \right] d\tau$$

and

$$N_2(t) = \int_0^t \left[(1+\tau)^{l_2+1-\bar{\epsilon}+\beta+\alpha(1+\beta)/(2-\alpha)} \int_{\mathbf{R}^N} e^{(2+\lambda)\psi} |u|^{2q} dx + (1+\tau)^{l_2-\bar{\epsilon}} \int_{\mathbf{R}^N} e^{2\psi} |u|^q |v| dx \right] d\tau.$$

Here $\lambda > 0$ is an arbitrary small number.

We now estimate $N_i(t)$ ($i = 1, 2$).

Lemma 2.4. *Assume $\Lambda < \frac{N-\alpha}{2}$ or $\Lambda_j < \frac{N-\alpha}{2}$ ($j = 1, 2$) and set*

$$l_j = -1 - (1+\beta) \frac{N-2}{2-\alpha} + \left(2(1+\beta) \frac{2}{2-\alpha} - \eta \right) \Lambda_j \quad (j = 1, 2) \quad (2.16)$$

for a small constant $\eta > 0$. Then $l_j \leq B$ ($j = 1, 2$) and

$$N_1(t) + N_2(t) \leq C \left(M(t)^p + M(t)^q + M(t)^{(p+1)/2} + M(t)^{(q+1)/2} \right). \quad (2.17)$$

To show (2.17) we use the lemma

Lemma 2.5 (Gagliardo-Nirenberg inequality). *When $p \leq \frac{N}{[N-2]_+}$, it holds that*

$$\begin{aligned} \|h\|_{L^{2p}} &\leq C \|\nabla h\|_{L^2}^{\sigma_{2p}} \|h\|_{L^2}^{1-\sigma_{2p}}, & \sigma_{2p} &= \frac{N(p-1)}{2p}, \\ \|h\|_{L^{p+1}} &\leq C \|\nabla h\|_{L^2}^{\sigma_{p+1}} \|h\|_{L^2}^{1-\sigma_{p+1}}, & \sigma_{p+1} &= \frac{N(p-1)}{2(p+1)}. \end{aligned}$$

Proof of Lemma 2.4. Choose λ in Lemma 2.2 satisfying $\frac{2+\lambda}{2p} < 1$, then

$$\begin{aligned} \int_{\mathbf{R}^N} e^{(2+\lambda)\psi} |v|^{2p}(s, x) \, dx &= \|e^{(2+\lambda)\psi/(2p)} v(s)\|_{L^{2p}}^{2p} \\ &\leq C \|\nabla(e^{(2+\lambda)\psi/(2p)} v)(s)\|_{L^2}^{2p\sigma_{2p}} \|e^{(2+\lambda)\psi/(2p)} v(s)\|_{L^2}^{2p(1-\sigma_{2p})}. \end{aligned}$$

Here

$$|\nabla(e^{(2+\lambda)\psi/(2p)} v)(s, x)|^2 \leq e^{(2+\lambda)\psi/p} \left(\frac{2+\lambda}{2p} |\nabla\psi| |v| + |\nabla v| \right)^2 (s, x)$$

and, since

$$\begin{aligned} |\nabla\psi|^2 e^{(2+\lambda)\psi/p} v^2(s, x) &= b(s, x)^{-1} |\nabla\psi|^2 e^{(2+\lambda)\psi/p} b(s, x) v^2 \\ &\leq C \langle x \rangle^\alpha (1+s)^\beta \frac{\langle x \rangle^{2(1-\alpha)}}{(1+s)^{2(1+\beta)}} e^{(2+\lambda)\psi/p} b(s, x) v^2 \\ &\leq C(1+s)^{-1} \frac{\langle x \rangle^{2-\alpha}}{(1+t)^{1+\beta}} e^{(2+\lambda)\psi/p} b(s, x) v^2 \leq C(1+s)^{-1} e^{2\psi} b(s, x) v^2, \end{aligned}$$

for $0 \leq s \leq t$

$$\begin{aligned} &\|\nabla(e^{(2+\lambda)\psi/(2p)} v)(s)\|_{L^2}^{2p\sigma_{2p}} \\ &\leq C \left((1+s)^{-1/2} \|e^\psi \sqrt{b(s, \cdot)} v\|_{L^2} + \|e^\psi \nabla v(s)\|_{L^2} \right)^{2p\sigma_{2p}} \\ &\leq C(1+s)^{-2p\sigma_{2p}(l_2+1-\varepsilon)/2} M(t)^{p\sigma_{2p}}. \end{aligned}$$

