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INTEGER POINTS IN ARITHMETIC SEQUENCES

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Abstract

We present a dynamical analog of the Mordell-Lang conjecture for integral points. We are able to prove this conjecture in the case of endomorphisms of semiabelian varieties.

1. Introduction

Faltings [5] proved the Mordell-Lang conjecture in the following form.

Theorem 1.1 (Faltings)**.** *Let G be an abelian variety defined over the field of complex numbers $\mathbb{C}$. Let $X \subset G$ be a closed subvariety and $\Gamma \subset G(\mathbb{C})$ a finitely generated subgroup of $G(\mathbb{C})$. Then $X(\mathbb{C}) \cap \Gamma$ is a finite union of cosets of subgroups of Γ.*

In a more general dynamical setting, one might consider analogous questions for an endomorphism $\Phi : X \longrightarrow X$ of a quasiprojective variety defined over $\mathbb{C}$ and the orbit of a point $\alpha \in X(\mathbb{C})$ under Φ. We let $Orb_\Phi(\alpha)$ denote set $\{\Phi^m(\alpha) \mid m \in \mathbb{N}\}$, where Φ^m denotes the m^{th} iterate $\Phi \circ \cdots \circ \Phi$. The Mordell-Lang conjecture describes the structure of the intersection of a finitely generated group with a subvariety; it seems natural to ask for a description of the structure of $Orb_\Phi(\alpha) \cap V(\mathbb{C})$ for V a subvariety of X. The Skolem-Mahler-Lech theorem for linear recurrences [12, 9, 10] suggests that $Orb_\Phi(\alpha) \cap V(\mathbb{C})$ can be described in terms of arithmetic progressions, that is sets of the form $\{k + \ell n \mid n \in \mathbb{N}\}$ for some $k, \ell \in \mathbb{N}$. We allow for the

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possibility that ℓ is 0, so that the progression is a single number. Various authors [4, 1, 6] have proposed the following conjecture.

Conjecture 1.2. *Let X be a quasiprojective variety defined over $\mathbb{C}$, let $V \subset X$ be any subvariety, let $\Phi : X \longrightarrow X$ be any endomorphism, and let $\alpha \in X(\mathbb{C})$. Then $\{n \in \mathbb{N} \mid \Phi^n(\alpha) \in V(\mathbb{C})\}$ is a union of finitely many arithmetic progressions.*

Note that here we think of the empty set as a union of finitely many arithmetic progressions. The conjecture does not assert that there are necessarily any n such that $\Phi^n(\alpha) \in V(\mathbb{C})$.

We would like to present a similar general question for integral points in orbits. We will prove it for self-maps of semiabelian varieties, present some counterexamples in other situations, and suggest a possible conjecture.

First, we describe our notation a little, following Vojta [15]. Let X be a variety defined over a number field K, D an effective divisor on X, and S a finite set of places of K including all the archimedean places. Let R_S denote the ring of integers of K localized away from S and let $\mathcal{X}$ be a model for X over R_S. Let $\iota : X \longrightarrow \mathcal{X}$ be the usual map coming from base extension from R_S to K. We say that a point $z \in X(K)$ is S-integral if z is the pull-back of a point $P \in \mathcal{X}(R_S)$. We say that $z \in X(K)$ is (S, D)-integral if there is a $P \in \mathcal{X}(R_S)$ such that $\iota(z) = P$ and such that P does not meet the Zariski the closure of D in $\mathcal{X}$.

Question 1.3. *Let X be a variety defined over a number field K, let S be a finite set of places of K that includes all the archimedean places, and let $\mathcal{X}$ be a model for X over R_S. Let $\Phi : X \longrightarrow X$ be a finite morphism that extends to a map from $\mathcal{X}$ to $\mathcal{X}$, let $\alpha \in X(K)$, and let D be a divisor on X. Is it true that*

$$\{n \in \mathbb{N} \mid \ \Phi^n(\alpha) \text{ is } (S, D)\text{-integral}\}$$

must form a finite union of arithmetic sequences?

2. Morphisms of Semiabelian Varieties

We show here that Question 1.3 has a positive answer when X is a semiabelian variety.

Theorem 2.1. *Let X be a semiabelian variety defined over a number field K, let S be a finite set of places of K that includes all the archimedean places, and let $\mathcal{X}$ be a model for X over R_S. Let $\Phi : X \longrightarrow X$ be an étale finite morphism that extends to a map from $\mathcal{X}$ to $\mathcal{X}$, let $\alpha \in X(K)$, and let D be a divisor on X. Then the set*

$$\{n \in \mathbb{N} \mid \ \Phi^n(\alpha) \textit{ is } (S, D)\textit{-integral}\}$$

forms a finite union of arithmetic sequences.

