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GLOBAL EXISTENCE THEORY FOR A GENERAL CLASS OF HYPERBOLIC BALANCE LAWS

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Abstract

We consider a general system of hyperbolic balance laws in m space dimensions ($m \geq 1$). Under a set of conditions we establish the existence of global solutions for the Cauchy problem when initial data are small perturbations of a constant equilibrium state. The proposed assumptions in this paper are different from those in literature for the system. Instead, our assumptions are parallel to those used in the study of hyperbolic-parabolic systems. In one space dimension our assumptions are natural extensions of those used in the study of the Green's function of the linearized system. They are also sufficient to the study of large time behavior in the pointwise sense for the nonlinear system, carried out in a different paper.

1. Introduction

We consider a general system of hyperbolic balance laws

$$u_t + \sum_{i=1}^{m} f_i(u)_{x_i} = r(u), \quad m \geq 1, \tag{1.1}$$

where $u, f_i, r \in \mathbb{R}^n$. Here u is the unknown density function, representing mass density, momentum density, etc; f_i, $1 \leq i \leq m$, are the flux functions; and r represents external force, relaxation, chemical reactions and so forth. We assume f_i and r are given smooth functions of u, while u is a function

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of the space variable $x = (x_1, \ldots, x_m)^t \in \mathbb{R}^m$ and time variable $t \in \mathbb{R}^+$. A constant state $\bar{u}$ is an equilibrium state if $r(\bar{u}) = 0$. We consider the Cauchy problem of (1.1) with initial data $u_0(x)$:

$$u(x, 0) = u_0(x), \tag{1.2}$$

where u_0 is a small perturbation of the constant equilibrium state $\bar{u}$.

There is an extensive literature about (1.1), on structural conditions, wave patterns, energy estimates, existence of global solutions, large time behavior, etc, for one dimensional and multidimensional general systems or special systems. See [15, 12, 7, 2, 6, 19, 1, 10, 11, 18, 20, 21, 22] and references therein. In this paper we propose a different set of hypotheses, which lead to the global existence of solution to (1.1), (1.2). As to be seen in the remarks following the hypotheses, they are parallel to those used to study the global existence of solution for hyperbolic-parabolic systems in [9] and other references by energy estimates. In the case of one space dimension, they are natural extensions to those used to study the Green's function of the linearized system of (1.1) in [20]. They are also sufficient to the study of large time behavior of (1.1), (1.2) in the pointwise sense in space and in time (a recent joint work with J. Chen). Our hypotheses in fact are slightly weaker than those proposed in [10] and in a more concise form, see Section 2 for details. We perform the energy estimate under the new assumptions to obtain a parallel result as stated in Theorem 3.2 of [11]. This gives us global existence, see Theorem 1.5. It is likely that the L^2 decay estimates in [11] and the global existence with critical regularity in [18] are still available under the new assumptions. These topics, however, are left for future study.

Now we prepare to state our assumptions. Since we are interested in small solutions, we consider a neighborhood of $\bar{u}$, denoted by O. Thus f_i, $1 \le i \le m$, and r are smooth functions of u in O. Define the equilibrium manifold E as

$$E = \{u \in O | r(u) = 0\}. \tag{1.3}$$

Assumption 1. (i) *Equation* (1.1) *can be written in the form*

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \sum_{i=1}^{m} \begin{pmatrix} f_{i1} \\ f_{i2} \end{pmatrix} (u)_{x_i} = \begin{pmatrix} 0 \\ r_2 \end{pmatrix} (u), \tag{1.4}$$

where $u_1,\ f_{i1} \in \mathbb{R}^{n_1}$, $u_2,\ f_{i2},\ r_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$ *with* $n_2 > 0$, *and* $(r_2)_{u_2} \in \mathbb{R}^{n_2 \times n_2}$ *is nonsingular.*

(ii) *There exists a strictly convex entropy function* U *(a scalar function of* u*) such that* $U'' f_i'$, $1 \le i \le m$, *are symmetric in* O *while* $U'' r'$ *is symmetric, semi-negative definite on* E. *Here* U'' *is the Hessian of* U *with respect to* u, *and* f_i' *and* r' *are the Jacobian matrices of* f_i *and* r *with respect to* u.

(iii) *The null space of* $r'(\bar{u})$ *contains no eigenvectors of*

$$A(\omega) = \sum_{i=1}^{m} \omega_i f_i'(\bar{u}) \tag{1.5}$$

for all unit vectors $\omega = (\omega_1, \ldots, \omega_m)^t \in \mathbb{R}^m$.

Remark 1.1. Assumption 1(i) is natural since many physical models come with a certain number of conservation laws. Here we set explicitly this number as n_1. For physical relevance $n_1 > 0$ although we have no need of such an assumption. (In fact, the case $n_1 = 0$ is an easier one since the solution decays exponentially in time.) However, we do need $n_2 > 0$ with $(r_2)_{u_2}$ nonsingular. Otherwise, (1.1) becomes n hyperbolic conservation laws and we do not expect global existence of classical solutions even when the initial data are smooth and small.

Remark 1.2. Assumption 1(ii) is an extension of the classical Lax entropy condition for hyperbolic conservation laws ($r = 0$ or $n_2 = 0$) to include the source term r. The Lax entropy condition states: There exists an entropy pair (U, F), where $U = U(u) \in \mathbb{R}$ is strictly convex in u, and $F = (F_1(u), \ldots, F_m(u))^t \in \mathbb{R}^m$ satisfies

$$U' f_i' = F_i', \quad 1 \le i \le m. \tag{1.6}$$

Such a condition implies that $U'' f_i'$, $1 \le i \le m$, are symmetric, [5]. On the other hand, if $U'' f_i'$, $1 \le i \le m$, are symmetric, then there exists an $F(u) \in \mathbb{R}^m$ such that (1.6) is satisfied, see Lemma 3.2 below. The symmetrization of

f_i' by U'' in O gives us the well-posedness of (1.1), (1.2). This is a classical result, e.g., see [4, 8], etc.

Remark 1.3. As the Lax entropy condition is extended to include the source term r, however, in Assumption 1(ii) we require that U'' symmetrizes r' on E rather than in O. This is demanded by physics since for several important models such as the dynamics of real gases, $U''r'$ is symmetric only on E but not in O. On the other hand, it might not be sufficient to require $U''r'$ to be symmetric at $\bar{u}$ only for the purpose of global existence, see more discussion after Theorem 1.5.

Remark 1.4. Assumption 1(iii) is known as Shizuta-Kawashima condition, originally introduced for hyperbolic-parabolic systems, which may include lower order terms. The Assumption implies the strong coupling between the source term and the flux functions. Thus even $n_1 > 0$, the source term induces dissipation to the whole solution and gives decay in time. This is clear through the study of Green's function in one space dimension: In conjunction with a linear version of Assumption 1(ii), Assumption 1(iii) implies that the Green's function of the linearized system consists of heat kernels and exponentially decaying δ-functions, [17, 20].

We introduce the following notations to abbreviate the norms of Sobolev spaces with respect to x:

$$\|\cdot\|_s = \|\cdot\|_{H^s(\mathbb{R}^m)}, \quad \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^m)}. \tag{1.7}$$

Theorem 1.5. *Let $\bar{u}$ be a constant equilibrium state, Assumption 1* (i)$-$(iii) *hold, $s > \frac{m}{2}+1$ $(m \geq 1)$ be an integer, and $u_0 - \bar{u} \in H^s(\mathbb{R}^m)$. Then there exists a constant $\varepsilon > 0$ such that if $\|u_0 - \bar{u}\|_s \leq \varepsilon$, the Cauchy problem* (1.1), (1.2) *has a unique global solution u. The solution satisfies $u - \bar{u} \in C([0,\infty); H^s(\mathbb{R}^m)) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}^m))$, $D_x u \in L^2([0,\infty); H^{s-1}(\mathbb{R}^m))$, $r(u) \in L^2([0,\infty); H^s(\mathbb{R}^m))$, and*

$$\sup_{t\geq 0} \|u - \bar{u}\|_s^2(t) + \int_0^\infty [\|D_x u\|_{s-1}^2(t) + \|r_2(u)\|_s^2(t)]\,dt \leq C\|u_0 - \bar{u}\|_s^2, \tag{1.8}$$

where $C > 0$ is a constant. Here $D_x u$ denotes first partial derivatives of u with respect to x.

Theorem 1.5 is parallel to Theorem 3.2 in [11], which was obtained under a slightly stronger set of assumptions, originally proposed in [10]. In Section 2 we discuss the relation between our set of hypotheses and the one in [11]. We see that ours is more concise. We want to point out that under slightly weaker assumptions, i.e., the symmetrization of r' by U'' is at $\bar{u}$ rather than on E, an energy estimate that implies global existence has been obtained for the case $m \geq 3$, [9]. The energy estimate in [9], however, contains the integral of $\|u - \bar{u}\|^2(t)$ with respect to t, which is unbounded for $m < 3$. Therefore, it seems that it is necessary to assume $U''r'$ to be symmetric, semi-negative definite on E, not just at $\bar{u}$. This is in any case a natural assumption. In Section 3 we have preliminaries for the proof of Theorem 1.5. In this section we also slightly relax Assumption 1(ii) to a weaker version, Assumption 3(ii$'$). Assumption 1(ii) is more concise in its statement. See Section 3 for details. In Section 4 we give the proof of Theorem 1.5 under Assumptions 1(i), 3(ii$'$) and 1(iii). Throughout this paper we use C for a universal positive constant.

