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WIRSING-TYPE INEQUALITIES

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Abstract

Wirsing's theorem on approximating algebraic numbers by algebraic numbers of bounded degree is a generalization of Roth's theorem in Diophantine approximation. We study variations of Wirsing's theorem where the inequality in the theorem is strengthened, but one excludes a certain easily-described special set of approximating algebraic points.

1. Introduction

Roth's fundamental result in Diophantine approximation describes how closely an algebraic number may be approximated by rational numbers:

Theorem 1.1 (Roth [9]). *Let $\alpha \in \overline{\mathbb{Q}}$ be an algebraic number. Let $\epsilon > 0$. Then there are only finitely many rational numbers $\frac{p}{q} \in \mathbb{Q}$ satisfying*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^{2+\epsilon}}.$$

Roth's theorem can be extended [6, 8] to an arbitrary fixed number field k (in place of $\mathbb{Q}$) and to allow finite sets of absolute values (including non-archimedean ones). A general statement of Roth's theorem, using the language of heights (see Section 2 for the definitions), is the following.

Theorem 1.2. *Let S be a finite set of places of a number field k. Let $P_1, \dots, P_q \in \mathbb{P}^1(k)$ be distinct points, $D = \sum_{i=1}^q P_i$, and $\epsilon > 0$. Then for all*

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but finitely many points $P \in \mathbb{P}^1(k) \setminus \mathrm{Supp} D$*,*

$$m_{D,S}(P) = \sum_{i=1}^{q} \sum_{v \in S} h_{P_i,v}(P) < (2+\epsilon)h(P).$$

We note that there is no loss of generality in the assumption that $P_1, \ldots, P_q$ are k-rational (see [13, Remark 2.2.3]).

Instead of taking the approximating elements from a fixed number field, a natural variation on Roth's theorem is to consider approximation by algebraic numbers of bounded degree. In this direction, Wirsing [14] proved a generalization of Roth's theorem, which we state in a general form.

Theorem 1.3 (Wirsing)**.** *Let* S *be a finite set of places of a number field* k*. Let* $P_1, \ldots, P_q \in \mathbb{P}^1(k)$ *be distinct points and let* $D = \sum_{i=1}^q P_i$*. Let* $\epsilon > 0$ *and let* d *be a positive integer. Then for all but finitely many points* $P \in \mathbb{P}^1(\overline{k}) \setminus \mathrm{Supp} D$ *satisfying* $[k(P):k] \leq d$*,*

$$m_{D,S}(P) < (2d+\epsilon)h(P).$$

Taking $d = 1$ in Wirsing's theorem recovers Roth's theorem. For $t \leq 2d$ and D, S, k, as in Theorem 1.3, the set

$$\{P \in \mathbb{P}^1(\overline{k}) \mid [k(P):k] = d, m_{D,S}(P) \geq th(P)\} \tag{1.1}$$

may be infinite. A natural way to obtain algebraic points $P \in \mathbb{P}^1(\overline{k})$ with $[k(P):k] = d$ is to pull back k-rational points via a degree d morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$. The following result may be used to classify those morphisms ϕ which contribute infinitely many points in this way to the set (1.1).

Theorem 1.4. *Let* S *be a finite set of places of a number field* k *containing the archimedean places. Let* $P_1, \ldots, P_q \in \mathbb{P}^1(k)$ *be distinct points and let* $D = \sum_{i=1}^q P_i$*. Let* $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ *be a morphism over* k *of degree* d*. Let* $\phi(\{P_1, \ldots, P_q\}) = \{Q_1, \ldots, Q_r\}$ *and let*

$$n_i = |\phi^{-1}(Q_i) \cap \{P_1, \ldots, P_q\}|, \quad i = 1, \ldots, r.$$

Rearrange the indices so that $n_1 \geq n_2 \geq \cdots \geq n_r$*.*

1. *Suppose that* $|S| > 1$. *For some constant* C, *the inequality*

$$m_{D,S}(P) > (n_1 + n_2)h(P) - C$$

holds for infinitely many points $P \in \phi^{-1}(\mathbb{P}^1(k))$.

2. *Let* $\epsilon > 0$. *The inequality*

$$m_{D,S}(P) < (n_1 + n_2 + \epsilon)h(P)$$

holds for all but finitely many points $P \in \phi^{-1}(\mathbb{P}^1(k))$ *with* $[k(P) : k] = d$.

After composing ϕ with an automorphism, we can always assume in Theorem 1.4 that $Q_1 = 0$ and $Q_2 = \infty$. Then Theorem 1.4 motivates making the following definitions. Let k be a number field, $P_1, \ldots, P_q \in \mathbb{P}^1(k)$ be distinct points, and $D = \sum_{i=1}^{q} P_i$. Let d be a positive integer and let t be a positive real number. Let $\mathrm{End}_k(\mathbb{P}^1)$ be the set of k-morphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^1$. Define

$$\begin{aligned} \Phi(D, d, t, k) &= \{\phi \in \mathrm{End}_k(\mathbb{P}^1) \mid \deg \phi \le d, |\phi^{-1}(\{0, \infty\}) \cap \mathrm{Supp} D| \ge t\}, \\ Z(D, d, t, k) &= \bigcup_{\phi \in \Phi(D,d,t,k)} \phi^{-1}(\mathbb{P}^1(k)). \end{aligned}$$

It is then natural to ask the following question.

Question 1.5. *Does the inequality*

$$m_{D,S}(P) < th(P) \tag{1.2}$$

hold for all but finitely many points $P \in \mathbb{P}^1(\overline{k}) \backslash Z(D, d, t, k)$ *satisfying* $[k(P) : k] \le d$?

We will show that Question 1.5 has a positive answer when $d = 2$.

Theorem 1.6. *Let* S *be a finite set of places of a number field* k. *Let* $P_1, \ldots, P_q \in \mathbb{P}^1(k)$ *be distinct points, let* $D = \sum_{i=1}^{q} P_i$, *and let* t *be a positive real number. Then the inequality*

$$m_{D,S}(P) < th(P)$$

holds for all but finitely many points $P \in \mathbb{P}^1(\overline{k}) \setminus Z(D, 2, t, k)$ *satisfying* $[k(P) : k] \le 2$.

More generally, we will show that Question 1.5 has a positive answer if either $t \leq d+1$ (Lemma 4.5) or $t > 2d-1$:

Theorem 1.7. *Let S be a finite set of places of a number field k. Let $P_1, \dots, P_q \in \mathbb{P}^1(k)$ be distinct points and let $D = \sum_{i=1}^q P_i$. Let d be a positive integer and let $t > 2d-1$ be a real number. Then the inequality*

$$m_{D,S}(P) < th(P)$$

holds for all but finitely many points $P \in \mathbb{P}^1(\overline{k}) \setminus Z(D,d,t,k)$ satisfying $[k(P) : k] \leq d$. Furthermore, in this case $\Phi(D,d,t,k)$ is a finite set and $Z(D,d,t,k) = \bigcup_{\phi \in \Phi(D,d,t,k)} \phi^{-1}(\mathbb{P}^1(k))$ is a finite union of sets of the form $\phi^{-1}(\mathbb{P}^1(k))$.

Thus, after excluding points of a special and easily described form, the inequality in Wirsing's theorem may be improved to

$$m_{D,S}(P) < (2d - 1 + \epsilon)h(P).$$

In general, we will see that Question 1.5 has a negative answer. By carefully studying the exceptional hyperplanes in the Schmidt Subspace Theorem in dimension three, we obtain a precise answer to Question 1.5 when $d = 3$, showing that in this case the question has a positive answer if $t > \frac{9}{2}$, but (at least for some choices of the parameters) it has a negative answer when $4 < t < \frac{9}{2}$.

