A CHARACTERIZATION OF HILBERT SPACES USING SUBLINEAR OPERATORS

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Abstract

In this paper we study a sublinear version of the Kwapień's Theorem. We show that X is isomorphic to a Hilbert space if, and only if, for every 2-summing sublinear operator $T : X \to l_2$, T is strongly positive p-summing.

1. Introduction

The concept of absolutely 1-summing linear operators was mainly introduced by Grothendieck in the 1950's and was generalized for all p by Pietsch in the 1967's. Cohen in [7] has used the 2-summing linear operators and their conjugates to give a characterization of inner product spaces. This result has been generalized by Kwapień [9] to normed spaces isomorphic to inner product spaces. In the same circle of ideas, the authors in [4] have given a characterization of Hilbert space by means of polynomial mappings. In this note we will give the same characterization by using the p-summing and strongly p-summing sublinear operators.

The paper is organized as follows. First we give some preliminaries where we fix some notations and recall some basic facts and properties concerning sublinear operators. Section 2 recalls certain definitions of summability in the category of sublinear operators which are basically the definition of p-summing, positive p-summing, strongly p-summing and strongly positive

p-summing sublinear operators. In Section 3, we give a sublinear characterization of Hilbert spaces. We will show that the Banach space X is isomorphic to a Hilbert space if, and only if, for every 2-summing sublinear operator T from X into l_2, T is strongly positive p-summing for all $p \geq 1$.

Firstly, we start by recalling the abstract definition of Banach lattices. A real Banach lattice (resp. a real complete Banach lattice) X is an ordered vector space equipped with a lattice (resp. a complete lattice) structure and a Banach space norm satisfying the following compatibility condition:

$$\forall x, y \in X, \quad |x| \leq |y| \Longrightarrow \|x\| \leq \|y\|,$$

where $|x| = \sup\{x, -x\}$. Obviously, x and $|x|$ have the same norm. Some example of these spaces can be mentioned such as: $C(K)$ space (Banach lattice), L_p and l_p ($1 \leq p \leq \infty$) spaces (complete Banach lattices). Note that any reflexive Banach lattice is a complete Banach lattice. We denote by $X_+ = \{x \in X : x \geq 0\}$. An element x of X is positive if $x \in X_+$. The dual X^* of a Banach lattice X is a complete Banach lattice endowed with the natural order

$$x_1^* \leq x_2^* \Longrightarrow \langle x_1^*, x \rangle \leq \langle x_2^*, x \rangle, \ \forall x \in X_+,$$

where $\langle , \rangle$ denotes the duality bracket. If we consider X as a sublattice of X^{**} we have for $x_1, x_2 \in X$

$$x_1 \leq x_2 \Longrightarrow \langle x^*, x_1 \rangle \leq \langle x^*, x_2 \rangle, \ \forall x^* \in X_+^*.$$

Definition 1.1. A mapping T from a Banach space X into a Banach lattice Y is considered to be sublinear if for all x, y in X and λ in $\mathbb{R}^+$, we have

(i) $T(\lambda x) = \lambda T(x)$ (i.e. positively homogeneous),
(ii) $T(x + y) \leq T(x) + T(y)$ (i.e. subadditive).

Note that the sum of two sublinear operators is a sublinear operator and the multiplication by a positive number is also a sublinear operator.

Let us denote by

$$\nabla T = \{u \in \mathcal{B}(X, Y) : u \leq T \text{ (i.e., } \forall x \in X, u(x) \leq T(x))\}.$$

Where $\mathcal{B}(X, Y)$ is the Banach space of all bounded (continuous) linear operators from X into Y.

We call now that a sublinear operator T is:

(a) symmetrical if for all x in X, $T(x) = T(-x)$,

(b) positive if for all x in X_+, $T(x) \geq 0$.