2. Structural Conditions

As mentioned in Section 1, Theorem 1.5 is parallel to Theorem 3.2 in [11], which was obtained under a set of assumptions proposed in [10]. In this section we compare the two sets of hypotheses. For this we introduce the following notation used in [10]:

$$M = \{\psi \in \mathbb{R}^n | \psi^t r(u) = 0, \, \forall u \in O\}. \tag{2.1}$$

Assumption 2 ([10])**.** (a) *There exists a strictly convex entropy function U such that $f_i'(U'')^{-1}$, $1 \leq i \leq m$, are symmetric in O, and $r'(U'')^{-1}$ is symmetric, semi-negative definite on E.*

(b) *On E the null space of $r'(U'')^{-1}$ coincides with M.*

(c) *The null space of $[r'(U'')^{-1}](\bar{u})$ contains no eigenvectors of $-U''(\bar{u})A(\omega)(U'')^{-1}(\bar{u})$, where $A(\omega)$ is defined in* (1.5)*.*

(d) *In O a state u is an equilibrium state ($u \in E$) if and only if $U'(u)^t \in M$.*

Proposition 2.1. *Under Assumption 2* (a)$-$(c)*, there exists a constant orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ such that the linear transformation $\tilde{u} = Qu$*

converts (1.1) *into an equivalent system*

$$\tilde{u}_t + \sum_{i=1}^{m} \tilde{f}_i(\tilde{u})_{x_i} = \tilde{r}(\tilde{u}) \tag{2.2}$$

that satisfies Assumption 1 (i)−(iii) *in a small neighborhood of* E.

Proof. Note that M is a subspace of $\mathbb{R}^n$, independent of u. By Gram-Schmidt process, there is an orthonormal basis $\{q_1, \ldots, q_{n_1}\}$ of M, which can be further extended to an orthonormal basis $\{q_1, \ldots, q_n\}$ of $\mathbb{R}^n$. Clearly, $\{q_{n_1+1}, \ldots, q_n\}$ is an orthonormal basis of $M^\perp$.

Let

$$Q^t = (q_1 \cdots q_n).$$

Then Q is a constant orthogonal matrix. Define the linear transformation

$$\tilde{u} = Qu, \quad \text{or} \quad u = Q^t \tilde{u}. \tag{2.3}$$

Multiply (1.1) by Q from the left we have

$$\tilde{u}_t + \sum_{i=1}^{m} \tilde{f}_i(\tilde{u})_{x_i} = \tilde{r}(\tilde{u}), \tag{2.4}$$

where

$$\tilde{f}_i(\tilde{u}) = Q f_i(Q^t \tilde{u}), \quad 1 \le i \le m, \qquad \tilde{r}(\tilde{u}) = Q r(Q^t \tilde{u}). \tag{2.5}$$

Our goal is to prove that under Assumption 2 (a)−(c), (2.4), (2.5) satisfy Assumption 1 (i)−(iii).

Consider the first n_1 components of $\tilde{r}$. For $1 \le i \le n_1$, by (2.5)

$$\tilde{r}_{1i} = q_i^t r(u) = 0$$

since $q_i \in M$. Therefore, (2.4) can be written as

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}_t + \sum_{i=1}^{m} \begin{pmatrix} \tilde{f}_{i1} \\ \tilde{f}_{i2} \end{pmatrix} (\tilde{u})_{x_i} = \begin{pmatrix} 0 \\ \tilde{r}_2 \end{pmatrix} (\tilde{u}),$$

where $\tilde{u}_1$, $\tilde{f}_{i1} \in \mathbb{R}^{n_1}$, $\tilde{u}_2$, $\tilde{f}_{i2}$, $\tilde{r}_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$. Here $n_2 > 0$ unless

$M = \mathbb{R}^n$ or $r = 0$. Noting

$$\tilde{r}_2(\tilde{u}) = (q_{n_1+1} \cdots q_n)^t r(Q^t \tilde{u}),$$

we have

$$(\tilde{r}_2)_{\tilde{u}_2} = (q_{n_1+1} \cdots q_n)^t r'(u)(q_{n_1+1} \cdots q_n). \tag{2.6}$$

To prove that (2.6) is nonsingular we first consider $u \in E$. For this we set $(\tilde{r}_2)_{\tilde{u}_2} v = 0$ for $v \in \mathbb{R}^{n_2}$ and argue that $v = 0$. From (2.6) and noting $\{q_{n_1+1}, \ldots, q_n\}$ is a basis of $M^\perp$, we conclude that if $(\tilde{r}_2)_{\tilde{u}_2} v = 0$ then

$$r'(u)(q_{n_1+1} \cdots q_n)v \in M.$$

By Assumption 2 (b), we further conclude that

$$r'(u)[U''(u)]^{-1} r'(u)(q_{n_1+1} \cdots q_n)v = 0. \tag{2.7}$$

Since $r'(u)[U''(u)]^{-1}$ is symmetric on E by Assumption 2 (a), (2.7) becomes

$$[U''(u)]^{-1}[r'(u)]^t r'(u)(q_{n_1+1} \cdots q_n)v = 0.$$

This further implies

$$v^t (q_{n_1+1} \cdots q_n)^t [r'(u)]^t r'(u)(q_{n_1+1} \cdots q_n)v = 0,$$

hence

$$r'(u)[U''(u)]^{-1} U''(u)(q_{n_1+1} \cdots q_n)v = 0.$$

Now Assumption (b) implies $U''(u)(q_{n_1+1} \cdots q_n)v \in M$. Since $(q_{n_1+1} \cdots q_n)v \in M^\perp$ we have

$$v^t (q_{n_1+1} \cdots q_n)^t U''(u)(q_{n_1+1} \cdots q_n)v = 0. \tag{2.8}$$

Noting $U''(u)$ is symmetric, positive definite from Assumption 2 (a), (2.8) implies $(q_{n_1+1} \cdots q_n)v = 0$. Since $\{q_{n_1+1}, \ldots, q_n\}$ is a linearly independent set, we must have $v = 0$. Therefore, we have proved that $(\tilde{r}_2)_{\tilde{u}_2}$ is nonsingular on E. As its determinant is a continuous function, it is nonsingular in a small neighborhood of E as well. This proves that (2.4), (2.5) satisfy Assumption 1(i) in a small neighborhood of E.

To verify that (2.4), (2.5) satisfy Assumption 1(ii) we set

$$\tilde{U}(\tilde{u}) = U(u) = U(Q^t\tilde{u}).$$

By Assumption 2 (a), $U''(u)$ is symmetric, positive definite in O. Thus

$$\tilde{U}_{\tilde{u}\tilde{u}} = QU''(Q^t\tilde{u})Q^t \tag{2.9}$$

is symmetric, positive definite in $\tilde{O} = \{Qu|u \in O\}$. From (2.5) and (2.9),

$$\tilde{U}_{\tilde{u}\tilde{u}}(\tilde{f}_i)_{\tilde{u}} = QU''(Q^t\tilde{u})f_i'(Q^t\tilde{u})Q^t,$$

which is symmetric in $\tilde{O}$ since $f_i'(U'')^{-1}$ is symmetric in O by Assumption 2 (a). Similarly,

$$\tilde{U}_{\tilde{u}\tilde{u}}\tilde{r}_{\tilde{u}} = QU''(Q^t\tilde{u})r'(Q^t\tilde{u})Q^t$$

is symmetric, semi-negative definite on $\tilde{E} = \{Qu|u \in E\}$ by Assumption 2 (a). Thus we have verified that (2.4), (2.5) satisfy Assumption 1(ii).

For Assumption 1(iii), by (2.5) the matrix $A(\omega)$ corresponding to (2.4) is

$$\tilde{A}(\omega) = \sum_{i=1}^{m} \omega_i Q f_i'(\bar{u})Q^t = QA(\omega)Q^t,$$

where $A(\omega)$ is defined in (1.5). Also from (2.5) the null space of $\tilde{r}_{\tilde{u}}(Q\bar{u}) = Qr'(\bar{u})Q^t$ is the null space of $r'(\bar{u})Q^t$. If there exist $\lambda \in \mathbb{R}$ and $\eta \in \mathbb{R}^n$ such that

$$r'(\bar{u})Q^t\eta = 0, \quad QA(\omega)Q^t\eta = \lambda\eta,$$

then

$$r'(\bar{u})[U''(\bar{u})]^{-1}U''(\bar{u})Q^t\eta = 0, \quad A(\omega)Q^t\eta = \lambda Q^t\eta. \tag{2.10}$$

The second equation in (2.10) is equivalent to

$$-U''(\bar{u})A(\omega)[U''(\bar{u})]^{-1}U''(\bar{u})Q^t\eta = -\lambda U''(\bar{u})Q^t\eta. \tag{2.11}$$

By Assumption 2 (c), the first equations of (2.10) and (2.11) imply

$$U''(\bar{u})Q^t\eta = 0.$$

Therefore, $\eta = 0$ and Assumption 1(iii) is satisfied by (2.4), (2.5). □

Proposition 2.1 shows that Assumption 2 (d) is redundant to obtain the global existence of small solutions and the energy estimate (1.8). Although Assumption 2 (d) may not be a true restriction as it is satisfied by some physical examples, Assumption 1 (i)−(iii) are more concise than Assumption 2 (a)−(d). This could help when we extend the study to more general systems, such as those violating the Shizuta-Kawashima condition.

