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GENERALIZED SKEW DERIVATIONS ON LIE IDEALS

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Abstract

Let R be a prime ring with center $Z(R)$, C its extended centroid, L a noncentral Lie ideal of R and $n, m \geq 1$ fixed integers. Suppose that F is a nonzero generalized skew derivation of R such that $F(u^n)u^m \in Z(R)$, for all $u \in L$. Then $dim_C RC = 4$.

1. Introduction

Let R be a prime ring with center $Z(R)$, extended centroid C, and right Martindale quotient ring Q_r. We mean by a derivation of R an additive map d from R into itself which satisifes the rule $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive map $g : R \longrightarrow R$ is called a generalized derivation of R if there exists a derivation d of R such that $g(xy) = g(x)y + xd(y)$, for all $x, y \in R$.

In [17] Lee and Shiue showed that if R is a non-commutative prime ring, I a nonzero left ideal of R and d is a derivation of R such that $[d(x^m)x^n, x^r]_k = 0$ for all $x \in I$, where k, m, n, r are fixed positive integers, then $d = 0$ unless $R \cong M_2(GF(2))$. Later in [1] Argaç and Demir proved the following result: Let R be a non-commutative prime ring, I a nonzero left ideal of R and k, m, n, r fixed positive integers. If there exists a generalized derivation g of R such that $[g(x^m)x^n, x^r]_k = 0$ for all $x \in I$, then there exists $a \in U$, the left Utumi quotient ring of R, such that $g(x) = xa$ for all $x \in R$, except when $R \cong M_2(GF(2))$ and $I[I, I] = 0$.

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Here we would like to continue on this line of investigation by considering generalized skew derivations defined on R. The definition of generalized skew derivation is a unified notion of skew derivation and generalized derivation, which are considered as classical additive mappings of non-associative algebras, have been investigated by many people from various views. Let R be an associative ring and α be an automorphism of R. An additive mapping $d : R \longrightarrow R$ is said to be a *skew derivation* of R if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

for all $x, y \in R$ and α is called an *associated automorphism* of d. An additive mapping $F : R \longrightarrow R$ is said to be a (right) *generalized skew derivation* of R if there exists a skew derivation d of R with associated automorphism α such that

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in R$, d is called an *associated skew derivation* of F and α is called an *associated automorphism* of F.

We will prove:

Theorem 1. *Let R be a prime ring with center $Z(R)$, C its extended centroid, L a noncentral Lie ideal of R and $n, m \geq 1$ fixed integers. Suppose that F is a nonzero generalized skew derivation of R such that $F(u^n)u^m \in Z(R)$, for all $u \in L$. Then $dim_C RC = 4$.*

In all that follows let Q_r be the right Martindale quotient ring, Q be the two-sided Martindale quotient ring of R and $C = Z(Q) = Z(Q_r)$ the center of Q and Q_r, $T = Q *_C C\{X\}$ the free product over C of the C-algebra Q and the free C-algebra $C\{X\}$, with X the countable set consisting of non-commuting indeterminates $x_1, x_2, \ldots, x_n, \ldots$. We refer the reader to [2] for the definitions and the related properties of these objects. Of course Q is a prime centrally closed C-algebra.

Moreover let s_4 be the standard polynomial of degree 4, in non-commting variables x_1, x_2, x_3, x_4.

It is known that automorphisms, derivations and skew derivations of R can be extended both to Q and Q_r. In [4] (Lemma 2), J.C. Chang extended the definition of a generalized skew derivation to the right Martindale quotient ring Q_r of R as follows: by a (right) generalized skew derivation we mean an additive mapping $F : Q_r \to Q_r$ such that $F(xy) = F(x)y + \alpha(x)d(y)$, for all $x, y \in Q$, where d is a skew derivation of R and α is an automorphism of R, moreover there exists $F(1) = a \in Q_r$ such that $F(x) = ax + d(x)$, for all $x \in R$. Moreover if $F(1) \in Q$, then F can be extended to Q.

Before starting with our proof, we also state the following well known result, which will be useful in the sequel:

Fact 1.1. *Let R be a prime ring and L a noncentral Lie ideal of R. Then either $char(R) = 2$ and $dim_C RC = 4$, or there exists a noncentral two-sided ideal I of R such that $0 \neq [I, R] \subseteq L$.*

Proof. If $char(R) \neq 2$, the result is contained in Lemma 2 of [3]. In case $char(R) = 2$ it follows from Theorem 4 of [15] and Lemma 2 of [10]. □

2. The Case of Inner Generalized Skew Derivations

In this section we consider the case when F is an inner generalized skew derivation induced by the elements $b, c \in R$ and $\alpha \in Aut(R)$, that is $F(x) = bx + \alpha(x)c$, for all $x \in R$. In this sense, our aim will be to prove the following:

Proposition 2.1. *Let R be a prime ring, I a noncentral two-sided ideal of R, $n, m \geq 1$ fixed integers, b, c nonzero elements of R, and $\alpha \in Aut(R)$ such that $(b[r_1, r_2]^n + \alpha([r_1, r_2]^n)c)[r_1, r_2]^m \in Z(R)$, for all $r_1, r_2 \in I$, then $dim_C RC = 4$.*

We begin with:

Fact 2.2. *Let R be a non-commutative prime ring and $s \geq 1$ be a fixed integer such that $[r_1, r_2]^s \in Z(R)$, for all $r_1, r_2 \in R$. Then $dim_C RC = 4$.*

Proof. The result is implicitly contained in Theorem 4 of [13]. □

Lemma 2.3. *Let R be a prime ring, I a noncentral two-sided ideal of R, $a, b \in R$, $n, m \geq 1$ fixed integer, such that $(au^n + u^n b)u^m \in Z(R)$, for all $u \in [I, I]$, then either $a = -b \in Z(R)$ or $dim_C RC = 4$.*

Proof. By our assumption we have that $(a[r_1,r_2]^n+[r_1,r_2]^nb)[r_1,r_2]^m \in Z(R)$ for all $r_1,r_2\in I$. Moreover I and R and Q_r satisfy the same generalized polynomial identities (see [5]), thus $(a[r_1,r_2]^n+[r_1,r_2]^nb)[r_1,r_2]^m\in C$ for all $r_1,r_2\in Q_r$. Hence we assume that Q_r satisfies the following generalized polynomial identity

$$P(x_1,x_2,x_3)=\big[(a[x_1,x_2]^n+[x_1,x_2]^nb)[x_1,x_2]^m,x_3\big] \tag{2.1}$$

and $P(x_1,x_2,x_3)$ is a generalized polynomial in the free product $Q_r *_C C\{x_1,x_2,x_3\}$ of the C-algebra Q_r and the free C-algebra $C\{x_1,x_2,x_3\}$.

2.1. Step 1: Here we prove that either $P(x_1,x_2,x_3)$ is a non-trivial generalized polynomial identity for R, or $a=-b\in C$.

Let $T=Q_r *_C C\{x_1,x_2,x_3\}$. For brevity we write $P(X)$ instead of $P(x_1,x_2,x_3)$ and $f(X)$ instead of $[x_1,x_2]$.

Now suppose that $P(X)\in Q_r *_C C\{X\}$ is a trivial generalized polynomial identity for Q_r, that is

$$P(X)=[(af(X)^n+f(X)^nb)f(X)^m,x_3]=0\in T.$$

Suppose that $\{a,1\}$ are linearly C-independent. By [5], it follows $af(X)^{n+m}x_3=0\in T$ which is a contradiction, since we suppose $a\notin C$. Therefore $\{a,1\}$ must be linearly C-dependent, that is $a\in C$ and

$$P(X)=[f(X)^n(a+b)f(X)^m,x_3]=0\in T.$$

Since $P(X)$ is trivial, again by [5], we have $a+b=0$ and the conclusion follows.

Therefore in all that follows we assume that $a\notin C$ and Q_r satisfies the non-trivial generalized polynomial identity $P(x_1,x_2,x_3)$. In case C is infinite, we have $P(r_1,r_2,r_3)=0$ for all $r_1,r_2,r_3\in Q_r\bigotimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q_r and $Q_r\bigotimes_C\overline{C}$ are centrally closed (theorems 2.5 and 3.5 in [11]) we may replace R by Q_r or $Q_r\bigotimes_C\overline{C}$ according as C is finite or infinite. Thus, without loss of generality, we may consider the case when R is centrally closed over C which is either finite or algebraically closed and $P(r_1,r_2,r_3)=0$, for all $r_1,r_2,r_3\in R$. By Martindale's theorem

[18] , R is a primitive ring having a nonzero socle with C as the associated division ring. In light of Jacobson's theorem (p. 75 in [12]) R is isomorphic to a dense ring of linear transformations on some vector space V over C.

2.2. Step 2: We prove that $dim_C V \leq 2$

Suppose by contradiction that $dim_C V \geq 3$. Of course under this assumption, R cannot satisfy the standard identity s_4. Suppose first that $dim_C V = l \geq 3$ is a finite integer, so that we may assume $Q_r = M_l(C)$, the ring of all $l \times l$ matrices over C. Denote e_{ij} the usual matrix unit, with 1 in the i,j-entry and zero elsewhere and let $[r_1, r_2] = [e_{ij}, e_{ji}] = e_{ii} - e_{jj}$, for any $j \neq i$. Therefore, by (2.1) and for $x_3 = e_{kk}$, with $k \neq i, j$, we have that

$$0 = \big[(a\big(e_{ii} - e_{jj})^n + (e_{ii} - e_{jj})^n b\big)(e_{ii} - e_{jj})^m, e_{kk}\big] = -e_{kk}a(e_{ii} - e_{jj})^{m+n} \tag{2.2}$$

that is a is a diagonal matrix in $M_l(C)$. Recall that for any $\sigma \in Aut(M_l(C))$, $M_l(C)$ satisfies

$$\big[(\sigma(a)[x_1, x_2]^n + [x_1, x_2]^n \sigma(b))[x_1, x_2]^m, x_3\big] \tag{2.3}$$

therefore $\sigma(a)$ is again a diagonal matrix. In particular we introduce some suitable automorphisms of $M_l(C)$. More precisely, let $i \neq j$ and

$$\lambda(x) = (1 + e_{ij})x(1 - e_{ij}) = x + e_{ij}x - xe_{ij} - e_{ij}xe_{ij}.$$

Hence $a + e_{ij}a - ae_{ij} - e_{ij}ae_{ij}$ is diagonal, that is the (i,i)-entry of a is equal to the (j,j)-one, which implies that a is a central matrix in $M_l(C)$. Thus Q_r satisfies

$$P(x_1, x_2, x_3) = \big[[x_1, x_2]^n c[x_1, x_2]^m, x_3\big]$$

where $c = a+b$. In case $c \in C$ we get $a, b \in C$ and Q_r satisfies $c[x_1, x_2]^{n+m} \in C$. Since Q_r does not satisfy s_4 and by Fact 2.2, we have that $c = 0$, that is $a = -b \in C$.