Let T be a sublinear operator from X into a Banach lattice Y. The operator T is bounded (continuous), if and only if, there is a positive constant C such that for all $x \in X$, $\|T(x)\| \leq C \|x\|$. In this case we say that T is bounded and we put

$$\|T\| = \sup\{\|T(x)\| : \|x\| = 1\}.$$

We will denote by $\mathcal{SB}(X, Y)$ the set of all bounded sublinear operators from X into Y. Let X be a Banach space and $n \in \mathbb{N}$. We denote by $l_p(X)$, $(1 \leq p \leq \infty)$, (resp. $l_p^n(X)$) the space of all sequences (x_i) in X equipped with the norm

$$\|(x_i)\|_{l_p(X)} = (\sum_{i=1}^{\infty} \|x_i\|^p)^{\frac{1}{p}} < \infty$$

$$\left(\text{resp.} \quad \left\|(x_i)_{1 \leq i \leq n}\right\|_{l_p^n(X)} = (\sum_{1}^{n} \|x_i\|^p)^{\frac{1}{p}} \right)$$

and by $l_p^\omega(X)$ (resp. $l_p^{n\,\omega}(X)$) the space of all sequences (x_i) in X equipped with the norm

$$\|(x_n)\|_{l_p^\omega(X)} = \sup_{\|x^*\|_{X^*}=1} \left(\sum_{1}^{\infty} |x^*(x_i)|^p \right)^{\frac{1}{p}} < \infty$$

$$\left(\text{resp.} \quad \|(x_n)\|_{l_p^{n\,\omega}(X)} = \sup_{\|x^*\|_{X^*}=1} \left(\sum_{1}^{n} |x^*(x_i)|^p \right)^{\frac{1}{p}} \right)$$

where X^* denotes the dual (topological) of X, with the usual modifications made when $p = \infty$. The closed unit ball of X will be noted by B_X.

Example 1.1.

1- If u is a linear operator from a Banach space X into a Banach lattice Y, then $|u|$ is sublinear and we have $u \in \nabla |u|$.

2- The operator $T(x) = \|x\|$ is sublinear.

3- Consider $T : l_1 \to l_p$ where $T\{(x_n)\} = \{(|x_n|)\}$. The operator T is sublinear and continuous with $\|T\| \le 1$.

Remark 1.1. Let $m \in \mathbb{N}$ and let X, E, Y, Z be Banach spaces such that Y, Z are Banach lattices.

(a) Consider T in $\mathcal{SB}(X, Y)$ and u in $\mathcal{B}(Y, Z)$ (assume that u is positive, i.e., $u(y) \ge 0$ if $y \in Y_+$). Then, $u \circ T \in \mathcal{SB}(X, Z)$.

(b) Consider u in $\mathcal{B}(E, X)$ and T in $\mathcal{SB}(X, Y)$. Then, $T \circ u \in \mathcal{SB}(E, Y)$.

(c) $\forall T \in \mathcal{SB}(X, Y)$ and $\forall \lambda \in \mathbb{R}$, we have $\lambda T(x) \le T(\lambda x)$.

Khinchin's Inequality. For any $0 < p < \infty$, there are positive constants A_p, B_p such that for any scalars $a_1, \ldots, a_n$, we have

$$A_p \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{\frac{1}{2}} \le \Big(\int_0^1 | \sum_{i=1}^{n} a_i r_i(t) |^p \, dt \Big)^{\frac{1}{p}} \le B_p \Big(\sum_{i=1}^{n} |a_i|^2 \Big)^{\frac{1}{2}},$$

where $r_i : [0, 1] \to \mathbb{R}$ is the Rademacher function defined by $r_i(t) := sign\left(\sin 2^i \pi t\right).$

Kahane's Inequality. Let X be a Banach space. If $0 < p, q < \infty$, then there is a constant $K_{p,q} > 0$ such that for every $x_1, \ldots, x_n \in X$, we have

$$\Big(\int_0^1 \| \sum_{i=1}^{n} r_i(t) x_i \|^q \, dt \Big)^{\frac{1}{q}} \le K_{p,q} \Big(\int_0^1 \| \sum_{i=1}^{n} r_i(t) x_i \|^p \, dt \Big)^{\frac{1}{p}}.$$

2. Some Classes of Sublinear Operators

In the linear case, the notion of positive p-summing operators has been extensively studied by Blasco in [5]. In this section we present the definition of p-summing and positive p-summing sublinear operators, where we refer the reader to the recent papers [1, 3] for further results.