Theorem 1.8. *Let k be a number field. Let $P_1, \dots, P_q \in \mathbb{P}^1(k)$ be distinct points and let $D = \sum_{i=1}^q P_i$. Let S be a finite set of places of k containing the archimedean places and let t be a real number.*

1. *If $t > \frac{9}{2}$, then the inequality*

$$m_{D,S}(P) < th(P)$$

   holds for all but finitely many points $P \in \mathbb{P}^1(\overline{k}) \setminus Z(D,3,t,k)$ satisfying $[k(P) : k] \leq 3$.
2. *If $4 < t < \frac{9}{2}$, $|S| > 2$, and $q = 6$, then there are infinitely many points $P \in \mathbb{P}^1(\overline{k}) \setminus Z(D,3,t,k)$ satisfying $[k(P) : k] = 3$ and*

$$m_{D,S}(P) > th(P).$$

From another viewpoint, Question 1.5 may be viewed as asking a quantitative generalization of results in [7], where integral points of bounded degree on affine curves were studied. In [7], affine curves with infinitely many integral points of degree d (over some number field) were characterized as follows.

Theorem 1.9. *Let $C \subset \mathbb{A}^n$ be a nonsingular affine curve defined over a number field k. Let $\tilde{C}$ be a nonsingular projective completion of C and let $(\tilde{C} \setminus C)(\overline{k}) = \{P_1, \ldots, P_q\}$. Let d be a positive integer. Let $\overline{\mathcal{O}}_{k,S}$ denote the integral closure of $\mathcal{O}_{k,S}$ in $\overline{k}$. Then there exists a finite extension L of k and a finite set of places S of L such that the set*

$$\{P \in C(\overline{\mathcal{O}}_{L,S}) \mid [L(P) : L] \le d\}$$

is infinite if and only if there exists a morphism $\phi : \tilde{C} \to \mathbb{P}^1$, over $\overline{k}$, with $\deg \phi \le d$ and $\phi(\{P_1, \ldots, P_q\}) \subset \{0, \infty\}$.

When $\tilde{C} = \mathbb{P}^1$ the following stronger result was proven.

Theorem 1.10. *Let S be a finite set of places of a number field k containing the archimedean places. Let $P_1, \ldots, P_q \in \mathbb{P}^1(k)$ be distinct points, let $D = \sum_{i=1}^q P_i$, and let $C = \mathbb{P}^1 \setminus \{P_1, \ldots, P_q\}$. Let d be a positive integer. For any set of (D, S)-integral points $R \subset \{P \in C(\overline{k}) \mid [k(P) : k] \le d\}$, the set $R \setminus Z(D, d, q, k)$ is finite.*

Note that $\Phi(D, d, q, k)$ is just the set of k-endomorphisms ϕ of $\mathbb{P}^1$ satisfying $\deg \phi \le d$ and $\phi(\{P_1, \ldots, P_q\}) \subset \{0, \infty\}$. In particular, $\Phi(D, d, q, k)$ is empty if $q \ge 2d + 1$. From the definition, R is a set of (D, S)-integral points if and only if

$$m_{D,S}(P) = (\deg D)h(P) + O(1)$$

for all $P \in R$. For some finite set of places $T \supset S$, we even have (using the definition of $m_{D,T}$ in Section 2) $m_{D,T}(P) = (\deg D)h(P)$ for all $P \in R$. Thus, Theorem 1.10 is equivalent to Question 1.5 having a positive answer for $t = \deg D$. In this sense, Question 1.5 asks a quantitative generalization of Theorem 1.9 (for the projective line) and Theorem 1.10.

Similar to Question 1.5, the analogue of Theorem 1.9 for algebraic points of bounded degree on curves holds only for small d ($d \le 3$) as we now discuss. Let C be a nonsingular projective curve defined over a number field k. Faltings' theorem asserts that $C(L)$ is infinite for some finite extension L of k if and only if the genus of C is zero or one. If C admits a degree d morphism to the projective line or an elliptic curve, then by pulling back k-rational points via this morphism one sees that, after possibly replacing k by a larger number field, the set

$$\{P \in C(\overline{k}) \mid [k(P) : k] \leq d\}$$

is infinite. Harris and Silverman [4] proved the converse in the case $d = 2$.

Theorem 1.11 (Harris, Silverman)**.** *Let C be a nonsingular projective curve defined over a number field k. Then the set*

$$\{P \in C(\overline{k}) \mid [L(P) : L] \leq 2\}$$

is infinite for some finite extension L of k if and only if C is hyperelliptic or bielliptic.

More generally, we have the following theorem of Abramovich and Harris [1].

Theorem 1.12 (Abramovich, Harris)**.** *Let $d \leq 4$ be a positive integer. Let C be a nonsingular projective curve over a number field k with genus not equal to 7 if $d = 4$. Then the set*

$$\{P \in C(\overline{k}) \mid [L(P) : L] \leq d\}$$

is infinite for some finite extension L of k if and only if C admits a map of degree $\leq d$, over $\overline{k}$, to $\mathbb{P}^1$ or an elliptic curve.

Given Theorem 1.12, Abramovich and Harris naturally conjectured that a similar result would hold for all d (this is the analogue of Theorem 1.9 for algebraic points). However, Debarre and Fahlaoui [3] gave counterexamples to the conjecture for all $d \geq 4$. The failure of this conjecture and the failure of Question 1.5 to always have a positive answer are somewhat analogous. Debarre and Fahlaoui's counterexamples rely on the fact that there may exist an elliptic curve E in the Jacobian of a curve C that is not induced by any morphism $C \to E$. To every morphism $\phi \in \Phi(D, d, t, k)$ of degree d, one may associate a line in $\mathrm{Sym}^d\mathbb{P}^1$ via the one-dimensional linear system associated to ϕ. Our examples rely on the fact that in a Diophantine approximation problem on $\mathrm{Sym}^d\mathbb{P}^1 \cong \mathbb{P}^d$ related to Question 1.5, there are exceptional hyperplanes in the Subspace Theorem that are not induced by the morphisms in $\Phi(D,d,t,k)$, i.e., that are not a Zariski closure of a union of lines associated to morphisms in $\Phi(D,d,t,k)$.

2. Diophantine Approximation on Projective Space: Definitions and Background Material

Let k be a number field and let $\mathcal{O}_k$ denote the ring of integers of k. Recall that we have a canonical set M_k of places (or absolute values) of k consisting of one place for each prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$, one place for each real embedding $\sigma : k \to \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \overline{\sigma} : k \to \mathbb{C}$. If S is a finite set of places of k containing the archimedean places, we let $\mathcal{O}_{k,S}$, and $\mathcal{O}_{k,S}^*$ denote the ring of S-integers of k and the group of S-units of k, respectively. If v is a place of k and w is a place of a field extension L of k, then we say that w lies above v, or $w|v$, if w and v define the same topology on k. We normalize our absolute values so that $|p|_v = \frac{1}{p}$ if v corresponds to $\mathfrak{p}$ and $\mathfrak{p}$ lies above a rational prime p, and $|x|_v = |\sigma(x)|$ if v corresponds to an embedding σ. For $v \in M_k$, let k_v denote the completion of k with respect to v. We set

$$\|x\|_v = |x|_v^{[k_v:\mathbb{Q}_v]/[k:\mathbb{Q}]}.$$

A fundamental equation is the product formula

$$\prod_{v \in M_k} \|x\|_v = 1,$$

which holds for all $x \in k^*$.

For a point $P = (x_0, \ldots, x_n) \in \mathbb{P}^n(k)$, we have the absolute logarithmic height

$$h(P) = \sum_{v \in M_k} \log \max\{\|x_0\|_v, \ldots, \|x_n\|_v\}.$$

Note that this is independent of the number field k and the choice of coordinates $x_0, \ldots, x_n \in k$. In general, one can define a height h_D (and local height $h_{D,v}$, $v \in M_k$), unique up to a bounded function, with respect to any Cartier divisor D on a projective variety (in fact, this can even be done with respect to an arbitrary closed subscheme [12]). If D and E are Cartier divisors on a projective variety X, then heights satisfy the additive relation

$$h_{D+E}(P) = h_D(P) + h_E(P) + O(1).$$

Let Supp$D$ denote the support of the divisor D. If $\phi : Y \to X$ is a morphism of projective varieties with $\phi(Y) \not\subset \mathrm{Supp}D$, then

$$h_D(\phi(P)) = h_{\phi^* D}(P) + O(1).$$

Similar relations hold for local heights. We refer the reader to [2, 5, 6, 13] for further details and properties of heights.

