AN EXTENDED PROPOSITIONAL LOGIC

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Abstract

Motivated by quantum mechanics we discuss an extended propositional logic (EPL) basing its syntax on partitions of sets. Every complete partition define a context and in every context EPL reduces to a classical propositional logic. Partitions lead to a notion of incompatibility expressing that knowledge cannot be refined without changing context. We also deal with interpretations, tautologies and semantic consequences in EPL.

1. Introduction

In his famous lecture notes on physics [6], see also [5], Richard Feynman describes the *two-slit experiment* as a rule of calculating probability in two different ways depending on the "context" of the experiment. Particles are emitted by the source, pass through the system of two slits ($i = 1, 2$) and arrive at the screen at some position $x \in X$ (we assume that X is finite). The elementary event e_{ix} $i = 1, 2$, $x \in X$ means that the particle passed through slit i and arrived at $x \in X$. With each event e_{ix} there is associated the complex number φ_{ix}, called the *amplitude* of e_{ix}.

The calculation of the probability of the event $e_x = e_{1x} \cup e_{2x}$ (= the arrival at x) depends, according to Feynman, on the "context" of the experiment. The decisive rôle is played by the so-called *which-way information*

(WWI). The fact that WWI is known means that it is known through which slit the particle has passed.

(1) If WWI does not exist, then $P(e_x) = |\varphi_{1x} + \varphi_{2x}|^2$ (add amplitudes),
(2) If WWI exists, then $P(e_x) = |\varphi_{1x}|^2 + |\varphi_{2x}|^2$ (add probabilities).

These are *Feynman's rules* for calculating the probability of events in quantum mechanics (QM), they depend on the "context".

There is an alternative description of this situation, which was proposed in the papers [11] [12] [13]. Instead of considering a unique event e_x and two different "contexts" (as Feynman suggested) it was proposed to consider these two situations as *two different events*:

(1) We have an event $e_{1x} \sqcup e_{2x}$ where $\sqcup$ describe the "indistinguishable" union of events e_{1x} and e_{2x}: We have no WWI.
(2) We have an event $e_{1x} \vee e_{2x}$ where $\vee$ denotes the (standard) distinguishable union of events: WWI is known.

Clearly, these two events are different

$$(e_{1x} \sqcup e_{2x}) \neq (e_{1x} \vee e_{2x}) \qquad \forall x \in X.$$

It is useful to represent them as subsets of $\Omega \times \Omega$, $\Omega := \{e_{ix} \mid i = 1, 2, \ x \in X\}$, as follows, see [11] [12] [13],

$$e_{1x} \sqcup e_{2x} \mapsto A_x := (e_{1x} \cup e_{2x}) \times (e_{1x} \cup e_{2x}),$$
$$e_{1x} \vee e_{2x} \mapsto B_x := (e_{1x} \times e_{1x}) \cup (e_{2x} \times e_{2x}).$$

The difference between these events is

$$A_x \setminus B_x = (e_{1x} \times e_{2x}) \cup (e_{2x} \times e_{1x}).$$

Then it is possible to introduce an appropriate measure μ on $\Omega \times \Omega$ such that $P(e_{1x} \sqcup e_{2x}) = \mu(A_x)$, $P(e_{1x} \vee e_{2x}) = \mu(B_x)$. Naturally, in general $\mu(A_x) \neq \mu(B_x)$, since $A_x \neq B_x$.

The relevant fact is that the two events $e_{1x} \sqcup e_{2x}$ and $e_{1x} \vee e_{2x}$ cannot be considered in the same experiment. This property is called *incompatibility*, [13]. The incompatibility has a purely logical origin: it is impossible in the same experiment "to have" and "not to have" the which-way information. This incompatibility has nothing to do with Bohr's complementarity, since

the incompatibility is a purely logical phenomenon. The compatibility of two events is denoted in [13] by $\pitchfork$ and so

$$(e_{1x} \sqcup e_{2x}) \not\pitchfork (e_{1x} \vee e_{2x})$$

and, moreover, $(e_{1x} \sqcup e_{2x}) \not\pitchfork e_{ix}$, $i = 1, 2$, while $(e_{1x} \vee e_{2x}) \pitchfork e_{ix}$, $i = 1, 2$, $e_{1x} \pitchfork e_{2x}$. Then the *context* was defined in [13] as a maximal set of mutually compatible events. The incompatibility of events A and B means the following: If A has happened, then B can be neither true nor false, since B is not a part of the context of A (see [13] for details).

The aim of this paper is to extend the classical propositional logic (CPL) to an extended propositional logic (EPL) which incorporates the ideas from [13]:

(1) the operator $\sqcup$ of indistinguishable disjunction,
(2) the extended set of propositions generated by the operators $\neg$, $\wedge$, $\vee$, $\sqcup$,
(3) the concept of incompatibility, $\pitchfork$.

A first rough step in such a description was done in [7], where the case of a finite set of propositional symbols is considered. In this paper, after introducing the *partitions* of a set that turns out to be an effective model of our EPL, in Section 2, we describe the syntax of our EPL. Then, in Section 3, we discuss the concept of *interpretation*. Section 4 is devoted to the concepts of *tautology* and of *semantic consequence*, and in Section 5 we show that *in every context our EPL reduces to a classical propositional logic*. Finally, Section 6 contains a few comments about some similarities and differences between our EPL and 3-valued logics or rough set theory. Essentially, in our EPL knowledge cannot be refined without changing context.

2. The Syntax: Partitions and Normal Formulas

In this section we illustrate the *syntax* of our *extended propositional logic*. In particular, we deal with a natural parallelism between formulas and partitions that is between language constructions and set-theoretical constructions that will lead us to the notion of *normal form* of a formula.

2.1. Partitions

First we illustrate a few properties of *partitions* of a set.

Let Ω be a nonempty set. We call *partition* of Ω a family $A = \{a_i\}_{i\in\mathcal{I}}$ of nonempty pairwise disjoint subsets of Ω. It is not requested that $\cup_i a_i = \Omega$. If this is the case, we speak of a *complete partition*. We say that a partition is an *atomic partition* if it consists of only one subset of Ω, $A = \{a\}$, $a \subset \Omega$, $a \neq \emptyset$. For instance, for two disjoint subsets $a, b \subset \Omega$, $\{a, b\}$ and $\{a \cup b\}$ are two distinct partitions of Ω. The latter is an atomic partition.

For convenience we also introduce the *empty partition* $\emptyset$ where the index set $\mathcal{I} = \emptyset$. The set of all partitions of Ω, augmented with the *empty partition* $\emptyset$, will be denoted by $\Pi(\Omega)$; of course, $\Pi(\Omega) \subset \mathcal{P}(\mathcal{P}(\Omega))$. We define the *support* of a partition $A \in \Pi(\Omega)$ as the set $spt A := \cup_i a_i \subset \Omega$. Two partitions are said to be *disjoint* or *orthogonal*, and we write $A \perp B$, if $spt A \cap spt B = \emptyset$. Of course, partitions are subsets of the set $\mathcal{P}(\Omega)$ of subsets of Ω thus an inclusion operator is inherited: let $A = \{a_i\}_{i\in I}$ and $B = \{b_j\}_{j\in J}$ be two partitions, we say that $A \subset B$ if for any $i \in I$ there exists $j \in J$ such that $a_i = b_j$. We say that two partitions $A = \{a_i\}$ and $B = \{b_j\} \in \Pi(\Omega)$ are *compatible*, and we write $A \pitchfork B$, if for any $i \in I$ and $j \in J$ either $a_i \cap b_j = \emptyset$ or $a_i = b_j$. Trivially, $A \perp B$ or $A \subset B$ or $B \subset A$ imply that A and B are compatible. Notice that compatibility is *not* an equivalence relation.

Let $A = \{a_i\}_{i\in I}$ and $B = \{b_j\}_{j\in J}$ be two partitions, then the set-theoretic intersection $A \cap B$ of A and B in $\mathcal{P}(\Omega)$, defined by

$$A \cap B = \left\{ c \subset \Omega \,\middle|\, c \in A \text{ and } c \in B \right\},$$

is a partition of Ω. In contrast, the set-theoretic union

$$A \cup B := \left\{ c \subset \Omega \,\middle|\, c \in A \text{ or } c \in B \right\}$$

in $\mathcal{P}(\Omega)$ is not a partition, in general. We now define the operators $\neg$, $\wedge$, $\vee$ and $\sqcup$ on $\Pi(\Omega)$ as follows. Let $A = \{a_i\}_{i\in I}$ and $B = \{b_j\}_{j\in J}$ be two partitions of Ω, then

(1) $\neg A := \{\Omega \setminus spt A\}$,
(2) $\sqcup A := \neg\neg A = \{spt A\}$,
(3) $A \sqcup B := \{spt A \cup spt B\}$,
(4) $\wedge A := A$,

(5) if for some i, j we have $a_i \cap b_j \neq \emptyset$, we set

$$A \wedge B := \left\{ a_i \cap b_j \,\middle|\, i \in I, \; j \in J, \; a_i \cap b_j \neq \emptyset \right\},$$

otherwise we set $A \wedge B = \emptyset$,

(6) $\vee A := A$,

(7) if the supports of A and B are disjoint, we set $A \vee B := A \cup B$, and in general

$$A \vee B = (A \wedge B) \vee (\neg A \wedge B) \vee (A \wedge \neg B).$$

since $A \wedge B$, $(\neg A) \wedge B$ and $A \wedge (\neg B)$ are pairwise disjoint partitions.

