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DEVIATIONS FROM S-INTEGRALITY IN ORBITS ON $\mathbb{P}^N$

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Abstract

Silverman proved that when the second iterate of a rational function ϕ is not a polynomial, there are only finitely many S-integral points in each orbit of a rational point. We will survey prior results that attempt to generalize this result to higher-dimensions, and then we will discuss some extensions. More specifically, one new result incorporates geometric properties from multiple iterates simultaneously, while another generalizes to maps with some indeterminancy. All of these general theories assume some version of a very deep Diophantine conjecture by Vojta, but we will give explicit examples for which this conjecture can be avoided. We will also give some examples of maps for which these general theories do not apply directly but for which deviations from S-integrality in orbits can be analyzed unconditionally. We will end by posing many questions still to be answered.

Given a self-map $\phi : X \longrightarrow X$, dynamics studies the asymptotic behaviors of the n-fold iteration $\phi^{(n)} = \underbrace{\phi \circ \cdots \circ \phi}_{n \text{ times}}$. One of the fundamental objects of interest in dynamics is the orbit $\mathcal{O}_\phi(P) = \{P, \phi(P), \phi^{(2)}(P), \ldots\}$ of a point P. For instance, P is called *preperiodic* for ϕ if $\mathcal{O}_\phi(P)$ is finite, and the behaviors around (pre)periodic points are crucial in understanding the dynamics. In arithmetic dynamics, we assume that X is an algebraic variety over a number field k and ϕ a morphism or a rational map over k. Here, some additional questions arise, such as whether there is a uniform bound for the number of preperiodic points defined over k as we vary ϕ in

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some family. Another natural object of study in the arithmetic setup is the intersection of an orbit with the set of integral points. This was completely analyzed by Silverman [9] in the case of morphisms on $\mathbb{P}^1$, proving that unless ϕ satisfies a special geometric property, the points in an orbit become further and further away from being integral.

Theorem 1 (Silverman). *Let $\phi \in \mathbb{Q}(z)$ be a rational function of degree ≥ 2.*

(i) *If $\phi^{(2)}$ is not a polynomial (i.e. not in $\mathbb{Q}[z]$), then $\mathcal{O}_\phi(P) \cap \mathbb{Z}$ is a finite set for any $P \in \mathbb{Q}$.*

(ii) *Assume that neither $\phi^{(2)}$ nor $\frac{1}{\phi^{(2)}(1/z)}$ is a polynomial. If we write $\phi^{(m)}(P) = a_m/b_m$ in a reduced form, then for any P with $|\mathcal{O}_\phi(P)| = \infty$,*

$$\lim_{m\to\infty} \frac{\log |a_m|}{\log |b_m|} = 1.$$

As an attempt to generalize Theorem 1 to higher dimensions, we pose the following question:

Question 2. Let $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a rational map of degree > 1 defined over a number field k, D be an effective divisor of $\mathbb{P}^N$, S be a finite set of places of k, and R_S be the set of S-integers of k.

(1) What kind of geometric condition pertaining to ramification forces $\mathcal{O}_\phi(P) \cap (\mathbb{P}^N \backslash D)(R_S)$ to be Zariski-non-dense in $\mathbb{P}^N$ for any $P \in \mathbb{P}^N(k)$ whose orbit avoids the indeterminancy locus of ϕ?

(2) More specifically, can we take the "geometric condition" in (1) to be not having any completely invariant subvarieties (i.e. possibly reducible subvariety V of $\mathbb{P}^N$ such that $\phi^{-1}(V) = V$ set-theoretically) defined over $\overline{\mathbb{Q}}$?

In this article, we will discuss previous and new results which can be viewed as potential answers to the above central question. There are three main difficulties in generalizing Silverman's result to higher dimension. First, Diophantine approximation results are not quite fully developed. Silverman used Roth theorem, which does not yet have a true analog in higher dimensions, as Schmidt's subspace theorem only deals with hyperplanes and the known approximation results for hypersurfaces are still weaker than optimal. Secondly, as discussed at the end of Section 4, the geometry of divisors is obviously much more complicated in higher dimensions. In dimension 1, all irreducible divisors are just points, while an irreducible divisor in higher dimensions can be non-smooth or can self-intersect. Thirdly, a self-map on $\mathbb{P}^N$ can be a rational map, and the presence of indeterminancy causes complications for degree growths and for height inequalities.

Because of these difficulties, we will resort to assuming a very deep Diophantine conjecture of Vojta for most of the theoretical results presented here. This has the unwanted consequence that we can only deal with normal-crossings divisors; this type of assumption is perhaps not necessary to answer Question 2, but it is unavoidable if one is using Vojta's conjecture. We also present specific examples where this conjecture can be avoided by using the known special cases of the conjecture. As for rational maps, the results presented here involve the notion of arithmetic degree, introduced recently by Silverman. As will be discussed in a remark after Theorem 12, Question 2 is nontrivial only when $\mathcal{O}_\phi(P)$ is Zariski-dense, in which case the arithmetic degree is conjecturally equal to the dynamical degree and thus a geometric notion.

We now discuss the results in a bit more detail. In Section 2, we quote the main theorems from [14]. They give some candidates for Question 2 (1) for *morphisms* on $\mathbb{P}^N$, assuming Vojta's conjecture. We also quote another result (Theorem 6) from [14] which avoids this conjecture, by instead using the known cases of the so-called Lang–Vojta conjecture for integral points. In Section 3, we prove several new results that generalize prior results. Theorem 7 incorporates geometric properties from pullbacks by multiple iterates simultaneously. Theorems 10 and 12 give two attempts at generalizing results of Section 2 to rational maps, using the notion of arithmetic degree. Section 4 goes back to analyzing $\mathbb{P}^1$, and we show that while there are examples which indicate that Theorem 7 for dimension 1 is not strong enough to give Silverman's theorem, we make a new observation that we can actually use Theorem 6 to recover the full-strength of the $\mathbb{P}^1$ result. Section 5 discusses a special family of rational maps on $\mathbb{P}^2$ that have a completely invariant point. Although Theorem 7 does not apply to these maps, we prove its analog (Proposition 15) without assuming any conjecture. This actually shows that the condition in Question 2 (2) is *not* a necessary condition, perhaps indicating that searching for the necessary and sufficient condition in (1) might be very difficult. In the last section, we mention some examples that demonstrate why the nonexistence of completely invariant subvarieties is a potential candidate for Question 2, and discuss many further related questions.

1. Background on Heights and Vojta's Conjecture

In this section, we will set some notations and introduce height functions and Vojta's conjecture. We will limit ourselves to working with projective spaces since this is all we need, but the height theory can be developed for projective varieties and Vojta's conjecture can be stated for any smooth projective variety. For a more complete account of height theory and Diophantine geometry in general, see [1] or [4], and for Vojta's conjecture specifically, see the original account [12] or a more recent survey [13].

Let k be a number field, and let M_k be the set of places of k. Given $v \in M_k$, we will use the convention that $|\cdot|_v$ restricted to $\mathbb{Q}$ is the $\frac{[k_v:\mathbb{Q}_v]}{[k:\mathbb{Q}]}$-th power of the normalized absolute value on $\mathbb{Q}$, namely the usual absolute value for archimedean and $|p|_p = \frac{1}{p}$. This convention makes the product formula simple: $\prod_{v \in M_k} |x|_v = 1$ for $x \in k^*$. We then define a (global) Weil height on $\mathbb{P}^N(\overline{\mathbb{Q}})$ by

$$h([x_0 : \cdots : x_N]) = \sum_{v \in M_k} \log \max(|x_0|_v, \ldots, |x_N|_v),$$

where k is any field containing the field of definition of the point $[x_0 : \cdots : x_N]$. We can check that this function is well-defined. When $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^M$ is a rational map defined by homogeneous polynomials of degree d in each coordinate, there exists a constant C_1 such that

$$h(\phi(P)) \leq dh(P) + C_1 \tag{1}$$

for all $P \in \mathbb{P}^N(\overline{\mathbb{Q}})$, and when ϕ is a morphism (i.e. well-defined everywhere) of degree d, there exists a constant C_2 such that

$$h(\phi(P)) \geq dh(P) - C_2$$

for all $P \in \mathbb{P}^N(\overline{\mathbb{Q}})$. For a divisor D on $\mathbb{P}^N$, we simply define $h_D(P) = (\deg D)h(P)$.