We will primarily use heights with respect to effective divisors on projective space. These can be explicitly described as follows. Let D be a hypersurface in $\mathbb{P}^n$ defined by a homogeneous polynomial $f \in k[x_0, \dots, x_n]$ of degree d. For $v \in M_k$, we let $|f|_v$ denote the maximum of the absolute values of the coefficients of f with respect to v. We define $\|f\|_v$ similarly. For $v \in M_k$ and $P = (x_0, \dots, x_n) \in \mathbb{P}^n(k) \setminus \mathrm{Supp}D$, $x_0, \dots, x_n \in k$, we define the local height function

$$h_{D,v}(P) = \log \frac{\|f\|_v \max_i \|x_i\|_v^d}{\|f(P)\|_v}.$$

Note that this definition is independent of the choice of the defining polynomial f and the choice of the coordinates for P. Let $h_D(P) = \sum_{v \in M_k} h_{D,v}(P)$. It follows from the product formula that $h_D(P) = (\deg D) h(P)$. Let S be a finite set of places of k. For $P \in \mathbb{P}^n(\overline{k}) \setminus \mathrm{Supp}D$ we define the proximity function $m_{D,S}(P)$ by

$$m_{D,S}(P) = \sum_{v \in S} \sum_{\substack{w \in M_{k(P)} \\ w | v}} h_{D,w}(P).$$

We will also have occasion to use heights associated to points in projective space. If $P = (x_0, \dots, x_n), Q = (y_0, \dots, y_n) \in \mathbb{P}^n(k)$, $x_i, y_i \in k$, $P \neq Q$, and $v \in M_k$, we define

$$h_{Q,v}(P) = \log \frac{\max_i \|x_i\|_v \max_i \|y_i\|_v}{\max_{i,j} \|x_i y_j - x_j y_i\|_v}.$$

If $D_1, \ldots, D_q$ are effective Cartier divisors on a projective variety X, then we say that $D_1, \ldots, D_q$ are in m-subgeneral position if for any subset $I \subset \{1, \ldots, q\}$, $|I| \leq m+1$, we have $\dim \cap_{i\in I} \mathrm{Supp} D_i \leq m - |I|$, where we set $\dim \emptyset = -1$. In particular, the supports of any $m+1$ divisors in m-subgeneral position have empty intersection. We say that the divisors are in general position if they are in $\dim X$-subgeneral position, i.e., for any subset $I \subset \{1, \ldots, q\}$, $|I| \leq \dim X + 1$, we have $\operatorname{codim} \cap_{i \in I} \mathrm{Supp} D_i \geq |I|$.

We now recall three fundamental results in Diophantine approximation on projective space: Roth's theorem, Schmidt's Subspace Theorem, and the Ru-Wong theorem.

To begin, we give a slightly more general version of Roth's theorem from the introduction.

Theorem 2.1 (Roth's theorem with multiplicities)**.** *Let S be a finite set of places of a number field k. Let $P_1, \ldots, P_q \in \mathbb{P}^1(k)$ be distinct points and let $c_1, \ldots, c_q$ be positive real numbers with $c_1 \geq c_2 \geq \cdots \geq c_q$. Let $\epsilon > 0$. Then*

$$\sum_{i=1}^{q} c_i m_{P_i,S}(P) < (c_1 + c_2 + \epsilon) h(P) + O(1)$$

for all points $P \in \mathbb{P}^1(k) \setminus \{P_1, \ldots, P_q\}$.

Proof. For all $P \in \mathbb{P}^1(k) \setminus \{P_1, \ldots, P_q\}$,

$$\begin{aligned}\sum_{i=1}^{q} c_i m_{P_i,S}(P) &\leq (c_1 - c_2) m_{P_1,S}(P) + c_2 \sum_{i=1}^{q} m_{P_i,S}(P) + O(1)\\ &\leq (c_1 - c_2) h(P) + c_2 \sum_{i=1}^{q} m_{P_i,S}(P) + O(1).\end{aligned}$$

Let $\epsilon > 0$. By the standard version of Roth's theorem (Theorem 1.2),

$$\sum_{i=1}^{q} m_{P_i,S}(P) \leq (2 + \epsilon) h(P) + O(1)$$

for all $P \in \mathbb{P}^1(k) \setminus \{P_1, \ldots, P_q\}$. So

$$\sum_{i=1}^{q} c_i m_{P_i,S}(P) \leq (c_1 - c_2) h(P) + c_2 (2 + \epsilon / c_2) h(P) + O(1)$$

$$\leq (c_1 + c_2 + \epsilon)h(P) + O(1)$$

for all $P \in \mathbb{P}^1(k) \setminus \{P_1, \ldots, P_q\}$. $\square$

Schmidt's Subspace Theorem is a powerful generalization of Roth's theorem to higher-dimensional projective space. We state a general version, including improvements due to Schlickewei [11].

Theorem 2.2 (Schmidt Subspace Theorem)**.** *Let S be a finite set of places of a number field k. For each $v \in S$, let $H_{0,v}, \ldots, H_{n,v} \subset \mathbb{P}^n$ be hyperplanes over k in general position. Let $\epsilon > 0$. Then there exists a finite union of hyperplanes $Z \subset \mathbb{P}^n$ such that the inequality*

$$\sum_{v \in S} \sum_{i=0}^{n} h_{H_{i,v},v}(P) < (n + 1 + \epsilon)h(P)$$

holds for all $P \in \mathbb{P}^n(k) \setminus Z$.

If $H_1, \ldots, H_q$ are hyperplanes over k in general position, then the Subspace Theorem easily implies that there exists a finite union of hyperplanes $Z \subset \mathbb{P}^n$ such that the inequality

$$\sum_{i=1}^{q} m_{H_i,S}(P) < (n + 1 + \epsilon)h(P)$$

holds for all $P \in \mathbb{P}^n(k) \setminus Z$. If one substitutes a weaker inequality, then the exceptional hyperplanes may be replaced by smaller-dimensional linear subvarieties. This is given in the Ru-Wong theorem [10], which we state more generally for hyperplanes in m-subgeneral position.

Theorem 2.3 (Ru-Wong)**.** *Let S be a finite set of places of a number field k. Let $H_1, \ldots, H_q \subset \mathbb{P}^n$ be hyperplanes over k in m-subgeneral position. Let $t > 2m - n + 1$ be a real number. Then there exists a finite union of linear subvarieties $Z \subset \mathbb{P}^n$ of dimension $\leq 2m + 1 - t$ such that*

$$\sum_{i=1}^{q} m_{H_i,S}(P) < th(P)$$

for all $P \in \mathbb{P}^n(k) \setminus (Z \cup H_1 \cup \cdots \cup H_q)$.

3. Points of Bounded Degree and Symmetric Powers

For a variety X, let $\mathrm{Sym}^d X$ denote the dth symmetric power of X. As is well known, $\mathrm{Sym}^d\mathbb{P}^1 \cong \mathbb{P}^d$. In this section we will explore the natural relationship between degree d points on $\mathbb{P}^1$ and rational points on $\mathrm{Sym}^d\mathbb{P}^1 \cong \mathbb{P}^d$.

Let d be a positive integer. Let

$$\prod_{i=1}^{d} b_i x - a_i y = \sum_{i=0}^{d} p_i(a_1, \ldots, a_d, b_1, \ldots, b_d) x^i y^{d-i},$$

where $p_0, \ldots, p_d$ are polynomials over $\mathbb{Z}$. We can define a morphism

$$\begin{aligned} \sigma : (\mathbb{P}^1)^d &\to \mathbb{P}^d \\ (a_1, b_1) \times \cdots \times (a_d, b_d) &\mapsto (p_0(a_1, \ldots, a_d, b_1, \ldots, b_d), \ldots, \\ &\qquad p_d(a_1, \ldots, a_d, b_1, \ldots, b_d)). \end{aligned}$$

The morphism σ is a realization of the natural map $(\mathbb{P}^1)^d \to \mathrm{Sym}^d\mathbb{P}^1 \cong \mathbb{P}^d$.

To a point $P = (a, b) \in \mathbb{P}^1(\overline{\mathbb{Q}})$ we associate the hyperplane H_P in $\mathbb{P}^d$ defined by $\sum_{i=0}^{d} a^i b^{d-i} x_i = 0$. Since the relevant Vandermonde determinants are nonzero, we find that

Lemma 3.1. *If $P_1, \ldots, P_q \in \mathbb{P}^1(\overline{\mathbb{Q}})$ are distinct points, then the hyperplanes $H_{P_1}, \ldots, H_{P_q}$ are in general position.*

Let $\pi_i : (\mathbb{P}^1)^d \to \mathbb{P}^1$ denote the natural projection map onto the ith factor.