By induction we then extend the operators $\vee$, $\wedge$ and $\sqcup$ to operators with arbitrary arity.

Example 1. Let $a, b, c \subset \Omega$ be three pairwise disjoint subsets of Ω and let $A = \{a\}$, $B = \{b\}$ and $C = \{c\}$, i.e., A, B and C are three atomic and pairwise disjoint partitions. Then $A \vee B = \{a, b\}$, $A \sqcup B = \{a \cup b\}$, $\sqcup(\sqcup A) = \sqcup A = A$, hence

$$
\begin{aligned}
A \wedge (A \vee B) &= A, \\
A \wedge (A \sqcup B) &= \{a\} \wedge \{a \cup b\} = \{a\} = A, \\
A \vee (A \vee B) &= A \vee B, \\
A \vee (A \sqcup B) &= \{a\} \vee \{a \cup b\} = \{a, b\} = A \vee B, \\
A \sqcup (A \vee B) &= \{a\} \sqcup \{a, b\} = \{a \cup b\} = A \sqcup B.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
(A \vee B) \sqcup (A \vee C) &= \{a, b\} \sqcup \{a, c\} = \{a \cup b \cup c\} = A \sqcup B \sqcup C, \\
(A \sqcup B) \vee (A \sqcup C) &= \{a \cup b\} \vee \{a \cup c\} = \{a\} \vee \{c\} \vee \{b\} = A \vee B \vee C, \\
(A \sqcup B) \wedge (A \sqcup C) &= \{a \cup b\} \wedge \{a \cup c\} = \{a\} = A, \\
(A \vee B) \wedge (A \vee C) &= \{a, b\} \wedge \{a, c\} = A.
\end{aligned}
$$

However, in general, $A \wedge (A \vee B) \neq A$ and $A \vee (A \wedge B) \neq A$. In fact, if $a, b, c \subset \Omega$ are pairwise disjoint and $A := \{a \cup b\}$, $B := \{b \cup c\}$ we have $A \wedge (A \vee B) = A \vee (A \wedge B) = \{a, b\}$.

Several properties of partitions are discussed in [7] [13]. Here we recall some of them.

Proposition 2. *Let Ω be a set. Then*

(1) *$\vee$, $\wedge$ and $\sqcup$ are associative and commutative on $\Pi(\Omega)$,*

(2) *the following distribution law holds*

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C) \qquad \forall A, B, C \in \Pi(\Omega),$$

while in general, $A \vee (B \wedge C)$ is not equal to $(A \vee B) \wedge (A \vee C)$,

(3) *the subset $\mathcal{A}(\Omega) \subset \Pi(\Omega)$ of atomic partitions augmented with the empty partition is a Boolean algebra with the operations $\neg$, $\wedge$ and $\sqcup$, and the elements 0 and 1 given by $\emptyset$ and $\{\Omega\}$, respectively.*

As a consequence, the following claims are equivalent:

(1) $A \pitchfork B$,

(2) $A \wedge B \subset A$ and $A \wedge B \subset B$,

(3) $A \vee B = A \cup B$,

(4) $A \cup B \in \Pi(\Omega)$.

Observe that if A and B are compatible, then the operations $A \vee B$ and $A \wedge B$ simply reduce to the set theoretic operations of union and intersection, respectively.

For two partitions A and B we define $B \setminus A := (\neg A) \wedge B$. Observe that the inclusion $B \setminus A \subset B$ is false, in general, and it is true if $A \subset B$.

A maximal class of compatible partitions is called a *context*, see [13]. Since for every $A \in \Pi(\Omega)$ the partition $A \vee (\neg A)$ is compatible with A, we conclude that every context $\mathcal{K}$ contains one (and only one) *complete partition* U of Ω, that we call the *universe* of $\mathcal{K}$. In particular, if $\mathcal{K}$ is a context with universe U, then a partition A belongs to $\mathcal{K}$ if and only if $A \subset U$.

Let $\mathcal{K}$ be a context with universe U. Then we may define a *negation operator within $\mathcal{K}$* by setting for $A \in \mathcal{K}$

$$\neg_{\mathcal{K}}(A) := (\neg A) \wedge U.$$

By the above, $\neg_{\mathcal{K}} A \in \mathcal{K}$.

Let $\mathcal{K}$ be a context with universe U. Since $A \in \mathcal{K}$ if and only if $A \subset U$, there is a bijection $j : \mathcal{K} \to \mathcal{P}(U)$, $j(A) := A$. Moreover,

Proposition 3. *j is an isomorphism between $\mathcal{K}$ with the three operations of union, intersection and negation $\neg_{\mathcal{K}}$ within $\mathcal{K}$ and the standard algebra of the parts $\mathcal{P}(U)$, so that $\mathcal{K}$ is a Boolean algebra.*

Therefore, the usual consequences holds. For instance, let $\mathcal{K}$ be a context and let $A, B \in \mathcal{K}$. Then

$$\begin{aligned}
\neg_{\mathcal{K}}(\neg_{\mathcal{K}}A) &= A, \\
\neg_{\mathcal{K}}(A \cap B) &= \neg_{\mathcal{K}}A \cup \neg_{\mathcal{K}}B, \\
\neg_{\mathcal{K}}(A \cup B) &= \neg_{\mathcal{K}}A \cap \neg_{\mathcal{K}}B,
\end{aligned}$$

and

$$(\neg_{\mathcal{K}}A) \cup B = U \qquad \text{if and only if} \qquad A \subset B. \tag{2.1}$$

2.2. The syntax of EPL

We begin by declaring our *alphabet* by means of which formulas are produced. It consists of

(1) a set $\mathcal{P} = \{p_1, p_2, \dots\}$ of at most denumerable *propositional symbols*,
(2) a set of symbols for logical operators, $\neg$, $\wedge$, $\vee$, $\sqcup$ (corresponding to the logic operators of *syntactical negation, conjunction, disjunction* and *irreducible* or *indistinguishable disjunction*), where arity of $\neg$ is 1 while the arity of $\wedge$, $\sqcup$ and $\vee$ is any arbitrary $n \geq 1$,
(3) auxiliary symbols (), {}, etc.

As usual, an *expression* is a concatenation of symbols of the alphabet. We write

$$\vee(f_1, f_2)$$

to denote the concatenation of the symbols "$\vee$", "(", "f_1", "f_2", ")" and similarly for $\wedge(f_1, f_2)$ and $\sqcup(f_1, f_2)$. We shall also write $f_1 \vee f_2$ instead of $\vee(f_1, f_2)$, and similarly for the expressions involving $\sqcup$ and $\wedge$.

Let $\mathcal{P}$ be a set of propositional symbols. We denote by $\mathcal{F}_{\mathcal{P}}$ the set of all *admissible expressions*, or *formulas*, of the language. These formulas are produced by the following *grammar*:

(1) if $p \in \mathcal{P}$, then $p \in \mathcal{F}_{\mathcal{P}}$,
(2) if $f \in \mathcal{F}_{\mathcal{P}}$, then $\neg f \in \mathcal{F}_{\mathcal{P}}$,
(3) if $f_1, f_2 \in \mathcal{F}_{\mathcal{P}}$, then $(f_1 \wedge f_2)$, $(f_1 \vee f_2)$, $(f_1 \sqcup f_2) \in \mathcal{F}_{\mathcal{P}}$.

By standard induction, each formula $f \in \mathcal{F}_\mathcal{P}$ involves a finite and ordered set of symbols from the alphabet. For f_1, f_2, ..., f_h in $\mathcal{F}_\mathcal{P}$ we denote in the sequel by $\wedge(f_1, \ldots, f_h)$ or by $f_1 \wedge \ldots \wedge f_h$ the formula

$$(\ldots((f_1 \wedge f_2) \wedge f_3) \wedge \ldots f_h)$$

A similar convention is also used for formulas involving $\sqcup$, $\wedge$ or $\vee$.