3. Preliminaries

In this section we prove some lemmas to prepare for the proof of our main result, Theorem 1.5. First we introduce new variables:

$$v \equiv \begin{pmatrix} u_1 \\ r_2(u) \end{pmatrix} = v(u) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (u). \tag{3.1}$$

It is straightforward to calculate the Jacobian matrices of the transformation,

$$\begin{aligned} v_u &= \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ (r_2)_{u_1} & (r_2)_{u_2} \end{pmatrix}, \\ u_v = (v_u)^{-1} &= \begin{pmatrix} I_{n_1\times n_1} & 0_{n_1\times n_2} \\ -[(r_2)_{u_2}]^{-1}(r_2)_{u_1} & [(r_2)_{u_2}]^{-1} \end{pmatrix}. \end{aligned} \tag{3.2}$$

As mentioned in Section 1, Theorem 1.5 is to be proved with Assumption 1(ii) slightly relaxed to the following weaker version:

Assumption 3. (ii′) *There exists a strictly convex entropy function U of u such that $U''f_i'$, $1 \le i \le m$, are symmetric in O and $U''(\bar{u})r'(\bar{u})$ is symmetric, semi-negative definite. Besides, if we partition the skew symmetric part of $U''(u)r'(u)$ as $n_1 + n_2$ in rows and columns:*

$$S(u) = \frac{1}{2}[U''r' - (U''r')^t](u) = \begin{pmatrix} S_1(u) & S_2(u) \\ -S_2^t(u) & S_3(u) \end{pmatrix}, \tag{3.3}$$

then in a small neighborhood of $\bar{u}$ we have

$$S_2(u) = O(1)|r_2(u)|, \quad S_3(u) = O(1)|r_2(u)|. \tag{3.4}$$

Lemma 3.1. *Under Assumption* 1(i), *Assumption* 1(ii) *implies Assumption* 3(ii$'$).

Proof. Assumption 1(i) implies that $u \to v$ is a diffeomorphism because $(r_2)_{u_2}$ is invertible. Thus we write

$$S = S(u) = S(u(v)) = S(u(v_1, v_2)) = S(u(v_1, 0)) + O(1)|v_2| \tag{3.5}$$

by Taylor expansion at $(v_1, 0)$. From (3.1) we see that $v_2 = r_2(u) = 0$ implies $u \in E$ if u is in O. That is, in a small neighborhood of $\bar{u}$ we have $u(v, 0) \in E$. Assumption 1(ii) then implies $S(u(v_1, 0)) = 0$, hence (3.5) becomes $S(u) = O(1)|v_2| = O(1)|r_2(u)|$. This further implies (3.4). The other statements in Assumption 3(ii$'$) are already included in Assumption 1(ii). □

For hyperbolic conservation laws ($r = 0$) it is a classical result that if there exist an entropy function U and entropy flux functions F_i, $1 \le i \le m$, such that

$$U' f_i' = F_i', \quad 1 \le i \le m,$$

then $U'' f_i'$, $1 \le i \le m$, are symmetric, [5]. Conversely, we have the following lemma.

Lemma 3.2. *If U and f_i are smooth functions of u such that U is strictly convex and $U'' f_i'$ are symmetric, $1 \le i \le m$, then there exist smooth functions F_i of u such that*

$$U' f_i' = F_i', \quad 1 \le i \le m. \tag{3.6}$$

Proof. A proof of Lemma 3.2 can be found in [10]. For completeness we include the proof here. Since U is strictly convex, $u \to \eta = [U'(u)]^t$ is a diffeomorphism. Thus we may consider

$$[f_i(u(\eta))]_\eta = f_i'(u) u_\eta = f_i'(u)[U''(u)]^{-1}, \quad 1 \le i \le m. \tag{3.7}$$

For each i by the assumption $U'' f_i'$ is symmetric, which implies that $f_i'(U'')^{-1}$ is symmetric. Therefore, the left-hand side of (3.7) is symmetric. Consequently,

$$\nabla_\eta \times f_i(u(\eta)) = 0,$$

and $f_i(u(\eta))$ is conservative. That is, there exists a scalar function $\tilde{F}_i(\eta)$ such that $f_i^t(u(\eta)) = \nabla_\eta \tilde{F}_i(\eta)$. Set

$$F_i(u) = \eta^t(u) f_i(u) - \tilde{F}_i(\eta(u)).$$

Then

$$F_i'(u) = f_i^t(u) U''(u) + U'(u) f_i'(u) - \nabla_\eta \tilde{F}_i(\eta(u)) U''(u) = U'(u) f_i'(u). \quad \square$$

By direct calculation we have

$$U''(u) r'(u) = \begin{pmatrix} U_{u_2 u_1} (r_2)_{u_1} & U_{u_2 u_1} (r_2)_{u_2} \\ U_{u_2 u_2} (r_2)_{u_1} & U_{u_2 u_2} (r_2)_{u_2} \end{pmatrix}, \tag{3.8}$$

$$r'(u)^t U''(u) = \begin{pmatrix} (r_2)_{u_1}^t U_{u_1 u_2} & (r_2)_{u_1}^t U_{u_2 u_2} \\ (r_2)_{u_2}^t U_{u_1 u_2} & (r_2)_{u_2}^t U_{u_2 u_2} \end{pmatrix}. \tag{3.9}$$

Lemma 3.3. *If $U''(\bar{u}) r'(\bar{u})$ is symmetric then*

$$[U_{u_2 u_2} (r_2)_{u_1}](\bar{u}) = [(r_2)_{u_2}^t U_{u_1 u_2}](\bar{u}), \tag{3.10}$$

$$[U_{u_2 u_2} (r_2)_{u_2}](\bar{u}) = [(r_2)_{u_2}^t U_{u_2 u_2}](\bar{u}). \tag{3.11}$$

If $U''(\bar{u}) r'(\bar{u})$ is semi-negative definite as well then $[U_{u_2 u_2} (r_2)_{u_2}](\bar{u})$ is semi-negative definite.

Proof. This is a direct consequence of (3.8) and (3.9). $\square$

With (3.2) by direct calculation we also have

$$\begin{aligned} U''(u) u_v(v) &= U''(u) v_u^{-1}(u) \\ &= \begin{pmatrix} U_{u_1 u_1} - U_{u_2 u_1} (r_2)_{u_2}^{-1} (r_2)_{u_1} & U_{u_2 u_1} (r_2)_{u_2}^{-1} \\ U_{u_1 u_2} - U_{u_2 u_2} (r_2)_{u_2}^{-1} (r_2)_{u_1} & U_{u_2 u_2} (r_2)_{u_2}^{-1} \end{pmatrix}. \end{aligned} \tag{3.12}$$

Lemma 3.4. *If $U''(\bar{u}) r'(\bar{u})$ is symmetric then*

$$U''(\bar{u}) u_v(\bar{v}) = \begin{pmatrix} U_{u_1 u_1} - U_{u_2 u_1} (r_2)_{u_2}^{-1} (r_2)_{u_1} & U_{u_2 u_1} (r_2)_{u_2}^{-1} \\ 0 & U_{u_2 u_2} (r_2)_{u_2}^{-1} \end{pmatrix} (\bar{u}), \tag{3.13}$$

where $\bar v = \begin{pmatrix} \bar u_1 \\ 0 \end{pmatrix}$.

Proof. From (3.10) and (3.11), when $u = \bar u$ we have

$$\begin{aligned} & U_{u_1u_2} - U_{u_2u_2}(r_2)_{u_2}^{-1}(r_2)_{u_1} \\ &= \big[(r_2)_{u_2}^{-1}\big]^t\big[(r_2)_{u_2}^t U_{u_1u_2} - (r_2)_{u_2}^t U_{u_2u_2}(r_2)_{u_2}^{-1}(r_2)_{u_1}\big] \\ &= \big[(r_2)_{u_2}^{-1}\big]^t\big[U_{u_2u_2}(r_2)_{u_1} - U_{u_2u_2}(r_2)_{u_2}(r_2)_{u_2}^{-1}(r_2)_{u_1}\big] = 0. \end{aligned}$$

Equation (3.13) follows from (3.12). □

Lemma 3.5. *Under Assumption* 3(ii′), *in a small neighborhood of* $\bar u$ *we have*

$$U_{u_1u_2} - U_{u_2u_2}(r_2)_{u_2}^{-1}(r_2)_{u_1} = O(1)|r_2(u)|. \tag{3.14}$$

Proof. We write the left-hand side of (3.14) as

$$\begin{aligned} & \big[(r_2)_{u_2}^{-1}\big]^t\big[(r_2)_{u_2}^t U_{u_1u_2} - (r_2)_{u_2}^t U_{u_2u_2}(r_2)_{u_2}^{-1}(r_2)_{u_1}\big] \\ &= \big[(r_2)_{u_2}^{-1}\big]^t\Big\{\big[(r_2)_{u_2}^t U_{u_1u_2} - U_{u_2u_2}(r_2)_{u_1}\big] \\ &\quad + \big[U_{u_2u_2}(r_2)_{u_2} - (r_2)_{u_2}^t U_{u_2u_2}\big](r_2)_{u_2}^{-1}(r_2)_{u_1}\Big\}. \end{aligned}$$