Proof. This is trivial when α is preperiodic for Φ, so we may assume that it is not. Furthermore, we may assume that there is some $m \in \mathbb{N}$ such that $\Phi^m(\alpha)$ is (S, D)-integral, since otherwise our assertion is vacuously true. Since proving the theorem for $\Phi^m(\alpha)$ is equivalent to proving it for α itself we may also suppose that α itself is (S, D)-integral. If Supp$D$ is empty, then the fact that Φ extends to a map from $\mathcal{X}$ to $\mathcal{X}$ means that $\Phi^n(\alpha)$ is then (S, D)-integral for all n. Hence, we may ssume that Supp$D$ is nonempty.

We will proceed by induction on the dimension of X. If X is 1-dimensional, then it is either the multiplicative group $\mathbb{G}_m$ or an elliptic curve. If X is an elliptic curve, then the fact that D is nonempty implies that there are finitely many n such that $\Phi^n(\alpha)$ is (S, D)-integral, by Siegel's theorem for integral points on curves of positive genus. If X is $\mathbb{G}_m$, then (S, D)-integral points on X correspond to points on the projective line that are S-integral relative to a divisor with support at three or more points; Siegel's theorem for integral points on curves of genus zero states there are finitely many such points.

By [16, Theorem 0.4], the set of (S, D)-integral points X is equal to all the S-integral points on some finite union of translates of semiabelian subvarieties B_i of X. If we have $B_i = X$ for some B_i, then $\Phi^n(\alpha)$ is (S, D)-integral for all n (since Φ sends S-integral points to S-integral points) and we are done. Thus, we may assume that the B_i all have dimension less than $\dim X$. In this case, either there are finitely many n such that $\Phi^n(\alpha)$ is (S, D)-integral or there is an infinite subset of $Orb_\Phi(\alpha)$ that is not dense in X. In the first case, our result follows trivially; in the second, [2, Corollary 1.4] implies that $Orb_\Phi(\alpha)$ is itself not dense in X (note that Φ is étale since by [8, Theorem 2] it is a composition of a translation and an algebraic group endomorphism). But then the closure of $Orb_\Phi(\alpha)$ is a proper subvariety W

of X. If Y is the union of the positive dimensional components of W, then $\Phi(Y) = Y$, because $W \setminus \Phi(W)$ consists of at most one point. Since Y has finitely many components, it follows that Φ must permute the components of Y; therefore, there is an m such that $\Phi^m(Y_i) = Y_i$ for each component Y_i. Applying [2, Corollary 1.4] to each $\Phi^m|_{Y_i}$ for each i, we see that since $Orb_\Phi(\alpha) \cap Y_i$ is dense in Y_i, any infinite subset of $Orb_\Phi(\alpha) \cap Y_i$ is dense in Y_i as well. Hence each Y_i contains either a dense set of (S, D)-integral points or contains at most finitely many (S, D)-integral points in $Orb_\Phi(\alpha) \cap Y_i$. If (S, D)-integral points are dense in Y_i, then Y_i is a finite union of translated semiabelian subvarieties of X, by [16, Theorem 0.4]. Since Y_i is irreducible, this means that Y_i itself is a translated semiabelian variety. We write $Y_i = B_i + t_i$, where B_i is a semiabelian subvariety of X and t_i is an (S, D)-integral point (we may choose t_i to be (S, D)-integral since (S, D)-integral points are dense in Y_i and any two points on Y_i are in the same coset of B_i). Hence the isomorphism, induced by translation, of Y_i with B_i extends to an isomorphism of R_S-schemes.

Now, there is an ℓ such that $\Phi^\ell(\alpha) \in Y$, since $W \setminus Y$ is finite. For $i = 1, \ldots, m$, let Y_i be a component of Y such that $\Phi^{\ell+i}(\alpha) \in Y_i$. Since $\Phi^m(Y_i) = Y_i$ and $\dim Y_i < X$, the inductive hypothesis implies that the set of k such that $\Phi^{mk}(\Phi^i(\alpha))$ is $(S, D|_{Y_i})$-integral is a finite union of arithmetic sequences. Since any finite union of finite unions of arithmetic sequences is itself a finite union of arithmetic sequences, taking the union over $i = 1, \ldots, m$ gives a finite union of arithmetic sequences, and completes our proof. □

3. Examples and Counterexamples

One might also imagine that if one takes two commuting morphisms Φ and Ψ on a semiabelian variety that the set of m, n such that $\Phi_1^m \Psi^n(\alpha)$ is (S, D)-integral will form a finite union of cosets of subsemigroups of $\mathbb{N} \times \mathbb{N}$. A special case of this question is treated in [3]. More generally, however, there are counterexamples such as the following, adapted from [7] shows.