Hence we assume $c \notin C$, that is there exists $v \in V$ such that v, cv are linearly C-independent. Moreover, since $dim_C V \geq 3$, there exists $w \in V$ such that v, cv, w are linearly C-independent. By the density of Q_r, there exist $r_1, r_2, r_3 \in Q_r$ such that

$$r_1 v = 0, \quad r_2 v = -w, \quad r_3 v = 0, \quad r_1(cv) = -v,$$
$$r_2(cv) = 0, \quad r_1 w = -v, \quad r_2 w = v, \quad r_3 w = -v.$$

Thus

$$[r_1, r_2]v = v, \quad [r_1, r_2](cv) = -w, \quad [r_1, r_2]w = -w$$

and we get the contradiction

$$0 = \big[[r_1, r_2]^n c[r_1, r_2]^m, r_3\big]v = (-1)^n v \neq 0.$$

Assume now that $dim_C V = \infty$. Suppose next that v and bv are linearly C-independent for some $v \in V$. There exist $w, u \in V$ such that v, bv, w, u are linearly independent over C. By the density of R there exist $x_1, x_2, x_3 \in R$ such that

$$x_1 v = 0, \qquad x_2 v = bv, \quad x_1 bv = v$$
$$x_3 w = v$$
$$x_1 w = w, \quad x_2 w = w$$
$$x_2 bv = u, \quad x_1 u = bv.$$

Then

$$[x_1, x_2]v = (x_1 x_2 - x_2 x_1)v = v$$
$$[x_1, x_2]w = (x_1 x_2 - x_2 x_1)w = 0$$

and

$$[x_1, x_2]bv = (x_1 x_2 - x_2 x_1)bv = 0.$$

Hence by (2.1)

$$0 = \big[(a[x_1, x_2]^n + [x_1, x_2]^n b)[x_1, x_2]^m, x_3\big]w = av.$$

Let $w \in V$ be such that $aw \neq 0$. Then $a(v - w) = -aw \neq 0$. Then by above argument w, bw are linearly C-dependent and $v - w, b(v - w)$ too. Therefore, there exist $\alpha, \beta \in C$ such that $bw = \alpha w$ and $b(v - w) = \beta(v - w)$. This gives $bv = \beta(v - w) + bw = \beta(v - w) + \alpha w$ that is $(\alpha - \beta)w = bv - \beta v$. Now, $\alpha = \beta$ implies bv, v are linearly C-dependent, a contradiction. Hence $\alpha \neq \beta$ and so $w \in Span_C\{v, bv\}$.

Finally consider $u \in V$ such that $au = 0$. In this case, $p(u+w) = pu + pw = pw \neq 0$ and then by previous argument, $u + w \in Span_C\{v, bv\}$. Since $w \in Span_C\{v, bv\}$, then also $u \in Span_C\{v, bv\}$.

As a consequence of the above two cases, we get $V = Span_C\{v, bv\}$ that is $dim_C V = 2$, a contradiction. This implies that v and bv are linearly C-dependent for all $v \in V$. Thus for each $v \in V$, $bv = \alpha_v v$ for some $\alpha_v \in C$. By using standard argument, it is easy to prove that α_v is independent of the choice of $v \in V$ and hence we can write $bv = \alpha v$ for all $v \in V$ and for a fixed $\alpha \in C$. Now let $r \in R$ and $v \in V$. Since $bv = \alpha v$, it follows

$$[b,r]v = (br)v - (rb)v = b(rv) - r(bv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[b,r]v = 0$ for all $v \in V$ i.e., $[b,r]V = 0$. Since $[b,r]$ acts faithfully as a linear transformation on the vector space V, $[b,r] = 0$ for all $r \in R$. Therefore, $b \in C$. Hence (2.1) reduces to $(a+b)[x_1,x_2]^{m+n} \in C$.

Denote $c = a + b$. As above, in case $c \in C$ we easily get $a = -b \in C$.