Local height functions are more arithmetic in nature. Given an effective divisor D on $\mathbb{P}^N$ and $v \in M_k$, choose a homogeneous polynomial F_v of degree

d defining D such that coefficients are v-adically integral, with one of them having valuation zero if v is non-archimedean. We then define the v-adic local height of D to be

$$\lambda_v(D,[x_0:\cdots:x_N]) = \log \frac{\max(|x_0|_v,\ldots,|x_N|_v)^d}{|F_v(x_0,\ldots,x_N)|_v}.$$

This function has a logarithmic pole along $|D|$, it is big when the point $[x_0:\cdots:x_N]$ is v-adically close to the divisor D, and is non-negative for any non-archimedean v. A key property of local heights is the decomposition

$$\sum_{v\in M_k} \lambda_v(D,P) = h_D(P) + O(1) \qquad P \notin |D|. \tag{2}$$

Note that the sum on the left-hand-side is always finite for all $P \notin |D|$ and it only changes by $O(1)$ when we choose different F_v's.

Given a rational map $\phi = [G_0 : \cdots : G_N] : \mathbb{P}^M \dashrightarrow \mathbb{P}^N$ defined by homogeneous polynomials G_j's in $M+1$ variables of degree e, we have by definition

$$\lambda_v(D,\phi(P)) = \log \frac{\max(|G_0(P)|_v,\ldots,|G_N(P)|_v)^d}{|F_v(G_0(P),\ldots,G_N(P))|_v}$$
$$\lambda_v(\phi^*(D),P) = \log \frac{\max(|x_0|_v,\ldots,|x_M|_v)^{de}}{|F_v(G_0(P),\ldots,G_N(P))|_v}$$

for $P = [x_0 : \cdots : x_M] \notin \phi^{-1}(|D|)$. Therefore, it immediately follows from the triangle inequality that

$$\lambda_v(D,\phi(P)) \le \lambda_v(\phi^*(D),P) + O(1) \qquad P \in \mathbb{P}^M(k)\backslash\phi^{-1}(|D|) \tag{3}$$

for any *rational* map $\phi : \mathbb{P}^M \dashrightarrow \mathbb{P}^N$, obtaining the local height version of (1). When ϕ is further a morphism, we have the full functoriality with respect to the pullback:

$$\lambda_v(\phi^* D,P) = \lambda_v(D,\phi(P)) + O(1) \qquad P \in \mathbb{P}^M(k)\backslash\phi^{-1}(|D|).$$

Another useful property of local heights as defined above is that when k' is a finite extension of k, we have

$$\lambda_v(D,P) = \sum_{\substack{w \in M_{k'} \\ w|v}} \lambda_w(D,P) \tag{4}$$

for all $P \in \mathbb{P}^N(k) \backslash |D|$. We also note that on $\mathbb{P}^1$, one can alternatively define local heights by

$$\lambda_v(\alpha, x) = \max(0, -\log|x-\alpha|_v), \tag{5}$$

as one can for example use Lemma 6.2 of [13] to show that the two definitions differ only by a bounded function and this new definition still satisfies (2).

When $D = (X_N = 0)$ and $S \subset M_{\mathbb{Q}}$ containing the archimedean place, writing $P \in \mathbb{P}^N(\mathbb{Q})$ as $[x_0 : \cdots : x_N]$ with $x_i \in \mathbb{Z}$ with gcd 1, we see from the definition that

$$\sum_{v \notin S} \lambda_v(D,P) = \log|x_N|'_S,$$

where $|\cdot|'_S$ is the prime-to-S part of an integer. More generally, a set of *S-integral points with respect to D* is a set of the form

$$\left\{ P : \sum_{v \notin S} \lambda_v(D,P) \le C \right\}$$

for some constant C, while a set of *quasi-(S,ϵ)-integral points* as defined in [5] is a set of the form

$$\left\{ P : \sum_{v \notin S} \lambda_v(D,P) \le (1-\epsilon) h_D(P) + C \right\} \tag{6}$$

for some $0 < \epsilon \le 1$ and a constant C.

Given a morphism $\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ of degree $d \ge 2$, we can mimic the Néron–Tate construction to give a (dynamical) canonical height. More precisely,

$$\lim_{m \to \infty} \frac{h(\phi^{(m)}([x_0 : \cdots : x_N]))}{d^m}$$

converges by a telescoping sum, and we call this the canonical height $\hat{h}_\phi([x_0 : \cdots : x_N])$. It is zero if and only if the point is preperiodic, just as the

Néron–Tate height distinguishes torsion points. There is a vast literature on canonical heights and their dynamical applications; see for example [10].

To introduce Vojta's conjecture, recall that a divisor is said to be *normal-crossings* if near each point it is defined by $x_1 \cdots x_k = 0$, where $x_1, \ldots, x_k$ is a subsystem of (analytic) coordinates. By definition, the multiplicity of each irreducible component of a normal-crossings divisor is 1. We are now ready to state Vojta's conjecture (restricted to $\mathbb{P}^N$):

Conjecture 3 (Vojta's Conjecture for $\mathbb{P}^N$)**.** *Let D be a normal-crossings divisor on $\mathbb{P}^N$ defined over k, and $S \subset M_k$ finite. Then given $\epsilon > 0$, there exist a finite union Z_ϵ of hypersurfaces and a constant C such that for $P \in \mathbb{P}^N(k)\backslash Z_\epsilon$,*

$$\sum_{v \in S} \lambda_v(D, P) < (N + 1 + \epsilon)h(P) + C. \tag{7}$$

This $N+1$ comes from the fact that the degree of the canonical divisor of $\mathbb{P}^N$ is $-(N+1)$. Since the local heights are big when P is v-adically close to D, the conjecture says that the arithmetic property of how close a rational point P can be to D is controlled by the global geometry of how negative the canonical divisor is. When N is equal to 1, this conjecture is equivalent to Roth's theorem [1, Proposition 14.2.7]. Thus, viewing Roth's theorem in the framework of Vojta's conjecture gives a geometric meaning to the approximation exponent 2. When D is a union of hyperplanes, this conjecture is equivalent to the Schmidt subspace theorem. The other cases are mostly unknown, and it is a very deep and a powerful conjecture.

2. Prior Results

In this section, we briefly quote results from [14] that are relevant for this article. There are other works dealing with integral points in orbits such as [2] treating maps on $\mathbb{P}^1 \times \mathbb{P}^1$, but we will omit the history here; see the introduction of [14] for a more thorough account.

Given an effective divisor D on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$, we consider all the normal-crossings subdivisors of D over $\overline{\mathbb{Q}}$ and the one with the highest degree will be called *a normal-crossings part* of D, denoted by D_{nc}. We denote the pullback $(\phi^{(n)})^*D$ by $D^{(n)}$ and its normal-crossings part by $D^{(n)}_{\mathrm{nc}}$ when the map ϕ is clear. The following theorem is an improvement of the main theorem of [15].

Theorem 4 (cf. Theorem 2 in [14]). *Let $\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ be a morphism defined over $\mathbb{Q}$ of degree $d \geq 2$, and let D be an effective divisor on $\mathbb{P}^N$. If $\deg D_{\mathrm{nc}}^{(n)} > N+1$ for some n, then Vojta's conjecture on $\mathbb{P}^N$ for the divisor $D_{\mathrm{nc}}^{(n)}$ implies that for any $P \in \mathbb{P}^N(\mathbb{Q})$, $\mathcal{O}_\phi(P) \cap (\mathbb{P}^N \backslash D)(\mathbb{Z})$ is Zariski-non-dense.*

The hypothesis in this theorem for $N = 1$ agrees with the hypothesis in (i) of Silverman's theorem (Theorem 1). Indeed, if a rational function satisfies $\phi^{(2)} \notin k[x]$, then it can be shown from the Riemann–Hurwitz formula that $\phi^{(4)}$ has at least 3 distinct poles. Alternatively, one can use a geometric lemma of Silverman (see Lemma 14 in Section 4) to show the existence of n such that $\phi^{(n)}$ has at least 3 distinct poles. Conversely, if $\phi^{(2)} \in k[x]$, then it is clear that the number of poles for any $\phi^{(n)}$ is exactly one. Therefore, Silverman's hypothesis is equivalent to the hypothesis in the above theorem for $N = 1$, assuming a geometric lemma based on Riemann–Hurwitz formula.

This theorem shows that the "existence of n for which $\deg D_{\mathrm{nc}}^{(n)} > N+1$" is one answer to Question 2 (1), under Vojta's conjecture. This condition is related to ramification, as highly ramified maps keep $\deg D_{\mathrm{nc}}^{(n)}$ low.