Also, since

$$\begin{aligned} e^{(2+\lambda)\psi/p} v^2(s, x) &= \langle x \rangle^\alpha (1+s)^\beta e^{(2+\lambda)\psi/p} b(s, x) v^2(s, x) \\ &\leq C(1+s)^{\beta+\alpha(1+\beta)/(2-\alpha)} \left(\frac{\langle x \rangle^{2-\alpha}}{(1+s)^{1+\beta}} \right)^{\alpha/(2-\alpha)} e^{(2+\lambda)\psi/p} b(s, x) v^2(s, x) \end{aligned}$$

$$\leq C(1+s)^{\beta+\alpha(1+\beta)/(2-\alpha)} e^{2\psi} b(s,x) v^2(s,x),$$

for $0 \leq s \leq t$

$$\begin{aligned} & \|e^{(2+\lambda)\psi/(2p)} v(s)\|_{L^2}^{2p(1-\sigma_{2p})} \\ & \leq C(1+s)^{2p(1-\sigma_{2p})(\beta+\alpha(1+\beta)/(2-\alpha))/2} \|e^{\psi} \sqrt{b(s,\cdot)} v(s)\|_{L^2}^{2p(1-\sigma_{2p})} \\ & \leq C(1+s)^{2p(1-\sigma_{2p})(\beta+\alpha(1+\beta)/(2-\alpha)-(l_2-\bar{\varepsilon}))/2} M(t)^{p(1-\sigma_{2p})}. \end{aligned}$$

Therefore

$$\begin{aligned} & \int_{\mathbf{R}^N} e^{(2+\lambda)\psi} |v|^{2p}(s,x) dx \\ & \leq C \|\nabla(e^{(2+\lambda)\psi/(2p)} v)(s)\|_{L^2}^{2p\sigma_{2p}} \|e^{(2+\lambda)\psi/(2p)} v(s)\|_{L^2}^{2p(1-\sigma_{2p})} \\ & \leq C(1+s)^{-2p\sigma_{2p}(l_2+1-\bar{\varepsilon})/2+2p(1-\sigma_{2p})(\beta+\alpha(1+\beta)/(2-\alpha)-(l_2-\bar{\varepsilon}))/2} M(t)^p. \end{aligned}$$

The second term of $N_1(t)$ can be estimated in a similar fashion to the above, and the following inequality holds:

$$N_1(t) \leq C \int_0^t \left[(1+s)^{\gamma_{11}} M(t)^p + (1+s)^{\gamma_{12}} M(t)^{(p+1)/2} \right] ds,$$

where by simple calculations

$$\begin{aligned} \gamma_{11} &= l_1 + 1 - \bar{\varepsilon} + \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) - 2p\sigma_{2p} \frac{1}{2}(l_2 + 1 - \bar{\varepsilon}) \\ & \quad - 2p(1-\sigma_{2p})(l_2 - \bar{\varepsilon}) + 2p(1-\sigma_{2p}) \frac{1}{2} \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) \\ &= \left\{ l_1 - pl_2 - \frac{N}{2}(p-1) + \frac{1}{2}(2(p+1) - N(p-1)) \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) \right\} \\ & \quad + 1 + (p-1)\bar{\varepsilon}, \\ \gamma_{12} &= \frac{1}{2} \left\{ l_1 - pl_2 - \frac{N}{2}(p-1) + \frac{1}{2}(2(p+1) - N(p-1)) \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) \right\} \\ & \quad + \frac{p-1}{2} \bar{\varepsilon} \\ &= \frac{1}{2}(\gamma_{11} - 1). \end{aligned}$$

Hence, we need $\gamma_{11}, \gamma_{12} < -1$ or

$$\left\{ l_1 - pl_2 - \frac{N}{2}(p-1) + \frac{1}{2}(2(p+1) - N(p-1)) \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) \right\} < -2. \quad (2.18)$$

By the complete same way for $N_2(t)$ we need $\gamma_{21}, \gamma_{22} < -1$ or

$$\left\{ l_2 - ql_1 - \frac{N}{2}(q-1) + \frac{1}{2}(2(q+1) - N(q-1)) \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) \right\} < -2. \quad (2.19)$$