Definition 2.1 ([1])**.** Let X be a Banach space, Y be a Banach lattice and $T \in \mathcal{SB}(X, Y)$. The sublinear operator T is p-summing for $1 \le p \le \infty$ if there is $C > 0$, such that: $\forall n \in \mathbb{N}, \forall \{x_1, \ldots, x_n\} \subset X$, we have

$$\Big(\sum_{i=1}^{n} \|T(x_i)\|^p \Big)^{\frac{1}{p}} \le C \sup_{x^* \in B_{X^\star}} \Big(\sum_{i=1}^{n} |x^*(x_i)|^p \Big)^{\frac{1}{p}}. \qquad (2.1)$$

We denote by $\pi_p\left(X,Y\right)$ the set of all p-summing sublinear operators and

$$\pi_p(T) = \inf\{C, \text{ verifying the equality (2.1) }\}.$$

For the definition of positive p-summing linear operator, we suppose that X is a Banach lattice and we replace $\forall\left\{x_1,\ldots,x_n\right\} \subset X$ by $\forall\left\{x_1,\ldots,x_n\right\} \subset X_+$, see ([8], p 343).

Ideal property. Let X,E be Banach spaces and Y, Z be two Banach lattices. Let $T \in \mathcal{SB}(X,Y)$, w be a positive operator in $\mathcal{B}(Y,Z)$ and $v \in \mathcal{B}(E,X)$.

(a) *If T is p-summing sublinear operator, then Tv is p-summing sublinear operator and $\pi_p\left(Tv\right) \leq \pi_p\left(T\right)\left\|v\right\|$.*

(b) *If T is p-summing sublinear operator, then wT is p-summing sublinear operator and $\pi_p\left(wT\right) \leq \left\|w\right\|\pi_p\left(T\right)$.*

Proposition 2.1 ([1])**.** *Let X be Banach space, Y be Banach lattice and $T : X \longrightarrow Y$ be sublinear operator. If $T \in \pi_p\left(X,Y\right)$, then $\forall u \in \nabla T$,*

$$u \in \pi_p\left(X,Y\right),$$

and we have $\pi_p\left(u\right) \leq 2\pi_p\left(T\right)$.

Now, we give the definition of strongly p-summing introduced by Cohen [6] for the linear case and generalized to sublinear operators in [3].

Definition 2.2. Let X be a Banach space and Y be a Banach lattice. A sublinear operator $T : X \to Y$ is strongly p-summing ($1 < p < \infty$), if there is a positive constant C such that for any $n \in \mathbb{N}, x_1,\ldots,x_n \in X$ and $y_1^*,\ldots,y_n^* \in Y^*$, we have

$$\left\|\left(\left\langle T\left(x_i\right),y_i^*\right\rangle\right)\right\|_{l_1^n} \leq C\left\|(x_i)\right\|_{l_p^n(X)}\left\|(y_i^*)\right\|_{l_{p^*}^n\omega}. \tag{2.2}$$

We denote by $\mathcal{D}_p(X,Y)$ the class of all strongly p-summing sublinear operators from X into Y and $d_p(T)$ the smallest constant C such that the inequality (2.2) holds. For $p = 1$, we have $\mathcal{D}_1(X,Y) = \mathcal{SB}(X,Y)$. For the definition of strongly positive p-summing sublinear operators, we replace Y^* by Y_+^*, $d_p(T)$ by $d_p^+(T)$ and $\mathcal{D}_p(X,Y)$ by $\mathcal{D}_p^+(X,Y)$.

The following proposition is due to ([3]).

Proposition 2.2. *Let X be a Banach space and let Y be a complete Banach lattice. Let T be a bounded sublinear operator from X into Y. Suppose that T is strongly (positive) p-summing $(1 < p < \infty)$. Then for all $u \in \nabla T$, u is strongly positive p-summing and u^* is positive p^*-summing.*

3. Main Result

In this section, we give a sublinear characterization of Hilbert spaces extending the linear result of Kwapień. We show that: X is isomorphic to a Hilbert space if, and only if, for every 2-summing sublinear operator T from X into l_2; T is strongly positive p-summing for all $p \geq 1$. Starting by the following lemmas.