We now state the analog of the second part of Silverman's theorem:

Theorem 5 (cf. Corollary 1 in [14]). *Let $c = \sup_n \frac{\deg D_{\mathrm{nc}}^{(n)} - (N+1)}{d^n \deg(D)}$, and let $S \subset M_k$ be finite. Then assuming Vojta's conjecture for $\mathbb{P}^N$, for all $\epsilon > 0$ and $P \in \mathbb{P}^N(k)$,*

$$\left\{ \phi^{(m)}(P) : \frac{\sum\limits_{v \notin S} \lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq c - \epsilon \right\} \tag{8}$$

is Zariski-non-dense.

When D is the point at infinity in $\mathbb{P}^1$ and S is just the archimedean place of $\mathbb{Q}$, $\sum_{v \notin S} \lambda_v(D, x)$ is the logarithm of the denominator of x, so this corresponds to the second part of Theorem 1. Using (6), this theorem states that the intersection of a set of quasi-$(S, 1-(c-\epsilon))$-integral points with an orbit is Zariski-non-dense. All the remaining results in this paper are similar in the sense that we show the Zariski-non-density of an appropriately-chosen set of quasi-S-integral points in orbits.

The obvious shortcoming of the above two theorems is the need to assume Vojta's conjecture. An irreducible divisor in higher dimension can be highly singular and can even self-intersect, unlike in dimension 1. Since it is difficult in general to control the pullback of a divisor by an iterate, this deep conjecture in Diophantine geometry is employed. To compensate for this problem, we discuss several results in [14] which avoid the usage of this conjecture. One case is when the normal-crossings part of $D^{(n)}$ is a union of hyperplanes, since then Schmidt's subspace theorem can be used in place of Vojta's conjecture. This has the additional advantage that the exceptional set to the height inequality (7) is a union of hyperplanes *defined over the same field as* the map and the initial point. This is exploited in Example 1 of [14], but we will not discuss it further as it is not needed in this article.

Instead, we quote the following, which takes advantage of another known special case of Vojta's conjecture, known as Lang–Vojta conjecture for integral points:

Theorem 6 (cf. Proposition 2 of [14]). *Let $\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ be a morphism of degree $d \geq 2$ defined over $\overline{\mathbb{Q}}$. Let D be an effective divisor on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$, and let $P \in \mathbb{P}^N(\overline{\mathbb{Q}})$. Let k be a number field that contains the fields of definition of ϕ, of irreducible components of D over $\overline{\mathbb{Q}}$, and of P, and let $S \subset M_k$ be a finite subset. Suppose there exists n with $(\phi^{(n)})^*D = D_1 + D_2$ such that*

- *D_1 contains $N+2$ distinct geometrically-irreducible components,*
- *$\exists \alpha < \deg(D_2)$ such that $\sum_{v \in S} \lambda_v(D_2, \phi^{(m)}(P)) \leq \alpha h(\phi^{(m)}(P)) + O(1)\ \forall m \gg 0$.*

$$\textit{Then } \left\{ \phi^{(m)}(P) : \frac{\sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P))}{(\deg D) h(\phi^{(m)}(P))} \leq \frac{\deg(D_2) - \alpha}{(\deg D) d^n} \right\} \textit{ is Zariski-non-dense.}$$

We now generalize these results, and use them to analyze in some special examples the deviation from S-integrality in orbits.

3. A Generalization of Theorems 4 and 5

In this section, we will discuss some generalizations of Theorems 4 and 5. First, we discuss a result that incorporates pullbacks by multiple iterates simultaneously.

Theorem 7. *Let $\phi : \mathbb{P}^N \longrightarrow \mathbb{P}^N$ be a morphism of degree $d \geq 2$ and let D be an effective divisor on $\mathbb{P}^N$, both defined over $\overline{\mathbb{Q}}$. Let $n_1, \dots, n_\ell$ be natural numbers and let D' be the normal-crossings part of the divisor $D^{(n_1)} + \cdots + D^{(n_\ell)}$. Let k be a number field containing the fields of definition of ϕ and D'. If $c := \deg D' - (N+1)$ is strictly positive, then Vojta's conjecture on $\mathbb{P}^N(k)$ for the divisor D' implies that given $S \subset M_k$ finite and $\epsilon > 0$,*

$$\left\{ \phi^{(m)}(P) : \frac{\sum_{v \notin S} \lambda_v(D, \phi^{(m+n_i)}(P))}{\deg(D) h(\phi^{(m+n_i)}(P))} \leq \frac{c}{(\deg D)\,(d^{n_1} + \cdots + d^{n_\ell})} - \epsilon \ \forall i = 1, \dots, \ell \right\} \tag{9}$$

is Zariski-non-dense for any $P \in \mathbb{P}^N(k)$.

Proof. The proof of this is very similar to the proof of Theorem 4 in [14]. This theorem is obvious if P is preperiodic, so we may assume that $|\mathcal{O}_\phi(P)| = \infty$. In particular, $h(\phi^{(m)}(P)) \to \infty$ as $m \to \infty$ since they are all defined over k. By applying Vojta's conjecture to the divisor D',

$$\sum_{v \in S} \lambda_v(D', Q) < \left(N + 1 + \frac{\epsilon}{2} \right) h(Q) + O(1) \tag{10}$$

for all $Q \in \mathbb{P}^N(k)$ except for points on a finite union Z_ϵ of hypersurfaces. Since global heights change only by bounded functions in linear equivalence,

$$\begin{aligned} &\sum_{v \in S} \lambda_v \left(D^{(n_1)} + \cdots + D^{(n_\ell)} - D', Q \right) \\ &\qquad \leq h_{D^{(n_1)} + \cdots + D^{(n_\ell)} - D'}(Q) + O(1) \\ &\qquad \leq \left((d^{n_1} + \cdots + d^{n_\ell}) \deg(D) - (c + N + 1) \right) h(Q) + O(1). \end{aligned}$$

Thus, by adding these two inequalities, for $Q \notin Z_\epsilon$,

$$\sum_{v \in S} \lambda_v(D^{(n_1)} + \cdots + D^{(n_\ell)}, Q) \leq \left((d^{n_1} + \cdots + d^{n_\ell}) \deg(D) - c + \frac{\epsilon}{2} \right) h(Q) + O(1).$$

Now, note that by (3), $\lambda_v(D, \phi^{(n_i)}(Q)) \le \lambda_v(D^{(n_i)}, Q) + O(1)$ for each i and $v \in S$. Therefore, plugging in $Q = \phi^{(m)}(P)$, we obtain

$$\begin{aligned}&\sum_{i=1}^{\ell}\sum_{v\in S}\lambda_v(D,\phi^{(m+n_i)}(P))\\ &\quad\le \left((d^{n_1}+\cdots+d^{n_\ell})\deg(D) - c + \frac{\epsilon}{2}\right) h(\phi^{(m)}(P)) + O(1) \qquad (11)\end{aligned}$$

as long as $\phi^{(m)}(P) \notin Z_\epsilon$. On the other hand, if m is such that $\phi^{(m)}(P)$ is in the set (9), we have

$$\begin{aligned}&\sum_{i=1}^{\ell}\sum_{v\notin S}\lambda_v(D,\phi^{(m+n_i)}(P))\\ &\le \sum_{i=1}^{\ell}\left(\frac{c}{(\deg D)\,(d^{n_1}+\cdots+d^{n_\ell})} - \epsilon\right)\cdot \deg D\cdot h(\phi^{(m+n_i)}(P)) + O(1)\\ &\le \sum_{i=1}^{\ell}\left(\frac{c}{(\deg D)\,(d^{n_1}+\cdots+d^{n_\ell})} - \epsilon\right)\cdot \deg D\cdot d^{n_i} h(\phi^{(m)}(P)) + O(1) \quad (12)\\ &< \left(\frac{c}{(\deg D)\,(d^{n_1}+\cdots+d^{n_\ell})} - \epsilon\right)\cdot \deg D\cdot (d^{n_1}+\cdots+d^{n_\ell})\, h(\phi^{(m)}(P)) + O(1)\\ &= (c - \deg D\cdot (d^{n_1}+\cdots+d^{n_\ell})\,\epsilon)\, h(\phi^{(m)}(P)) + O(1),\end{aligned}$$

where we used (1) in (12). Combining with (11) and using (2), we conclude that

$$\begin{aligned}&\sum_{i=1}^{\ell} h_D(\phi^{(m+n_i)}(P)) - O(1) \qquad (13)\\ &\le \left((d^{n_1}+\cdots+d^{n_\ell})\deg(D) - \left(\deg D\cdot (d^{n_1}+\cdots+d^{n_\ell}) - \frac{1}{2}\right)\epsilon\right) h(\phi^{(m)}(P))\end{aligned}$$

if $\phi^{(m)}(P)$ is in (9) but outside of Z_ϵ. We now use the assumption that ϕ is a morphism, so that we have the height inequality $h(\phi^{(m+n_i)}(P)) \ge d^{n_i} h(\phi^{(m)}(P)) + O(1)$ for each i. As $h(\phi^{(m)}(P)) \to \infty$ as $m \to \infty$, we have a contradiction for sufficiently large m. Thus, $\phi^{(m)}(P)$ in (9) either comes from a finite set or is in Z_ϵ, finishing the proof. $\square$