Therefore (2.18) and (2.19) are equivalent to the vector form of $l = {}^t(l_1, l_2)$

$$\begin{aligned} & -(P-I)l - \frac{N}{2}(P-I) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2}(2(P+I) - N(P-I)) \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= -(2+\eta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

for $\eta > 0$ (see Remark 1.1) and, since $P+I = (P-I) + 2I$,

$$\begin{aligned} & -(P-I)l - \frac{N}{2}(P-I)(1+\beta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (P-I) \left(\beta + \frac{\alpha(1+\beta)}{2-\alpha} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= - \left(2(1+\beta) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \eta \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Multiplying this by $(P-I)^{-1}$ and noting $\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = (P-I)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we obtain

$$l = - \begin{pmatrix} 1 \\ 1 \end{pmatrix} - (1+\beta) \frac{N-2}{2-\alpha} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(2(1+\beta) \frac{2}{2-\alpha} + \eta \right) \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix}.$$

If $\Lambda < \frac{N-\alpha}{2}$ or $\Lambda_j < \frac{N-\alpha}{2}$ ($j = 1, 2$), then

$$\begin{aligned} l &= \left(\beta + \frac{(1+\beta)(N-\alpha)}{2-\alpha} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \eta \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} - 2(1+\beta) \frac{2}{2-\alpha} \begin{pmatrix} \frac{N-\alpha}{2} - \Lambda_1 \\ \frac{N-\alpha}{2} - \Lambda_2 \end{pmatrix} \\ &< B \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

for sufficiently small η (“ $<$ ” means that both components satisfy (“ $<$ ”). Thus, under the assumption $\Lambda < \frac{N-\alpha}{2}$, the decay rates $l_j \leq B$ ($j = 1, 2$) yield (2.18) and (2.19), which completes the proof. \square

Proof of Theorem 1.1. By Lemma 2.4

$$M(t) \leq C\varepsilon^2 I_0 + CM(t)^r$$

with $r = \min(2p, 2q, \frac{p+1}{2}, \frac{q+1}{2}) > 1$. Hence $M(t) \leq C\varepsilon^2 I_0$ for small $\varepsilon > 0$, which is the desired a priori estimate.

3. Blow-up of Solutions in the Critical and Subcritical Exponents

For the proof of blow-up of solutions in a finite time we apply the test function method, which has been developed for several evolution equations by Mitidieri and Pohozaev [17]. Here the method by Zhang [36] for the damped wave equation is used (cf. Ikeda and Wakasugi [9]).

For this, define the functions

$$\eta(t) = \begin{cases} 1 & (0 \leq t \leq \frac{1}{2}), \\ \frac{e^{-1/(1-t^2)}}{e^{-1/(t^2-1/4)} + e^{-1/(1-t^2)}} & (\frac{1}{2} < t < 1), \\ 0 & (t \geq 1), \end{cases}$$

$$\phi(x) = \eta(|x|), \quad x \in \mathbf{R}^N, \quad |x| = \sqrt{x_1^2 + \cdots + x_N^2}. \tag{3.1}$$

Note that

$$|\eta'(t)| \leq C\eta(t)^{1/r}, \quad |\eta''(t)| \leq C\eta(t)^{1/r} \text{ for any } r > 1 \tag{3.2}$$

for some positive constant C .

Proof of Theorem 1.2 (I). We assume the non-trivial solution $(u, v) \in [C([0, \infty); H^1) \cap C^1([0, \infty); L^2)]^2$ satisfies

$$\begin{cases} u_{tt} - \Delta u + \langle x \rangle^{-\alpha} u_t = |v|^p, \\ v_{tt} - \Delta v + \langle x \rangle^{-\alpha} v_t = |u|^q, \end{cases} \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N \tag{3.3}$$

and derive the contradiction. By (3.1) define the test function

$$\psi_R(t, x) = \eta\left(\frac{t}{R^{2-\alpha}}\right)\phi\left(\frac{x}{R}\right) := \eta_R(t)\phi_R(x) \tag{3.4}$$

for a large $R \in \mathbf{R}$, and set

$$V_R = \int_{Q_R} |v|^p \psi_R(t, x) dx dt, \quad U_R = \int_{Q_R} |u|^q \psi_R(t, x) dx dt,$$

with

$$Q_R = \{(t, x); 0 \leq t \leq R^{2-\alpha}, |x| \leq R\} =: [0, R^{2-\alpha}] \times B_R. \tag{3.5}$$

By (3.3)₁ (denote the first equation in (3.3))

$$\begin{aligned} & V_R + \varepsilon \int_{B_R} (\langle x \rangle^{-\alpha} u_0(x) + u_1(x)) \phi_R(x) dx \\ &= \int_{Q_R} u \{(\psi_R)_{tt} - \Delta(\psi_R) - \langle x \rangle^{-\alpha} (\psi_R)_t\}(t, x) dx dt =: I_1 + I_2 + I_3. \end{aligned} \tag{3.6}$$