For $f \in \mathcal{F}_\mathcal{P}$ the smallest subset $\mathcal{Q} \subset \mathcal{P}$ such that $f \in \mathcal{F}_\mathcal{Q}$ is called the *basis*, or the *basic alphabet*, of f and it is denoted by $bas(f)$. Since $\mathcal{F}_\mathcal{Q} \cap \mathcal{F}_\mathcal{R} = \mathcal{F}_{\mathcal{Q} \cap \mathcal{R}}$, we have

$$bas(f) = \bigcap \left\{ \mathcal{Q} \subset \mathcal{P} \,\middle|\, f \in \mathcal{F}_\mathcal{Q} \right\}.$$

Consider sequences of literals $(q_1, q_2, \ldots)$ where for each i we have $q_i = p_i$ or $\neg p_i$ for some $p_i \in \mathcal{F}$. One can identify such a sequence to a binary sequence indexed by $\mathcal{P}$ that at the place p has 1 if $q_i = p$ and 0 if $q_i = \neg p$. This way we have a natural identification of the set of sequences of literals in $\mathcal{P}$ with the set of binary sequences indexed by $\mathcal{P}$

$$\Omega_\mathcal{P} := 2^\mathcal{P} := \left\{ \sigma \,\middle|\, \sigma : \mathcal{P} \to \{0, 1\} \right\}.$$

As usual, every formula $f \in \mathcal{F}_\mathcal{P}$ is made by a finite sequence of propositional symbols, the elements of $bas(f)$. These propositional symbols can appear as itself or as their negation. We can see a formula as a binary tuple indexed by the symbols involved. Since each formula $f \in \mathcal{F}_\mathcal{P}$ has a finite basis $\mathcal{Q} = bas(f) \subset \mathcal{P}$, we identify f with the subset of all binary sequences indexed by $\mathcal{F}$ that have the components indexed by the symbols in $bas(f)$ fixed.

Definition 4. Let $\mathcal{Q} \subset \mathcal{P}$. For any $E \subset \Omega_\mathcal{Q}$, the set

$$C_\mathcal{Q}(E) = \left\{ \alpha : \mathcal{P} \to \{0, 1\} \,\middle|\, \alpha_{|\mathcal{Q}} \in E \right\}$$

is said to be a $\mathcal{Q}$-*cylinder* over E. We also say that the cylinder $C_\mathcal{Q}(E)$ is $\mathcal{Q}$-*based and over* E.

The $\mathcal{Q}$-cylinders inherit the properties of the parts of the finite set $\Omega_\mathcal{Q}$. For instance, $\Omega_\mathcal{P} = C_\mathcal{Q}(\Omega_\mathcal{Q})$ for all $\mathcal{Q} \subset \mathcal{P}$, and

$$
\begin{aligned}
C_\mathcal{Q}(E \cap F) &= C_\mathcal{Q}(E) \cap C_\mathcal{Q}(F), \\
C_\mathcal{Q}(E \cup F) &= C_\mathcal{Q}(E) \cup C_\mathcal{Q}(F), \\
C_\mathcal{Q}(E \setminus F) &= C_\mathcal{Q}(E) \setminus C_\mathcal{Q}(F)
\end{aligned}
\tag{2.2}
$$

for all $E, F \subset \Omega_\mathcal{Q}$. If we set $C_\mathcal{Q}(\emptyset) := \emptyset$, then the map $\Omega_\mathcal{Q} \to \Omega_\mathcal{P}$ defined by $E \mapsto C_\mathcal{Q}(E)$ is a one-to-one correspondence between the algebra of parts of $\Omega_\mathcal{Q}$ and the algebra of $\mathcal{Q}$-based cylinders in $\Omega_\mathcal{P}$ In particular, if $\mathcal{Q}$ is finite, then the family of $\mathcal{Q}$-based cylinders is finite with cardinality $\Omega_{|\mathcal{Q}|}$. Moreover, $\mathcal{Q}_1$-based cylinders are also $\mathcal{Q}_2$-based cylinders if $\mathcal{Q}_1 \subset \mathcal{Q}_2$.

A set $A \subset \Omega_\mathcal{P}$ is said to be *finitely generated* if A is a $\mathcal{Q}$-based cylinder $A = C_\mathcal{Q}(E)$ for some finite $\mathcal{Q} \subset \mathcal{P}$ and $E \subset \Omega_\mathcal{Q}$. The smallest $\mathcal{Q}$ for which A is $\mathcal{Q}$-based is called the *basis* of A and we have

$$
bas(A) = \bigcap \left\{ \mathcal{Q} \,\middle|\, A \text{ is } \mathcal{Q}\text{-based} \right\}.
$$

Finite partitions $A = \{a_i\}$ *of* $\Omega_\mathcal{P}$ *consisting of* $\mathcal{Q}_i$-*based cylinders for some finite* $\mathcal{Q}_i \subset \mathcal{P}$ will play a relevant role for us. The class of such partitions will be denoted by

$$
\Pi^f(\Omega_\mathcal{P}).
$$

For every $\mathcal{Q} \subset \mathcal{P}$ the map $\Omega_\mathcal{Q} \to \Omega_\mathcal{P}$ given by $E \mapsto C_\mathcal{Q}(E)$ is a homomorphism of algebras; consequently, the operations $\neg$, $\cup$, $\cap$ of $\Omega_\mathcal{Q}$ extends to operations on the finite partitions of $\Omega_\mathcal{P}$ into $\mathcal{Q}$-based cylinders, and, when $\mathcal{Q}$ varies, to operations on $\Pi^f(\Omega_\mathcal{P})$.

We now define a map $part : \mathcal{F}_\mathcal{P} \to \Pi^f(\Omega_\mathcal{P})$ as follows:

(1) if $p \in \mathcal{P}$, then $part(p) := \left\{ C_{\{p\}}(\{1\}) \right\}$,

(2) if $f \in \mathcal{F}_\mathcal{P}$, then $part(\neg f) := \neg part(f)$,

(3) if $f_1, \ldots, f_k \in \mathcal{F}_\mathcal{P}$, then

$$
\begin{aligned}
part(\vee(f_1, \ldots, f_k)) &:= part(f_1) \vee \cdots \vee part(f_k), \\
part(\wedge(f_1, \ldots, f_k)) &:= part(f_1) \wedge \ldots \wedge part(f_k), \\
part(\sqcup(f_1, \ldots, f_k)) &:= part(f_1) \sqcup \cdots \sqcup part(f_k).
\end{aligned}
$$

Notice that *part* is well-defined and that the image partitions are always finite partitions into finite generated cylinders; also

(1) If $p \in \mathcal{P}$, then $part(\neg p) = \neg part(p) = \{C_{\{p\}}(\{0\})\}$.
(2) $\forall f \in \mathcal{F}_\mathcal{P}$ we have $part(\vee(f)) = \cup \, part(f) = part(f)$.
(3) $\forall f \in \mathcal{F}_\mathcal{P}$ we have $part(\wedge(f)) = \cap \, part(f) = part(f)$.
(4) For every simple conjunction f we have $part(\sqcup(f)) = part(f)$.
(5) $\forall f \in \mathcal{F}_\mathcal{P}$ we have $bas(f) = bas(part(f))$.

The map *part* splits $\mathcal{F}_\mathcal{P}$ into equivalence classes, setting for $f, g \in \mathcal{F}_\mathcal{P}$ $f \simeq g$ iff $part(f) = part(g)$. We assume that *part* defines the rules of calculus of our logic by considering equivalent the formulas which give rise to the same partition. Then it readily follows that logic operators preserve the equivalence classes, that the logic operators $\wedge$, $\vee$ and $\sqcup$ are associative and commutative, and, moreover, that $\vee(f) \simeq f$, $\wedge(f) \simeq f \; \forall f \in \mathcal{F}_\mathcal{P}$.

2.3. Normal formulas

In this section we prove that in fact the map *part* from $\mathcal{F}_\mathcal{P}$ into $\Pi^f(\Omega_\mathcal{P})$ is onto. Actually, we construct a map $\psi : \Pi^f(\Omega_\mathcal{P}) \to \mathcal{F}_P$ such that $part(\psi(A)) = A \; \forall A \in \Pi^f(\Omega_\mathcal{P})$. Therefore ψ is injective and *part* is surjective.

In order to define the map ψ, let us make some observation about the ordering of the elements of a $\mathcal{Q}$-cylinder.

The elements of $\mathcal{P}$ are naturally ordered, so are the elements of any finite family $\mathcal{Q}$ of symbols. In contrast, there is no natural order of the parts of a partition. Moreover, there is no natural order on the set of finite tuples in $\mathcal{O}_\mathcal{Q}$. Of course, one can introduce the *lexicographic order* on tuples of $\Omega_\mathcal{Q}$. This induces a corresponding order among the elements of a $\mathcal{Q}$-cylinder $C_\mathcal{Q}(E)$, $E \subset \mathcal{Q}$. More explicitly, if $\mathcal{Q} = \{p_{i_1}, \ldots, p_{i_s}\}$, $i_1 < i_2 < \cdots < i_s$, then the maps in $\Omega_\mathcal{Q}$, i.e., the binary s-uples, are ordered as follows: For $\sigma, \tau : \mathcal{Q} \to \{0,1\}$, we have $\sigma < \tau$ if there is k such that $\sigma(q_i) = \tau(q_i)$ for $i = 1, \ldots, k-1$ and $\sigma(q_k) = 0 < 1 = \tau(q_k)$. It is easily seen that $<$ defines a total order on $\Omega_\mathcal{Q}$. In particular, every subset $M \subset \Omega_\mathcal{Q}$ has a minimum point, denoted $\min(M)$.