Example 3.1. Let $X = \mathbb{G}_m^3$. Fix a set of coordinates (x, y, z) for X (which gives us a model to define integrality with). Let D be the divisor consisting of all $(2, x, y) \in \mathbb{G}_m^3$ and all $(-2, x, y) \in \mathbb{G}_m^3$ (thus D has two irreducible components). Let $S = \{\infty, 3\}$, where ∞ denotes the single archimedean place on $\mathbb{Q}$. Then a point in $X(\mathbb{Q})$ is (S, D)-integral when its first coordinate is ± 1 and its other coordinates are in $\mathbb{Z}$ localized away from 3. This is easy to check: the first coordinate must have the form $\pm 3^n$ for some $n \in \mathbb{Z}$. If $n < 0$ or $n > 1$, then $\pm 3^n - 2$ has a factor in its numerator other than 3. That leaves only 3, -3, and ± 1 as possibilities. But 3 meets -2 at 5 and -3 meets 2 at 5. So we are left with ± 1 as the only possibility for the first coordinate. Let $\alpha = (1, 1/3, 9)$ and let

$$\Phi(x, y, z) = \left(x^2 y^{-1}, y^2 z^{-2}, z^2\right)$$

and

$$\Psi(x, y, z) = \left(x^2 y^2, y^2 z^4, z^2\right).$$

Any image of α under $\Phi^m \Psi^n$ has all of its coefficients in $\mathbb{Z}$ localized away from 3. Thus, $\Phi^m \Psi^n(\alpha)$ is (S, D)-integral if and only if its first coefficient is ± 1. In [7], it is shown that happens exactly when (m, n) is in the set

$$\{(3k^2, 3(k^2 + k)/2) : k \in \mathbb{Z}\}$$

which is not a coset of a subsemigroup of $\mathbb{N}^2$. We see then that $\Phi^n(\alpha)$ is not (S, D)-integral for any positive integer n except $n = 3$. Similarly, there is no positive integer m such that $\Psi^m(\alpha)$ is (S, D)-integral. Thus the conclusion of Theorem 2.1 is met for each individual map Φ and Ψ.

Example 3.1 also gives rise to counterexamples on elliptic curves E^3 by replacing the powering maps with multiplication-by-m maps. More exotic counterexamples can likely be obtained via the more general methods of Scanlon and Yasufuku [13].

Question 1.3 itself has a negative answer for some individual maps.

Example 3.2. Consider the map $\varphi : z \mapsto z + 1$ on $\mathbb{P}^1$ with the divisor D taken to be $[0]$ and $\alpha = 0$. Let $K = \mathbb{Q}$ and let $S = \{\infty, p\}$ for a prime number p. Then $\varphi^n(\alpha)$ is (S, D)-integral exactly when n is a power of p.

4. A Conjecture

By imposing extra conditions, we can eliminate maps like $\varphi(z) = z + 1$ from consideration. One natural condition is Zhang's notion of "polarization" (see [18]): a map $\Phi : X \longrightarrow X$ on a projective variety X is said to be *polarized* by an ample divisor D if $\Phi^* D \cong mD$ for some $m > 1$.

Conjecture 4.1. *Let X be a projective variety defined over a number field K, let S be a finite set of places of K including all the archimedean places, and let $\mathcal{X}$ be a model for X over R_S. Let $\Phi : X \longrightarrow X$ be a morphism polarized by an ample divisor D, and let $\alpha \in X(K)$. Suppose that Φ extends to a morphism $\mathcal{X} \longrightarrow \mathcal{X}$. Then the set*

$$\{n \in \mathbb{N} \mid \ \Phi^n(\alpha) \text{ is } (S,D)\text{-integral}\}$$

forms a finite union of arithmetic sequences.

While we do not yet know how to prove this, one possible approach goes as follows, though obstacles remain (even assuming other conjectures). If there is some m such that $\Phi^{-m}(\mathrm{Supp} D) = \mathrm{Supp} D$, then one naturally gets arithmetic sequences $\ell, \ell+m, \ldots, \ell+km, \ldots$. Otherwise, one might expect that for large n, the reduced divisor R such that $\mathrm{Supp} R = \mathrm{Supp}(\Phi^n)^* D$ will have the property that $R + K_X$ is ample, where K_X is the canonical divisor of X. Then a conjecture of Vojta [14] would imply that the (S,R)-integral points are not dense as long as the singularities of R are not too bad (this approach is used to treat similar questions in [17] and [11]). Using the conjectural general form of [2, Corollary 1.4], one sees that if there are infinitely many (S,R)-integral points in the orbit of α, then since these points are not dense, the entire orbit is not dense. Then one applies the inductive hypothesis (on dimension) to the Zariski closure of the orbit of α, as in Theorem 2.1. We note that one might expect to obtain a finiteness result unless there is an m such that $\Phi^{-m}(D \cap W) = D \cap W$, where W is the union of the positive-dimensional components of the Zariksi closure of $Orb_\Phi(\alpha)$.

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