Calculate $I_i (i = 1, 2, 3)$. First,

$$\begin{aligned} I_3 &= \int_{Q_R} \{-\langle x \rangle^{-\alpha} u \eta'\left(\frac{t}{R^{2-\alpha}}\right) \phi\left(\frac{x}{R}\right) \cdot R^{-(2-\alpha)}\} dx dt \\ &\leq C R^{-(2-\alpha)} \left(\int_{\hat{Q}_{R,t}} |u|^q \psi_R dx dt\right)^{\frac{1}{q}} \left(\int_{\hat{Q}_{R,t}} \langle x \rangle^{-\alpha q'} dx dt\right)^{\frac{1}{q'}}, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$ and $\hat{Q}_{R,t} = [\frac{1}{2}R^{2-\alpha}, R^{2-\alpha}] \times B_R$. Since

$$\int_{B_R} \langle x \rangle^{-\alpha q'} dx \leq \begin{cases} C & \alpha q' > N, \\ C \log R & \alpha q' = N, \\ C R^{N-\alpha q'} & \alpha q' < N, \end{cases}$$

and $\alpha q' = N$ is equivalent to $q = 1 + \frac{\alpha}{N-\alpha}$,

$$\begin{aligned} & C R^{-(2-\alpha)} \left(\int_{\hat{Q}_{R,t}} \langle x \rangle^{-\alpha q'} dx dt\right)^{\frac{1}{q'}} \\ &\leq \begin{cases} C R^{-(2-\alpha)(1-\frac{1}{q'})} & q < 1 + \frac{\alpha}{N-\alpha} \text{ or } \alpha q' > N, \\ C R^{-(2-\alpha)(1-\frac{1}{q'})+\delta_q} & q = 1 + \frac{\alpha}{N-\alpha} \text{ or } \alpha q' = N, \\ C R^{-(2-\alpha)(1-\frac{1}{q'})+\frac{N-\alpha q'}{q'}} & q > 1 + \frac{\alpha}{N-\alpha} \text{ or } \alpha q' < N \end{cases} \tag{3.7} \\ &=: C R^{-e(\alpha,q)}, \end{aligned}$$

where $0 < \delta_q \ll 1$. Hence,

$$I_3 \leq CR^{-e(\alpha,q)}(\hat{U}_{R,t})^{\frac{1}{q}} \quad (3.8)$$

with $\hat{U}_{R,t} = \int_{\hat{Q}_{R,t}} |u|^q \psi_R(t, x) dx dt$. By similar calculations on I_1, I_2 ,

$$I_1 \leq CR^{-2(2-\alpha) + \frac{N+2-\alpha}{q'}} (\hat{U}_{R,t})^{\frac{1}{q}}, \quad (3.9)$$

$$I_2 \leq CR^{-2 + \frac{N+2-\alpha}{q'}} (\hat{U}_{R,x})^{\frac{1}{q}}, \quad (3.10)$$

with $\hat{U}_{R,x} = \int_{\hat{Q}_{R,x}} |u|^q \psi_R(t, x) dx dt$, $\hat{Q}_{R,t} = [0, R^{2-\alpha}] \times (B_R \setminus B_{R/2})$. Combining (3.6) with (3.8)–(3.10), we have

$$\begin{aligned} V_R + \varepsilon \int_{B_R} (\langle x \rangle^{-\alpha} u_0(x) + u_1(x)) \phi_R(x) dx \\ \leq CR^{-2 + \frac{N+2-\alpha}{q'}} (\hat{U}_{R,t} + \hat{U}_{R,x})^{\frac{1}{q}} + CR^{-e(\alpha,q)} (\hat{U}_{R,t})^{\frac{1}{q}} \end{aligned} \quad (3.11)$$

since $2(2-\alpha) > 2$. Moreover, by simple calculations, $2 - \frac{N+2-\alpha}{q'} \geq e(\alpha, q)$.