From (3.8) and (3.9), and using the notations in (3.3), the right-hand side is

$$2\big[(r_2)_{u_2}^{-1}\big]^t\big\{S_2^t(u) + S_3(u)(r_2)_{u_2}^{-1}(r_2)_{u_1}\big\}.$$

Thus by Assumption 3 (ii′), in a small neighborhood of $\bar u$ we have (3.14). □

Lemma 3.6. *If* $U''(\bar u)r'(\bar u)$ *is symmetric then*

$$(u_v^t U'' r' u_v)(\bar u) = \begin{pmatrix} 0 & 0 \\ 0 & \big[(r_2)_{u_2}^{-1}\big]^t U_{u_2u_2} \end{pmatrix}(\bar u) = \begin{pmatrix} 0 & 0 \\ 0 & U_{u_2u_2}(r_2)_{u_2}^{-1} \end{pmatrix}(\bar u). \tag{3.15}$$

Proof. We write

$$u_v^t U'' r' u_v = (U'' u_v)^t r' u_v.$$

Applying (3.13) and (3.2), and noting $r = \begin{pmatrix} 0 \\ r_2 \end{pmatrix}$, (3.15) is verified by direct calculation. □

Let D_x^l be the partial derivatives $(\partial/\partial x)^\alpha$ with a multi index α such that $|\alpha| = l$. For D_x^1 we write D_x, which has been introduced in Theorem 1.5. The following lemma states the two special cases of Gagliardo-Nirenberg inequality needed later.

Lemma 3.7 ([14]). *If $w \in H^s(\mathbb{R}^m)$ with $s > m/2$ then*

$$\|w\|_{L^\infty} \leq C_1 \|w\|_s, \tag{3.16}$$

where C_1 is a constant depending on m and s only. If $w \in L^\infty(\mathbb{R}^m)$ and $D_x^l w \in L^2(\mathbb{R}^m)$, then for $1 \leq i \leq l$,

$$\|D_x^i w\|_{L^{2l/i}} \leq C_2 \|w\|_{L^\infty}^{1-i/l} \|D_x^l w\|^{i/l}, \tag{3.17}$$

with C_2 depending only on i, l and m.

The following lemmas are Moser-type calculus inequalities, e.g. see [9, 13] and references therein. Here for completeness we include their proofs.

Lemma 3.8. *Let w be a given smooth function of u in a neighborhood of $\bar{u}$. If $u - \bar{u} \in H^s(\mathbb{R}^m)$ with $\|u - \bar{u}\|_s \leq \varepsilon$ and $s > m/2$ then*

$$\|D_x w\|_{s-1} \leq C \|D_x u\|_{s-1}, \tag{3.18}$$

where C is a constant depending only on m, s and ε.

Proof. We only need to prove

$$\|D_x^l w\| \leq C \|D_x^l u\|, \quad 1 \leq l \leq s. \tag{3.19}$$

By the chain rule

$$|D_x^l w| \leq C \sum_{\alpha_1, \ldots, \alpha_l} |D_x u|^{\alpha_1} \cdots |D_x^l u|^{\alpha_l},$$

where $\alpha_1, \ldots, \alpha_l \geq 0$ are integers satisfying

$$\alpha_1 + 2\alpha_2 + \cdots + l\alpha_l = l, \tag{3.20}$$

and the constant C depends on l and the partial derivatives of w with respect to u, thus on s, m and ε if we apply (3.16) to $u - \bar{u}$. By the triangle inequality and the generalized Hölder's inequality, with

$$p_i = \frac{l}{i\alpha_i}, \quad \frac{1}{p_1} + \cdots + \frac{1}{p_l} = 1,$$

we have

$$\|D_x^l w\| \le C \sum_{\alpha_1,\dots,\alpha_l} \||D_x u|^{\alpha_1} \cdots |D_x^l u|^{\alpha_l}\| \le C \sum_{\alpha_1,\dots,\alpha_l} \|D_x u\|_{L^{2l/1}}^{\alpha_1} \cdots \|D_x^l u\|_{L^{2l/l}}^{\alpha_l}.$$

Applying (3.17) to the right-hand side and using (3.20) we have

$$\begin{aligned}\|D_x^l w\| &\le C \sum_{\alpha_1,\dots,\alpha_l} \|u-\bar{u}\|_{L^\infty}^{(1-\frac{1}{l})\alpha_1+\cdots+(1-\frac{l}{l})\alpha_l} \|D_x^l u\|^{\frac{\alpha_1}{l}+\cdots+\frac{l\alpha_l}{l}} \\ &= C \sum_{\alpha_1,\dots,\alpha_l} \|u-\bar{u}\|_{L^\infty}^{\alpha_1+\cdots+\alpha_l-1} \|D_x^l u\|.\end{aligned}$$

Noting the assumption $\|u-\bar{u}\|_s \le \varepsilon$ and by (3.16) we obtain (3.19). □

Lemma 3.9. *If $D_x w_1 \in H^{l-1}(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ and $w_2 \in H^{l-1}(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ then*

$$\|D_x^l(w_1 w_2) - w_1 D_x^l w_2\| \le C(\|D_x w_1\|_{L^\infty} \|D_x^{l-1} w_2\| + \|D_x^l w_1\| \|w_2\|_{L^\infty}), \tag{3.21}$$

where $C > 0$ is a constant depending only on m and l.

Proof. By the product rule

$$\begin{aligned}|D_x^l(w_1 w_2) - w_1 D_x^l w_2| &\le C \sum_{1\le i\le l} |D_x^i w_1||D_x^{l-i} w_2| \\ &= C \sum_{0\le i\le l-1} |D_x^i D_x w_1||D_x^{l-1-i} w_2|.\end{aligned}$$

Let $p = \frac{l-1}{i}$ and $q = \frac{l-1}{l-1-i}$ and apply Hölder's inequality. We have

$$\begin{aligned}\|D_x^l(w_1 w_2) - w_1 D_x^l w_2\| &\le C \sum_{0\le i\le l-1} \||D_x^i D_x w_1||D_x^{l-1-i} w_2|\| \\ &\le C \sum_{0\le i\le l-1} \|D_x^i D_x w_1\|_{L^{2(l-1)/i}} \|D_x^{l-1-i} w_2\|_{L^{2(l-1)/(l-1-i)}}.\end{aligned}$$

Applying (3.17), which is also true for $i = 0$, to the right-hand side, we obtain

$$\|D_x^l(w_1w_2) - w_1D_x^lw_2\| \\ \leq C \sum_{0\leq i\leq l-1} \|D_xw_1\|_{L^\infty}^{1-\frac{i}{l-1}} \|D_x^lw_1\|^{\frac{i}{l-1}} \|w_2\|_{L^\infty}^{\frac{i}{l-1}} \|D_x^{l-1}w_2\|^{1-\frac{i}{l-1}}.$$

By comparing $\|D_xw_1\|_{L^\infty}\|D_x^{l-1}w_2\|$ with $\|D_x^lw_1\|\|w_2\|_{L^\infty}$, we obtain (3.21). □

Lemma 3.10. *Let $A \in \mathbb{R}^{n\times n}$ be a given smooth function of u in a neighborhood of $\bar{u}$, and w be an n-vector valued function. If $u - \bar{u} \in H^s(\mathbb{R}^m)$ with $\|u - \bar{u}\|_s \leq \varepsilon$, $s > m/2 + 1$, and $w \in H^{s-1}(\mathbb{R}^m)$, then for $1 \leq l \leq s$ we have*

$$\left\|D_x^l[A(u)w] - A(u)D_x^lw\right\| \leq C\|D_xu\|_{s-1}\|w\|_{s-1}, \tag{3.22}$$

where $C > 0$ is a constant depending only on m, s and ε.

Proof. Equation (3.22) is a direct consequence of (3.21), (3.18) and (3.16). □

The next lemma concerns the symmetric linear system

$$A_0w_t + \sum_{j=1}^m A_jw_{x_j} = Bw, \tag{3.23}$$

where

$$A_0 = (u_v^tU''u_v)(\bar{u}), \quad A_j = (u_v^tU''f_j'u_v)(\bar{u}), \quad B = (u_v^tU''r'u_v)(\bar{u}). \tag{3.24}$$

Under Assumptions 1(i) and 3(ii′) A_0 is symmetric, positive definite, B is symmetric, semi-negative definite, and A_j, $1 \leq j \leq m$, are symmetric. Set

$$\tilde{A}(\omega) = \sum_{j=1}^m \omega_jA_j = u_v^t(\bar{u})U''(\bar{u})A(\omega)u_v(\bar{u}), \tag{3.25}$$

where $\omega = (\omega_1, \ldots, \omega_m)^t$ is a unit vector in $\mathbb{R}^m$. Here $A(\omega)$ is defined in (1.5). Clearly, Assumption 1(iii) is equivalent to the following statement: The null space of B contains no eigenvectors of $A_0^{-1}\tilde{A}(\omega)$ for all unit vectors $\omega \in \mathbb{R}^m$. The following lemma is a special case of Theorem 1.1 in [16].