Lemma 3.2. *Let $P \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Then for any i, $\sigma_* \pi_i^*(P)$ is the hyperplane H_P.*

Proof. By symmetry, it suffices to the prove the lemma for $i = 1$. Let $P = (a, b)$. Setting $x = a$ and $y = b$, for any $a_2, \ldots, a_d, b_2, \ldots, b_d \in \overline{\mathbb{Q}}$ we have

$$(bx - ay) \prod_{i=2}^{d} b_i x - a_i y = \sum_{i=0}^{d} p_i(a, a_2, \ldots, a_d, b, b_2, \ldots, b_d) a^i b^{d-i} = 0.$$

So $\sigma((a,b)\times(a_2,b_2)\times\cdots\times(a_d,b_d))\in H_P$. Conversely, if

$$\sum_{i=0}^{d} c_i a^i b^{d-i} = 0,$$

then

$$\sum_{i=0}^{d} c_i x^i y^{d-i} = (bx-ay)\prod_{i=2}^{d} b_i x - a_i y$$

for some $a_2,\dots,a_d,b_2,\dots,b_d \in \overline{\mathbb{Q}}$, and hence $\sigma((a,b)\times(a_2,b_2)\times\cdots\times(a_d,b_d)) = (c_0,\dots,c_d)$. It follows that $\sigma_*\pi_1^*(P) = H_P$. $\square$

Let k be a number field. For $Q\in\{P\in\mathbb{P}^1(\overline{k}) \mid [k(P):k]=d\}$, let $Q=Q_1,\dots,Q_d\in\mathbb{P}^1(\overline{k})$ be the d conjugates of Q over k (in some order) and let $\rho(Q)=(Q_1,\dots,Q_d)\in(\mathbb{P}^1)^d$. Let $\psi=\sigma\circ\rho:\{P\in\mathbb{P}^1(\overline{k}) \mid [k(P):k]=d\}\to\mathbb{P}^d(\overline{k})$. Explicitly, if $P=(\alpha,1)$ and $[k(P):k]=d$, then $\psi(P)=(c_0,\dots,c_d)$ where $\sum_{i=0}^d c_i x^i$ is the minimal polynomial of α over k. The next lemma relates Diophantine approximation on $\mathbb{P}^1$ with respect to $P_1,\dots,P_q$ and Diophantine approximation on $\mathbb{P}^d$ with respect to $H_{P_1},\dots,H_{P_q}$.

Lemma 3.3. *Let $P_1,\dots,P_q\in\mathbb{P}^1(k)$. Then for $Q\in\{P\in\mathbb{P}^1(\overline{k}) \mid [k(P):k]=d\}$, the point $\psi(Q)$ is k-rational and*

$$\sum_{i=1}^{q} m_{H_{P_i},S}(\psi(Q)) = d\sum_{i=1}^{q} m_{P_i,S}(Q)+O(1),$$
$$h(\psi(Q)) = dh(Q)+O(1).$$

Proof. Let $Q\in\{P\in\mathbb{P}^1(\overline{k}) \mid [k(P):k]=d\}$ and let $Q=Q_1,\dots,Q_d\in\mathbb{P}^1(\overline{k})$ be the d conjugates of Q over k. It's clear from the definitions (or the remark before Lemma 3.3) that $\psi(Q)$ is k-rational. We have, up to $O(1)$,

$$\begin{aligned}\sum_{i=1}^{q} m_{H_{P_i},S}(\psi(Q)) &= \sum_{i=1}^{q} m_{\sigma_*\pi_1^*(P_i),S}(\sigma(\rho(Q))) = \sum_{i=1}^{q} m_{\sigma^*\sigma_*\pi_1^*(P_i),S}(\rho(Q))\\ &= \sum_{i=1}^{q} m_{\sum_{j=1}^d \pi_j^*(P_i),S}(\rho(Q)) = \sum_{i=1}^{q}\sum_{j=1}^{d} m_{\pi_j^*(P_i),S}(\rho(Q))\end{aligned}$$

$$= \sum_{i=1}^{q}\sum_{j=1}^{d} m_{P_i,S}(\pi_j(\rho(Q))) = \sum_{i=1}^{q}\sum_{j=1}^{d} m_{P_i,S}(Q_j)$$
$$= d\sum_{i=1}^{q} m_{P_i,S}(Q).$$

A similar calculation shows that $h(\psi(Q)) = dh(Q) + O(1)$. $\square$

We end by discussing the relationship between lines in $\mathrm{Sym}^d\mathbb{P}^1$ and morphisms $\phi : \mathbb{P}^1 \to \mathbb{P}^1$.

Lemma 3.4. *Let $P = (a_0, \ldots, a_d), Q = (b_0, \ldots, b_d) \in \mathbb{P}^d$, $P \neq Q$. Let L be the line through P and Q and let $\phi_{PQ} = \frac{\sum_{i=0}^{d} a_i x^i}{\sum_{i=0}^{d} b_i x^i}$. Then*

$$\psi^{-1}(L(k)) \subset \phi_{PQ}^{-1}(\mathbb{P}^1(k)).$$

Proof. If $d = 1$ then the lemma is essentially trivial. Suppose that $d > 1$. Let $P' \in L(k)$, $P' \neq Q$. Then

$$P' = (a_0 + tb_0, \ldots, a_d + tb_d)$$

for some $t \in k$. Let $f(x) = \sum_{i=0}^{d}(a_i + tb_i)x^i$. If P' is in the image of ψ, then f must be irreducible over k and $\psi^{-1}(P') = \{\alpha_1, \ldots, \alpha_d\}$ is the set of roots of f (identifying $\mathbb{A}^1 \subset \mathbb{P}^1$ as usual). We finish by noting that $\{\alpha_1, \ldots, \alpha_d\} = \phi_{PQ}^{-1}(-t)$. $\square$

4. Proof of Theorems 1.4, 1.6, 1.7

We begin by proving Theorem 1.4.

Proof. [Proof of Theorem 1.4] We first prove part (1). After an automorphism of $\mathbb{P}^1$, we can assume that $Q_1 = 0$ and $Q_2 = \infty$. Let $R = \phi^{-1}(\mathcal{O}_{k,S}^*)$. Since $|S| > 1$, the set R is infinite. From the definitions, for all $P \in R$,

$$m_{Q_1+Q_2,S}(\phi(P)) = 2h(\phi(P)),$$

and by functoriality,

$$m_{\phi^*(Q_1)+\phi^*(Q_2),S}(P) = 2dh(P) + O(1).$$

For any point $Q \in \mathbb{P}^1(\overline{k})$, $m_{Q,S}(P) \leq h(P) + O(1)$. It follows that for any point $Q \in \phi^{-1}(\{Q_1, Q_2\})$, $m_{Q,S}(P) = h(P) + O(1)$ for all $P \in R$ (i.e., R is a set of $(\phi^*(Q_1) + \phi^*(Q_2), S)$-integral points). Thus,

$$m_{D,S}(P) \geq (n_1 + n_2)h(P) + O(1)$$

for all $P \in R$, proving part (1).