In fact,

$$2 - \frac{N+2-\alpha}{q'} - e(\alpha, q) = \begin{cases} \frac{\alpha q' - N}{q'} > 0 & (\alpha q' > N), \\ \delta_q > 0 & (\alpha q' = N), \\ 0 & (\alpha q' < N). \end{cases}$$

Hence, by (3.11)

$$V_R + \varepsilon \int_{B_R} (\langle x \rangle^{-\alpha} u_0(x) + u_1(x)) \phi_R(x) dx \leq CR^{-e(\alpha,q)} (\hat{U}_{R,t} + \hat{U}_{R,x})^{\frac{1}{q}}. \quad (3.12)$$

By (3.3)₂ (the second equation in (3.3)) we also obtain

$$U_R + \varepsilon \int_{B_R} (\langle x \rangle^{-\alpha} v_0(x) + v_1(x)) \phi_R(x) dx \leq CR^{-e(\alpha,p)} (\hat{V}_{R,t} + \hat{V}_{R,x})^{\frac{1}{p}}, \quad (3.13)$$

where

$$\hat{V}_{R,t} = \int_{\hat{Q}_{R,t}} |v|^p \psi_R(t, x) dx dt, \quad \hat{V}_{R,x} = \int_{\hat{Q}_{R,x}} |v|^p \psi_R(t, x) dx dt.$$

Since $\hat{U}_{R,t} \leq U_R$ etc. and $\int_{B_R} (\langle x \rangle^{-\alpha} u_0(x) + u_1(x)) \phi_R(x) dx > 0$ for

$R \gg 1$, both (3.12) and (3.13) yield

$$\begin{aligned}
 U_R + \varepsilon \int_{B_R} (\langle x \rangle^{-\alpha} v_0(x) + v_1(x)) \phi_R(x) dx \\
 \leq CR^{-e(\alpha,p) - \frac{e(\alpha,q)}{p}} (\hat{U}_{R,t} + \hat{U}_{R,x})^{\frac{1}{pq}} \leq CR^{-e(\alpha,p) - \frac{e(\alpha,q)}{p}} U_R^{\frac{1}{pq}}.
 \end{aligned}
 \tag{3.14}$$

Therefore, if

$$P := e(\alpha,p) + \frac{e(\alpha,q)}{p} > 0,
 \tag{3.15}$$

then $U_R^{1 - \frac{1}{pq}} \rightarrow 0$ and $U_R \rightarrow 0$ ($R \rightarrow \infty$), which contradicts that the solution (u, v) is non-trivial.

Assume $q \geq p$ without loss of generality, and the subcritical condition is

$$\Lambda = \max\left(\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\right) = \frac{q+1}{pq-1} > \frac{N-\alpha}{2}$$

or

$$q\left(p - \frac{2}{N-\alpha}\right) < 1 + \frac{2}{N-\alpha}.
 \tag{3.16}$$

Noting (3.16), we show (3.15) in each case.

(i) When $q < 1 + \frac{\alpha}{N-\alpha}$, also $p < 1 + \frac{\alpha}{N-\alpha}$. Hence,

$$P = (2-\alpha)\left(1 - \frac{1}{p'}\right) + \frac{1}{p}(2-\alpha)\left(1 - \frac{1}{q'}\right) > 0.$$

(ii) When $q = 1 + \frac{\alpha}{N-\alpha}$,

$$\begin{aligned}
 P = \begin{cases} (2-\alpha)\left(1 - \frac{1}{p'}\right) + \frac{1}{p}(2-\alpha)\left(1 - \frac{1}{q'}\right) - \delta_p - \frac{1}{p}\delta_q & \text{if } p = 1 + \frac{\alpha}{N-\alpha}, \\ (2-\alpha)\left(1 - \frac{1}{p'}\right) + \frac{1}{p}(2-\alpha)\left(1 - \frac{1}{q'}\right) - \frac{1}{p}\delta_q & \text{if } p < 1 + \frac{\alpha}{N-\alpha} \end{cases} \\
 > 0 \quad (\text{by taking } \delta_p, \delta_q \ll 1).
 \end{aligned}$$

(iii) When $q > 1 + \frac{\alpha}{N-\alpha}$.

(iii)_{1,2} If $p < 1 + \frac{\alpha}{N-\alpha}$, then

$$P = \frac{N+2-\alpha}{p} \left(\frac{1}{q} - \frac{N-2}{N+2-\alpha}\right)$$

(when $p = 1 + \frac{\alpha}{N-\alpha}$, $P = \frac{N+2-\alpha}{p} \left(\frac{1}{q} - \frac{N-2}{N+2-\alpha}\right) - \delta_p$). Hence, for $N = 1, 2$, clearly $P > 0$. For $N \geq 3$, $P > 0$ is equivalent to $q < \frac{N+2-\alpha}{N-2}$. Since

$$q \leq \frac{N}{N-2}, P > 0.$$

(The discussion here is the reason why Remark 1.2 is necessary.)