Lemma 3.11. *Assume that* A_0 *is symmetric, positive definite,* B *is symmetric, semi-negative definite, and* A_j, $1 \le j \le m$, *are symmetric. Also assume that the null space of* B *contains no eigenvectors of* $A_0^{-1}\tilde{A}(\omega)$, $\tilde{A}(\omega) = \sum_{j=1}^{m} \omega_j A_j$, *for all unit vectors* $\omega = (\omega_1, \ldots, \omega_m)^t \in \mathbb{R}^m$. *Then there exists a compensating matrix* $K(\omega)$ *for the system* (3.23), *satisfying*

(i) $K(\omega)$ *is smooth on the unit sphere* $\mathbb{S}^{m-1}$, *and* $K(-\omega) = -K(\omega)$ *for each* $\omega \in \mathbb{S}^{m-1}$.

(ii) $K(\omega)A_0$ *is skew symmetric for each* $\omega \in \mathbb{S}^{m-1}$.

(iii) $\frac{1}{2}[K(\omega)\tilde{A}(\omega) + \tilde{A}(\omega)^t K(\omega)^t] - B$ *is symmetric, positive definite for each* $\omega \in \mathbb{S}^{m-1}$.

4. Energy Estimate

In this section we prove our main result, Theorem 1.5. By Lemma 3.1 we may replace Assumption 1(ii) by Assumption 3(ii′). Thus throughout this section we assume Assumptions 1(i), 3(ii′) and 1(iii).

Local existence of solutions for symmetric hyperbolic systems with small Cauchy data is classical. For instance, see Theorem 5.1.1 in [3] and references in Section 5.6 therein. Here we cite Theorem 2.9 in [9]:

Theorem 4.1. *Suppose that in* O *there is a symmetric, positive definite matrix* $A_0 \in \mathbb{R}^{n\times n}$ *such that* $A_0(u)f_i'(u)$, $1 \le i \le m$, *are symmetric. Let* $m \ge 1$ *and* $s > \frac{m}{2} + 1$ *be integers. If* $u_0 - \bar{u} \in H^s(\mathbb{R}^m)$ *and* $u_0(x)$ *takes value in a compact subset of* O *for all* $x \in \mathbb{R}^m$, *then there exists a positive constant* T *such that the Cauchy problem* (1.1), (1.2) *has a unique solution* u *satisfying*

$$u - \bar{u} \in C([0,T]; H^s(\mathbb{R}^m)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m)).$$

Clearly, under Assumption 3(ii′) and by setting $A_0(u) = U''(u)$ we have local existence theory for (1.1), (1.2). To prove Theorem 1.5 we need to perform a priori energy estimates. We introduce the following notation:

$$N_s^2(t) = \sup_{0\le\tau\le t} \|u - \bar{u}\|_s^2(\tau) + \int_0^t \left[\|D_x u\|_{s-1}^2(\tau) + \|r_2(u)\|_s^2(\tau)\right] d\tau. \qquad (4.1)$$

By standard continuity argument, all we need is to prove the following proposition.

Proposition 4.2. *Let Assumptions* 1(i), 3(ii′) *and* 1(iii) *hold,* $s > \frac{m}{2} + 1$ *(*$m \geq 1$*) be an integer, and* $T > 0$ *be a constant. Let* u *be a solution to* (1.1), (1.2), *satisfying* $u - \bar{u} \in C([0,T]; H^s(\mathbb{R}^m)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m))$, $D_x u \in L^2([0,T]; H^{s-1}(\mathbb{R}^m))$, *and* $r_2(u) \in L^2([0,T]; H^s(\mathbb{R}^m))$. *If* $N_s(T)$ *is bounded by a small positive constant independent of* T*, then* $N_s(T) \leq C\|u_0 - \bar{u}\|_s$*, where* C *is a constant independent of* T.

Proof. We set $0 \leq t \leq T$. Let

$$\mathscr{E}(u) = U(u) - U(\bar{u}) - U'(\bar{u})(u - \bar{u}). \tag{4.2}$$

Differentiating (4.2) with respect to t and substituting (1.1) into the result, we have

$$\begin{aligned}\mathscr{E}_t &= [U'(u) - U'(\bar{u})]u_t \\ &= -U'(u)\sum_{i=1}^m f_i'(u)u_{x_i} + U'(\bar{u})\sum_{i=1}^m f_i(u)_{x_i} + [U'(u) - U'(\bar{u})]r(u). \end{aligned}\tag{4.3}$$

Now we apply Lemma 3.2 to the right-hand side of (4.3) to obtain

$$\mathscr{E}_t = \sum_{i=1}^m [-F_i(u) + U'(\bar{u})f_i(u)]_{x_i} + [U'(u) - U'(\bar{u})]r(u). \tag{4.4}$$

For the second term on the right-hand side of (4.4) we consider u as a function of v defined in (3.1). Setting

$$\bar{v} \equiv v(\bar{u}) = \begin{pmatrix}\bar{v}_1 \\ \bar{v}_2\end{pmatrix} = \begin{pmatrix}\bar{u}_1 \\ 0\end{pmatrix}, \tag{4.5}$$

we write

$$\begin{aligned}&[U'(u) - U'(\bar{u})]r(u) = r^t(u)\int_0^1 \frac{d}{d\theta}U'(u(\bar{v} + \theta(v - \bar{v})))^t d\theta \\ &= \; r^t(u)\int_0^1 U''(u(\bar{v} + \theta(v - \bar{v})))u_v(\bar{v} + \theta(v - \bar{v}))\, d\theta(v - \bar{v}) \\ &= \; \mathscr{T}_1 + \mathscr{T}_2, \end{aligned}\tag{4.6}$$

where

$$\mathscr{T}_1 = r^t(u)U''(\bar{u})u_v(\bar{v})(v-\bar{v}), \tag{4.7}$$

$$\mathscr{T}_2 = r^t(u)\int_0^1 \big[U''(u(\bar{v}+\theta(v-\bar{v})))u_v(\bar{v}+\theta(v-\bar{v}))-U''(\bar{u})u_v(\bar{v})\big]\, d\theta(v-\bar{v}).$$

Under Assumption 3(ii′) we may apply (3.13) to $\mathscr{T}_1$:

$$\mathscr{T}_1 = r_2^t(u)\big[U_{u_2u_2}(r_2)_{u_2}^{-1}\big](\bar{u})r_2(u). \tag{4.8}$$

From Lemma 3.3 $\big[(r_2)_{u_2}^t U_{u_2u_2}\big](\bar{u})$ is symmetric, semi-negative definite, while from Assumptions 1(i) and 3(ii′) $(r_2)_{u_2}^t$ and $U_{u_2u_2}$ are nonsingular. Therefore, $\big[(r_2)_{u_2}^t U_{u_2u_2}\big](\bar{u})$ is symmetric, negative definite. This implies $\big[U_{u_2u_2}(r_2)_{u_2}^{-1}\big](\bar{u})$ is symmetric, negative definite as well. Thus (4.8) implies that there exists a constant $c_1 > 0$ such that

$$\mathscr{T}_1 \le -c_1|r_2(u)|^2. \tag{4.9}$$

From (3.12), (3.13), (3.1), (4.5) and Assumption 1(i),

$$\begin{aligned} & r^t(u)\big[U''(u(\bar{v}+\theta(v-\bar{v})))u_v(\bar{v}+\theta(v-\bar{v})) - U''(\bar{u})u_v(\bar{v})\big](v-\bar{v}) \\ = \;& r_2^t(u)\big[U_{u_1u_2} - U_{u_2u_2}(r_2)_{u_2}^{-1}(r_2)_{u_1}\big](u(\bar{v}+\theta(v-\bar{v})))(u_1-\bar{u}_1) \\ & + r_2^t(u)\Big\{\big[U_{u_2u_2}(r_2)_{u_2}^{-1}\big](u(\bar{v}+\theta(v-\bar{v}))) - \big[U_{u_2u_2}(r_2)_{u_2}^{-1}\big](\bar{u})\Big\}r_2(u). \end{aligned} \tag{4.10}$$

By Gagliardo-Nirenberg inequality, Lemma 3.7, if $N_s(T)$ is bounded by a sufficiently small positive constant (independent of T), then u is in a small neighborhood of $\bar{u}$, and we may apply Lemma 3.5. Thus (3.14) implies that the right-hand side of (4.10) is reduced to

$$O(1)|r_2(u)||r_2(u(\bar{v}+\theta(v-\bar{v})))||u_1-\bar{u}_1| + O(1)|r_2(u)|^2|u-\bar{u}| \tag{4.11}$$

for $0 \le \theta \le 1$. From (4.7) we integrate (4.11) with respect to θ to obtain

$$\begin{aligned} \mathscr{T}_2 = \;& O(1)|r_2(u)||u_1-\bar{u}_1|\int_0^1 |r_2(u(\bar{v}+\theta(v-\bar{v})))|\, d\theta \\ & + O(1)|r_2(u)|^2|u-\bar{u}|. \end{aligned} \tag{4.12}$$