Hence we assume $a + b = c \notin C$, that is there exists $v \in V$ such that v, cv are linearly C-independent. Moreover, since $dim_C V = \infty$, there exist $w, u \in V$ such that v, cv, w, u are linearly C-independent. By the density of Q_r, there exist $r_1, r_2, r_3 \in Q_r$ such that

$$r_1 w = w, \quad r_2 w = w$$
$$r_3 w = v$$
$$r_1 v = 0, \quad r_2 v = u, \quad r_1 u = v.$$

Thus

$$[r_1, r_2]v = v, \quad [r_1, r_2]w = 0$$

and we get the contradiction

$$0 = \left[c[r_1, r_2]^{n+m}, r_3\right]w = cv \neq 0.$$

Therefore $dim_C V \leq 2$ and R is a noncommutative prime ring satisfying the standard identity of degree 4, which implies that $dim_C RC = 4$. □

Lemma 2.4. *Let R be a dense subring of the ring of linear transformations of a vector space V over a division ring D, and let R contain nonzero linear tranformations of finite rank. Let I be a noncentral two-sided ideal of R,*

$n, m \geq 1$ fixed integers, α be an automorphism of R and suppose $b, c \in R$ and $F(x) = bx + \alpha(x)c$ such that $F(x^n)x^m \in Z(R)$, for all $x \in [I, I]$. If $F \neq 0$ and R does not satisfy s_4, then $dim_D V \leq 2$.

Proof. We assume $dim_D V \geq 3$ and prove that a number of contradictions follows.

Since R is a primitive ring with nonzero socle, by [12] (p.79) there exists a semi-linear automorphism $T \in End(V)$ such that $\alpha(x) = TxT^{-1}$ for all $x \in R$, hence $(bx^n + Tx^nT^{-1}c)x^m \in Z(R)$, for all $x \in [I, I]$. Assume first that v and $T^{-1}cv$ are D-dependent for all $v \in V$. By Lemma 1 in [8], there exists $\lambda \in D$ such that $T^{-1}cv = v\lambda$, for all $v \in V$. In this case, for all $x \in R$,

$$\begin{aligned} F(x)v &= (bx + TxT^{-1}c)v = bxv + TxT^{-1}cv = bxv + T(xv\lambda) \\ &= bxv + T((xv)\lambda) = bxv + T(T^{-1}c)(xv) = bxv + cxv = (b + c)xv. \end{aligned}$$

This means that $(F(x)-(b+c)x)V = (0)$, for all $x \in R$ and since V is faithful, it follows that $F(x) = (b + c)x$, for all $x \in R$, and $(b + c)x^nx^m \in Z(R)$, for all $x \in [I, I]$. By Lemma 2.3 either R satisfies s_4 or $b + c = 0$ and $F = 0$, a contradiction again.

Thus there exists $v_0 \in V$ such that v_0 and $T^{-1}cv_0$ are linearly D-independent. Since $dim_D V \geq 3$, then there exists $w \in V$ such that w, v_0 and $T^{-1}cv_0$ are linearly D-independent (denote for clearness $T^{-1}cv_0 = u$). By the density of R, there exist $r_1, r_2, r_3 \in I$ such that

$$r_1v_0 = w, r_1w = v_0, r_1u = w, r_2v_0 = w, r_2w = 0, r_2u = 0, r_3u = v_0.$$

Thus

$$[r_1, r_2]u = 0, \quad [r_1, r_2]v_0 = v_0$$

and

$$0 = \big[(b[r_1, r_2]^n + T[r_1, r_2]^nT^{-1}c)[r_1, r_2]^m, r_3\big]u = bv_0.$$

Since v_0+w is D-independent of v_0 and u, in the same way we get $b(v_0+w) = 0$, that is $bw = 0$. Analogously, $u + w$ is D-independent of v_0 and u, and $b(u + w) = 0$ implies $bu = 0$. Therefore $bV = (0)$ and so $b = 0$.

Hence $[Tx^nT^{-1}cx^m, r_3] = 0$, for all $x \in [I, I]$, $r_3 \in R$. As above, by the density of R there exist $s_1, s_2, s_3 \in I$, such that

$$s_1v_0 = w, s_1w = w, s_1u = v_0, s_2v_0 = u, s_2w = 0, r_2u = 0, s_3w = v_0.$$

Thus

$$[s_1, s_2]v_0 = v_0, \quad [s_1, s_2]w = 0, \quad [s_1, s_2]u = -u$$

and

$$0 = \big[T[s_1, s_2]^n (T^{-1}c)[s_1, s_2]^m, s_3\big] w = (-1)^n c v_0.$$

Following the same above argument, we get $c = 0$. Therefore we have the contradiction $F = 0$. □

2.3. Proof of Proposition 2.1

Suppose first that α is X-inner. Thus there exists an invertible element $q \in Q_r$ such that $\alpha(x) = qxq^{-1}$, for all $x \in R$. Thus $(bu^n + qu^n q^{-1}c)u^m \in Z(R)$, for all $u \in [I, I]$. Since I, R and Q_r satisfy the same generalized polynomial identities with coefficients in Q_r (see [5]), it follows that $(bu^n + qu^n q^{-1}c)u^m \in Z(R)$, for all $u \in [Q_r, Q_r]$. If $q^{-1}c \in C = Z(Q_r)$, then $F(x) = (b + c)x$, for all $x \in R$ and $(b + c)u^n u^m \in Z(R)$, for all $u \in [Q_r, Q_r]$. Again by Lemma 2.3 either R satisfies s_4 or $b + c = 0$ and $F = 0$, a contradiction. So we may assume that $q^{-1}c \notin C$, and

$$[(b[x_1, x_2]^n + q[x_1, x_2]^n q^{-1}c)[x_1, x_2]^m, x_3] \tag{2.4}$$

is a non-trivial generalized polynomial identity for Q_r. By Martindale's theorem [18], Q_r is isomorphic to a dense subring of the ring of linear tranformations of a vector space V over D, where D is a finite dimensional division ring over C. By Lemma 2.4 we have that either $dim_C RC = 4$ or $dim_D V \leq 2$. In this last case it follows that either $Q_r \cong D$ or $Q_r \cong M_2(D)$, the ring of 2×2 matrices over D. More generally we assume $Q_r \cong M_k(D)$, for $k \leq 2$.