So, if $E, F \subset \Omega_\mathcal{Q}$ with $\min(E) < \min(F)$, we say $E \prec F$ and, similarly, that $C_\mathcal{Q}(E) \prec C_\mathcal{Q}(F)$. Again $\prec$ is an order relation that is a total order on

disjoint sets of $\Omega_{\mathcal{Q}}$ and also on the corresponding cylinders in $\Omega_{\mathcal{P}}$. This way we can order the elements of a finite partition in $\Pi^f(\Omega_{\mathcal{P}})$.

Define now $\psi : \Pi^f(\Omega_{\mathcal{P}}) \to \mathcal{F}_{\mathcal{P}}$ by means of the following syntactic rules:

(1) If $A := \{C_{\{p\}}(\{1\})\}$, $p \in \mathcal{P}$, we set $\psi(A) := p$, and, if $A = \{C_{\{p\}}(\{0\})\}$, we set $\psi(A) := \neg p$. In other words, we associate to the atomic partition containing the $\{p\}$-cylinder $C_{\{p\}}(\{1\})$ ($C_{\{p\}}(\{0\}$, respectively) the symbol p ($\neg p$, respectively). Trivially, we have $part(\psi(A)) = A$.

(2) Let $A = \{C_{\mathcal{Q}}(E)\}$ be an atomic partition, where $C_{\mathcal{Q}}(E)$ is a $\mathcal{Q}$-cylinder, $\mathcal{Q}$ is finite and $E := \{\alpha\}$, $\alpha : \mathcal{Q} \to \{0,1\}$. By enumerating the elements of $\mathcal{Q}$ by using the order in $\mathcal{Q}$, $\mathcal{Q} = \{p_{i_1}, \ldots p_{i_k}\}$, we have $E = \{\delta_{i_1}, \ldots, \delta_{i_k}\}$ and

$$C_{\mathcal{Q}}(E) = \bigcap_{j=1}^{k} C_{\{p_{i_j}\}}(\{\delta_{p_{i_j}}\}).$$

If f_j denotes the literal $f_j := \psi\left(\left\{C_{\{p_{i_j}\}}(\{\delta_{p_{i_j}}\})\right\}\right)$, we then set

$$\psi(A) := \wedge\left(f_1, f_2, \ldots, f_k\right),$$

i.e., we associate to the atomic partition A, the element of which is a $\mathcal{Q}$-cylinder over E, a *simple conjunction*. Again $part(\psi(A)) = A$.

(3) Let $A = \{C_{\mathcal{Q}}(E)\}$ be an atomic partition where $C_{\mathcal{Q}}(E)$ is a $\mathcal{Q}$-cylinder, $\mathcal{Q}$ is finite and $E \subset \Omega_{\mathcal{Q}}$. Since E is a finite set of tuples, we may enumerate the tuples in E according to the lexicographic order we introduced above so that $E = \{\alpha_1, \ldots, \alpha_s\}$, $\alpha_i : \mathcal{Q} \to \{0,1\}$ and $C_{\mathcal{Q}}(E)$ decomposes as union of disjoint $\mathcal{Q}$-cylinders over a single tuple, $C_{\mathcal{Q}}(E) = \cup_{i=1}^{s} C_{\mathcal{Q}}(\{\alpha_i\})$. If

$$q_i := \psi\left(\{C_{\mathcal{Q}}(\{\alpha_i\})\}\right), \qquad i = 1, \ldots, s,$$

we then set

$$\psi(A) := \sqcup(q_1, q_2, \ldots, q_s).$$

One sees that $part(\psi(A)) = A$.

(4) Finally, if $A = \{C_{Q_i}(E_i) \,|\, i = 1, \ldots, s\}$ is a finite partition into disjoint cylinders in $\Pi^f(\Omega_{\mathcal{P}})$, denoting $g_i := \psi(\{C_{Q_i}(E_i)\})$, we set

$$\psi(A) := \vee(g_1, \ldots, g_s),$$

and again we have $part(\psi(A)) = A$.

The image of ψ is contained in a special class of formulas that we call *normal formulas*. They are generated starting from the propositional symbols by the following rules:

(1) A *literal* is a formula of the type p or $\neg p$ with $p \in \mathcal{P}$; of course, $bas(p) = bas(\neg p) = \{p\}$. The *literals* are normal formulas.
(2) If $\{q_i\}$, $i = 1, 2, \ldots, h$ is a finite set of literals with disjoint basis sets, $bas(q_i) \cap bas(q_j) = \emptyset \;\forall i, j, \, i \neq j$, then $f = \wedge(q_1, \ldots, q_h)$ is called a *simple conjunction*; of course, $bas(f) = \cup_{i=1}^h bas(q_i)$. The simple conjunctions are normal formulas. Notice that $p \wedge p$, $p \wedge \neg p$ are not simple conjunctions.
(3) If $\{f_i\}$ is a finite set of simple conjunctions, then the formula

$$g = \sqcup(f_1, \ldots, f_k)$$

is called an *atomic formula*; of course, $bas(g) = \cup_{i=1}^h bas(f_i)$. Atomic formulas are *normal formulas*.
(4) Finally, if $g_1, \ldots, g_h$ are atomic formulas with $bas(g_i) \cap bas(g_j) = \emptyset \;\forall i \neq j$, $i, j = 1, \ldots, h$, then $F = \vee(g_1, \ldots, g_h)$ is a normal formula. Of course $bas(f) = \cup_{i=1}^h bas(g_i)$.

Notice that for two simple conjunctions f_1 and f_2, $\vee(f_1, f_2)$ is not a normal formula, while $\vee(\sqcup(f_1), \sqcup(f_2))$ and $\vee(\sqcup(f_2), \sqcup(f_1))$ are equivalent normal formulas.

The image $\psi(\Pi^f(\Omega_{\mathcal{P}}))$ we have constructed above does not coincide with the full class of normal forms, since for a given partition A, $\psi(A)$ depend on A and on the order relation used to enumerate the elements of each cylinder of the partition A. However, albeit changing the orderings changes the map ψ and the normal formula $\psi(A)$, the equivalence class of $\psi(A)$ never changes. Thus *part* is a one to one correspondence between $\mathcal{F}_{\mathcal{P}}/\simeq$ and $\Pi^f(\Omega_{\mathcal{P}})$ and ψ is a bijection between $\Pi^f(\Omega_{\mathcal{P}})$ and $\mathcal{N}_{\mathcal{P}}/\simeq$. Therefore, even if $\psi \circ part : \mathcal{F}_{\mathcal{P}} \to \mathcal{N}_{\mathcal{P}}$ is not an intrinsic map, $\psi \circ part$ factorizes to an intrinsic bijection from $\mathcal{F}_{\mathcal{P}}/\simeq$ and $\mathcal{N}_{\mathcal{P}}/\simeq$, which in fact associate to a generic formula $f \in \mathcal{F}_{\mathcal{P}}$ a normal form $g := \psi \circ part(f)$ which is uniquely

defined up to a permutation of the addends. Finally, it is worth noticing that the definition of the map $\psi \circ part$ is purely syntactic and independent of any interpretation of the formulas.

Example 5. It may be worth illustrating the above by means of a few examples.

(1) We have $\wedge(f, f) \simeq f$ and $\vee(f, f) \simeq f$ for all $f \in \mathcal{F}_\mathcal{P}$. In fact, we have $part(\vee(f, f)) = part(f) \vee part(f) = part(f)$ and, similarly, $part(\wedge(f, f)) = part(f) \wedge part(f) = part(f)$. In particular, literals are equivalent to simple conjunctions.

(2) If f is a simple conjunction, then $\sqcup(f) \simeq f$ since $part(\sqcup(f)) = \sqcup(part f) = part(f)$. In other words, simple conjunctions are equivalent to atomic formulas.

(3) If f is an atomic formula, then $part(f)$ is an atomic partition; then it follows that $\sqcup(part(f)) = part(f)$, hence $\sqcup(f) \simeq f$.