(iii)₃ $p > 1 + \frac{\alpha}{N-\alpha}$, then, by simple but tedious calculations,

$$P = \frac{N - \alpha}{pq} \left(1 + \frac{2}{N - \alpha} - q \left(p - \frac{2}{N - \alpha} \right) \right) > 0$$

by (3.16).

Thus, in the subcritical exponent we have had a contradiction.

Next, the critical exponent case is equivalent to $P = 0$ by (iii)₃. Then, back to (3.14), $U_R \leq C$ and $\int_0^\infty \int_{\mathbf{R}^N} |u|^q(t, x) dx dt \leq C$ by $R \rightarrow \infty$. Hence, thanks to the Lebesgue convergence theorem, both $\hat{U}_{R,t}$ and $\hat{U}_{R,x}$ tend to zero as $R \rightarrow \infty$, which gives the contradiction by (3.14) again.

We have now completed the proof of Theorem 1.2 (I).

To apply the test function method to (3.3), the key points are that the left-hand sides have divergence forms and that the right-hand sides are non-negative. But, in the system of damped wave equations with time-dependent damping

$$\begin{cases} u_{tt} - \Delta u + (t + 1)^{-\beta} u_t = |v|^p, \\ v_{tt} - \Delta v + (t + 1)^{-\beta} v_t = |u|^q, \end{cases} \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N \tag{3.17}$$

with $b(t) = (t + 1)^{-\beta}$ ($-1 < \beta < 1$) (for simplicity, drop the suffix 2 of b), the left-hand sides are not divergence forms. To overcome this, we apply the idea in Lin, Nishihara and Zhai [14]. Following [14], multiply (3.17) by a suitable nonnegative function $g(t)$:

$$\begin{cases} (g(t)u_t)_t - \Delta(g(t)u) + (-g'(t) + b(t)g(t))u_t = g(t)|v|^p, \\ (g(t)v_t)_t - \Delta(g(t)v) + (-g'(t) + b(t)g(t))v_t = g(t)|u|^q. \end{cases} \tag{3.18}$$

We now choose $g(t)$ as

$$-g'(t) + b(t)g(t) = 1, \quad g(t) > 0. \tag{3.19}$$

Solving (3.19), we define $g(t)$ by

$$g(t) = \exp \left(\int_0^t b(\tau) d\tau \right) \cdot \int_t^\infty \exp \left(- \int_0^\tau b(s) ds \right) d\tau > 0 \tag{3.20}$$

with

$$g(0) = \int_0^\infty \exp\left(-\int_0^\tau b(s) ds\right) d\tau > 0. \tag{3.21}$$

Note that, by the l'Hôpital's rule,

$$\begin{aligned} \lim_{t \rightarrow \infty} b(t)g(t) &= \lim_{t \rightarrow \infty} \frac{\int_t^\infty \exp\left(-\int_0^\tau b(s) ds\right) d\tau}{(t+1)^\beta \exp\left(-\int_0^t b(\tau) d\tau\right)} \\ &= \lim_{t \rightarrow \infty} \frac{-\exp\left(-\int_0^t b(s) ds\right)}{(\beta(t+1)^{\beta-1} - (t+1)^\beta b(t)) \exp\left(-\int_0^t b(\tau) d\tau\right)} = 1, \end{aligned}$$

and hence

$$C^{-1}b(t)^{-1} \leq g(t) \leq Cb(t)^{-1}, \quad 0 \leq t < \infty. \tag{3.22}$$

Thus, (3.18) is reduced to

$$\begin{cases} (g(t)u_t)_t - \Delta(g(t)u) + u_t = g(t)|v|^p, \\ (g(t)v_t)_t - \Delta(g(t)v) + u_t = g(t)|u|^q, \quad (t, x) \in \mathbf{R}_+ \times \mathbf{R}^N. \end{cases} \tag{3.23}$$

We are now ready to apply the test function method.