Lemma 3.1 ([2], Proposition 2.1)**.** *Let T be a sublinear operator between a Banach space X and a Banach lattice Y.*

(1) *For all x in X, we put $\varphi(x) = \sup\{T(x),\ T(-x)\}$. Then, φ is symmetrical sublinear operator. In addition,*

$$|T| \leq \varphi \ \text{ and } \ \|\varphi(x)\| \leq \sup\{\|T(x)\|, \|T(-x)\|\}. \qquad (3.1)$$

(2) *For every $(\alpha_i)_{i=1}^n \subset \mathbb{R}$, we have $T(\sum\limits_{i=1}^n \alpha_i x_i) \leq \sum\limits_{i=1}^n |\alpha_i|\, \varphi(x_i)$. In addition*

$$\left\| T\left(\sum_{i=1}^n \alpha_i x_i \right) \right\| \leq \sum_{i=1}^n \|\alpha_i\| \|\varphi(x_i)\|.$$

Lemma 3.2. *The sublinear operator $T : X \to Y$ is 2-summing if, and only if, the sublinear operator φ is 2-summing.*

Proof. If φ is 2-summing, then T is also 2-summing by (3.1). Conversely, let $x_1, \ldots, x_n \in X$. Again, we have by (3.1)

$$\sum_{i=1}^n \|\varphi(x_i)\|^2 \ \leq \ \sum_{i=1}^n \sup\left\{ \|T(x_i)\|^2, \|T(-x_i)\|^2 \right\}$$

$$\leq \ \sum_{i=1}^n \|T(x_i)\|^2 + \sum_{i=1}^n \|T(-x_i)\|^2$$

$$\leq\ 2\pi_2\left(T\right)^2 \sup_{\|x^*\|\leq 1} \sum_{i=1}^{n} |x^*\left(x_i\right)|^2.$$

Hence, φ is 2-summing and

$$\pi_2\left(\varphi\right) \leq \sqrt{2}\pi_2\left(T\right). \tag{3.2}$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The main result in this paper is the next extension of "Kwapień's theorem" for the class of sublinear operators. For the proof, we will use the Khintchin's inequality and the Kahane's inequality.

Theorem 3.1. *Let X be a Banach space. The following properties are equivalent.*

(1) *The space X is isomorphic to a Hilbert space.*

(2) *If $T \in \Pi_2(X, l_2)$, then $T \in \mathcal{D}_p^+(X, l_2)$ for all $p \geq 1$.*

(3) *If $T \in \Pi_2(X, l_2)$, then $T \in \mathcal{D}_2^+(X, l_2)$.*

Proof. (1) $\implies$ (2). First let $X = H$ be a Hilbert space and we put $Y = l_2$. Without restricting the generality, we may replace H by l_2^m for $m \in \mathbb{N}$. Because, if we take $(x_i)_{i=1}^n \subset H$, then there is $m \in \mathbb{N}$ such that $span\,\{x_i\}_{i=1}^n$ is isomorphic l_2^m. We use the projection $P : H \to l_2^m$ and we return to the initial case. Now, let $T \in \Pi_2(l_2^m, Y)$, $(x_i)_{i=1}^n \subset l_2^m$ and $(y_i^*)_{i=1}^n \subset Y_+^*$. We have

$$x_i = \sum_{j=1}^{m} \alpha_i^j e_j, \ i = 1, 2, \ldots, n.$$

By Lemma 3.1, we have

$$-\sum_{j=1}^{m} \left|\alpha_i^j\right| \varphi(e_j) \leq -T\Big(\sum_{j=1}^{m} \alpha_i^j e_j\Big) \leq T\Big(\sum_{j=1}^{m} \alpha_i^j(-e_j)\Big) \leq \sum_{j=1}^{m} \left|\alpha_i^j\right| \varphi(e_j),$$

and then for every $y_i^* \in Y_+^*$

$$\Big\langle -\sum_{j=1}^{m} \left|\alpha_i^j\right| \varphi(e_j), y_i^* \Big\rangle \leq \Big\langle -T\Big(\sum_{j=1}^{m} \alpha_i^j e_j\Big), y_i^* \Big\rangle \leq \Big\langle \sum_{j=1}^{m} \left|\alpha_i^j\right| \varphi(e_j), y_i^* \Big\rangle.$$

So, we have

$$|\langle T(x_i), y_i^* \rangle| \leq \Big\langle \sum_{j=1}^{m} \Big| \alpha_i^j \Big| \varphi(e_j), y_i^* \Big\rangle.$$