(4) If f_1, f_2, f_3 are atomic formulas, then

$$f_1 \vee (f_1 \sqcup f_2 \sqcup f_3) \simeq f_1 \vee (f_2 \sqcup f_3),$$
$$f_1 \wedge (f_1 \sqcup f_2 \sqcup f_3) \simeq f_1.$$

(5) $f_1 \sqcup (f_1 \vee f_2) \simeq f_1 \sqcup f_2$ since, if $part(f_1) = \{a\}$ and $part(f_2) = \{b\}$, then $part(f_1 \vee f_2) = \{a, b\}$ consequently

$$part(f_1 \sqcup (f_1 \vee f_2)) = \{a\} \sqcup \{a, b\} = \{a \cup b\}.$$

(6) Let f_1, f_2, f_3 be three simple conjunction that are pairwise orthogonal. Then

$$(f_1 \sqcup f_2) \vee (f_1 \sqcup f_3) \simeq f_1 \vee f_2 \vee f_3,$$
$$(f_1 \vee f_2) \sqcup (f_1 \vee f_3) \simeq f_1 \sqcup f_2 \sqcup f_3,$$
$$(f_1 \sqcup f_2) \wedge (f_1 \sqcup f_3) \simeq f_1.$$

Finally, notice that, if $f = e_1 \sqcup e_2$ and $g = e_1 \sqcup e_3$, then $f \vee (f \wedge g) \neq f$ and $f \wedge (f \vee g) \neq f$. In fact, $f \vee (f \wedge g) = f \wedge (f \vee g) = e_1 \vee e_2$.

3. Interpretations

In the classical propositional logic, an *interpretation*, or a *valuation*, is a map $I : \mathcal{F}_{\mathcal{P}} \to \{0, 1\}$ that associates a truth-value 0 or 1 to each formula $f \in \mathcal{F}_{\mathcal{P}}$ and, in turn, fix the *meaning* of all formulas according to the rules

(1) $I(\neg f) = 1 - I(f)$,

(2) $I(f \wedge g) = \min(I(f), I(g))$,

(3) $I(f \vee g) = \max(I(f), I(g))$,

(4) $I(f \to g) = \max(1 - I(f), I(g))$,

(5) $I(f \leftrightarrow g) = |I(f) + I(g) - 1|$.

In particular, the truth-values of the composition of formulas depend *functionally* on the truth-values of the addends. This property is called *functionality*.

Let us simplify the notations writing $\mathcal{F}$, Ω and $\Pi(\Omega)$ instead of $\mathcal{F}_{\mathcal{P}}$, $\Omega_{\mathcal{P}}$ and $\Pi^f(\Omega_{\mathcal{P}})$, respectively.

In our case, as we already stated, the rules of computations of our logic system are modeled on the rules for partitions, since we have set for $f, g \in \mathcal{F}$ $f \simeq g$ iff $part(f) = part(g)$. So we assume that interpretations $I(f)$, $I(g)$ of two equivalent formulas ought to agree,

(E1) If $f, g \in \mathcal{F}$ and $f \simeq g$ then $I(f) = I(g)$.

Equivalently, the interpretations of f depend only on the normal form of f.

A basic goal in our logic is to implement in it the *principle of incompatibility*. This dictates a different set of rules for meaningful interpretations. What we want is that if the normal form of $f \in \mathcal{F}$ is atomic, $I(f) = 1$ and g is not "compatible" with f, neither $I(g) = 0$ nor $I(g) = 1$ be possible. Of course, this implies that an interpretation cannot be defined on the whole set $\mathcal{F}$ of formulas, but only on a proper subset $\mathcal{D} \subset \mathcal{F}$ of formulas that are compatible with f.

We deal with the requirement of incompatibility by saying that two formulas $f, g \in \mathcal{F}$ are *compatible*, and we write $f \pitchfork g$, if $part(f) \pitchfork part(g)$. The principle of incompatibility is then given in terms of partitions as follows:

(E2) Let $b \subset \Omega$ be a nonempty and proper subset of Ω and let $A \in \Pi(\Omega)$. If $I(b) = 1$ and $A \npitchfork \{b\}$, then neither $I(A) = 0$ nor $I(A) = 1$ is possible.

We are led to the following definition.

Definition 6. Let $h \subset \Omega$ be a proper, nonempty subset of Ω. A *standard interpretation* parametrized by h is a couple $(\mathcal{D}_h, I_h)$ where $\mathcal{D}_h \subset \Pi(\Omega)$ is such that $A \in \mathcal{D}_h$ if and only if $A \pitchfork \{h\}$, and I_h is a map $I_h : \mathcal{D}_h \to \{0, 1\}$ such that for every $A = \{a_1, a_2, \ldots a_n\} \in \mathcal{D}_h$ we have

$$I(A) = \begin{cases} 1 & \text{if } a_i = h \text{ for some } i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

We repeat again: Let $b \subset \Omega$ be a nonempty proper subset of Ω, and $(\mathcal{D}_b, I_b)$ the interpretation associated to b. Then for $A \in \Pi(\Omega)$ we have $I_b(A) = 1$ if and only if $b \in A$.

Example 7. (i) Let $\Omega = \{a, b, c, d\}$ and let $h = \{a\}$. Then the partitions that are compatible with $\{\{a\}\}$ are the nonempty disjoint families of subsets chosen from

$$\{a\}, \; \{b\}, \; \{c\}, \{d\}, \; \{b, c\}, \; \{b, d\}, \; \{c, d\}, \; \{b, c, d\}$$

and, for $A \in \mathcal{D}_h$, $I_h(A) = 1$ if and only if $\{a\} \in A$.

(ii) Let $\Omega = \{a, b, c, d\}$ and let $h = \{a, b\}$. Then the partitions compatible with $\{\{a, b\}\}$ are the nonempty families of subsets chosen from

$$\{a, b\}, \; \{c\}, \{d\}, \; \{c, d\}$$

and for $A \in \mathcal{D}_{\{a,b\}}$, $I_{\{a,b\}}(A) = 1$ if and only if $\{a, b\} \in A$.

Remark 8. In terms of formulas, the incompatibility principle says

> Let $f \in \mathcal{F}$ be such that its normal form is atomic and $I(f) = 1$, and let $g \in \mathcal{F}$ be incompatible with f, $g \not\pitchfork f$, then neither $I(g) = 0$ nor $I(g) = 1$ is possible.

and Definition 6 reads as: Let h be an atomic formula. We call *standard interpretation* parametrized by h a couple $(\mathcal{D}_h, I_h)$ where $\mathcal{D}_h \subset \mathcal{F}$ is the set of all formulas compatible with h, and $I_h : \mathcal{D}_h \to \{0, 1\}$ has the following

property. For $f \in \mathcal{D}_h$, let $f_1 \vee f_2 \vee \cdots \vee f_s$ be the normal form of f where $f_1, \ldots, f_s$ are atomic formulas. Then

$$I_h(f) = \begin{cases} 1 & \text{if } f_i = h \text{ for some } i = 1, \ldots, s, \\ 0 & \text{otherwise.} \end{cases}$$

When necessary we write also $(\mathcal{D}_h, I_h)$ to denote the interpretation parametrized by h.

One sees that standard interpretations $(\mathcal{D}, \mathcal{I})$ on $\Pi(\Omega)$ have the following properties:

(a) (SUPPORT PRINCIPLE) Let $b \subset \Omega$ be a nonempty proper subset of Ω and let $A \in \mathcal{D}$ be such that $A \perp \{b\}$. If $I(A) = 1$ then $I(b) = 0$.
(b) For every nonempty $b \subset \Omega$ and for every $A \in \Pi(\Omega)$ such that $A \perp \{b\}$ we have $A \in \mathcal{D}$ if $A \vee \{b\} \in \mathcal{D}$ equivalently, if $A \notin \mathcal{D}$ then $A \vee \{b\} \notin \mathcal{D}$.
(c) For every nonempty set $b \subset \Omega$ with $I(b) = 0$ and for every $A \in \mathcal{D}$ such that $A \perp \{b\}$ we have $I(A \vee \{b\}) = I(A)$.

Definition 9. A couple $(\mathcal{D}, I)$ is called an *interpretation* on $\Pi(\Omega)$ if it satisfies (E2) and (a) (b) and (c) above.

We have

Theorem 10. *Let $(\mathcal{D}, I)$ be an interpretation such that there exist $A, B \in \mathcal{D}$ such that $I(A) = 1$ and $I(B) = 0$. Then there exists a nonempty proper subset $h \subset \Omega$ such that $(\mathcal{D}, I)$ is the standard interpretation parametrized by h.*

Lemma 11. *Let $(\mathcal{D}, I)$ be as in the statement of Theorem 10. For any partition $A = \{a_1, \ldots, a_s\} \in \Pi(\Omega)$, set $A_i := \{a_i\}$. We have*

(1) $A \in \mathcal{D}$ *iff* $A_i \in \mathcal{D}$ *for every* $i = 1, \ldots, s$,
(2) $I(A) = 1$ *iff there exists* i_0 *such that* $I(A_{i_0}) = 1$,
(3) $I(A) = 0$ *iff* $I(A_i) = 0$ *for every* $i = 1, \ldots, s$,
(4) *Let $B \in \Pi(\Omega)$ be orthogonal to A, $A \perp B$. Then*
   (a) *if $A, B \in \mathcal{D}$, $I(A) = 0$ and $I(B) = 0$, then $A \vee B \in \mathcal{D}$ and $I(A \vee B) = 0$,*
   (b) *if $A, B \in \mathcal{D}$ and $I(A) = 1$, then $I(B) = 0$ and $I(A \vee B) = 1$,*
   (c) *if $A \notin \mathcal{D}$, then $A \vee B \notin \mathcal{D}$.*

Proof.