We now prove part (2). We note the symmetry $h_{P,v}(Q) = h_{Q,v}(P)$ for $P, Q \in \mathbb{P}^1(k)$, $P \neq Q$, and $v \in M_k$. Let $P' \in \phi^{-1}(\mathbb{P}^1(k))$ with $[k(P') : k] = d$. Let $P'_1, \ldots, P'_d$ be the d conjugates of P' over k. Let $i \in \{1, \ldots, q\}$ and let $\phi(P_i) = Q_j$. Then

$$\begin{aligned} m_{P_i,S}(P') &= \frac{1}{d}\sum_{j=1}^{d} m_{P_i,S}(P'_j) = \frac{1}{d}\sum_{j=1}^{d} m_{P'_j,S}(P_i) = \frac{1}{d} m_{\phi^*(\phi(P')),S}(P_i) + O(1) \\ &= \frac{1}{d} m_{\phi(P'),S}(\phi(P_i)) + O(1) = \frac{1}{d} m_{\phi(P'),S}(Q_j) + O(1) \\ &= \frac{1}{d} m_{Q_j,S}(\phi(P')) + O(1). \end{aligned}$$

Note also that $h(\phi(P')) = dh(P') + O(1)$. Let $\epsilon > 0$. Then by Theorem 2.1,

$$\begin{aligned} m_{D,S}(P') &= \frac{1}{d}\sum_{j=1}^{r} n_j m_{Q_j,S}(\phi(P')) + O(1) \leq \frac{n_1 + n_2 + \epsilon}{d} h(\phi(P')) + O(1) \\ &\leq (n_1 + n_2 + \epsilon)h(P') + O(1). \end{aligned}$$

□

The proof of Theorem 1.7 proceeds by first transporting the problem to $\mathrm{Sym}^d\mathbb{P}^1 \cong \mathbb{P}^d$. We then use the Ru-Wong theorem to reduce to considering lines in $\mathbb{P}^d$, where Roth's theorem is applicable.

Proof. [Proof of Theorem 1.7] Let $t > 2d - 1$ be a real number. If $t > 2d$, then the statement in the theorem is an immediate consequence of Wirsing's theorem. Assume now that $2d - 1 < t \leq 2d$. By Wirsing's theorem, inequality (1.2) holds for all but finitely many points $P \in \mathbb{P}^1(\overline{k}) \setminus \mathrm{Supp} D$ satisfying $[k(P) : k] < d$. So we need only consider points $P \in \mathbb{P}^1(\overline{k})$ with $[k(P) : k] = d$. Let

$$R = \{P \in \mathbb{P}^1(\overline{k}) \mid [k(P) : k] = d, m_{D,S}(P) \geq th(P)\}.$$

By Lemma 3.3, for some constant C we have

$$\sum_{i=1}^{q} m_{H_{P_i},S}(\psi(P)) \geq th(\psi(P)) + C$$

for all points $P \in R$. Let $\epsilon > 0$ be such that $2d-1+\epsilon < t$. By the Ru-Wong theorem,

$$\sum_{i=1}^{q} m_{H_{P_i},S}(Q) < (2d-1+\epsilon)h(Q) + C$$

for all $Q \in \mathbb{P}^d(k) \setminus (Z' \cup H_{P_1} \cup \cdots \cup H_{P_q})$, where Z' is a finite union of lines and points in $\mathbb{P}^d$ not contained in any of the hyperplanes H_{P_i}, $i = 1, \ldots, q$. If $P \in \mathbb{P}^1(\overline{k})$ and $[k(P) : k] = d$, then $\psi(P) \notin H_{P_i}$ for all i. Thus, $\psi(R) \subset Z'$ and we need only analyze the set Z'. Let L be a line in the exceptional set Z'. If L is not defined over k, then $L(k)$ is finite and may be replaced by a finite number of points in Z'. Assume now that L is defined over k. Let $D = \sum_{i=1}^{q} H_{P_i}|_L = \sum_{i=1}^{s} c_i Q_i$, a divisor on $L \cong \mathbb{P}^1$, where $Q_1, \ldots, Q_s \in L(k)$ are distinct points. Since the hyperplanes H_{P_i} are in general position, $c_i \leq d$ for all i. By Theorem 2.1, if there are not two distinct indices $j, j' \in \{1, \ldots, s\}$ with $c_j = c_{j'} = d$, then for all $Q \in L(k) \setminus \mathrm{Supp} D$,

$$\sum_{i=1}^{q} m_{H_{P_i},S}(Q) = m_{D,S}(Q) + O(1) < \left(2d - 1 + \frac{\epsilon}{2}\right) h(Q) + O(1).$$

It follows that again L may be replaced in Z' by a finite number of points. So assume now that $c_j = c_{j'} = d$ for distinct $j, j' \in \{1, \ldots, s\}$.

Let

$$I_1 = \{i \in \{1, \ldots, q\} \mid Q_j \in H_{P_i}\},$$
$$I_2 = \{i \in \{1, \ldots, q\} \mid Q_{j'} \in H_{P_i}\}.$$

Then by our assumptions, $|I_1| = |I_2| = d$. Let $P_i = (a_i, b_i)$, $i = 1, \ldots, q$. Let $Q_j = (c_0, \ldots, c_d)$ and $Q_{j'} = (c'_0, \ldots, c'_d)$. Let $f_1(x, y) = \sum_{i=0}^{d} c_i x^i y^{d-i}$ and $f_2(x, y) = \sum_{i=0}^{d} c'_i x^i y^{d-i}$. Since $Q_j \in \cap_{i \in I_1} H_{P_i}$,

$$f_1(a_i, b_i) = \sum_{l=0}^{d} c_l a_i^l b_i^{d-l} = 0$$

for all $i \in I_1$. Similarly, f_2 vanishes at P_i for all $i \in I_2$. Thus, if $\phi = (f_1, f_2)$, then $\phi \in \Phi(D, d, t, k)$. It follows from Lemma 3.4 that if $\psi(P) \in L(k)$, then $P \in \phi^{-1}(k)$. Therefore $R \setminus Z(D, d, t, k)$ is a finite set.

Finally, we note that $Z(D, d, t, k)$ admits a simple description. If $\deg D = q < 2d$ then $Z(D, d, t, k) = \emptyset$. Otherwise, let $I = (I_1, I_2)$, where I_1 and I_2 are nonempty disjoint subsets of $\{1, \ldots, q\}$ of cardinality d. Then we define $\phi_I = (\prod_{i \in I_1} b_i x - a_i y, \prod_{i \in I_2} b_i x - a_i y)$. Let $\mathcal{I}$ be the set of all such I. If $2d - 1 < t \leq 2d$, then

$$Z(D, d, t, k) = \bigcup_{I \in \mathcal{I}} \phi_I^{-1}(\mathbb{P}^1(k)).$$

Note that $|\mathcal{I}| = \binom{q}{d, d, q-2d} = \frac{q!}{d!d!(q-2d)!}$ and $\mathcal{I}$ is a finite set. □

Finally, we note that Theorem 1.6 is an immediate consequence of Theorem 1.7 and the following lemma showing that Question 1.5 has a positive answer for trivial reasons when $t \leq d + 1$.

Lemma 4.5. *Let k be a number field. Let $P_1, \ldots, P_q \in \mathbb{P}^1(k)$ be distinct points, let $D = \sum_{i=1}^q P_i$, and let t be a positive real number.*

(1) *Let S be a finite set of places of k. If $\deg D < t$, then*

$$m_{D,S}(P) < th(P)$$

for all but finitely many points $P \in \mathbb{P}^1(\overline{k})$.

(2) *If $t \leq d + 1$ and $t \leq \deg D$, then*

$$Z(D, d, t, k) = \{P \in \mathbb{P}^1(\overline{k}) \mid [k(P) : k] \leq d\}.$$

Proof. Part (1) follows from the trivial observation that if $\deg D < t$, then

$$m_{D,S}(P) \leq h_D(P) + O(1) = (\deg D)h(P) + O(1) < th(P)$$

for all but finitely many $P \in \mathbb{P}^1(\overline{k})$.