Remark. Of course, one would like to conclude that

$$\left\{\phi^{(m)}(P) : \frac{\sum\limits_{v\notin S}\lambda_v(D,\phi^{(m)}(P))}{\deg(D)\cdot h(\phi^{(m)}(P))} \le \frac{c}{(\deg D)d^{n_\ell}} - \epsilon\right\}$$

is Zariski-non-dense even when we use the normal-crossings part of $D^{(n_1)} + \cdots + D^{(n_\ell)}$; this could mean that

$$\text{the existence of } n_1,\ldots,n_\ell \text{ such that } \deg\left(D^{(n_1)}+\cdots+D^{(n_\ell)}\right)_{\mathrm{nc}} > N+1$$

is an answer for Question 2 (1). On the other hand, this type of conclusion seems beyond reach by our method: to get a contradiction at the end, we must have $\sum\limits_{v\notin S}\lambda_v(D,\phi^{(m+n_i)}(P))$ to be sufficiently small for each i, necessitating a condition for each $\phi^{(m+n_i)}(P)$ in (9).

Example 8. Let $\phi = [Z^2(X+Y+Z) : F_1 : XY^2]$ be a morphism on $\mathbb{P}^2$, where $F_1 \in \mathbb{Q}[X,Y,Z]$ is homogeneous of degree 3 such that F_1 at $[0:1:0]$, $[0:1:-1]$, $[1:0:0]$, and $[1:0:-1]$ are nonzero. For $D=(Z=0)$, $\phi^*(D)$ is defined by XY^2 and $(\phi^{(2)})^*(D)$ is defined by $Z^2(X+Y+Z)F_1^2$. Assume that F_1 is geometrically irreducible, and $F_1=0$ goes through the point $[1:-1:0]$ and has a cusp there. Then $D_{\mathrm{nc}}^{(1)} = (XY=0)$ and $D_{\mathrm{nc}}^{(2)} = (Z(X+Y+Z)=0)$. Neither of these produce enough normal-crossings parts, and in general, $D_{\mathrm{nc}}^{(n)}$ for $n\ge 3$ will be a divisor for which Vojta's conjecture is not yet known. Therefore, Theorems 4 and 5 do not give us unconditional results. On the other hand, $D^{(1)}+D^{(2)}$ contains the divisor $D' = (XYZ(X+Y+Z)=0)$, which is linear of degree 4. Since Vojta's conjecture for D' is known by the Schmidt subspace theorem, given $\epsilon>0$ and $S\subset M_{\mathbb{Q}}$ finite, we can *unconditionally* conclude from Theorem 7 that

$$\left\{\phi^{(m)}(P) : \frac{\log|c_m|_S'}{\log\max(|a_m|,|b_m|,|c_m|)} \le \frac{1}{12}-\epsilon \right.$$
$$\left.\text{and} \quad \frac{\log|c_{m+1}|_S'}{\log\max(|a_{m+1}|,|b_{m+1}|,|c_{m+1}|)} \le \frac{1}{12}-\epsilon\right\}$$

is Zariski-non-dense, where we write $\phi^{(m)}(P) = [a_m : b_m : c_m]$ with integer coordinates without common divisor. Further, since the exception to the inequality of the subspace theorem is contained in a finite union of hyperplanes defined over $\mathbb{Q}$, we can even conclude *finiteness* of the above set if no line defined over $\mathbb{Q}$ contains an infinite subset of the orbit of P (cf. Proposition 1 and Example 1 of [14]).

Next, we discuss some extensions of Theorem 7 to rational maps. For this, we will first recall the notion of dynamical degree and arithmetic degree. Given a rational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$, $\deg \phi^{(n)}$ may not be equal to $(\deg \phi)^n$. The (first) *dynamical degree*, introduced by Russakovskii and Shiffmann [8], is

$$\delta_\phi = \lim_{n\to\infty} (\deg(\phi^{(n)}))^{1/n}.$$

When ϕ is a morphism, $\delta_\phi = \deg \phi$, so δ_ϕ gives some indication of how far ϕ is from being a morphism.

The *arithmetic degree* $\alpha_\phi(P)$ of a point P is defined to be

$$\alpha_\phi(P) = \limsup_{m\to\infty} h(\phi^{(m)}(P))^{1/m}$$

whenever the orbit of P does not intersect the indeterminancy locus of ϕ. This is a very new concept introduced by Silverman [11], intending to capture the arithmetic complexity of the orbit, and it has already been intensely studied. Many conjectures have been raised by Silverman in connection with the arithmetic degree, including the conjecture that we can use the limit rather than limsup in the definition of $\alpha_\phi(P)$. In this article, what is relevant is the following.

Conjecture 9 (Silverman)**.** *Let $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a dominant rational map defined over $\overline{\mathbb{Q}}$. If $P \in \mathbb{P}^N(\overline{\mathbb{Q}})$ is such that $\mathcal{O}_\phi(P)$ is Zariski-dense, then $\alpha_\phi(P) = \delta_\phi$.*

This conjecture is nontrivial even for morphisms. It has been verified for monomial maps and regular affine automorphisms [11], and its generalization to projective varieties has been proved for endomorphisms on abelian varieties [6].

Now we are ready to state the extensions of Theorems 4 and 5 to rational maps. In fact, we will prove two results. Theorem 10 is actually a special case of Theorem 12, but the conditions in Theorem 12 are difficult to check in explicit examples, so we highlight one specific case in Theorem 10 and then discuss an explicit example (Example 11).

Note that one extension to rational maps was already proposed in Theorem 5 of [14], using the notion of D-ratio introduced by Lee [7]. The D-ratio gives a useful height inequality, but its drawback is that it only applies to rational maps whose indeterminancy is contained in a hyperplane. The condition in the following theorems involves the arithmetic degree, and thus it might be possible to prove deviations from S-integrality in orbits for certain points even when the map overall is very complicated. The multiple-iterate version similar to Theorem 7 is also possible, but to keep the statement relatively simple, we only state here for one iterate.

Theorem 10. *Let $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a rational map of degree $d \geq 2$ defined over a number field k such that $\deg(\phi^{(n)}) = d^n$. Let D be an effective divisor on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$. Let us suppose that $P \in \mathbb{P}^N(k)$ and a natural number n satisfy:*

(i) *$\mathcal{O}_\phi(P)$ does not intersect the indeterminancy locus of ϕ.*

(ii) *Writing $h(\phi^{(m)}(P)) = (da_m)^m$, $\lim a_m = 1$ (in particular, $\alpha_\phi(P)$ can be defined by the limit and is equal to $d = \delta_\phi$).*

(iii) *$c := \deg(D_{\mathrm{nc}}^{(n)}) - (N + 1)$ is strictly positive.*

(iv) *For any $\delta > 0$, $\frac{a_{m+n}^{m+n}}{a_m^m} > 1 - \delta$ for all sufficiently large m.*

Then Vojta's conjecture on $\mathbb{P}^N$ for the divisor $D_{\mathrm{nc}}^{(n)}$ implies that given $S \subset M_k$ finite and $\epsilon > 0$,

$$\left\{ \phi^{(m)}(P) : \frac{\sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq \frac{c}{(\deg D) d^n} - \epsilon \right\} \tag{14}$$

is Zariski-non-dense.

Remark. Of course, as a corollary of Theorem 10, we now have Theorem 4 for *rational* maps on $\mathbb{P}^N$ and points P satisfying (i)–(iv). Indeed, if $\mathcal{O}_\phi(P)$ is not Zariski-dense, then the statement is obvious, so we may assume that $h(\phi^{(m)}(P)) \to \infty$ as $m \to \infty$. The sum of local heights outside S are bounded for S-integral points, so by taking $\epsilon = \frac{c}{2(\deg D) d^n}$, every $\phi^{(m)}(P)$ which is S-integral will be in (14) for all sufficiently large m. Thus, we can conclude that S-integral points in an orbit are Zariski-non-dense.