If C is finite, then D is a field by Wedderburn's Theorem. On the other hand, if C is infinite, let $\overline{C}$ be the algebraic closure of C, then by the van der Monde determinant argument, we see that $Q_r \bigotimes_C \overline{C}$ satisfies the same generalized polynomial identity (2.4). Moreover $Q_r \bigotimes_C \overline{C} \cong M_k(D) \bigotimes_C \overline{C} \cong M_k(D \bigotimes_C \overline{C}) \cong M_t(\overline{C})$, for some $t \geq 1$.

By using again the result in Lemma 2.4 and since Q_r is not commutative, we get $t = 2$. Hence R is an order in a 4-dimensional central simple algebra, as required.

Hence we may assume that α is X-outer. By Theorem 1 in [6], Q_r satisfies

$$(b[x_1,x_2]^n + \alpha([x_1,x_2]^n)c)[x_1,x_2]^m \in C \tag{2.5}$$

moreover by Main Theorem in [6] Q_r is a GPI-ring. Thus Q_r is a primitive ring having nonzero socle and its associated division ring D is a finite-dimensional over C. If C is finite, then it follows that D is also finite. By Wedderburn's Theorem D is a field and by Lemma 2.4 we also have $dim_D V \leq 2$. Hence from now on we assume that C is infinite.

If α is not Frobenius, then by main Theorem in [7] Q_r satisfies

$$(b[x_1,x_2]^n + [y_1,y_2]^n c)[x_1,x_2]^m \in C$$

and in particular Q_r satisfies both

$$b[x_1,x_2]^{n+m} \ \in \ C \tag{2.6}$$

and

$$[y_1,y_2]^n c[x_1,x_2]^m \ \in \ C. \tag{2.7}$$

By applying Lemma 2.3 to (2.6) and (2.7) it follows that Q_r satisfies s_4 (and also $b,c \in C$).

On the other hand, if α is Frobenius, then $char(Q_r) = p > 0$ (if not $\alpha(\lambda) = \lambda$ for all $\lambda \in C$ and α must be X-inner by Theorem 4.7.4 in [2]). Moreover $\alpha(\lambda) = \lambda^{p^t}$ for all $\lambda \in C$, where t is some fixed integer, and there exists $\mu \in C$ such that $\mu^{p^t} \neq \mu$. In (2.5) replace x_1 by λx_1 and get $\lambda^m(\lambda^n b[x_1,x_2]^n + \lambda^{np^t}\alpha([x_1,x_2]^n)c)[x_1,x_2]^m \in C$ that is

$$\lambda^n b[x_1,x_2]^n + \lambda^{np^t}\alpha([x_1,x_2]^n)c)[x_1,x_2]^m \in C. \tag{2.8}$$

Comparing (2.5) with (2.8) it follows that Q_r satisfies

$$\alpha([x_1,x_2]^n)c[x_1,x_2]^m - \lambda^{n(p^t-1)}\alpha([x_1,x_2]^n)c[x_1,x_2]^m \in C. \tag{2.9}$$

Since (2.9) holds for all $\lambda \in C$, if we choose λ such that $\lambda\mu^n = 1$, then $(\lambda^n)^{p^t} \neq \lambda^n$ and it follows from (2.9) that $\alpha([x_1,x_2]^n)c[x_1,x_2]^m \in C$. From this and (2.5) we also have $b[x_1,x_2]^{n+m} \in C$. As a consequence of Lemma 2.3, Q_r satisfies s_4, unless $b = 0$.

Thus, in the following we will consider $b = 0$ and Q_r satisfies

$$\alpha([x_1, x_2]^n)c[x_1, x_2]^m \in C. \tag{2.10}$$

Again by Lemma 2.4, we get $dim_D V \leq 2$. Notice that if $dim_D V = 1$, then Q_r is a domain; moreover if Q_r is not commutative then both $\alpha([x_1, x_2])$ and $\alpha([x_1, x_2]^n)$ are not identities for Q_r. In this case, by (2.10) we have that

$$0 = \big[\alpha([x_1, x_2]^n)c[x_1, x_2]^m, \alpha([x_1, x_2])\big] = \alpha([x_1, x_2]^n)\big[c[x_1, x_2]^m, \alpha([x_1, x_2])\big].$$

Since Q_r is a domain, it follows that $[c[x_1, x_2]^m, \alpha([x_1, x_2])]$ is an identity for Q_r. Moreover any $\alpha(x_i)$-word degree is 1, so that, by Theorem 3 in [7], Q_r satisfies the identity $[c[x_1, x_2]^m, [y_1, y_2]]$, that is $c[x_1, x_2]^m \in C$. Once again by Lemma 2.3 it follows either $c = 0$, which implies $F = 0$, or Q_r satisfies s_4.