From (3.1) and since $u \to v$ is a diffeomorphism, see the proof of Lemma

3.1,

$$\begin{pmatrix} \bar{u}_1+\theta(u_1-\bar{u}_1) \\ \theta r_2(u) \end{pmatrix} = \bar{v}+\theta(v-\bar{v}) = v(u(\bar{v}+\theta(v-\bar{v}))) = \begin{pmatrix} u_1(\bar{v}+\theta(v-\bar{v})) \\ r_2(u(\bar{v}+\theta(v-\bar{v}))) \end{pmatrix}.$$

That is,

$$r_2(u(\bar{v}+\theta(v-\bar{v}))) = \theta r_2(u). \tag{4.13}$$

Substituting (4.13) into (4.12) we have

$$\mathscr{T}_2 = O(1)|r_2(u)|^2|u-\bar{u}|. \tag{4.14}$$

Equations (4.4), (4.6), (4.9) and (4.14) give us

$$\begin{aligned} \mathscr{E}_t &= \sum_{i=1}^{m}[-F_i(u)+U'(\bar{u})f_i(u)]_{x_i} + \mathscr{T}_1 + \mathscr{T}_2 \\ &\le \sum_{i=1}^{m}[-F_i(u)+U'(\bar{u})f_i(u)]_{x_i} - c_1|r_2(u)|^2 + O(1)|r_2(u)|^2|u-\bar{u}| \\ &\le \sum_{i=1}^{m}[F_i(\bar{u})-F_i(u)+U'(\bar{u})f_i(u)-U'(\bar{u})f_i(\bar{u})]_{x_i} - \frac{c_1}{2}|r_2(u)|^2 \end{aligned} \tag{4.15}$$

if $|u-\bar{u}|$ is sufficiently small, or if $N_s(T)$ is sufficiently small. Integrating (4.15) on $\mathbb{R}^m \times [0,t]$, we have

$$\int_{\mathbb{R}^m} \mathscr{E}(u(x,t))\,dx + \frac{c_1}{2}\int_0^t \int_{\mathbb{R}^m} |r_2(u)|^2(x,\tau)\,dxd\tau \le \int_{\mathbb{R}^m} \mathscr{E}(u(x,0))\,dx. \tag{4.16}$$

From (4.2), $\mathscr{E}(u) = \frac{1}{2}U''(\bar{u}+\theta(u-\bar{u}))(u-\bar{u},u-\bar{u})$ for some $0 \le \theta \le 1$. By Assumption 3(ii′), there exist constants $c_2 > c_3 > 0$ such that

$$c_3|u-\bar{u}|^2 \le \mathscr{E}(u) \le c_2|u-\bar{u}|^2. \tag{4.17}$$

Substituting (4.17) into (4.16) gives us

$$\|u-\bar{u}\|^2(t) + \int_0^t \|r_2(u)\|^2(\tau)\,d\tau \le C\|u_0-\bar{u}\|^2, \tag{4.18}$$

where $C > 0$ is a constant independent of t.

Next we consider derivatives of u. This part is similar to the proofs of

Lemma 3.1 and Lemma 3.2 in [9], and the corresponding part in the proof of Proposition 3.1 in [11]. We include this part for completeness.

For $1 \le l \le s$ we apply D_x^l to (1.1) and multiply the result by $(D_x^l u)^t U''(u)$. This gives us

$$(D_x^l u)^t U''(u) D_x^l u_t + (D_x^l u)^t U''(u) \sum_{i=1}^m D_x^l f_i(u)_{x_i} = (D_x^l u)^t U''(u) D_x^l r(u). \tag{4.19}$$

Noting $U''(u)$ is symmetric, we have

$$(D_x^l u)^t U''(u) D_x^l u_t = \frac{1}{2}\big[(D_x^l u)^t U''(u) D_x^l u\big]_t - \frac{1}{2}(D_x^l u)^t U''(u)_t D_x^l u. \tag{4.20}$$

Similarly, $U''(u)f_i'(u)$, $1 \le i \le m$, are symmetric by Assumption 3(ii′). Thus

$$\begin{aligned}
&(D_x^l u)^t U''(u) D_x^l f_i(u)_{x_i}\\
&= (D_x^l u)^t U''(u) f_i'(u) D_x^l u_{x_i} + (D_x^l u)^t U''(u)\big\{D_x^l[f_i'(u)u_{x_i}] - f_i'(u) D_x^l u_{x_i}\big\}\\
&= \frac{1}{2}\big[(D_x^l u)^t U''(u) f_i'(u) D_x^l u\big]_{x_i} - \frac{1}{2}(D_x^l u)^t [U''(u) f_i'(u)]_{x_i} D_x^l u\\
&\quad +(D_x^l u)^t U''(u)\big\{D_x^l[f_i'(u)u_{x_i}] - f_i'(u) D_x^l u_{x_i}\big\}.
\end{aligned} \tag{4.21}$$

For the right-hand side of (4.19), since $l \ge 1$ we may write $D_x^l = D_x^{l-1} D_{x_k}$ for some $1 \le k \le m$. Thus

$$\begin{aligned}
&(D_x^l u)^t U''(u) D_x^l r(u)\\
&\quad = (D_x^l u)^t (U''r')(u) D_x^l u + (D_x^l u)^t U''(u)\big\{D_x^{l-1}[r'(u)u_{x_k}] - r'(u) D_x^l u\big\}.
\end{aligned}$$

For the first term on the right-hand side we change variables and use v as defined in (3.1), and linearize the leading term around $\bar{u}$. This gives us

$$\begin{aligned}
&(D_x^l u)^t U''(u) D_x^l r(u)\\
&= \big[D_x^{l-1}(u_v v_{x_k})\big]^t (U''r')(u) D_x^{l-1}(u_v v_{x_k})\\
&\quad +(D_x^l u)^t U''(u)\big\{D_x^{l-1}[r'(u)u_{x_k}] - r'(u) D_x^l u\big\}\\
&= (D_x^l v)^t (u_v^t U'' r' u_v)(\bar{u}) D_x^l v + (D_x^l v)^t \big[(u_v^t U'' r' u_v)(u) - (u_v^t U'' r' u_v)(\bar{u})\big] D_x^l v\\
&\quad +\Big\{\big[D_x^{l-1}(u_v v_{x_k})\big]^t (U''r')(u) D_x^{l-1}(u_v v_{x_k}) - (D_x^l v)^t (u_v^t U'' r' u_v)(u) D_x^l v\Big\}\\
&\quad +(D_x^l u)^t U''(u)\big\{D_x^{l-1}[r'(u)u_{x_k}] - r'(u) D_x^l u\big\}.
\end{aligned} \tag{4.22}$$

We apply (3.15) to the first term on the right-hand side, and note (3.1). Then we compare the result with (4.8) and follow the argument that leads to (4.9) to obtain

$$\begin{aligned}(D_x^l v)^t(u_v^t U'' r' u_v)(\bar{u}) D_x^l v &= [D_x^l r_2(u)]^t \big[U_{u_2u_2}(r_2)_{u_2}^{-1}\big](\bar{u}) D_x^l r_2(u) \\ &\leq -c_1 |D_x^l r_2(u)|^2, \qquad (4.23)\end{aligned}$$

where $c_1 > 0$ is the same constant as in (4.9).

Now we integrate (4.19) on $\mathbb{R}^m \times [0,t]$ and apply (4.20)−(4.23):

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^m} \big[(D_x^l u)^t U''(u) D_x^l u\big](x,t)\,dx + c_1 \int_0^t \int_{\mathbb{R}^m} |D_x^l r_2(u)|^2(x,\tau)\,dxd\tau \\ &\quad \leq \frac{1}{2}\int_{\mathbb{R}^m} \big[(D_x^l u_0)^t U''(u_0) D_x^l u_0\big](x)dx + \mathscr{T}_3 + \mathscr{T}_4 + \mathscr{T}_5 + \mathscr{T}_6 + \mathscr{T}_7 + \mathscr{T}_8, \quad (4.24)\end{aligned}$$

where

$$\begin{aligned}\mathscr{T}_3 &= \frac{1}{2}\int_0^t \int_{\mathbb{R}^m} \big[(D_x^l u)^t U''(u)_t D_x^l u\big](x,t)\,dxd\tau, \\ \mathscr{T}_4 &= \frac{1}{2}\int_0^t \int_{\mathbb{R}^m} \Big\{\sum_{i=1}^m (D_x^l u)^t \big[U''(u) f_i'(u)\big]_{x_i} D_x^l u\Big\}(x,\tau)\,dxd\tau, \qquad (4.25) \\ \mathscr{T}_5 &= -\int_0^t \int_{\mathbb{R}^m} \Big\{\sum_{i=1}^m (D_x^l u)^t U''(u)\big[D_x^l(f_i'(u)u_{x_i}) - f_i'(u) D_x^l u_{x_i}\big]\Big\}(x,\tau)\,dxd\tau, \\ \mathscr{T}_6 &= \int_0^t \int_{\mathbb{R}^m} \Big\{(D_x^l v)^t\big[(u_v^t U'' r' u_v)(u) - (u_v^t U'' r' u_v)(\bar{u})\big] D_x^l v\Big\}(x,\tau)\,dxd\tau, \\ \mathscr{T}_7 &= \int_0^t \int_{\mathbb{R}^m} \Big\{\big[D_x^{l-1}(u_v v_{x_k})\big]^t (U'' r')(u) D_x^{l-1}(u_v v_{x_k}) \\ &\qquad - (D_x^l v)^t (u_v^t U'' r' u_v)(u) D_x^l v\Big\}(x,\tau)\,dxd\tau, \\ \mathscr{T}_8 &= \int_0^t \int_{\mathbb{R}^m} \Big\{(D_x^l u)^t U''(u)\big[D_x^{l-1}(r'(u)u_{x_k}) - r'(u) D_x^l u\big]\Big\}(x,\tau)dxd\tau.\end{aligned}$$