To prove (2), suppose now that $t \leq d + 1$ and $t \leq \deg D$. Without loss of generality we can assume that t is a positive integer. One of the set inclusions in the statement is trivial. For the other, let $P \in \mathbb{P}^1(\overline{k})$ with $[k(P) : k] \leq d$. Let $P_i = (\alpha_i, 1)$ and $P = (\alpha, 1)$, where $\alpha_i \in k$, $i = 1, \ldots, t$, and $\alpha \in \{x \in \overline{k} \mid [k(x) : k] \leq d\}$ (after an automorphism, we can assume that none of the points are the point at infinity). If $\alpha \in \{\alpha_1, \dots, \alpha_t\}$, then it is easy that $P \in Z(D, d, t, k)$. Otherwise, let $\phi_0 = \frac{\prod_{i=1}^{t-1} x-\alpha_i}{x-\alpha_t}$. Since $[k(\alpha) : k] \leq d$ and $\phi_0(\alpha) \in k(\alpha)$, we can write $\phi_0(\alpha) = \sum_{i=0}^{d-1} c_i \alpha^i$ with $c_i \in k$, $i = 0, \dots, d-1$. If $[k(\alpha) : k] < d$, then we have some freedom in choosing the c_i. In any case, we can ensure that none of $\alpha_1, \dots, \alpha_t$ are roots of $\sum_{i=0}^{d-1} c_i x^i$. Now let $\phi = \phi_0 / \sum_{i=0}^{d-1} c_i x^i$. Then $\phi(\alpha) = 1$, $\deg \phi \leq d$, $\phi \in \mathrm{End}_k(\mathbb{P}^1)$, and $|\phi^{-1}(\{0, \infty\}) \cap \mathrm{Supp} D| \geq t$. So $\phi \in \Phi(D, d, t, k)$ and $P \in Z(D, d, t, k)$. $\square$

5. Exceptional Subspaces in $\mathbb{P}^3$

In order to prove Theorem 1.8 we need to study the exceptional hyperplanes that appear in the Schmidt Subspace Theorem for hyperplanes $H_1, \dots, H_q$ in $\mathbb{P}^3$ in general position. If $H_1, \dots, H_q$ are hyperplanes in $\mathbb{P}^3$ in general position and H is a hyperplane in $\mathbb{P}^3$ distinct from $H_1, \dots, H_q$, then $H_1 \cap H, \dots, H_q \cap H$ are lines in $H \cong \mathbb{P}^2$ in 3-subgeneral position. Thus, we are reduced to studying Diophantine approximation in the plane with respect to lines in 3-subgeneral position.

Let $L_1, \dots, L_q$ be lines in $\mathbb{P}^2$ in 3-subgeneral position. We say that $L_1, \dots, L_q$ is of:

1. Type I if $q > 4$ and
   (a) $L_i = L_j$ for some $i \neq j$.
   (b) There is a point in $\mathbb{P}^2$ that is contained in three distinct lines in $\{L_1, \dots, L_q\}$.
2. Type II if $q > 4$ and
   (a) The lines $L_1, \dots, L_q$ are distinct.
   (b) There are at least three noncollinear points in $\mathbb{P}^2$ that are each contained in three distinct lines in $\{L_1, \dots, L_q\}$.
3. Type III otherwise.

Define

$$c(L_1, \dots, L_q) = \begin{cases} 5 & \text{if } L_1, \dots, L_q \text{ is of Type I,} \\ \frac{9}{2} & \text{if } L_1, \dots, L_q \text{ is of Type II,} \\ 4 & \text{if } L_1, \dots, L_q \text{ is of Type III.} \end{cases}$$

Theorem 5.1. *Let k be a number field and let S be a finite set of places of k. Let $L_1,\dots,L_q \subset \mathbb{P}^2$ be lines over k in 3-subgeneral position. Let $c = c(L_1,\dots,L_q)$ and let $\epsilon > 0$. Then there exists a finite union of lines Z in $\mathbb{P}^2$ such that*

$$\sum_{i=1}^{q} m_{L_i,S}(P) \leq (c+\epsilon)h(P)$$

for all points $P \in \mathbb{P}^2(k) \setminus Z$.

Proof. By the Ru-Wong theorem, there exists a finite union of lines Z in $\mathbb{P}^2$ such that

$$\sum_{i=1}^{q} m_{L_i,S}(P) \leq (5+\epsilon)h(P)$$

for all points $P \in \mathbb{P}^2(k) \setminus Z$. So if $L_1,\dots,L_q$ is of Type I we are done. Suppose now that $L_1,\dots,L_q$ is of Type II. Since the lines $L_1,\dots,L_q$ are in 3-subgeneral position, any point can be v-adically close to at most three of the lines $L_1,\dots,L_q$. It follows that

$$\sum_{i=1}^{q} m_{L_i,S}(P) = \sum_{v\in S}\sum_{i=1}^{q} h_{L_i,v}(P) \leq \sum_{v\in S}\sum_{i=1}^{3} h_{L_{i,v},v}(P) + O(1),$$

where for each $v \in S$, $L_{1,v}, L_{2,v}, L_{3,v}$ are some choice of distinct lines in $\{L_1,\dots,L_q\}$. Then by the Schmidt Subspace Theorem, for all $\epsilon > 0$, there exists a finite union of lines Z in $\mathbb{P}^2$ such that

$$\begin{aligned}
&\sum_{v\in S} h_{L_{1,v},v}(P) + h_{L_{2,v},v}(P) \leq (3+\epsilon)h(P),\\
&\sum_{v\in S} h_{L_{1,v},v}(P) + h_{L_{3,v},v}(P) \leq (3+\epsilon)h(P),\\
&\sum_{v\in S} h_{L_{2,v},v}(P) + h_{L_{3,v},v}(P) \leq (3+\epsilon)h(P),
\end{aligned}$$

for all $P \in \mathbb{P}^2(k) \setminus Z$. Adding the three equations and dividing by 2 yields that for all $\epsilon > 0$, there exists a finite union of lines Z in $\mathbb{P}^2$ such that

$$\sum_{i=1}^{q} m_{L_i,S}(P) \leq \left(\frac{9}{2}+\epsilon\right) h(P)$$

for all $P \in \mathbb{P}^2(k) \setminus Z$, as desired.

Finally, suppose that $L_1, \dots, L_q$ is of Type III. If $q \le 4$, then it is trivial that

$$\sum_{i=1}^{q} m_{L_i,S}(P) \le (4+\epsilon)h(P).$$

Suppose now that $q > 4$. Suppose that some line appears twice in $L_1, \dots, L_q$. Then there must be exactly one such line (from 3-subgeneral position) and since $L_1, \dots, L_q$ is not of Type I, no three distinct lines in $\{L_1, \dots, L_q\}$ meet at a point. After reindexing, we may assume that $L_{q-1} = L_q$. Then it follows that the lines $L_1, \dots, L_{q-1}$ are in general position. Let $\epsilon > 0$. Then by the Schmidt Subspace Theorem, there exists a finite union of lines Z in $\mathbb{P}^2$ such that

$$\begin{aligned}\sum_{i=1}^{q} m_{L_i,S}(P) &= \sum_{i=1}^{q-1} m_{L_i,S}(P) + m_{L_q,S}(P) \le \sum_{i=1}^{q-1} m_{L_i,S}(P) + h(P)\\ &\le (4+\epsilon)h(P)\end{aligned}$$

for all $P \in \mathbb{P}^2(k) \setminus Z$.

We now assume that $L_1, \dots, L_q$ are distinct lines. Let $P_1, \dots, P_n$ be the points in $\mathbb{P}^2$ that are contained in three distinct lines in $\{L_1, \dots, L_q\}$. Then since $L_1, \dots, L_q$ is not of Type II, $P_1, \dots, P_n$ all lie on a line L. Let $v \in S$ and $P \in \mathbb{P}^2(k) \setminus \cup_{i=1}^{q} L_i$. For simplicity, rearrange the indices so that

$$h_{L_1,v}(P) \ge h_{L_2,v}(P) \ge \cdots \ge h_{L_q,v}(P).$$

If $L_1 \cap L_2 \ne \{P_i\}$, $i = 1, \dots, n$, then

$$\sum_{i=1}^{q} h_{L_i,v}(P) \le h_{L_1,v}(P) + h_{L_2,v}(P) + O(1).$$

If $L_1 \cap L_2 \cap L_j = \{P_i\}$ for some $i \in \{1, \dots, n\}$ and $j \in \{3, \dots, q\}$, then from the theory of heights associated to closed subschemes [12], we have

$$\min\{h_{L_1,v}(P), h_{L_2,v}(P), h_{L_{j'},v}(P)\} = \begin{cases} h_{P_i,v}(P) + O(1) & \text{if } j' = j, \\ O(1) & \text{if } j' \notin \{1,2,j\}, \end{cases}$$

and if $P \notin L$,

$$h_{P_i,v}(P) \leq h_{L,v}(P) + O(1).$$

Then if $P \notin L$,

$$\begin{aligned}\sum_{i=1}^{q} h_{L_i,v}(P) &\leq h_{L_1,v}(P) + h_{L_2,v}(P) + h_{P_i,v}(P) + O(1)\\ &\leq h_{L_1,v}(P) + h_{L_2,v}(P) + h_{L,v}(P) + O(1).\end{aligned}$$

It follows that if $P \notin L$,

$$\sum_{i=1}^{q} m_{L_i,S}(P) = \sum_{v\in S}\sum_{i=1}^{q} h_{L_i,v}(P) \leq \sum_{v\in S}\sum_{i=1}^{2} h_{L_{i,v},v}(P) + \sum_{v\in S} h_{L,v}(P) + O(1)$$

for some lines $L_{i,v}$, $v \in S$. Then by the Schmidt Subspace Theorem and the trivial estimate $\sum_{v\in S} h_{L,v}(P) \leq h(P) + O(1)$, we find that there exists a finite union of lines Z in $\mathbb{P}^2$ such that

$$\sum_{v\in S}\sum_{i=1}^{q} h_{L_i,v}(P) \leq (4+\epsilon)h(P)$$

for all $P \in \mathbb{P}^2(k) \setminus Z$. □

We now show that the previous theorem is essentially sharp.