Hence we now assume $dim_D V = 2$ that is $Q_r \cong M_2(D)$, the ring of 2×2 matrices over D.

Let $h \neq k$ be any element of D such that $[h, k] \neq 0$, and choose in (2.10)

$$[r_1, r_2] = \left[\begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}, \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\right].$$

Moreover use the following notations:

$$c = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \qquad \gamma = [h, k], \qquad \alpha([r_1, r_2]^n) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Since by (2.10) we have $[\alpha([r_1, r_2]^n)c[r_1, r_2]^m, e_{22}] = 0$, by calculations it follows

$$\begin{bmatrix} 0 & (b_{11}c_{12} + b_{12}c_{22})\gamma^m \\ (b_{21}c_{11} + b_{22}c_{21})\gamma^m & 0 \end{bmatrix} = 0$$

which implies both $b_{11}c_{12} + b_{12}c_{22} = 0$ and $b_{21}c_{11} + b_{22}c_{21} = 0$, that is $\alpha([r_1, r_2]^n)c = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$ for suitable $s_1, s_2 \in D$.

Starting from this, and using again (2.10), we also have $[\alpha([r_1, r_2]^n)c[r_1, r_2]^m, e_{12}] = 0$ and by calculations we get $\begin{bmatrix} 0 & (s_1 - s_2)\gamma^m \\ 0 & 0 \end{bmatrix} =$

0, which implies $s_1 = s_2$.

Finally for any $s_3 \in D$ and from $[\alpha([r_1,r_2]^n)c[r_1,r_2]^m, s_3e_{11}+s_3e_{22}] = 0$ we have $\begin{bmatrix} [s_1,s_3]\gamma^m & 0 \\ 0 & [s_1,s_3]\gamma^m \end{bmatrix} = 0$, which implies $[s_1,s_3] = 0$, that is $s_1 \in Z(D)$ and $\alpha([r_1,r_2]^n)c \in Z(M_2(D))$.

In case $\alpha([r_1,r_2]^n)c = 0$, then also $[r_1,r_2]^n\alpha^{-1}(c) = 0$. If denote $\alpha^{-1}(c) = \begin{bmatrix} c'_{11} & c'_{12} \\ c'_{21} & c'_{22} \end{bmatrix}$ this implies that

$$0 = [r_1,r_2]^n\alpha^{-1}(c) = \begin{bmatrix} \gamma^n c'_{11} & \gamma^n c'_{12} \\ \gamma^n c'_{21} & \gamma^n c'_{22} \end{bmatrix}$$

and since $\gamma^n \neq 0$, it follows $\alpha^{-1}(c) = 0$ and also $c = 0$. In this case we conclude $F = 0$.

Thus we may assume that $0 \neq \alpha([r_1,r_2]^n)c \in Z(M_2(D))$ and by (2.10) also $[r_1,r_2]^m \in Z(M_2(D))$.

Moreover by (2.10) we also have

$$[x_1,x_2]^n\alpha^{-1}(c)\alpha^{-1}([x_1,x_2]^m) \in C \tag{2.11}$$

and using the same above argument, one has that: if $c \neq 0$ then $[r_1,r_2]^n \in Z(M_2(D))$.

All the previous argument says that: if $h,k \in D$ and

$$[r_1,r_2] = \left[\begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix}, \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\right]$$

then either $[h,k]=0$ or both $[r_1,r_2]^m \in Z(M_2(D))$ and $[r_1,r_2]^n \in Z(M_2(D))$. In particular, for $[x_1,x_2] = [r_1,r_2]$ in (2.10), it follows $0 \neq c \in C$. Finally, by using again (2.10), we have that Q_r satisfies $\alpha([x_1,x_2]^n)[x_1,x_2]^m \in C$.

Moreover, since $[h,k]$ is either zero or both $[h,k]^n$ and $[h,k]^m$ are central in D, for all $h,k \in D$, by Fact 2.2 it follows that D satisfies the standard identity s_4, that is $[h,k]^2$ is central in D for all $h,k \in D$. Moreover, either D is commutative, or both n and m are even integers. Our aim is to prove that also in this case D must be commutative.

Suppose on the contrary that there exist $h, k \in D$, such that $\gamma = [h,k] \neq 0$. Let $[he_{11}, ke_{11}] = \gamma e_{11} \in [Q_r, Q_r]$ and denote $\alpha(\gamma^n e_{11}) = c_1 e_{11} + c_2 e_{12} + c_3 e_{21} + c_4 e_{22}$ (where $c_i \in D$). By our hypothesis, it follows that $\alpha(\gamma^n e_{11})(\gamma^m e_{11}) \in Z(Q_r)$, and by calculations we get $c_1 = c_3 = 0$.