(i) follows applying inductively (b).

(ii) Assume that there exists i_0 such that $I(A_{i_0}) = 1$, say $i_0 = 1$. Then by (c) $I(A_1 \vee A_2) = 1$, $I((A_1 \vee A_2) \vee A_3) = 1$ and so on. After a finite number of steps we get $I(A) = 1$. Conversely, assuming $I(A) = 1$, in particular $A \in \mathcal{D}$ and by (b) $A_i \in \mathcal{D}$ for every i. Were $I(A_i) = 0 \; \forall i$, (c) would give $I(A) = 0$, a contradiction, and therefore there exists i_0 such that $I(A_{i_0}) = 1$.

(iii) If $I(A_{i_0}) = 1$ for some i_0, then $I(A) = 1$ by (c). If for all i we have $I(A_i) = 0$, then $I(A) = 0$ by (c).

(iv) Trivial.

Proof of Theorem 10. From the assumptions there exists a partition A such that $I(A) = 1$. Decomposing A as $A = \{a_1, \ldots, a_s\}$, one gets by (i) and (ii) of Lemma 11 that $\{a_i\} \in \mathcal{D} \; \forall i$ and that $I(\{a_i\}) = 1$ for some i_0. Set $h := a_{i_0}$. We now prove that $(\mathcal{D}, I)$ is the interpretation parametrized by h.

Let $B = \{b_1, \ldots, b_r\} \in \Pi(\Omega)$. Let us look at the relations between B and $\{h\}$. We have only three possibilities;

(1) We have $b_{j_0} = h$ for some j_0. In this case $I(\{b_{j_0}\}) = 1$ hence $B \in \mathcal{D}$ and $I(B) = 1$ by (i) and (ii) of Lemma 11. On the other hand, $B \in \mathcal{D}_h$ and $I_h(B) = 1$ by definition.

(2) We have $b_j \cap h = \emptyset \; \forall j$. In this case by (i) and (iii) of Lemma 11 we get $I(\{b_j\}) = 0 \; \forall j$ and (i) and (iii) of Lemma 11 yields $I(B) = 0$. On the other hand, by definition $B \in \mathcal{D}_h$ and $I_h(B) = 0$.

(3) $b_{j_1} \cap h \neq \emptyset$. This means that $\{b_{j_1}\} \not\pitchfork \{h\}$ hence by (E2) $\{b_{j_1}\} \notin \mathcal{D}$, hence, by (i) of Lemma 11, $B \notin \mathcal{D}$. On the other hand, $B \notin \mathcal{D}_h$ by definition.

Thus, for any partition B we have proved that $B \in \mathcal{D}$ if and only if $B \in \mathcal{D}_h$ and in this case $I(B) = I(B_h)$. This completes the proof.

We formulate the relations of interpretations $(\mathcal{D}, I)$ on $\mathcal{F}$ in terms of the logical operators $\neg, \sqcup, \wedge$ and $\vee$ passing from formulas to partitions and then at their interpretations. The results are summarized in the tables below. We observe that no functional dependence exists. The result of the logic operations is often determined by the internal structure of the propositions.

A	$\neg A$	$\sqcup A$
$I(A) = 1$	$I(\neg A) = 0$	$I(\sqcup A) = 1$ or $\sqcup A \notin \mathcal{D}$
$I(A) = 0$	$I(\neg A) = 1$ or $\neg A \notin \mathcal{D}$	$I(\sqcup A) = 0$
$A \notin \mathcal{D}$	$I(\neg A) = 0$ or $\neg A \notin \mathcal{D}$	$I(\sqcup A) = 1$ or $\sqcup A \notin \mathcal{D}$

This can be shortened as

A	$\neg A$	$\sqcup A$
1	0	1 or $\notin \mathcal{D}$
0	1 or $\notin \mathcal{D}$	0
$\notin \mathcal{D}$	0 or $\notin \mathcal{D}$	1 or $\notin \mathcal{D}$

Moreover for $A \neq \emptyset$, $B \neq \emptyset$, $A \neq B$ we have

A	B	$A \wedge B$	$A \vee B$	$A \sqcup B$
1	1	1	1	$\notin \mathcal{D}$
1	0	0	1	$\notin \mathcal{D}$
0	0	0	0	0
$\notin \mathcal{D}$	1	1 or $\notin \mathcal{D}$	1 or $\notin \mathcal{D}$	1 or $\notin \mathcal{D}$
$\notin \mathcal{D}$	$\notin \mathcal{D}$	1 or 0 or $\notin \mathcal{D}$	1 or $\notin \mathcal{D}$	1 or $\notin \mathcal{D}$
$\notin \mathcal{D}$	0	0	1 or $\notin \mathcal{D}$	$\notin \mathcal{D}$

Remark 12. An interpretation $(\mathcal{D}, \mathcal{I})$ can be seen as a three-valued map $\widetilde{I} : \Pi(\Omega) \to \{0, 1, NULL\}$ setting $\widetilde{I}(A) = 0$ (1) if $I(A) = 0$ (1, respectively) and $\widetilde{I}(A) = NULL$ if $A \notin \mathcal{D}$. In particular, the previous tables can be rewritten using $\widetilde{I}$ instead of I and $NULL$ instead of $\notin \mathcal{D}$ as in Bochvar's logic. However, our logic differs from Bochvar's three valued logic, see [3]. For instance, in the contest of Bochvar logic we have

$$\neg NULL = NULL,$$
$$(A = NULL \text{ or } B = NULL) \text{ implies } A \wedge B = A \vee B = NULL,$$

while in our logic, we may have

$$\neg NULL = 0, \qquad NULL \wedge 1 = 1, \qquad NULL \wedge NULL = 1,$$
$$NULL \wedge NULL = 0, \qquad NULL \vee 1 = 1, \qquad \text{etc.}$$

Since our logic differs drastically from databases's (or Bochvar's) logic, we preferred to present the interpretations as two valued maps defined in a subdomain of $\Pi(\Omega)$.

4. Semantics

As we stated, the incompatibility principle implies that for any $f \in \mathcal{F}$ there exists an interpretation $(\mathcal{D}, I)$ such that $f \notin \mathcal{D}$. This means that no classical tautology exists, and we have to redefine this concept as well as the related concept of *(semantic) consequence*.

Definition 13. We define the following.

(1) A formula $f \in \mathcal{F}$ is a *tautology*, and we write $\models f$, if for every interpretation $(\mathcal{D}, I)$ such that $f \in \mathcal{D}$ we have $I(f) = 1$.
(2) Let $f, g \in \mathcal{F}$. We say that g is a *(semantic) consequence* of f, and we write $f \models g$, if for every interpretation $(\mathcal{D}, I)$ for which $I(f) = 1$ we have $g \in \mathcal{D}$ and $I(g) = 1$.
(3) Let $f, g \in \mathcal{F}$. We say that f and g are *(semantically) equivalent*, and we write $f \equiv g$, if for every interpretation $(\mathcal{D}, I)$ we have $f \in \mathcal{D}$ if and only if $g \in \mathcal{D}$ and, in this case, $I(f) = I(g)$.

The following proposition translates the previous relations on formulas into properties of the corresponding partitions and states a few useful properties.

Proposition 14. *Let $f, g \in \mathcal{F}$. Then*

(1) $\models f$ *iff* $spt(part(f)) = \Omega$,
(2) $f \models g$ *iff* $part(f) \subset part(g)$,
(3) $f \equiv g$ *iff* $f \models g$ *and* $g \models f$,
(4) $f \models g$ *implies* $f \pitchfork g$,
(5) *if* $f \models g$ *and* $g \models h$, *then* $f \models h$,
(6) *if* $f \equiv g$ *and* $g \equiv h$, *then* $f \equiv h$.

Proof. All the statements are simple consequences of definitions. Through the proof we set $A = part(f)$ and $B = part(g)$.

(i) Assume that $I(f) = 1$ for every interpretation $(\mathcal{D}, I)$ such that $f \in \mathcal{D}$. We want to prove that $spt A = \Omega$. Assuming by contradiction $spt A \neq \Omega$, one finds a nonempty proper subset $b \subset \Omega$ such that $spt A \cap b = \emptyset$. Consider the interpretation $(\mathcal{D}_b, I_b)$ where $\mathcal{D}_b$ is the set of partitions compatible with $\{b\}$. Then $A \in \mathcal{D}_b$ and $I_b(A) = 0$, a contradiction.

Conversely, for any nonempty proper subset $b \subset \Omega$ such that $A \in \mathcal{D}_b$, we must have $\{b\} \perp A$. Since we assume $spt A = \Omega$, we may have $\{b\} \subset A$, hence $I_b(A) = 1$.

(ii) Let $b \subset \Omega$ be a nonempty proper subset $b \subset \Omega$ and let $(\mathcal{D}, I)$ be the interpretation parametrized by b. Recall that for any $C \in \Pi(\Omega)$ $I_b(C) = 1$ if and only if $b \in C$.