Proof of Theorem 1.2 (II). Define the test function by

$$\psi_{R,\beta}(t, x) = \eta\left(\frac{t}{R^{2/(1+\beta)}}\right)\phi\left(\frac{x}{R}\right) =: \eta_{R,\beta}(t)\phi_R(x) \tag{3.24}$$

and

$$\begin{aligned} V_{R,g} &= \int_{Q_{R,\beta}} g(t)|v|^p \psi_{R,\beta}(t, x) dx dt, \\ U_{R,g} &= \int_{Q_{R,\beta}} g(t)|u|^q \psi_{R,\beta}(t, x) dx dt, \end{aligned} \tag{3.25}$$

with

$$Q_{R,\beta} = \{(t, x) | 0 \leq t \leq R^{\frac{2}{1+\beta}}, |x| \leq R\} = [0, R^{\frac{2}{1+\beta}}] \times B_R. \tag{3.26}$$

By (3.23)₁,

$$\begin{aligned} V_{R,g} &= \int_{Q_{R,\beta}} \{(g(t)u_t)_t - \Delta(g(t)u) + u_t\} \psi_{R,\beta}(t, x) dx dt \\ &=: J_1 + J_2 + J_3 \end{aligned}$$

(This equation is interpreted that ψ is taken as $\psi = g(t)\psi_{R,\beta}(t, x)$ in the

definition (1.2) of the weak solution). By almost similar calculations of $I_i (i = 1, 2, 3)$, $J_j (j = 1, 2, 3)$ are calculated as

$$\begin{aligned} J_1 + \varepsilon \int_{B_R} g(0)u_1(x)\phi(x) dx &\leq CR^{-\bar{e}(\beta,q)}(\hat{U}_{R,g,t})^{\frac{1}{q}}, \\ J_2 &\leq CR^{-\bar{e}(\beta,q)}(\hat{U}_{R,g,x})^{\frac{1}{q}}, \\ J_3 + \varepsilon \int_{B_R} u_0(x)\phi_R(x)dx &\leq CR^{-\bar{e}(\beta,q)}(\hat{U}_{R,g,t})^{\frac{1}{q}}, \end{aligned}$$

where

$$\bar{e}(\beta, q) = 2 - \frac{N+2}{q'} \quad (3.27)$$

and

$$\begin{aligned} \hat{U}_{R,g,t} &= \int_{\hat{Q}_{R,\beta,t}} g(t)|u|^q \psi_{R,\beta}(t, x) dx dt, \\ \hat{U}_{R,g,x} &= \int_{\hat{Q}_{R,\beta,x}} g(t)|u|^q \psi_{R,\beta}(t, x) dx dt \end{aligned} \quad (3.28)$$

with $\hat{Q}_{R,\beta,t} = [\frac{1}{2}R^{\frac{2}{1+\beta}}, R^{\frac{2}{1+\beta}}] \times B_R$, $\hat{Q}_{R,\beta,x} = [0, R^{\frac{2}{1+\beta}}] \times (B_R \setminus B_{R/2})$. In fact, for an example,

$$\begin{aligned} J_3 + \varepsilon \int_{B_R} u_0(x)\phi_R(x) dx &= \int_{Q_{R,\beta}} \{-u\eta'_{R,\beta}(t)\phi_R(x) \cdot R^{-\frac{2}{1+\beta}}\} dx dt \\ &\leq CR^{-\frac{2}{1+\beta}} \left(\int_{\hat{Q}_{R,\beta}} g(t)|u|^q \psi_{R,\beta} dx dt \right)^{\frac{1}{q}} \left(\int_{\hat{Q}_{R,\beta}} g(t)^{-\frac{q'}{q}} dx dt \right)^{\frac{1}{q'}} \\ &\leq CR^{-(2-\frac{N+2}{q'})}, \end{aligned}$$

since $-\frac{2}{1+\beta} + \frac{1}{q'}(N + \frac{2}{1+\beta} - \frac{2}{1+\beta}\frac{\beta q'}{q}) = -2 + \frac{N+2}{q'}$. Therefore, we have

$$V_{R,g} + \varepsilon \int_{B_R} (u_0(x) + g(0)u_1(x))\phi_R(x) dx \leq CR^{-\bar{e}(\beta,q)}(\hat{U}_{R,g,t} + \hat{U}_{R,g,x})^{\frac{1}{q}}. \quad (3.29)$$

By (3.2)₂, we also have

$$U_{R,g} + \bar{\varepsilon} \int_{B_R} (v_0(x) + g(0)v_1(x))\phi_R(x) dx \leq CR^{-\bar{e}(\beta,p)}(\hat{V}_{R,g,t} + \hat{V}_{R,g,x})^{\frac{1}{p}}, \quad (3.30)$$

$\hat{V}_{R,g,t}$ etc. are, respectively, defined in a similar fashion to $\hat{U}_{R,g,t}$ etc.