Remark. When $\mathcal{O}_\phi(P)$ is Zariski-dense, (ii) should be automatic according to the conjecture of Silverman. Therefore, assumption (ii) is quite natural. On the other hand, assumption (iv) is unwanted.

If we assume (ii) *and* $\{a_m\}$ is an eventually-monotone increasing sequence, then (iv) is unnecessary. Indeed, for large enough m, $a_m > \sqrt[n]{1-\delta}$, and so

$$\frac{a_{m+n}^{m+n}}{a_m^m} = \left(\frac{a_{m+n}}{a_m}\right)^m \cdot a_{m+n}^n > 1-\delta.$$

On the other hand, there are sequences that satisfy (ii) but not (iv): let $n=1$, and define the sequence $\{a_m\}$ inductively by

$$\begin{aligned} a_{2m} &= e^{-\frac{1}{20m}} a_{2m-1} \\ a_{2m+1} &= e^{\frac{1}{20m}} b_m a_{2m}, \end{aligned}$$

where $a_1 \prod b_m$ converges to 1. This sequence satisfies (ii) but

$$\frac{a_{2m}^{2m}}{a_{2m-1}^{2m-1}} = \left(\frac{a_{2m}}{a_{2m-1}}\right)^{2m-1} \cdot a_{2m} = \exp\left(-\frac{1}{10} + \frac{1}{20m}\right) \cdot a_{2m},$$

which goes to $\exp(-1/10) < 1$. We would of course like to remove assumption (iv), but the condition $\alpha_\phi(P) = \delta_\phi$ does not seem to be quite enough to obtain the result. We note that Theorem 12 below can handle a sequence like this.

Proof. We note that the proof of Theorem 7 holds for rational maps until the last paragraph when we obtain the contradiction. Everything before uses just (1) and (3), which are true for rational maps. So we still have (13), using $Q = \phi^{(m-n)}(P)$:

$$h_D(\phi^{(m)}(P)) \le \left(d^n \deg(D) - \left(\deg D \cdot d^n - \frac{1}{2}\right)\epsilon\right) h(\phi^{(m-n)}(P)) + O(1) \tag{15}$$

for $\phi^{(m-n)}(P) \notin Z_\epsilon$. By assumption (ii), we have

$$\frac{h(\phi^{(m)}(P))}{h(\phi^{(m-n)}(P))} = d^n \frac{a_m^m}{a_{m-n}^{m-n}}.$$

By (iv), for sufficiently large m, this is greater than $d^n - \epsilon$, so by dividing (15) by $\deg(D)$ throughout, we see that $h(\phi^{(m)}(P))$ must be bounded. Hence, we conclude that the set (14) comes from a finite set of m's plus $\phi^{(n)}(Z_\epsilon)$. $\square$

Example 11. Let $\phi = [256X^2 : (X+Y)(X+2Y+Z) : YZ]$ on $\mathbb{P}^2$. It is a rational map undefined at $[0:0:1]$, but because of the first coordinate, it is clear that $\deg \phi^{(n)} = 2^n$. The divisor $(\phi^{(2)})^*(Z=0)$ is defined by $(X+Y)(X+2Y+Z)YZ$, so it is a normal-crossings union of four lines. Note that $[a:a:1]$ gets mapped to $[256a : 6a+2 : 1]$, so there is no constant C such that $h(\phi(P)) \geq 2h(P) - C$ for all P. On the other hand, if we look at $P = [4:1:1]$, $\phi(P) = [4^6 : 35 : 1]$, and it is easy to see by induction that the first coordinate of $\phi^{(m)}(P)$ is always a power of 2 while the last two coordinates are odd. In particular, there is never a common factor when computing the orbit. Moreover, one can easily prove by induction that the first coordinate of $\phi^{(m)}(P)$ is always at least six times as large as the second coordinate for all $m \geq 1$. Thus, the height comes from the first coordinate, and we see by induction that

$$\begin{aligned} a_m^m = \frac{h(\phi^{(m)}(P))}{2^m} &= \frac{((1+2+\cdots+2^{m-1})\cdot 8 + 2^m \cdot 2)\log 2}{2^m} \\ &= \left(6+2+1+\frac{1}{2}+\cdots+\frac{1}{2^{m-3}}\right)\log 2 \leq 10\log 2. \end{aligned}$$

It is now easy to see that both (ii) and (iv) are satisfied in this case. Moreover, we can use Schmidt's subspace theorem in place of Vojta's conjecture, so we *unconditionally* conclude that for any finite set S of primes and $\epsilon > 0$,

$$\left\{\phi^{(m)}(P) : \frac{\log |c_m|_S'}{\log \max(|a_m|, |b_m|, |c_m|)} \leq \frac{1}{4} - \epsilon\right\}$$

is Zariski-non-dense, where we write $\phi^{(m)}(P) = [a_m : b_m : c_m]$ with integers with gcd 1. In this example, from the analysis of the 2-adic behavior of orbit points (together with global height computations), we obtain information about arithmetic of orbit points with respect to other primes.

As mentioned earlier, we end this section with a generalization of Theorem 10. There are fewer hypotheses, but as a consequence, the conclusion might be vacuous in certain cases. Since one needs to know the Weil height of orbit points in advance, it is probably of little practical use, but this indicates what we can say using the arguments of this article.

Theorem 12. *Let $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ be a rational map of degree $d \geq 2$ defined over a number field k. Let D be an effective divisor on $\mathbb{P}^N$ defined over $\overline{\mathbb{Q}}$. Let n be such that $\deg(D_{\mathrm{nc}}^{(n)}) > (N+1)$. Let $P \in \mathbb{P}^N(k)$ be a non-preperiodic point such that $\mathcal{O}_\phi(P)$ does not intersect the indeterminancy locus of ϕ, and let $\alpha = \alpha_\phi(P)$. Define the sequence $\{a_m\}$ by $h(\phi^{(m)}(P)) = (\alpha a_m)^m$. Finally, let σ be such that*

$$\sigma < 1 - \frac{\deg(\phi^{(n)}) \cdot \deg(D) - \deg(D_{\mathrm{nc}}^{(n)}) + (N+1)}{\alpha^n \deg(D) \liminf_m a_{m+n}^{m+n}/a_m^m}. \tag{16}$$

Then Vojta's conjecture on $\mathbb{P}^N$ for the divisor $D_{\mathrm{nc}}^{(n)}$ implies that for any $S \subset M_k$ finite,

$$\left\{ \phi^{(m)}(P) : \frac{\sum_{v \notin S} \lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq \sigma \right\} \tag{17}$$

is Zariski-non-dense.

Remark. Theorem 10 is a special case. Indeed, we have $\liminf a_{m+n}^{m+n}/a_m^m = 1$: ≥ 1 follows immediately from (iv), while if the liminf is $\beta > 1$, then one can easily show by induction that there exists a nonzero constant C such that $a_m^m > C\left(\sqrt[n]{1 + \frac{\beta-1}{2}}\right)^m$ for all sufficiently large m, contradicting (ii). Therefore, using $\deg(\phi^{(n)}) = d^n = \alpha^n$ and dividing both the numerator and the denominator of (16) by $d^n \deg(D)$,

$$\sigma < 1 - \frac{1 - \frac{\deg(D_{\mathrm{nc}}^{(n)}) - (N+1)}{d^n \deg(D)}}{\liminf} = \frac{\deg(D_{\mathrm{nc}}^{(n)}) - (N+1)}{d^n \deg(D)}.$$

More generally, (16) indicates that if $\deg(\phi^{(n)})$ is sufficiently smaller than α^n, then σ can be chosen positive and this theorem becomes nontrivial. So in this sense, the deviation from S-integrality in orbits is connected to questions about the arithmetic and the dynamical degrees.

Remark. As noted in the introduction, the condition (16) can be viewed as a first step toward answering Question 2 for *rational* maps. Indeed, when the entire orbit $\mathcal{O}_\phi(P)$ is not Zariski-dense, the theorem is trivial; otherwise, α is conjecturally equal to the δ_ϕ, which is a global geometric invariant that does not depend on the arithmetic of the initial point. On the other hand, because of the liminf term in (16), the condition as of right now depends heavily on the arithmetic of the initial point and thus is not geometric.