If $f \models g$ then we have $I_b(B) = 1$ if $I_b(A) = 1$ for any nonempty proper subset b of Ω, hence $part(f) \subset part(g)$. Conversely, if $I_b(A) = 1$, then $b \in A$, hence $b \in B$ thus $I_b(B) = 1$.

(iii), (iv), (v) and (vi) then follows from (i) and (ii).

In classical propositional logic, one introduces the *implication* $f \rightarrow g$ as the formula $(\neg f) \vee g$ and one proves the *(semantic) deduction theorem*:

$$f \models g \qquad \text{if and only if} \qquad \models (f \rightarrow g).$$

In our extended propositional logic, the semantic deduction theorem is not true. Therefore we begin by analyzing the formula $\models (f \rightarrow g)$.

Definition 15. Let $f, g \in \mathcal{F}$.

(1) $(f \rightarrow g)$ denotes the formula $(\neg f \vee g)$,
(2) $(f \leftrightarrow g)$ denotes the formula $(f \wedge g) \vee (\neg f \wedge \neg g)$.

In terms of partitions, we have

Proposition 16. *Let $f, g \in \mathcal{F}$. Then*

(1) $\models (f \rightarrow g)$ *if and only if* $spt(part(f)) \subset spt(part(g))$,
(2) $\models (f \leftrightarrow g)$ *if and only if* $spt(part(f)) = spt(part(g))$,
(3) $f \models g$ *if and only if* $f \pitchfork g$ *and* $\models (f \rightarrow g)$,
(4) $f \equiv g$ *if and only if* $f \pitchfork g$ *and* $\models (f \leftrightarrow g)$,
(5) *the relations* $\models (f \rightarrow g)$ *and* $\models (f \leftrightarrow g)$ *are transitive and reflexive; moreover,* $\models (f \leftrightarrow g)$ *is a symmetric relation,*
(6) $\models (f \leftrightarrow g)$ *if and only if* $\models (f \rightarrow g)$ *and* $\models (g \rightarrow f)$.

Proof. Set $A := part(f)$ and $B = part(g)$.

(i) By definition $\models (g \to f)$ means that $\neg f \vee g$ is a tautology, hence by (i) Proposition 14 $spt(\neg A \vee B) = \Omega$. This last statement is equivalent to $\Omega \setminus spt(A) \cup sptB = \Omega$, that is, $sptA \subset sptB$,

(ii) By definition $\models (f \leftrightarrow g)$ means that the formula $(f \wedge g) \vee (\neg f \wedge \neg g)$ is a tautology, so that by (i)

$$spt((A \wedge B) \vee (\neg A \wedge \neg B)) = \Omega$$

or, equivalently,

$$(sptA \cap sptB) \cup ((\Omega \setminus sptA) \cap (\Omega \setminus sptB)) = \Omega$$

which in turn is equivalent to $sptA = sptB$.

(iii), (iv) and (v) then follows from (i) and (ii) taking also into account Proposition 14.

Propositions 14 and 16 allow us to express the differences between the two semantic consequence operators $\models$ and $\models (\cdot \to \cdot)$ in terms of the associated partitions. We have $f \models g$ if and only if $part(f) \subset part(g)$, while $\models (f \to g)$ if and only if $spt(part(f)) \subset spt(part(g))$. Notice that the relation $f \models g$ is false for incompatible formulas while the relation $\models (f \to g)$ may be true. On the other hand, the relation $\models (f \to g)$ is, in a sense, almost trivial if compared with $f \models g$. In fact, since for any partition $A \in \Pi(\Omega)$ we have $sptA = spt(\sqcup A)$, hence

Proposition 17. $\models (f \leftrightarrow \sqcup f) \; \forall f \in \mathcal{F}$.

Let us introduce the subclass $\mathcal{F}^a \subset \mathcal{F}$ of formulas for which the associated normal formulas are atomic. By definition, $f \in \mathcal{F}^a$ if and only if f does not contain the operator $\vee$. In terms of partitions, $f \in \mathcal{F}^a$ if and only if $part(f)$ is an atomic partition. Observe that for an atomic partition $A = \{a\}$ we have

$$A = \{sptA\}, \qquad \text{and} \qquad a = spt(\{a\}).$$

so that for $A = \{a\}$ and $B = \{b\}$ we have

$$A = B \qquad \text{if and only if} \qquad sptA = sptB.$$

Therefore, compare Propositions 14 and 16,

Proposition 18. $\forall f, g \in \mathcal{F}^a$

$$f \equiv g \qquad \text{if and only if} \qquad \models (f \leftrightarrow g).$$

Moreover, our propositional logic behaves on $\mathcal{F}^a$ as a classical propositional logic with respect to the operations $\neg$, $\wedge$ and $\sqcup$. This follows looking once more at the associated partitions that are atomic. Denote by $\Pi^a(\Omega)$ the class of atomic partitions. The map i:$\mathcal{P}(\Omega) \to \Pi(\Omega)$ given by $a \mapsto \{a\}$, which is a trivially a bijection onto $\Pi^a(\Omega)$, is in fact an isomorphism between the algebra $\mathcal{P}(\Omega)$ with the standard operations of $\neg$, $\cap$ and $\cup$ and $\Pi^a(\Omega)$ with the operations $\neg$, $\wedge$ and $\sqcup$, respectively.

5. Contexts

Recall that a *context* of partitions is a maximal set $\mathcal{H} \subset \Pi(\Omega)$ of compatible partitions. Similarly, one defines a *context of formulas* $\mathcal{K} \subset \mathcal{F}$ as a maximal set of compatible formulas. Of course, the two definitions above are related by the map *part*. In fact, $part(\mathcal{K})$ is a context of partitions if $\mathcal{K}$ is a context of formulas, and $part^{-1}(\mathcal{H})$ is a context of formulas if $\mathcal{H}$ is a context of partitions. Moreover, by maximality,

$$\mathcal{K} = part^{-1}(part(\mathcal{K}))$$

for any context of formulas $\mathcal{K}$.

We now show that in every context our propositional logic behaves as a classical propositional logic referring once again at the associated partitions.

A context of partitions $\mathcal{K}$ contains a unique complete (in general, denumerable) partition $\mathcal{U}_\mathcal{K}$, that we called the *universe* of $\mathcal{K}$. By comparing the definitions,

$$A \in \mathcal{K}, \qquad A \pitchfork \mathcal{U}_\mathcal{K} \qquad and \qquad A \subset \mathcal{U}_\mathcal{K},$$

are equivalent. Moreover for $A, B \in \mathcal{K}$, we have

$$A \subset B \qquad \text{if and only if} \qquad sptA \subset sptB. \tag{5.1}$$

There is a one-to-one correspondence $j : \mathcal{K} \to \mathcal{P}(\mathcal{U}_\mathcal{K})$ between $\mathcal{K}$ and the parts of $\mathcal{U}_\mathcal{K}$. Moreover, for $A, B \in \mathcal{K}$, we have

$$j(A \wedge B) = j(A) \cap j(B), \qquad j(A \vee B) = j(A) \cup j(B)$$

so that the two operations $\wedge$ and $\vee$ are each other distributive within $\mathcal{K}$.

In general, for $A \in \mathcal{K}$ it may happen $\neg A \notin \mathcal{K}$, in particular, $\neg$ is not defined within $\mathcal{K}$. We introduce a new operator, denoted by $\neg_\mathcal{K}$ and called the *negation operator within* $\mathcal{K}$, by

$$\neg_\mathcal{K} A := \neg A \wedge \mathcal{U}_\mathcal{K} \qquad \forall A \in \mathcal{K}.$$

It is easy to check that $\neg_\mathcal{K} A \in \mathcal{K} \; \forall A \in \mathcal{K}$ and that

$$\neg_\mathcal{K} \neg_\mathcal{K} A = A \qquad \text{and} \qquad \neg A = \sqcup(\neg_\mathcal{K} A) \qquad \forall A \in \mathcal{K}.$$

Moreover,

$$\neg_\mathcal{K}(A \wedge B) = (\neg_\mathcal{K} A) \vee (\neg_\mathcal{K} B),$$
$$\neg_\mathcal{K}(A \vee B) = (\neg_\mathcal{K} A) \wedge (\neg_\mathcal{K} B).$$

Thus $\mathcal{K}$ with the three operations of $\neg_K$, $\wedge$, $\vee$ becomes an algebra isomorphic to the standard algebra of the subsets of the universe $\mathcal{U}_\mathcal{K}$. Thus we conclude

Proposition 19. *Let $\mathcal{K} \subset \mathcal{F}$ be a context of formulas. Then the operations $\neg_\mathcal{K}$, $\wedge$ and $\vee$ acts on $\mathcal{K}$ as in a classical propositional logic.*

It remains to discuss the meaning of a *semantic interpretation within a context* $\mathcal{K}$.