Applying (1.1) we have

$$\begin{aligned}\mathscr{T}_3 &= O(1)\int_0^t \int_{\mathbb{R}^m} \big[|D_x^l u|^2(|D_x u| + |r(u)|\big](x,\tau)\,dxd\tau \\ &= O(1)\int_0^t \|D_x^l u\|^2(\tau)\big(\|D_x u\|_{L^\infty}(\tau) + \|r(u)\|_{L^\infty}(\tau)\big)\,d\tau.\end{aligned}$$

Noting $\|r(u)\|_{L^\infty} = \|r(u) - r(\bar{u})\|_{L^\infty} = O(1)\|u - \bar{u}\|_{L^\infty}$, and by Lemma 3.7, $\|D_x u\|_{L^\infty}(\tau) + \|u - \bar{u}\|_{L^\infty}(\tau) \le C\|u - \bar{u}\|_s(\tau) \le CN_s(\tau)$, we have

$$\mathscr{T}_3 = O(1)N_s(t)\int_0^t \|D_x^l u\|^2(\tau)\,d\tau = O(1)N_s^3(t). \tag{4.26}$$

Similarly,

$$\mathscr{T}_4 + \mathscr{T}_6 = O(1)N_s^3(t) + O(1)N_s(t)\int_0^t \|D_x^l v\|^2(\tau)\,d\tau = O(1)N_s^3(t), \tag{4.27}$$

where we have applied (3.18).

For $\mathscr{T}_5$ we use Lemma 3.10 to bound the commutator. From (4.25), (3.16) and (3.22), and by Hölder's inequality,

$$\begin{aligned}\mathscr{T}_5 &= O(1)\int_0^t \sum_{i=1}^m \|D_x^l u\|\big\|D_x^l[f_i'(u)u_{x_i}] - f_i'(u)D_x^l u_{x_i}\big\|(\tau)\,d\tau \\ &= O(1)\int_0^t \|D_x^l u\|\|D_x u\|_{s-1}^2(\tau)\,d\tau = O(1)N_s^3(t). \end{aligned}\tag{4.28}$$

Similarly,

$$\begin{aligned}\mathscr{T}_7 &= \int_0^t\int_{\mathbb{R}^m}\Big\{\big[D_x^{l-1}(v_{x_k}^t u_v^t) - (D_x^{l-1}v_{x_k})^t u_v^t\big](U''r')(u)D_x^l u\Big\}(x,\tau)\,dxd\tau \\ &\quad + \int_0^t\int_{\mathbb{R}^m}\Big\{(D_x^l v)^t(u_v^t U'' r')(u)\big[D_x^{l-1}(u_v v_{x_k}) - u_v D_x^{l-1}v_{x_k}\big]\Big\}(x,\tau)dxd\tau \\ &= O(1)\int_0^t\big[\|D_x^l u\|\|D_x u\|_{s-1}\|D_x v\|_{s-1} + \|D_x^l v\|\|D_x u\|_{s-1}\|D_x v\|_{s-1}\big](\tau)d\tau \\ &= O(1)N_s^3(t), \end{aligned}\tag{4.29}$$

where we have noted that (3.22) is trivial when $l = 0$, and we have used (3.18). $\mathscr{T}_8$ is similar to $\mathscr{T}_7$ hence

$$\mathscr{T}_8 = O(1)N_s^3(t). \tag{4.30}$$

Combining (4.24) and (4.26)-(4.30), and noting U'' is symmetric, positive definite by Assumption 3(ii$'$), we obtain

$$\|D_x^l u\|^2(t) + \int_0^t \|D_x^l r_2(u)\|^2(\tau)\,d\tau \le C\|D_x^l u_0\|^2 + CN_s^3(t) \tag{4.31}$$

for $1 \le l \le s$. By taking summation of (4.31) with respect to l and (4.18) we further obtain

$$\|u-\bar{u}\|_s^2(t)+\int_0^t\|r_2(u)\|_s^2(\tau)\,d\tau\le C\|u_0-\bar{u}\|_s^2+CN_s^3(t). \tag{4.32}$$

To obtain the other term in the integral for $N_s^2(t)$ we consider the equation for v. Thus we multiply (1.1) by u_v^{-1} from the left:

$$v_t+\sum_{j=1}^m u_v^{-1}f_j'(u)u_v v_{x_j}=u_v^{-1}r(u(v)). \tag{4.33}$$

Next we linearize (4.33) around $\bar{v}=v(\bar{u})$, and set $w=v-\bar{v}$:

$$w_t+\sum_{j=1}^m(u_v^{-1}f_j'u_v)(\bar{u})w_{x_j}=(u_v^{-1}r'u_v)(\bar{u})w+\tilde{R},$$

where

$$\begin{aligned}\tilde{R} &= \sum_{j=1}^m\big[(u_v^{-1}f_j'u_v)(\bar{u})-(u_v^{-1}f_j'u_v)(u)\big]v_{x_j}\\ &\quad+\big[(u_v^{-1}r)(u)-(u_v^{-1}r'u_v)(\bar{u})(v-\bar{v})\big].\end{aligned} \tag{4.34}$$

This further gives us

$$(u_v^tU''u_v)(\bar{u})w_t+\sum_{j=1}^m(u_v^tU''f_j'u_v)(\bar{u})w_{x_j} = (u_v^tU''r'u_v)(\bar{u})w+R,$$

or

$$A_0w_t+\sum_{j=1}^m A_jw_{x_j} = Bw+R, \tag{4.35}$$

where A_0, A_j and B are as defined in (3.24) and

$$R=(u_v^tU''u_v)(\bar{u})\tilde{R}. \tag{4.36}$$

Note that under Assumptions 1(i), 3(ii′) and 1(iii) we may apply Lemma 3.11 to (4.35), see the discussion above Lemma 3.11. That is, there exists a compensating matrix $K(\omega)$, which is smooth on $\omega\in\mathbb{S}^{m-1}$, $K(\omega)A_0$ is skew symmetric on $\mathbb{S}^{m-1}$, and $\frac{1}{2}\big[K(\omega)\tilde{A}(\omega)+\tilde{A}(\omega)^tK(\omega)^t\big]-B$ is symmetric, positive definite on $\mathbb{S}^{m-1}$. Recall that $\tilde{A}(\omega)=\sum_{j=1}^m\omega_jA_j$.

We take Fourier transform of (4.35) with respect to x, and denote the transform of w as $\hat{w}$. Then we multiply the result by $-i|\xi|\hat{w}^*K(\omega)$ from the left, where ξ is the Fourier variable, $\hat{w}^*$ is the conjugate transpose of $\hat{w}$, and $\omega = \xi/|\xi| \in \mathbb{S}^{m-1}$:

$$-i|\xi|\hat{w}^*K(\omega)A_0\hat{w}_t + |\xi|^2\hat{w}^*K(\omega)\tilde{A}(\omega)\hat{w} = -i|\xi|\hat{w}^*K(\omega)(B\hat{w}+\hat{R}). \quad (4.37)$$

Noting $K(\omega)A_0$ is real, skew symmetric, we take the real part of (4.37) to arrive at

$$\begin{aligned} -\frac{i}{2}|\xi|\big[\hat{w}^*K(\omega)A_0\hat{w}\big]_t + |\xi|^2\hat{w}^*\Big\{\frac{1}{2}\big[K(\omega)\tilde{A}(\omega)+\tilde{A}(\omega)^tK(\omega)^t\big]-B\Big\}\hat{w} \\ = -|\xi|^2\hat{w}^*B\hat{w} - \mathrm{Re}\,\big\{i|\xi|\hat{w}^*K(\omega)B\hat{w}\big\} - \mathrm{Re}\,\big\{i|\xi|\hat{w}^*K(\omega)\hat{R}\big\}. \quad (4.38) \end{aligned}$$

Next we multiply the equation by $|\xi|^{2l}$, $0 \le l \le s-1$, and integrate the result over $\mathbb{R}^m \times [0,t]$. Since $\frac{1}{2}\big[K(\omega)\tilde{A}(\omega)+\tilde{A}(\omega)^tK(\omega)^t\big]-B$ is symmetric, positive definite on $\mathbb{S}^{m-1}$, there exists a constant $c_4>0$ such that the second term on the left-hand side of (4.38) is bounded below by $c_4|\xi|^2|\hat{w}|^2$. Thus we have

$$c_4\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat{w}|^2(\xi,\tau)\,d\xi d\tau \le \mathscr{T}_9+\mathscr{T}_{10}+\mathscr{T}_{11}+\mathscr{T}_{12}, \quad (4.39)$$

where

$$\begin{aligned} \mathscr{T}_9 &= \frac{i}{2}\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+1}\big[\hat{w}^*K(\omega)A_0\hat{w}\big]_t(\xi,\tau)\,d\xi d\tau, \\ \mathscr{T}_{10} &= -\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}\big(\hat{w}^*B\hat{w}\big)(\xi,\tau)\,d\xi d\tau, \\ \mathscr{T}_{11} &= -\int_0^t\int_{\mathbb{R}^m}\mathrm{Re}\,\big\{i|\xi|^{2l+1}\hat{w}^*K(\omega)B\hat{w}\big\}(\xi,\tau)\,d\xi d\tau, \\ \mathscr{T}_{12} &= -\int_0^t\int_{\mathbb{R}^m}\mathrm{Re}\,\big\{i|\xi|^{2l+1}\hat{w}^*K(\omega)\hat{R}\big\}(\xi,\tau)\,d\xi d\tau, \end{aligned} \quad (4.40)$$