Theorem 5.2. *Let k be a number field and let S be a finite set of places of k containing the archimedean places. Let $L_1, \ldots, L_q \subset \mathbb{P}^2$, $q > 3$, be lines over k in 3-subgeneral position, but not in general position. Let $c = c(L_1, \ldots, L_q)$. Suppose that*

$$\begin{cases} |S| > 1 & \text{if } L_1, \ldots, L_q \text{ is of Type I or III,}\\ |S| > 2 & \text{if } L_1, \ldots, L_q \text{ is of Type II.}\end{cases}$$

Then there exists a Zariski dense set of points $R \subset \mathbb{P}^2(k)$ such that

$$\sum_{i=1}^{q} m_{L_i,S}(P) \geq (c-\epsilon)h(P)$$

for all $P \in R$.

Proof. Suppose first that $L_1, \dots, L_q$ is of Type I. Then after reindexing, we can assume that $L_1 \cap L_2 \cap L_3 = \{Q\}$ is nonempty and $L_4 = L_5$. Let L be a k-rational line through Q distinct from $L_1, \dots, L_q$. Then $\cup_{i=1}^5 L \cap L_i = \{Q, Q'\}$ consists of two points. Since $|S| > 1$, there exists an infinite set R of k-rational $(Q + Q', S)$-integral points on L, i.e.,

$$m_{Q+Q',S}(P) = 2h(P) + O(1)$$

for all $P \in R$. Then for all $P \in R$,

$$\begin{aligned}\sum_{i=1}^{q} m_{L_i,S}(P) &\geq \sum_{i=1}^{5} m_{L_i,S}(P) + O(1) = 3m_{Q,S}(P) + 2m_{Q',S}(P) + O(1)\\ &= 5h(P) + O(1).\end{aligned}$$

Thus, there are infinitely many points $P \in L(k)$ satisfying

$$\sum_{i=1}^{q} m_{L_i,S}(P) \geq (5 - \epsilon)h(P).$$

Since the union of k-rational lines L through Q is Zariski dense in $\mathbb{P}^2$, this proves the result in the Type I case.

Suppose now that $L_1, \dots, L_q$ is of Type III. Since $L_1, \dots, L_q$ are not in general position, after reindexing we can assume that $L_1 \cap L_2 \cap L_3 = \{Q\}$ is nonempty. Let L be a k-rational line through Q distinct from $L_1, \dots, L_q$ and let $\{Q'\} = L \cap L_4$. Then by the same argument as above, taking $R \subset L(k)$ to be an infinite set of $(Q + Q', S)$-integral points on L, for all $P \in R$ we have

$$\begin{aligned}\sum_{i=1}^{q} m_{L_i,S}(P) &\geq \sum_{i=1}^{4} m_{L_i,S}(P) + O(1) = 3m_{Q,S}(P) + m_{Q',S}(P) + O(1)\\ &= 4h(P) + O(1).\end{aligned}$$

Thus, there are infinitely many points $P \in L(k)$ satisfying

$$\sum_{i=1}^{q} m_{L_i,S}(P) \geq (4 - \epsilon)h(P).$$

Since the union of such lines L is Zariski dense in $\mathbb{P}^2$, this proves the result in the Type III case.

Finally, suppose that $L_1, \dots, L_q$ is of Type II. Let Q_1, Q_2, and Q_3 be three noncollinear points in $\mathbb{P}^2(k)$ that are each contained in three distinct lines in $\{L_1, \dots, L_q\}$. After an automorphism of $\mathbb{P}^2$ we may assume that $Q_1 = (1,0,0)$, $Q_2 = (0,1,0)$, and $Q_3 = (0,0,1)$. Let $S = \{v_1, \dots, v_s\}$, where by assumption $s \geq 3$. By the Dirichlet unit theorem, the image of $\mathcal{O}_{k,S}^*$ under the map

$$\begin{aligned} \mathcal{O}_{k,S}^* &\to \mathbb{R}^s, \\ u &\mapsto (\log \|u\|_{v_1}, \dots, \log \|u\|_{v_s}), \end{aligned}$$

is a (full) lattice in the subspace of $\mathbb{R}^s$ defined by $x_1 + \cdots + x_s = 0$. It follows that for each positive integer m, there exist units $u_{1,m}, u_{2,m} \in \mathcal{O}_{k,S}^*$ such that

$$\begin{aligned} \log \|u_{1,m}\|_{v_1} &= m + O(1), \log \|u_{1,m}\|_{v_2} = O(1), \\ \log \|u_{1,m}\|_{v_i} &= -\frac{m}{s-2} + O(1), \quad i = 3, \dots, s, \\ \log \|u_{2,m}\|_{v_1} &= O(1), \log \|u_{2,m}\|_{v_2} = m + O(1), \\ \log \|u_{2,m}\|_{v_i} &= -\frac{m}{s-2} + O(1), \quad i = 3, \dots, s. \end{aligned}$$

Let $P_m = (u_{1,m}, u_{2,m}, 1) \in \mathbb{P}^2(k)$. Let L_x, L_y, and L_z be the three lines in $\mathbb{P}^2$ defined by $x=0$, $y=0$, and $z=0$, respectively. Then $h(P_m) = 2m + O(1)$ and

$$\begin{aligned} h_{L_x,v_1}(P_m) &= O(1), h_{L_x,v_2}(P_m) = m + O(1), \\ h_{L_x,v_i}(P_m) &= \frac{m}{s-2} + O(1), \qquad i = 3, \dots, s, \\ h_{L_y,v_1}(P_m) &= m + O(1), h_{L_y,v_2}(P_m) = O(1), \\ h_{L_y,v_i}(P_m) &= \frac{m}{s-2} + O(1), \qquad i = 3, \dots, s, \\ h_{L_z,v_1}(P_m) &= m + O(1), h_{L_z,v_2}(P_m) = m + O(1), \\ h_{L_z,v_i}(P_m) &= O(1), \quad i = 3, \dots, s. \end{aligned}$$

For $v \in S$, we have (see [12])

$$h_{Q_1,v} = \min\{h_{L_y,v}, h_{L_z,v}\} + O(1),$$

$$h_{Q_2,v} = \min\{h_{L_x,v}, h_{L_z,v}\} + O(1),$$
$$h_{Q_3,v} = \min\{h_{L_x,v}, h_{L_y,v}\} + O(1),$$

where the functions are defined. It follows that

$$\begin{aligned}
h_{Q_1,v_1}(P_m) &= m + O(1), h_{Q_1,v_2}(P_m) = O(1),\\
h_{Q_1,v_i}(P_m) &= O(1), \qquad i = 3, \dots, s,\\
h_{Q_2,v_1}(P_m) &= O(1), h_{Q_2,v_2}(P_m) = m + O(1),\\
h_{Q_2,v_i}(P_m) &= O(1), \qquad i = 3, \dots, s,\\
h_{Q_3,v_1}(P_m) &= O(1), h_{Q_3,v_2}(P_m) = O(1),\\
h_{Q_3,v_i}(P_m) &= \frac{m}{s-2} + O(1),\ i = 3, \dots, s.
\end{aligned}$$

Then for all m such that $P_m \notin L_1 \cup \cdots \cup L_q$,

$$\sum_{i=1}^{q} m_{L_i,S}(P_m) \geq 3 \sum_{v\in S} \sum_{i=1}^{3} h_{Q_i,v}(P_m) = 9m + O(1) = \frac{9}{2}h(P_m) + O(1).$$

To complete the proof, it remains to show that the set $R = \{P_m \mid m \in \mathbb{N}\}$ is Zariski dense in $\mathbb{P}^2$. Suppose that there exists a homogeneous polynomial $p \in k[x, y, z]$ that vanishes on R. Looking at the valuations of $u_{1,m}^i u_{2,m}^j$ with respect to v_1 and v_2, this is plainly impossible. Thus, we arrive at a contradiction and the set R is Zariski dense in $\mathbb{P}^2$. $\square$

6. Proof of Theorem 1.8

Using the results of the previous section we now prove Theorem 1.8.