Definition 20. Let $\mathcal{K} \subset \Pi(\Omega)$ be a context of partitions. We say that an interpretation $(\mathcal{D}, I)$ parametrized by a nonempty proper subset $b \subset \Omega$ is *compatible* with $\mathcal{K}$, and we write $(\mathcal{D}, I) \pitchfork \mathcal{K}$, if $\{b\} \in \mathcal{K}$.

Notice that an interpretation $(\mathcal{D}, I)$ parametrized by a nonempty proper subset $b \subset \Omega$ is compatible with $\mathcal{K}$ if and only if one of the following two conditions holds:

(1) $\mathcal{K} \subset \mathcal{D}$,
(2) $I(\mathcal{U}_\mathcal{K}) = 1$, where $\mathcal{U}_\mathcal{K}$ is the universe of $\mathcal{K}$.

Definition 21. Let $\mathcal{K} \subset \mathcal{F}$ be a context. We say that

(1) $f \in \mathcal{K}$ is a $\mathcal{K}$-tautology, and we write $\models_{\mathcal{K}} f$, if $I(f) = 1$ for every interpretation $(\mathcal{D}, I)$ compatible with $\mathcal{K}$, $I \Vdash \mathcal{K}$,

(2) g is a consequence of f within $\mathcal{K}$, $f \models_{\mathcal{K}} g$, if $I(f) = 1$ implies $I(g) = 1$ for every interpretation $(\mathcal{D}, I)$ compatible with $\mathcal{K}$,

(3) f implies g within $\mathcal{K}$, and we write $f \to_{\mathcal{K}} g$ if the formula $(\neg_{\mathcal{K}} f) \lor g$ is a $\mathcal{K}$-tautology.

Proposition 22. *Let $\mathcal{K} \subset \mathcal{F}$ be a context and let $f, g \in \mathcal{K}$. Then we have the following.*

(1) *f is a $\mathcal{K}$-tautology if and only if $spt(part(f)) = \Omega$.*
(2) *$f \models_{\mathcal{K}} g$ if and only if $part(f) \subset part(g)$.*

Therefore for $f, g \in \mathcal{K}$

(a) *$\models f$ if and only if $\models_{\mathcal{K}} f$,*
(b) *$f \models g$ if and only if $f \models_{\mathcal{K}} g$,*
(c) *The deduction theorem holds within $\mathcal{K}$: $f \models g$ if and only if $f \to_{\mathcal{K}} g$ is a tautology.*

Finally, for every $f, g \in \mathcal{K}$ we have $f \models g$ if and only if $f \models_{\mathcal{K}} g$.

Proof. Through the proof let denote by $\mathcal{H} := part(\mathcal{K})$ and by $\mathcal{U}_{\mathcal{H}} = \{u_i\}_{i \in \mathcal{I}}$ the universe of $\mathcal{H}$.

(i) If $spt(B) \neq \Omega$, there exists i_0 such that $u_{i_0} \cap spt(B) = \emptyset$. Thus, for the interpretation $(\mathcal{D}, I)$ parametrized by u_{i_0} we have $I(B) = 0$. Conversely, if $spt(B) = \Omega$ we get $B = \mathcal{U}_{\mathcal{H}}$ since $B \subset \mathcal{U}_{\mathcal{K}}$, hence $I(B) = 1$ for any $(\mathcal{D}, \mathcal{I}) \Vdash \mathcal{H}$.

(ii) Let $A = part(f)$ and $B = part(g)$. Since $f, g \in \mathcal{K}$, $A, B \subset \mathcal{U}_{\mathcal{H}}$. For any proper nonempty subset $b \in A$, let $(\mathcal{D}, I)$ the interpretation parametrized by b. Then $\{b\} \subset \mathcal{K}$ and $I(A) = 1$. Therefore $f \models_{\mathcal{K}} g$ yields $I(B) = 1$ which in turns means $b \in B$. Thus we conclude that $A \subset B$.

Conversely, assume $A \subset B$. For any nonempty subset $b \subset \Omega$ such that $\{b\} \subset \mathcal{K}$, let $(\mathcal{D}, I)$ the interpretation parametrized by b. If $I(A) = 1$, then $b \in A$ and, consequently, $b \in B$ which in fact is equivalent to $I(B) = 1$.

Claims (a) (b) and (c) then follow from (i) and (ii) taking also into account Propositions 14 and 16 and equation (5.1).

Finally, let us prove the last statement. Let $A := part(f)$, $B = part(g)$. By (ii) we have $A \subset B$. Choose

$$\mathcal{U} := B \cup \neg B, \qquad \mathcal{K} := \left\{ f \in \mathcal{F} \,\middle|\, part(f) \subset \mathcal{U} \right\}.$$

Then $f, g \in \mathcal{K}$ and $A \subset B \subset \mathcal{U}$, i.e., $f \models_{\mathcal{K}} g$.

6. Comments

As stated in the introduction, our extended propositional logic (EPL) has similarities to other logical systems, for instance it is non functional and may be seen as a 3-valued logic. We would like to conclude with a few remarks concerning our logic in comparison to some of the most-known logics, [9], [3], [8], [4], [10].

Distinctive features of our extended propositional logic are:

- The logical operators in EPL are not functionally defined (there are no truth-tables for them).

- EPL contains a new type of logical operator $\sqcup$ of "indistinguishable" (irreducible) disjunction, which does not exists in other logics known to the authors. In this way completely new propositions can be constructed, e.g.,

$$(e_1 \sqcup e_2) \vee (e_3 \sqcup e_4) \vee e_5 = \{e_1, e_2, e_3, e_4, e_5\}$$

which do not seem to be considered in other logic systems, again to the author's knowledge.

- In EPL there exist incompatible propositions. If f and g are two incompatible propositions and if f is atomic and is evaluated as true, $I(f) = 1$, then g can be neither true nor false, i.e., g cannot be evaluated. This concept is borrowed from quantum mechanics where situations with the "which way information " being available or not available are not compatible. The examples of incompatible propositions are simple: $f \sqcup g$ is incompatible with each of f, g and $f \vee g$.

• The negation $\neg$ has several unusual properties in EPL:

$$\neg 1 = 0,$$
$$\neg 0 = 1 \text{ or } NULL,$$
$$\neg NULL = 0 \text{ or } NULL.$$

This may hint to the intuitionist logic where the main assumption is that the law of excluded middle does not hold, i.e., that $f \vee \neg f$ is not a tautology. In general, in EPL $f \vee \neg f$ is a tautology. Moreover, models of intuitionistic logic (Kripke's models) appears to us to be quite different from our model based on the concept of partition.

With respect to 3-valued logics (such that Lukasiewitz's or Bochvar's logics), first, most of them are functional and even for the non-functional ones known to the authors, see e.g. [2], the usual properties of the negation appear to be different from the negation operator $\neg$ in EPL.

Finally, let us compare our logic to Rough Set Theory that has been developed to tackle the idea of limited knowledge, see e.g. [1]. Roughly, let us assume that we have a set M, the elements of which have some properties (called attributes). It is clear that the system of properties (the "knowledge") relates to a decomposition of M into parts (of elements with the same properties), say $M = \cup_{i=1}^{k} M_i$, called a knowledge base. For any subset of M then one defines the lower and upper approximations of A by

$$A^{low} = \bigcup \left\{ M_i \,\Big|\, M_i \subset A \right\},$$
$$A^{upp} = \bigcup \left\{ M_i \,\Big|\, M_i \cap A \neq \emptyset \right\},$$

The couple (A^{low}, A^{upp}) can be considered as the image of A through the knowledge $M = \cup_i M_i$.

In our EPL we have a similar construction. Let us consider Ω and a decomposition $\Omega = \cup_{i=1}^{k} u_i$ into disjoint sets. Then the context (as a partition) created by this decomposition is

$$\mathcal{U} = \{u_1, \ u_2, \ \ldots, \ u_k\}.$$

But the meaning of $\mathcal{U}$ in EPL is different compared to the meaning of the partition $\{M_i\}$ in the rough set theory.

Let us consider a simple example. Set $m_1 = \{e_1, e_2\}$, $m_2 = \{e_3, e_4\}$, $M = m_1 \cup m_2$. A better knowledge may produce

$$\{e_1\} \cup \{e_2\} \cup \{e_3, e_4\}$$

and this better knowledge is compatible with the previous knowledge. In other words, if the initial knowledge cannot find the difference between e_1 and e_2, we are assuming that a better knowledge can differentiate between them. A better knowledge thus extends the original knowledge and is compatible with it. The partial knowledge is described by the distinguishable disjunction $e_1 \vee e_2$. In EPL the situation is different, there is in principle no better knowledge of $\{e_1, e_2\} \cup \{e_3, e_4\}$ since the indistinguishable disjunction $e_1 \sqcup e_2$ is incompatible with e_1 and e_2 (and also with $e_1 \vee e_2$), and this is one of the main features of the logic we discussed here.

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