Noting $\hat{V}_{R,g,t} \leq V_{R,g}$ etc., by (3.29)–(3.30) we get

$$\begin{aligned} U_{R,g} + \varepsilon \int_{B_R} (v_0(x) + g(0)v_1(x))\phi_R(x) dx \\ \leq CR^{-\bar{e}(\beta,p) - \frac{\bar{e}(\beta,q)}{p}} (\hat{U}_{R,g,t} + \hat{U}_{R,g,x})^{\frac{1}{pq}} \\ \leq CR^{-\bar{e}(\beta,p) - \frac{\bar{e}(\beta,q)}{p}} (U_{R,g})^{\frac{1}{pq}}. \end{aligned} \quad (3.31)$$

Since $\bar{e}(\beta, p) + \frac{\bar{e}(\beta, q)}{p} \geq 0$ is equivalent to

$$q\left(p - \frac{2}{N}\right) \leq 1 + \frac{2}{N},$$

and, when $q \geq p$ (WLOG), to

$$\Lambda = \frac{q+1}{pq-1} \geq \frac{N}{2}.$$

Thus, (3.31) derives the contradiction in a similar way to the case of Theorem 1.2 (I).

4. Further Discussions

Let us state the motivations of our problems. From the viewpoint of the diffusion phenomena of solutions of damped wave equations, compared with the result [1] for (1.10), what are the critical exponents for (1.1)? This is the first motivation. However, our final goal is to solve

Problem. What are the critical exponents for the system

$$\begin{cases} u_{tt} - \Delta u + u_t = |v|^p, \\ v_{tt} - \Delta v = |u|^q, \quad x \in \mathbf{R}^n, \quad t > 0. \end{cases} \quad (4.1)$$

This problem was proposed to the authors by Professor Mitsuhiro Nakao, Emeritus of Kyushu University, Japan. By the diffusion phenomena, the first equation is quantitatively the diffusive equation, while the second one is the wave equation. Of course, each solution of two equations has essentially different properties from each other. The semilinear terms are coupled with different properties. So, if we could determine the critical exponents, then we may know in some sense or explain how both equations influence with

each other. Therefore, we believe that Problem 1 is so interesting, but it seems so difficult.

As a first step, we have considered the Cauchy problem (1.1) as our Problem, which is another motivation. So, the next problem is

Problem 2. Determine the critical exponents for

$$\begin{cases} u_{tt} - \Delta u + b_1(t, x)u_t = |v|^p, \\ v_{tt} - \Delta v + b_2(t, x)v_t = |u|^q, \end{cases} \quad (4.2)$$

where

$$b_1(t, x) \neq b_2(t, x) \text{ with } b_i(t, x) \sim \langle x \rangle^{-\alpha_i} (t+1)^{-\beta_i}, \quad (\alpha_i, \beta_i \geq 0, \alpha_i + \beta_i < 1) \quad (4.3)$$

that is, the case that both dissipations are effective. Here, $f \sim g$ means $C^{-1}g \leq f \leq Cg$ for some positive constant C . The typical example in Problem 2 is

$$\begin{cases} u_{tt} - \Delta u + \langle x \rangle^{-\alpha} u_t = |v|^p \quad (0 \leq \alpha < 1), \\ v_{tt} - \Delta v + (t+1)^{-\beta} v_t = |u|^q \quad (0 \leq \beta < 1). \end{cases} \quad (4.4)$$

If $0 \leq b(t, x) \leq C \langle x \rangle^{-\alpha} (t+1)^{-\beta}$ ($\alpha, \beta \geq 0, \alpha + \beta > 1$), then the dissipation is non-effective, or the equation essentially becomes the wave equation. Therefore, the third problem is

Problem 3. Determine the critical exponents for (4.2) with

$$b_i(t, x) \sim \langle x \rangle^{-\alpha_i} (t+1)^{-\beta_i}, \quad (\alpha_i, \beta_i \geq 0, \alpha_1 + \beta_1 < 1, \alpha_2 + \beta_2 > 1). \quad (4.5)$$

The typical system is

$$\begin{cases} u_{tt} - \Delta u + \langle x \rangle^{-\alpha} u_t = |v|^p \quad (0 \leq \alpha < 1), \\ v_{tt} - \Delta v + (t+1)^{-\beta} v_t = |u|^q \quad (\beta > 1). \end{cases} \quad (4.6)$$

Thus, the first equation is essentially parabolic and the second one hyperbolic, and we finally reach to *Problem*.

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