Proof. The structure of the proof is very close to the proofs of Theorems 7 and 10, but we need to be more careful with the inequality estimates. Since the condition on σ is a *strict* inequality, we can choose $\epsilon > 0$ so that

$$1-\sigma > \frac{\deg(\phi^{(n)}) \cdot \deg(D) - \deg(D_{\mathrm{nc}}^{(n)}) + (N+1) + \epsilon}{\deg(D)\alpha^n \liminf_m a_{m+n}^{m+n}/a_m^m}.$$

As a result, a similar adjustment shows that there exists an M such that for all $m \geq M$, we have

$$1-\sigma > \frac{\deg(\phi^{(n)}) \cdot \deg(D) - \deg(D_{\mathrm{nc}}^{(n)}) + (N+1) + \epsilon}{\deg(D)\alpha^n a_{m+n}^{m+n}/a_m^m}.$$

Therefore, by the definition of $\{a_m\}$, it immediately follows that

$$\begin{aligned}(1-\sigma)&\deg(D)h(\phi^{(m+n)}(P)) \\ &> \left(\deg(\phi^{(n)}) \cdot \deg(D) - \deg(D_{\mathrm{nc}}^{(n)}) + (N+1) + \epsilon\right) h(\phi^{(m)}(P)).\end{aligned} \tag{18}$$

Now, we argue as before. From Vojta's conjecture on $D_{\mathrm{nc}}^{(n)}$ and degree considerations (cf. the argument up to (11)), we have

$$\begin{aligned}&\sum_{v\in S}\lambda_v(D, \phi^{(m+n)}(P)) \\ &\quad\leq \left(\deg(\phi^{(n)}) \deg(D) - \deg(D_{\mathrm{nc}}^{(n)}) + (N+1) + \frac{\epsilon}{2}\right) h(\phi^{(m)}(P)) + O(1)\end{aligned} \tag{19}$$

if $\phi^{(m)}(P) \notin Z_\epsilon$. On the other hand, if $\phi^{(m+n)}(P)$ is in the set (17), we have

$$\sum_{v\notin S}\lambda_v(D, \phi^{(m+n)}(P)) \leq \sigma \deg(D)h(\phi^{(m+n)}(P)).$$

Combining with (19) and rearranging using (2), we obtain

$$\begin{aligned}&(1-\sigma)\deg(D)h(\phi^{(m+n)}(P)) \\ &\leq \left(\deg(\phi^{(n)}) \deg(D) - \deg(D_{\mathrm{nc}}^{(n)}) + (N+1) + \frac{\epsilon}{2}\right) h(\phi^{(m)}(P)) + O(1).\end{aligned}$$

By possibly making M larger, this contradicts (18) for all $m \geq M$. Thus, if $\phi^{(m+n)}(P)$ is in the set (17), then $m < M$ or $\phi^{(n+m)}(P) \notin \phi^{(n)}(Z_\epsilon)$. $\square$

4. The Case of $\mathbb{P}^1$

We have noted after Theorem 4 that its hypothesis for $N = 1$ agrees with Silverman's Theorem (i). In contrast, Silverman's theorem (ii) is different from Theorem 5 for $N = 1$. We now discuss an example that demonstrates that Silverman's conclusion is strictly stronger. To remedy this situation, we will show after the example that Theorem 6 for $N = 1$ can be used to show the full strength of Silverman's Theorem (ii). This enables us to view Silverman's result in the framework of a higher-dimensional theory.

Example 13. We create rational functions of degree $d \geq 3$ which do not become polynomials upon iterations but for which

$$\sup_n \frac{\deg\left(\left(\phi^{(n)}\right)^*(a)\right)_{\mathrm{nc}} - 2}{d^n} = \sup_n \frac{(\#\text{ of distinct preimages of } a \text{ via } \phi^{(n)}) - 2}{d^n}$$

is not 1. Intuitively, although Silverman's lemma (see Lemma 14 below) shows that the maximum ramification is not very big, the number of critical points can also grow exponentially upon iteration, slowing the growth of the number of distinct preimage points.

More specifically, suppose that a rational map only has 0 and ∞ as the preimages of 1. Since preimages of any point via ϕ are at most d points, the number of distinct preimages of 1 via $\phi^{(n)}$ is at most $2d^{n-1}$. Then the above expression is at most

$$\frac{2d^{n-1} - 2}{d^n} < \frac{2}{d} < 1.$$

We can also create ϕ such that the number of distinct preimages of 1 by $\phi^{(n)}$ is precisely $2d^{n-1}$. For example, let ϕ be a degree-4 rational map of the form $\frac{g(x)+x^2}{g(x)}$. This ϕ has just 0 and ∞ as preimages of 1. Differentiating, we obtain

$$\frac{(g'(x) + 2x)g(x) - (g(x) + x^2)g'(x)}{g(x)^2} = \frac{2xg(x) - x^2g'(x)}{g(x)^2}.$$

Thus, we find that $x = \infty$ and $x = 0$ are both critical points, and there should be four others (counting with multiplicity). Note that since $2g(x) - xg'(x)$ does not have a quadratic term, the location of three other critical points determines the last. Letting 2, 3, and 4 to be critical, we see that the last one is $-\frac{26}{9}$. Solving

$$2g(x) - xg'(x) = (x-2)(x-3)(x-4)\Big(x + \frac{26}{9}\Big),$$

$g(x) = -\frac{1}{2}x^4 + \frac{55}{9}x^3 + \frac{460}{9}x - \frac{104}{3}$. If P is a preimage of 1 via $\phi^{(k-1)}$ for $k \geq 2$ and $\phi^{-1}(P)$ does not split into d points, then there is a $Q \in \phi^{-1}(P)$ which is a critical point. So Q is $0, 2, 3, 4, -\frac{26}{9}$ or ∞. In particular, Q is a rational point, so so are all the points in the orbit of Q. Since $\phi^{(k)}(Q) = 1$, $\phi^{(k-1)}(Q)$ must be 0 or ∞. On the other hand, neither $g(x)$ nor $g(x)+x^2$ has a rational root, contradicting the fact that $\phi^{(k-2)}(Q)$ is rational. Thus, after the first step, all the preimages split up completely, making the number of distinct preimages of 1 via $\phi^{(n)}$ to be precisely $2d^{n-1}$, as desired.

Thus, Theorem 5 for $N = 1$ does not directly show Theorem 1 (ii). However, we will now derive this using Theorem 6. In fact, we will prove the S-version: if ϕ is a rational function such that $\phi^{(2)} \notin k[x]$ and $P \in k$ is not preperiodic, then

$$\lim_{m\to\infty} \frac{\displaystyle\sum_{v \notin S} \lambda_v(\infty, \phi^{(m)}(P))}{h(\phi^{(m)}(P))} = 1. \tag{20}$$

We will still use the following geometric lemma of Silverman [10, Lemma 3.52], giving a bound on the worst ramification. This lemma is based on the Riemann–Hurwitz formula and combinatorics, and it does not involve any arithmetic.

Lemma 14 (Silverman). *Suppose that ϕ is a rational function of degree $d \geq 2$ such that $\phi^{(2)} \notin k[x]$. Then letting e_n denote the maximum of the ramification indices at the poles of $\phi^{(n)}$, we have*

$$\lim_{n\to\infty} \frac{e_n}{d^n} = 0.$$

Given $\epsilon > 0$, Lemma 14 allows us to fix n such that $\frac{3e_n}{d^n} < \epsilon$. Because of (4), we can extend k and S if necessary to assume that k contains the fields of definition of $Q_1, \ldots, Q_\ell$, the distinct poles of $\phi^{(n)}$. Given $v \in S$, let C_v be the half of the minimum v-adic distance between Q_i and Q_j; here we use the v-adic absolute value that is an extension of the normalized one on $\mathbb{Q}$, so that we are guaranteed to have the triangle inequality. Given a point $Q \in \mathbb{P}^1(k)$, there is a Q_{i_v} which is the closest to Q in the v-adic distance, and this may occur with multiplicity e_n in $(\phi^{(n)})^*(\infty)$. The rest of the points Q_j are at least C_v away from Q, so using the definition of local heights (5), we can estimate

$$\lambda_v(Q_j, Q) \leq \max\left(0,\ -\frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]}\log C_v\right).$$