Analogously, if denote $\alpha(\gamma^n e_{22}) = d_1 e_{11} + d_2 e_{12} + d_3 e_{21} + d_4 e_{22}$ (where $d_i \in D$), and since $\alpha(\gamma^n e_{22})(\gamma^m e_{22}) \in Z(Q_r)$, it follows that $d_2 = d_4 = 0$. This implies that

$$\alpha(\gamma^n e_{11}) = \begin{bmatrix} 0 & c_2 \\ 0 & c_4 \end{bmatrix}, \quad \alpha(\gamma^n e_{22}) = \begin{bmatrix} d_1 & 0 \\ d_3 & 0 \end{bmatrix}.$$

Moreover, since n is even, we also have $\alpha(\gamma^n e_{11} + \gamma^n e_{22}) \in Z(Q_r)$, which implies $c_2 = d_3 = 0$ and $d_1 = c_4$, so that we may write

$$\alpha(\gamma^n e_{11}) = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}, \quad \alpha(\gamma^n e_{22}) = \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix}, \qquad \lambda \in D.$$

Let now $[h(e_{12} + e_{22}), k(e_{12} + e_{22})] = \gamma(e_{12} + e_{22}) \in [Q_r, Q_r]$ and denote $\alpha(\gamma^n(e_{12} + e_{22}) = t_1 e_{11} + t_2 e_{12} + t_3 e_{21} + t_4 e_{22}$ (where $t_i \in D$). Therefore, by the hypothesis,

$$\alpha(\gamma^n(e_{12} + e_{22})) \cdot \gamma^m(e_{12} + e_{22}) = \begin{bmatrix} 0 & (t_1 + t_2)\gamma^m \\ 0 & (t_3 + t_4)\gamma^m \end{bmatrix} \in Z(Q_r)$$

which implies $t_1 + t_2 = 0$ and $t_3 + t_4 = 0$, since $\gamma \neq 0$. Hence

$$\alpha(\gamma^n(e_{12} + e_{22})) = \begin{bmatrix} t_1 & -t_1 \\ t_3 & -t_3 \end{bmatrix}, \quad t_1, t_3 \in D \tag{2.12}$$

and this means that

$$\alpha(\gamma^n e_{12}) = \begin{bmatrix} t_1 & -t_1 \\ t_3 & -t_3 \end{bmatrix} - \alpha(\gamma^n e_{22}) = \begin{bmatrix} t_1 - \lambda & -t_1 \\ t_3 & -t_3 \end{bmatrix}. \tag{2.13}$$

On the other hand $\alpha(\gamma^n e_{12}) = \alpha(\gamma^n e_{12} e_{22}) = \alpha(e_{12})\alpha(\gamma^n e_{22})$. If denote $\alpha(e_{12}) = p_1 e_{11} + p_2 e_{12} + p_3 e_{21} + p_4 e_{22}$ (where $p_i \in D$), it follows

$$\alpha(\gamma^n e_{12}) = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \cdot \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} p_1\lambda & 0 \\ p_3\lambda & 0 \end{bmatrix}. \tag{2.14}$$

Finally, by comparing (2.13) and (2.14) we get $t_1 = t_3 = 0$, that is, by (2.12),

$$\alpha(\gamma^n(e_{12}+e_{22})) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which is a contradiction if $\gamma \neq 0$. □

3. The Proof of Theorem 1

As remarked in the Introduction we can write $F(x) = bx + d(x)$ for all $x \in R$, $b \in Q_r$ and d is a skew derivation of R (see [4]).

Since L is a noncentral Lie ideal, by Fact 1.1 we have that either $char(R) = 2$ and $dim_C RC = 4$, or there exists a noncentral two-sided ideal I of R such that $[I,I] \subseteq L$. In this last case we get that $F(u^n)u^m \in Z(R)$, for all $u \in [I,I]$ for I a noncentral two-sided ideal of R. By Theorem 2 in [9] I, R and Q_r satisfy the same generalized polynomial identities with a single skew derivation, then $F(u^n)u^m \in C$, for all $u \in [Q_r, Q_r]$. Suppose that d is X-inner, then there exist $c \in Q_r$ and $\alpha \in Aut(Q_r)$ such that $d(x) = cx - \alpha(x)c$, for all $x \in R$. In this case $F(x) = (b+c)x - \alpha(x)c$ and by Proposition 2.1 it follows that Q_r satisfies s_4 and $dim_C RC = 4$.

Assume finally that d is X-outer. Since Q_r satisfies

$$\big(b[x_1,x_2]^n + d([x_1,x_2]^n)\big)[x_1,x_2]^m \in C \tag{3.1}$$

and recalling that

$$d(x^n) = \sum_{i=0}^{n-1} \alpha(x^i)d(x)x^{n-i-1}$$

then Q_r satisfies

$$b[x_1,x_2]^{n+m} + \left(\sum_{i=1}^{n-1} \alpha([x_1,x_2]^i)\big(d(x_1)x_2 + \alpha(x_1)d(x_2)\big)[x_1,x_2]^{n-i-1}\right)[x_1,x_2]^m$$
$$+\left(\sum_{i=1}^{n-1} \alpha([x_1,x_2]^i)\big(-d(x_2)x_1 - \alpha(x_2)d(x_1)\big)[x_1,x_2]^{n-i-1}\right)[x_1,x_2]^m \in C. \tag{3.2}$$

By Theorem 1 in [9] and (3.2), Q_r satisfies

$$b[x_1,x_2]^{n+m}+\sum_{i=1}^{n-1}\alpha([x_1,x_2]^i)\Big(y_1x_2+\alpha(x_1)y_2-y_2x_1 \\ -\alpha(x_2)y_1\big)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^m\in C. \tag{3.3}$$