Therefore,

$$
\begin{aligned}
\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| &\leq \sum_{i=1}^{n} \Big\langle \sum_{j=1}^{m} \Big| \alpha_i^j \Big| \varphi(e_j), y_i^* \Big\rangle \ \text{(By Hölder)} \\
&\leq \sum_{i=1}^{n} \Big(\sum_{j=1}^{m} \Big| \alpha_i^j \Big|^2 \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{m} |\langle \varphi(e_j), y_i^* \rangle|^2 \Big)^{\frac{1}{2}} \\
&= \sum_{i=1}^{n} \|x_i\| \Big(\sum_{j=1}^{m} |\langle \varphi(e_j), y_i^* \rangle|^2 \Big)^{\frac{1}{2}}.
\end{aligned}
$$

By Khinchin's inequality, $\sum\limits_{i=1}^{n} |\langle T(x_i), y_i^* \rangle|$ can be estimated by

$$
\begin{aligned}
&\sum_{i=1}^{n} \|x_i\| \frac{1}{A_{p^*}} \Big(\int_0^1 \Big| \sum_{j=1}^{m} r_j(t) \langle \varphi(e_j), y_i^* \rangle \Big|^{p^*} dt \Big)^{\frac{1}{p^*}} \qquad \text{(By Hölder)} \\
&\leq \frac{1}{A_{p^*}} \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} \int_0^1 \Big| \sum_{j=1}^{m} r_j(t) \langle \varphi(e_j), y_i^* \rangle \Big|^{p^*} dt \Big)^{\frac{1}{p^*}} \\
&= \frac{1}{A_{p^*}} \|(x_i)_{i=1}^n\|_{l_p^n(X)} \Big(\int_0^1 \sum_{i=1}^{n} \Big| \sum_{j=1}^{m} r_j(t) \langle \varphi(e_j), y_i^* \rangle \Big|^{p^*} dt \Big)^{\frac{1}{p^*}} \\
&\leq \frac{1}{A_{p^*}} \|(x_i)_{i=1}^n\|_{l_p^n(X)} \Big(\int_0^1 \Big\| \sum_{j=1}^{m} r_j(t) \varphi(e_j) \Big\|^{p^*} \sup_{\|y\|=1} \sum_{i=1}^{n} |y_i^*(y)|^{p^*} dt \Big)^{\frac{1}{p^*}},
\end{aligned}
$$

and by Kahane's inequality

$$
\leq \frac{1}{A_{p^*}} K_{2,p^*} \|(x_i)_{i=1}^n\|_{l_p^n(X)} \|(y_i^*)_{i=1}^n\|_{l_{p^*}^{n,w}(Y^*)} \Big(\int_0^1 \Big\| \sum_{j=1}^{m} r_j(t) \varphi(e_j) \Big\|^2 \Big)^{\frac{1}{2}}.
$$

In a Hilbert space, it is not difficult to see that

$$
\Big(\int_0^1 \Big\| \sum_{j=1}^{m} r_j(t) \varphi(e_j) \Big\|^2 \Big)^{\frac{1}{2}} = \Big(\sum_{j=1}^{m} \|\varphi(e_j)\|^2 \Big)^{\frac{1}{2}}.
$$

As φ is 2-summing, it follows that

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| \leq \frac{K_{2,p^*} \pi_2(\varphi)}{A_{p^*}} \|(x_i)_{i=1}^{n}\|_{l_p^n(X)} \|(y_i^*)_{i=1}^{n}\|_{l_{p^*}^{n,w}(Y^*)} \| (e_j)_{j=1}^{m} \|_{l_2^{m,w}}$$

$$= \frac{K_{2,p^*} \pi_2(\varphi)}{A_{p^*}} \|(x_i)_{i=1}^{n}\|_{l_p^n(X)} \|(y_i^*)_{i=1}^{n}\|_{l_{p^*}^{n,w}(Y^*)}.$$