We have a maximum of $d^n - 1$ of these in $(\phi^{(n)})^*(\infty)$ for each v, excluding the closest Q_{i_v}. We next deal with the closest one. For each $\{Q_{i_v}\}_{v\in S} \in \{Q_1, \ldots, Q_\ell\}^{|S|}$, we apply (with $\epsilon = 1$) the following Lang's version of Roth's theorem [1, Theorem 6.2.3], which is equivalent to Vojta's conjecture on $\mathbb{P}^1$:

$$\sum_{v\in S}\lambda_v(Q_{i_v}, Q) < 3h(Q) + O(1), \qquad \forall Q \in \mathbb{P}^1(k)\backslash\{Q_1, \ldots, Q_\ell\}.$$

By making $O(1)$ larger, we have the above inequality for any choice of $\{Q_{i_v}\}_{v\in S}$ and Q. Hence, in total we must have

$$\begin{aligned}
\sum_{v\in S}\lambda_v(\infty, \phi^{(m)}(P)) &\leq \sum_{v\in S}\lambda_v((\phi^{(n)})^*(\infty), \phi^{(m-n)}(P)) + O(1)\\
&\leq 3e_n h(\phi^{(m-n)}(P)) + (d^n - 1)\sum_{v\in S}\max\left(0,\ -\frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]}\log C_v\right) + O(1)\\
&\leq 3e_n h(\phi^{(m-n)}(P)) + O(1),
\end{aligned}$$

where this $O(1)$ only depends on n (not on m). Therefore,

$$\begin{aligned}
\sum_{v\notin S}\lambda_v(\infty, \phi^{(m)}(P)) &\geq h(\phi^{(m)}(P)) - 3e_n h(\phi^{(m-n)}(P)) - O(1)\\
&\geq \left(1 - \frac{3e_n}{d^n}\right) h(\phi^{(m)}(P)) - O(1) > (1-\epsilon)h(\phi^{(m)}(P)) - O(1).
\end{aligned}$$

Since $h(\phi^{(m)}(P)) \to \infty$ as $m \to \infty$, this finishes the proof that Vojta's conjecture on $\mathbb{P}^1$ together with a geometric lemma (Lemma 14) shows the S-version of Theorem 1 (ii). $\square$

Remark. Of course, the above discussion did not prove anything new. The hope is that it illustrated what is special about dimension 1, namely Lemma 14. The Diophantine content even in dimension 1 is precisely Vojta's conjecture, agreeing with the framework set forth for higher dimensions in Theorems 4, 5, and 7. To elucidate this point further, one can actually use Theorem 6 to prove (20): we let D_1 be three distinct points of $(\phi^{(n)})^*(\infty)$ and D_2 be the rest and employ the above argument with D_2. What distinguishes dimension 1 is that divisors are simply bunch of points with various multiplicities, and that the multiplicity can be controlled via Riemann–Hurwitz. In contrast, in higher dimensions, irreducible divisors are much more complicated geometrically, and one cannot for example estimate the distance of a point to a divisor by the "worst" contribution. As a result, it is more difficult to obtain a sharp result in higher-dimensions.

5. Maps on $\mathbb{P}^2$ with a Totally Ramified Fixed Point

In this section, we analyze rational maps on $\mathbb{P}^2$ which have a totally ramified fixed point, that is, a point P which is completely invariant. In Example 5 of [14], we treated an example of this type to demonstrate that the hypothesis in Theorem 4 is not a necessary condition. In fact, Theorems 4, 5, and 7 do not apply to these maps because there are not enough normal-crossings part. Instead, we will use Silverman's theorem (Theorem 1) to obtain the following result on deviations from integrality in orbits, without needing to assume Vojta's conjecture. Note that we can even conclude *finiteness* rather than just Zariski-non-density in this case.

Proposition 15. *Let $\phi = [F_0(X,Y,Z) : F_1(Y,Z) : F_2(Y,Z)]$ be a rational map on $\mathbb{P}^2$ defined over $\mathbb{Q}$, where F_1 and F_2 are homogeneous polynomials in just Y and Z without a common factor over $\overline{\mathbb{Q}}$. Write $P \in \mathbb{P}^2(\mathbb{Q})$ as $[a : b : c]$, where $a, b, c \in \mathbb{Z}$ with gcd 1, and similarly $\phi^{(m)}(P) = [a_m : b_m : c_m]$. Let $\psi = [F_1(Y,Z) : F_2(Y,Z)]$ be the map on the $\mathbb{P}^1$ defined by $X = 0$, and assume that $\mathcal{O}_\psi([b : c])$ is infinite and that $\psi^{(2)}$ is not a polynomial, i.e. does not* *have a totally ramified fixed point at* $[1:0]$. *Let* S *be a finite set of primes. Then there exists a positive constant* $c>0$ *such that*

$$\frac{\log|c_m|'_S}{\log\max(|a_m|,|b_m|,|c_m|)}>c$$

for all m *sufficiently large.*

Remark. We can easily conclude even *finiteness* of $(\mathbb{P}^2\backslash(Z=0))(\mathbb{Z})\cap\mathcal{O}_\phi(P)$ from Silverman's theorem. Indeed, since $\psi^{(2)}$ is not a polynomial, $\mathcal{O}_\psi([b:c])$ only has finitely many integral points. Since the orbit of $[b:c]$ under ψ is infinite, this means that $c_m|b_m$ for only finitely many m's, so we do not even need to consider the first coordinate to conclude finiteness of integral points. On the other hand, the comparison of the number of digits of coordinates is not entirely obvious. There can be less cancelation for $[a_m:b_m:c_m]$ compared with points of $\mathcal{O}_\psi([b:c])$, and the ratio of the number of digits is affected by the amount of cancelation between coordinates.

Remark. It is necessary to assume that $\mathcal{O}_\psi([b:c])$ is infinite. If $\phi=[X^2:YZ:Y^2-3Z^2]$ and $P=[4:2:1]$, then ψ fixes $[2:1]$ while $h(\phi^{(m)}(P))=2^m\log 4$. Of course, in this case, the orbit points are on the line $Y=2Z$. The assumption of $|\mathcal{O}_\psi([b:c])|=\infty$ can be removed if we change the conclusion to just Zariski-non-density. This theorem thus shows that the nonexistence of completely invariant subvarieties is *not* a necessary condition for having Zariski-non-dense integral points in orbits, so even if Question 2 (2) is proved in the affirmative, this geometric condition will not be the necessary and sufficient condition.

Remark. We can generalize this theorem from $\mathbb{Q}$ to a number field k, replacing $\log|c_m|'_S$ by $\sum_{v\notin S}\lambda_v((Z=0),\phi^{(m)}(P))$.

Proof. Since $\mathcal{O}_\psi([b:c])$ is infinite, $\hat{h}_\psi([b:c])\neq 0$. We also note that $\deg\psi=d$, as F_1 and F_2 do not have a common factor. Let us denote $\psi^{(m)}([b:c])$ by $[b'_m:c'_m]$ in the reduced form. Then Silverman's theorem in the form of (20) tells us that

$$\lim_{m\to\infty}\frac{\log|c'_m|'_S}{\log\max(|b'_m|,|c'_m|)}=1,$$

while the definition of canonical height tells us that

$$\lim_{m\to\infty} \frac{\log \max(|b'_m|, |c'_m|)}{d^m} = \hat{h}_\psi([b:c]).$$

Combined, $\frac{\log |c'_m|'_S}{d^m}$ goes to $\hat{h}_\psi([b:c])$, so for sufficiently small $\delta > 0$, we have

$$\log |c'_m|'_S > d^m(\hat{h}_\psi([b:c]) - \delta) \qquad m \gg 0. \tag{21}$$

We note that because there can be less cancelation among coordinates of $\phi^{(m)}(P)$ compared with $\psi^{(m)}([b:c])$, $\log |c_m|'_S \geq \log |c'_m|'_S$.

From the height inequality $h(\phi(R)) \leq dh(R) + C$, we obtain

$$h(\phi^{(m)}(P)) \leq d^m h(P) + d^{m-1}C + d^{m-2}C + \cdots + C \qquad \text{for all } m. \tag{22}$$

Therefore, using (21), for all sufficiently large m,

$$\begin{aligned} \frac{\log |c_m|'_S}{h(\phi^{(m)}(P))} &\geq \frac{\log |c'_m|'_S}{h(\phi^{(m)}(P))} = \frac{\log |c'_m|'_S}{d^m} \cdot \frac{d^m}{h(\phi^{(m)}(P))} \\ &> \frac{\hat{h}_\psi([b:c]) - \delta}{h(P) + \frac{C}{d} + \cdots + \frac{C}{d^m}} > \frac{\hat{h}_\psi([b:c]) - \delta}{h(P) + \frac{C}{d-1}} =: c, \end{aligned}$$

where c is visibly independent of m. Since C can be assumed to be positive, we note that $c > 0$. This concludes the proof. $\square$

Remark. By choosing δ appropriately, we can actually make c in the statement of the theorem to be any number strictly less than $\frac{\hat{h}_\psi([b:c])(d-1)}{(d-1)h(P)+C}$, where C is the minimum of $h(\phi(R)) - dh(R)$ as R runs through $\mathbb{P}^2(\mathbb{Q})$ (or actually $\mathcal{O}_\phi(P)$).