Now we estimate each term in (4.40). Clearly

$$\begin{aligned} \mathscr{T}_9 &= \frac{i}{2}\int_{\mathbb{R}^m}|\xi|^{2l+1}\big[\hat{w}^*K(\omega)A_0\hat{w}\big](\xi,t)\,d\xi \\ &\quad -\frac{i}{2}\int_{\mathbb{R}^m}|\xi|^{2l+1}\big[\hat{w}^*K(\omega)A_0\hat{w}\big](\xi,0)\,d\xi \\ &= O(1)\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat{w}|^2(\xi,t)\,d\xi + O(1)\int_{\mathbb{R}^m}|\xi|^{2l}|\hat{w}|^2(\xi,t)\,d\xi \end{aligned}$$

$$
\begin{aligned}
&+O(1)\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat{w}|^2(\xi,0)\,d\xi+O(1)\int_{\mathbb{R}^m}|\xi|^{2l}|\hat{w}|^2(\xi,0)\,d\xi\\
=\;& O(1)\big[\|D_x^{l+1}w\|^2(t)+\|D_x^l w\|^2(t)+\|D_x^{l+1}w\|^2(0)+\|D_x^l w\|^2(0)\big]. \qquad (4.41)
\end{aligned}
$$

From (3.24), (3.15) and the definition of w,

$$
\begin{aligned}
\mathscr{T}_{10} &= -\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}\Big\{\hat{r}_2^*\big[U_{u_2u_2}(r_2)_{u_2}^{-1}\big](\bar{u})\hat{r}_2\Big\}(\xi,\tau)\,d\xi d\tau,\\
&= O(1)\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat{r}_2|^2(\xi,\tau)\,d\xi d\tau\\
&= O(1)\int_0^t\|D_x^{l+1}r_2(u)\|^2(\tau)\,d\tau. \qquad (4.42)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\mathscr{T}_{11} &\le C\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+1}\big(|\hat{w}||\hat{r}_2|\big)(\xi,\tau)\,d\xi d\tau\\
&\le \frac{c_4}{4}\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat{w}|^2(\xi,\tau)\,d\xi d\tau+C\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l}|\hat{r}_2|^2(\xi,\tau)\,d\xi d\tau\\
&\le \frac{c_4}{4}\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat{w}|^2(\xi,\tau)\,d\xi d\tau+C\int_0^t\|D_x^l r_2(u)\|^2(\tau)\,d\tau, \qquad (4.43)\\
\mathscr{T}_{12} &\le C\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+1}\big(|\hat{w}||\hat{R}|\big)(\xi,\tau)\,d\xi d\tau\\
&\le \frac{c_4}{4}\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat{w}|^2(\xi,\tau)\,d\xi d\tau+C\int_0^t\|D_x^l R\|^2(\tau)\,d\tau. \qquad (4.44)
\end{aligned}
$$

From (4.34) and (4.36) we write

$$
R=R_1+R_2+R_3, \qquad (4.45)
$$

where

$$
\begin{aligned}
R_1 &= (u_v^tU''u_v)(\bar{u})\sum_{j=1}^m\big[(u_v^{-1}f_j'u_v)(\bar{u})-(u_v^{-1}f_j'u_v)(u)\big]v_{x_j},\\
R_2 &= (u_v^tU''u_v)(\bar{u})(u_v^{-1}r)(u), \quad R_3=-(u_v^tU''u_v)(\bar{u})(v-\bar{v}).
\end{aligned} \qquad (4.46)
$$

For $0\le l\le s-1$, from (3.22), (3.16), (3.19) and (3.15),

$$
\|D_x^lR_1\|^2 \le C\sum_{j=1}^m\big\|D_x^l\big(u_v^{-1}f_j'u_vv_{x_j}\big)-\big(u_v^{-1}f_j'u_v\big)(u)D_x^lv_{x_j}\big\|^2
$$

$$
\begin{aligned}
&+C\sum_{j=1}^{m}\|u-\bar u\|_{L^\infty}^2\|D_x^l v_{x_j}\|^2\\
&\le C\|D_xu\|_{s-1}^2\|D_xv\|_{s-1}^2+C\|u-\bar u\|_s^2\|D_xv\|_{s-1}^2\\
&\le C\|u-\bar u\|_s^2\|D_xu\|_{s-1}^2,\\
\|D_x^lR_2\|^2 &\le C\big\|D_x^l\big(u_v^{-1}r\big)(u)\big\|^2\\
&\le C\big\|D_x^l\big(u_v^{-1}r\big)(u)-u_v^{-1}(u)D_x^lr(u)\big\|^2+C\|D_x^lr(u)\|^2\\
&\le C\|D_xu\|_{s-1}^2\|r_2(u)\|_{s-1}^2+C\|r_2(u)\|_l^2,\\
\|D_x^lR_3\|^2 &=\big\|\big[U_{u_2u_2}(r_2)_{u_2}^{-1}\big](\bar u)D_x^lr_2(u)\big\|^2\le C\|D_x^lr_2(u)\|^2\le C\|r_2(u)\|_l^2.
\end{aligned}
$$

Therefore,

$$\|D_x^lR\|^2\le C\big(\|u-\bar u\|_s^2\big(\|D_xu\|_{s-1}^2+\|r_2(u)\|_{s-1}^2\big)+C\|r_2(u)\|_{s-1}^2.$$

Equation (4.44) becomes

$$\mathscr{T}_{12}\le\frac{c_4}{4}\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat w|^2(\xi,\tau)\,d\xi d\tau+CN_s^4(t)+C\int_0^t\|r_2(u)\|_{s-1}^2(\tau)\,d\tau. \tag{4.47}$$

Combining (4.39), (4.41)-(4.43) and (4.47), and noting $0\le l\le s-1$, we arrive at

$$\int_0^t\int_{\mathbb{R}^m}|\xi|^{2l+2}|\hat w|^2(\xi,\tau)\,d\xi d\tau\ \le\ C\big[\|w\|_s^2(t)+\|w\|_s^2(0)+\int_0^t\|r_2(u)\|_s^2(\tau)\,d\tau\big]+CN_s^4(t). \tag{4.48}$$

Since the left-hand side can be replaced by $\int_0^t\|D_x^{l+1}w\|^2(\tau)\,d\tau$, we have

$$\int_0^t\|D_xw\|_{s-1}^2(\tau)\,d\tau\le C\big[\|w\|_s^2(t)+\|w\|_s^2(0)+\int_0^t\|r_2(u)\|_s^2(\tau)\,d\tau\big]+CN_s^4(t). \tag{4.49}$$

Noting $w=v(u)-\bar v$ and $u=u(v)=u(\bar v+w)$, form (3.19) we have

$$\|w\|_s\le C\|u-\bar u\|_s,\quad \|D_xu\|_{s-1}\le C\|D_xw\|_{s-1}. \tag{4.50}$$

Applying (4.50) to (4.49) we obtain

$$\int_0^t\|D_xu\|_{s-1}^2(\tau)\,d\tau$$

$$\leq C_1\Big[\|u-\bar{u}\|_s^2(t)+\int_0^t \|r_2(u)\|_s^2(\tau)\,d\tau\Big]+C_1\|u_0-\bar{u}\|_s^2+C_1N_s^4(t), \quad (4.51)$$

where $C_1 > 0$ is a constant independent of t.

We multiply (4.51) by a small positive constant c, and add it to (4.32):

$$\|u-\bar{u}\|_s^2(t)+\int_0^t \|r_2(u)\|_s^2(\tau)\,d\tau + c\int_0^t \|D_x u\|_{s-1}^2(\tau)\,d\tau$$
$$\leq C\|u_0-\bar{u}\|_s^2+CN_s^3(t)+cC_1\Big[\|u-\bar{u}\|_s^2(t)+\int_0^t \|r_2(u)\|_s^2(\tau)\,d\tau\Big].$$

We choose $c > 0$ such that $cC_1 \leq \frac{1}{2}$. This gives us

$$\|u-\bar{u}\|_s^2(t)+\int_0^t \|r_2(u)\|_s^2(\tau)\,d\tau+\int_0^t \|D_x u\|_{s-1}^2(\tau)\,d\tau \leq C\|u_0-\bar{u}\|_s^2+CN_s^3(t).$$

Noting $0 \leq t \leq T$ and the definition of $N_s(t)$ in (4.1), we have

$$N_s^2(T) \leq C\|u_0-\bar{u}\|_s^2+CN_s^3(t),$$

which implies that if $N_s(T)$ is bounded by a small positive constant independent of T, then

$$N_s^2(T) \leq C\|u_0-\bar{u}\|_s^2. \qquad \square$$

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