This implies that, $T \in \mathcal{D}_p^+(l_2^m, Y)$ and by (3.2)

$$d_p(T) \leq \frac{\sqrt{2}}{A_{p^*}} K_{2,p^*} \pi_2(T).$$

Now, consider a Banach space X which is isomorphic to a Hilbert space H and $T : X \to Y$ be 2-summing sublinear operator ($Y = l_2$). Then, there is an isomorphism I from H onto X. By ideal property, the sublinear operator $T \circ I$ is 2-summing and by the above is strongly positive p-summing. Let $(x_i)_{i=1}^{n} \subset X$ and $(y_i^*)_{i=1}^{n} \subset Y_+^*$. Then,

$$\sum_{i=1}^{n} |\langle T(x_i), y_i^* \rangle| = \sum_{i=1}^{n} |\langle T \circ I(I^{-1}(x_i)), y_i^* \rangle|$$

$$\leq d_p(T \circ I) \|((I^{-1}(x_i)))_{i=1}^{n}\|_{l_p^n(X)} \|(y_i^*)_{i=1}^{n}\|_{l_{p^*}^{n,w}(Y^*)}$$

$$\leq \frac{\sqrt{2} K_{2,p^*}}{A_{p^*}} \pi_2(T \circ I) \|I^{-1}\| \|(x_i)_{i=1}^{n}\|_{l_p^n(X)} \|(y_i^*)_{i=1}^{n}\|_{l_{p^*}^{n,w}(Y^*)}$$

$$\leq \frac{\sqrt{2} K_{2,p^*}}{A_{p^*}} \pi_2(T) \|I\| \|I^{-1}\| \|(x_i)_{i=1}^{n}\|_{l_p^n(X)} \|(y_i^*)_{i=1}^{n}\|_{l_{p^*}^{n,w}(Y^*)}.$$

This shows that $T \in \mathcal{D}_p^+(X, Y)$ with

$$d_p(T) \leq \frac{\sqrt{2} K_{2,p^*}}{A_{p^*}} \pi_2(T) \|I\| \|I^{-1}\|.$$

$(2) \implies (3)$ Obvious.

$(3) \implies (1)$ Let $K = B_{X^*}$ endowed with the weak* topology. Let $(x_i^*)_{i \in \mathbb{N}} \subset C(K)^*$ such that $\|(x_i^*)_{i \in \mathbb{N}}\|_{l_2^w(C(K)^*)} \leq 1$. We associate it the linear application

$$u : C(K) \to l_2$$

defined by $u(x) = (\langle x, x_i^* \rangle)_{i \in \mathbb{N}}$. By ([8], Theorem 4.19) the operator

$$u \circ i_X : X \to C(K) \to l_2$$

is 2-summing, where i_X is the natural embedding. Then, the sublinear operator

$$T = |u \circ i_X|$$

is also 2-summing. By (3), it is strongly positive 2-summing and by Proposition 2.2, every element of ∇T is also strongly positive 2-summing. The linear operator $u \circ i_X$ belongs to ∇T, so, we conclude that $u \circ i_X$ is strongly positive 2-summing and its adjoint operator $i_X^* \circ u^*$ is positive 2-summing. Now, we use the same argument given in the proof of ([8], Theorem 4.19), one can show that $i_X^* \in \Pi_2(C(K)^*, X^*)$. Indeed, let $(x_i^*) \in l_2^w(C(K)^*)$ and let u the associated operator of (x_i^*) as in the above. Then, for every $1 \le i \le n$ and $f \in C(K)$,

$$u^*(e_i)(f) = \langle e_i, u(f) \rangle = \langle e_i, (\langle f, x_k^* \rangle)_{k \in \mathbb{N}} \rangle = x_i^*(f),$$

therefore $x_i^* = u^*(e_i)$. We have

$$\left(\sum_{i \in \mathbb{N}} \|i_X^*(x_i^*)\|^2 \right)^{\frac{1}{2}} = \left(\sum_{i \in \mathbb{N}} \|i_X^* \circ u^*(e_i)\|^2 \right)^{\frac{1}{2}} \le \pi_2^+(i_X^* \circ u^*) < \infty,$$

consequently i_X^* is 2-summing. By the Pietsch's factorization theorem for 2-summing operators ([8], Corollary 2.16), there is a Hilbert space H such that

$$i_X^* = v_1 \circ v_2 : C(K)^* \xrightarrow{v_2} H \xrightarrow{v_1} X^*.$$

The surjectivity of i_X^* transfers to v_1, so the Open Mapping Theorem tells us that X^* is isomorphic to a Hilbert space and then X has the same property.   □

Acknowledgment

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