Proof of Theorem 1.8. We first prove part (1). If $t > 5$, then part (1) follows immediately from Theorem 1.7. Suppose now that $\frac{9}{2} < t \leq 5$. By Wirsing's theorem, the set of points $P \in \mathbb{P}^1(\overline{k}) \setminus \mathrm{Supp}D$ satisfying $[k(P) : k] \leq 2$ and

$$m_{D,S}(P) \geq th(P)$$

is finite, and so we may ignore such points. Let R be the set

$$R = \{P \in \mathbb{P}^1(\overline{k}) \mid [k(P) : k] = 3, m_{D,S}(P) \geq th(P)\}.$$

Then by Lemma 3.3,

$$\sum_{i=1}^{q} m_{H_{P_i},S}(\psi(P)) \geq th(\psi(P)) + O(1)$$

for all $P \in R$. Let $R' = \psi(R)$. Since $t > 4$, by the Schmidt Subspace Theorem, R' lies in a finite union of hyperplanes of $\mathbb{P}^3$. Let H be one of the hyperplanes.

Suppose first that $H_{P_1}|_H, \ldots, H_{P_q}|_H$ is not of Type I. Then by Theorem 5.1, $R' \cap H$ lies in a finite union of lines (with no line contained in any of the hyperplanes $H_{P_1}, \ldots, H_{P_q}$). Let L be one of these lines and let $\sum_{i=1}^{q} H_{P_i}|_L = \sum_{i=1}^{r} c_i Q_i$, where $Q_1, \ldots, Q_r \in L(k)$ are distinct points and $c_1 \geq c_2 \geq \cdots \geq c_r$. Then for all $P \in R' \cap L$,

$$\sum_{i=1}^{r} c_i m_{Q_i,S}(P) \geq th(P) + O(1).$$

If $R' \cap L$ is infinite, then by Theorem 2.1, we must have $c_1 + c_2 \geq t > \frac{9}{2}$. Since c_1 and c_2 are integers and $c_1, c_2 \leq 3$, we must have that $c_1 = 3$ and $c_2 \geq 2$. After reindexing, we can assume that $H_{P_1} \cap H_{P_2} \cap H_{P_3} \cap L = \{Q_1\}$ and $H_{P_4} \cap H_{P_5} \cap L = \{Q_2\}$. By Lemma 3.4, $\psi^{-1}(L(k)) \subset \phi_{Q_1Q_2}^{-1}(\mathbb{P}^1(k))$. From the definitions, $\phi_{Q_1Q_2}^{-1}(0) = \{P_1, P_2, P_3\}$ and $\phi_{Q_1Q_2}^{-1}(\infty) \supset \{P_4, P_5\}$. Thus, since $t \leq 5$, $\phi_{Q_1Q_2} \in \Phi(D, 3, t, k)$ and $\psi^{-1}(L(k)) \subset Z(D, 3, t, k)$. It follows that all but finitely many points of $\psi^{-1}(R' \cap H)$ are contained in $Z(D, 3, t, k)$.

Suppose now that $H_{P_1}|_H, \ldots, H_{P_q}|_H$ is of Type I. After reindexing, we can assume that $H_{P_1} \cap H_{P_2} \cap H_{P_3} \cap H = \{Q\}$, for some point $Q \in H(k)$, and $H_{P_4} \cap H = H_{P_5} \cap H$. Let $P \in H(k) \setminus (H_{P_1} \cup H_{P_2} \cup H_{P_3})$, and let L be the line through P and Q. Let $L \cap H_4 \cap H = L \cap H_5 \cap H = \{Q'\}$. By Lemma 3.4, $\psi^{-1}(L(k)) \subset \phi_{QQ'}^{-1}(\mathbb{P}^1(k))$. From the definitions, $\phi_{QQ'}^{-1}(0) = \{P_1, P_2, P_3\}$ and $\phi_{QQ'}^{-1}(\infty) \supset \{P_4, P_5\}$. Thus, since $t \leq 5$, $\phi_{QQ'} \in \Phi(D, 3, t, k)$ and $\psi^{-1}(L(k)) \subset Z(D, 3, t, k)$. Since $P \in H(k) \setminus (H_{P_1} \cup H_{P_2} \cup H_{P_3})$ was arbitrary, in particular $\psi^{-1}(R' \cap H) \subset Z(D, 3, t, k)$. Combining this fact with the previous case above, we have shown that $R \setminus Z(D, 3, t, k)$ is a finite set, proving part (1).

Suppose now that $4 < t < \frac{9}{2}$, $|S| > 2$, and $q = 6$. Let

$$\begin{aligned}\{Q_1\} &= H_{P_1} \cap H_{P_2} \cap H_{P_3},\\ \{Q_2\} &= H_{P_1} \cap H_{P_4} \cap H_{P_5},\\ \{Q_3\} &= H_{P_2} \cap H_{P_4} \cap H_{P_6}.\end{aligned}$$

The line through Q_1 and Q_2 lies in H_{P_1}. Since the hyperplanes H_{P_i} are in general position, $Q_3 \notin H_{P_1}$ and it follows that Q_1, Q_2, and Q_3 are not collinear. Let $H \subset \mathbb{P}^3$ be the unique hyperplane through Q_1, Q_2, and Q_3. Since the hyperplanes H_{P_i} are in general position, it follows easily that all of the lines $H_{P_i}|_H$ are distinct (otherwise there would be four hyperplanes H_{P_i} containing some point Q_j). Then $H_{P_1}|_H, \ldots, H_{P_6}|_H$ is of Type II. Let $0 < \epsilon < \frac{1}{4}$ be such that $t < \frac{9}{2} - \epsilon$. By Theorem 5.2, there exists a set of points $R' \subset H(k)$ that is Zariski dense in H and such that

$$\sum_{i=1}^{6} m_{H_{P_i},S}(P) > \left(\frac{9}{2} - \epsilon\right) h(P)$$

for all $P \in R'$. Let $P \in R'$ and let $\sigma((Q_1', Q_2', Q_3')) = P$. Then by the same calculation as in the proof of Lemma 3.3, we have

$$\sum_{i=1}^{6} \sum_{j=1}^{3} m_{P_i,S}(Q_j') > \left(\frac{9}{2} - \epsilon\right) \sum_{j=1}^{3} h(Q_j') + O(1).$$

If $[k(Q_j') : k] \leq 2$ for some j (and hence all j), then by Wirsing's theorem, $\sum_{i=1}^{6} m_{P_i,S}(Q_j') < (4+\epsilon)h(Q_j') + O(1)$. It follows that for all but finitely many points $P \in R'$, $P \in \mathrm{im}\psi$. Let $R = \psi^{-1}(R')$. By Lemma 3.3,

$$\sum_{i=1}^{6} m_{P_i,S}(P) > \left(\frac{9}{2} - \epsilon\right) h(P) + O(1) > th(P)$$

for all but finitely many $P \in R$. From the definitions and the proof of Lemma 3.4, every point in $\psi(R \cap Z(D, 3, t, k))$ lies on a line L through points P and Q in $\mathbb{P}^3$, where P lies in the intersection of three distinct hyperplanes $H_{P_{i_1}}, H_{P_{i_2}}, H_{P_{i_3}}$, and Q lies in the intersection of two other distinct hyperplanes $H_{P_{i_4}}$ and $H_{P_{i_5}}$. The set of such lines L does not intersect H in a Zariski dense set in H. It follows that $R \setminus Z(D, 3, t, k)$ is infinite. □

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