For $y_1 = y_2 = 0$ we have $b[x_1,x_2]^{n+m} \in C$ and by Lemma 2.3 either $dim_C RC = 4$, or $b = 0$. In this last case Q_r satisfies

$$\Big(\sum_{i=1}^{n-1}\alpha([x_1,x_2]^i)\big(y_1x_2+\alpha(x_1)y_2-y_2x_1-\alpha(x_2)y_1\Big)[x_1,x_2]^{n-i-1}\big)[x_1,x_2]^m\in C. \tag{3.4}$$

Assume α is X-outer. By Theorem 1 in [9] and (3.4) we have that Q_r satisfies

$$\Big(\sum_{i=1}^{n-1}\alpha([t_1,t_2]^i)\big(y_1x_2+t_1y_2-y_2x_1-t_2y_1\big)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^m\in C.$$

and in particular for $t_1 = t_2 = 0$ and $y_1 = x_1$, $y_2 = x_2$, it satisfies $[x_1,x_2]^{n+m} \in C$, and $dim_C RC = 4$ follows from Fact 2.2.

Finally consider the case α is X-inner, then there exists an invertible element q of Q_r, such that $\alpha(x) = qxq^{-1}$, for all $x \in Q_r$. Consider first the simplest case when $q \in C$, that is α is the identity map on Q_r and d is an usual derivation of R. Then by (3.1) and $b = 0$, Q_r satisfies the polynomial identity

$$\Big[\Big(\sum_{i+j=n-1}[x_1,x_2]^i d([x_1,x_2])[x_1,x_2]^j\Big)[x_1,x_2]^m,x_3\Big]$$

that is

$$\Big[\Big(\sum_{i+j=n-1}[x_1,x_2]^i([d(x_1),x_2]+[x_1,d(x_2)])[x_1,x_2]^j\Big)[x_1,x_2]^m,x_3\Big]$$

and since d is X-outer, by Kharchenko's result in [14], Q_r satisfies the identity

$$\Big[\Big(\sum_{i+j=n-1}[x_1,x_2]^i([y_1,x_2]+[x_1,y_2])[x_1,x_2]^j\Big)[x_1,x_2]^m,x_3\Big]$$

in particular it satisfies

$$\Big[\Big(\sum_{i+j=n-1}[x_1,x_2]^i[y_1,x_2][x_1,x_2]^j\Big)[x_1,x_2]^m,x_3\Big]. \tag{3.5}$$

It is well known that in this situation there exists a suitable field K such that Q_r and the matrix ring $M_t(K)$ satisfy the same polynomial identities. Then suppose $t\geq 3$ and in (3.5) let $x_1=e_{12}$, $x_2=e_{21}$, $y_1=e_{32}$, $x_3=e_{13}$. By calculation it follows from (3.5) the contradiction $0=e_{33}$. Therefore $t\leq 2$ and Q_r satisfies s_4. Moreover, since R is not commutative, then Q_r is also not commutative and $t=2$, that is $dim_C RC=4$.

In light of this, we may consider $q\notin C$. From (3.4) and $y_1=0$, Q_r satisfies

$$\Big(\sum_{i=1}^{n-1}q[x_1,x_2]^iq^{-1}\big(qx_1q^{-1}y_2-y_2x_1\big)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^m\in C.$$

and replacing y_2 by qy_2, we have that Q_r satisfies

$$\Big[q\Big(\sum_{i=1}^{n-1}[x_1,x_2]^i\big(x_1y_2-y_2x_1\big)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^m,x_3\Big]. \tag{3.6}$$

Here we denote by $g(x_1,x_2,y_2)$ the following polynomial

$$\Big(\sum_{i=1}^{n-1}[x_1,x_2]^i\big(x_1y_2-y_2x_1\big)[x_1,x_2]^{n-i-1}\Big)[x_1,x_2]^m.$$

Hence the generalized polynomial identity $[qg(x_1,x_2,y_2),x_3]$ is satisfied by Q_r. In particular, for $x_3=q$, it follows that $q[q,g(x_1,x_2,y_2)]$ is a generalized polynomial identity for Q_r. Moreover $0\neq q$ is an invertible element of Q_r, then Q_r satisfies $[q,g(x_1,x_2,y_2)]$. Therefore, by Theorem 6 in [16] and since $q\notin C$, we have that either $dim_C RC=4$, or the polynomial $g(x_1,x_2,y_2)$ is central-valued on Q_r. In this last case

$$\Big[\Big(\sum_{i=1}^{n-1}[r_1,r_2]^i\big(r_1s_2-s_2r_1\big)[r_1,r_2]^{n-i-1}\Big)[r_1,r_2]^m,r_3\Big]=0 \tag{3.7}$$

for all $r_1,r_2,r_3,s_2\in Q_r$. As above, Q_r is a PI-ring and there exists a suitable field K such that Q_r and the matrix ring $M_t(K)$ satisfy the same polynomial identities. Notice that, if $t \geq 3$ and for $[r_1, r_2] = [e_{12}, e_{21}] = e_{11} - e_{22}$, $r_3 = e_{11}$ and $s_2 = e_{31}$ in relation (3.7), it follows the contradiction $e_{31} = 0$. Hence $t \leq 2$ and $dim_C RC = 4$. □

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