6. Question 2 and Further Problems

In this final section, we discuss Question 2 and pose a few related problems for further study. Some of them are completely new and some of them are more refined versions of the questions raised at the end of [14].

6.1. Completely invariant subvarieties

We now discuss why the nonexistence of completely invariant subvarieties can be a candidate for Question 2. Although the normal-crossings part of $(\phi^{(n)})^*(D)$ discussed in this article is a geometric and dynamical notion, it is in some sense more natural dynamically to think of completely invariant subvarieties, i.e. (possibly reducible) subvarieties V such that $\phi^{-1}(V) = V$ set-theoretically (in the terminology of classical dynamics, V is an exceptional set). In dimension 1, $\phi^{(2)} \in k[x]$ is equivalent to $\phi^{(n)} \in k[x]$ for some n, which in turn is equivalent to the condition that $\{\infty, \phi(\infty)\}$ is completely invariant.

We now discuss some examples of maps which have completely invariant subvarieties and whose orbits can contain Zariski-dense integral points. Let $\phi_1 = [F_0 : \cdots : F_3]$ on $\mathbb{P}^3$ be such that F_1, F_2 are degree-d homogeneous polynomials in $\mathbb{Z}[X_1, X_2]$ without any common factors and $F_3 = X_2X_3^{d-1}$. In this case, the line defined by $X_1 = X_2 = 0$ is completely invariant. When $F_1 = X_1^d$,

$$|\mathcal{O}_{\phi_1}([a_0 : 1 : a_2 : a_3]) \cap (\mathbb{P}^3 \backslash (X_1 = 0))(\mathbb{Z})|$$

is obviously infinite when $a_i \in \mathbb{Z}$. This set is also Zariski-dense when F_0 and F_2 are chosen generically.

Some may give an objection to the above example that the hyperplane $X_1 = 0$ with respect to which integrality is defined is completely invariant, so we provide another example. As discussed in Example 6 of [14], the orbit of $[2^N : 2^{N-1} : \cdots : 2 : 1]$ under the rational map $\phi_2 = [X_0^3 : X_1^3 : X_1X_2^2 : X_2X_3^2 : \cdots : X_{N-1}X_N^2]$ has Zariski-dense (in particular, infinitely many) integral points with respect to $X_N = 0$. Here, the hyperplane $X_N = 0$ with respect to which integrality is defined is *not* completely invariant, though hyperplanes $X_0 = 0$ and $X_1 = 0$ are completely invariant.

As a further example, monomial maps can have infinitely many S-integral points in orbits. In fact, based on some explicit conditions on the eigenvalues of the tangent map at the identity, monomial maps on $\mathbb{P}^2$ having just finitely many integral points in orbits have been classified completely [3, Theorem 2]. In dimensions greater than 2, a complete classification becomes harder as the number of possible Jordan decompositions grows rapidly, but it is easy to create monomial maps having Zariski-dense integral points in orbits. As an example, $\psi = (x_1/x_2, x_1^5x_2)$ on $(\mathbb{G}_m)^2$ has some orbit containing infinitely many integral points because it corresponds to a matrix with complex eigenvalues [3, Theorem 2], and so $\phi = (x_1/x_2, x_1^5x_2, x_3^2, x_4^2, \ldots, x_N^2)$

on $(\mathbb{G}_m)^N \subset \mathbb{P}^N$ has the property that $\mathcal{O}_\phi(P) \cap (\mathbb{P}^N \backslash (X_0 = 0))(\mathbb{Z})$ is Zariski-dense for some P. Of course, all monomial maps on $\mathbb{P}^N$ satisfy $\phi^{-1}(D) = D$ set-theoretically for the divisor $D = (X_0 \cdots X_N = 0)$, so they always contain a (reducible) completely invariant subvariety.

We have so far discussed maps having completely invariant subvarieties and having Zariski-dense integral points in orbits. The examples like ϕ_1 and ϕ_2 above seem possible only when some completely invariant subvarieties are present, serving as evidences for Question 2 (2). We can even go further, and expect the true analog of the quasi-S-integral version of Silverman's Theorem (Theorem 1 (ii)) to hold in higher-dimensions: if ϕ does not have any completely invariant subvarieties over $\overline{\mathbb{Q}}$, then for all $\epsilon > 0$,

$$\left\{ \phi^{(m)}(P) : \frac{\sum\limits_{v \notin S} \lambda_v(D, \phi^{(m)}(P))}{\deg(D) h(\phi^{(m)}(P))} \leq 1 - \epsilon \right\}$$

is Zariski-non-dense. Either an affirmative or a negative answer to this statement would be an important step in understanding integrality in orbits in higher-dimensions.

Of course, as indicated already, having a completely invariant subvariety will not prevent orbits from being Zariski-non-dense. Proposition 15 is one example, and there are many monomial maps on $\mathbb{P}^2$ having just finitely many integral points in orbits. Even in dimensions higher than 2, the monomial map $(x_2/x_1, x_1, x_3^2, \ldots, x_N^2)$ on $\mathbb{P}^N$ for example has just finitely many integral points in orbits. Further, when the second iterate of $[F_1 : F_2]$ of ϕ_1 above does not have a totally ramified fixed point on the line defined by $X_0 = X_3 = 0$, it follows immediately from Silverman's theorem that orbit points in $(\mathbb{P}^3 \backslash (X_1 = 0))(R_S)$ are finite (just by looking at the middle two coordinates). So even if Question 2 (2) were answered in the affirmative, this will not be a necessary condition in higher-dimensions.

6.2. Extensions and complements

Here we collect together various possible extensions and complements to the results of this article. Since Theorem 4 shows that the existence of n such that $\deg D_{\mathrm{nc}}^{(n)} > N+1$ is a candidate for Question 2 (1), it is natural to ask what happens when $\sup_n \deg D_{\mathrm{nc}}^{(n)}$ is less than or equal to $N+1$. This would be complementary to Theorem 4, and would help us with the search for a necessary condition for the Zariski-non-density of integral points in orbits. We will assume that D is geometrically irreducible; we are mostly interested in the case when D is a hyperplane.

We can create a map ϕ on $\mathbb{P}^N$ satisfying $\sup_n \deg D_{\mathrm{nc}}^{(n)} = 2$ as follows, just as in Proposition 15: we let F_{N-1} and F_N be homogeneous of X_{N-1} and X_N, where $\phi = [F_0 : \cdots : F_N]$. Note that for these maps, even *finiteness* of integral points in orbits is immediate from Silverman's theorem as long as the second iterate of the restriction to the $\mathbb{P}^1$ defined by $X_0 = \cdots = X_{N-2} = 0$ is not a polynomial. In contrast, as remarked earlier, Theorem-5-like results analyzing the deviation from S-integrality in orbits are not as obvious. Proposition 15 is an example of what we can show in this case.

To create a map with $\sup \deg D_{\mathrm{nc}}^{(n)} = 3$, we let F_N be the product of powers of X_{N-1} and X_N, and then let F_{N-2} and F_{N-1} be homogeneous in X_{N-2} and X_{N-1}. Then $(\phi^{(n)})^*(X_N = 0)$ is defined by the product of a power of X_N and a homogeneous polynomial in X_{N-2} and X_{N-1}. Thus, this divisor always factors into linear components over $\overline{\mathbb{Q}}$ and the degree of the normal-crossings part is 3 for all n. It would be interesting to determine whether S-integral points in orbits are Zariski-dense. We would also like results pertaining to deviation from S-integrality in orbits for maps of this type.

So far, the author has not been able to construct a map on $\mathbb{P}^N$ with $N \geq 4$ such that $D_{\mathrm{nc}}^{(n)}$ is always linear and $\sup \deg D_{\mathrm{nc}}^{(n)} = N$. The author believes that this is possible, while there might not exist morphisms on $\mathbb{P}^N$ with $\sup \deg D_{\mathrm{nc}}^{(n)} = N+1$. In fact, even for $N = 1$, $\sup \deg D_{\mathrm{nc}}^{(n)} = 2$ never happens: when $\phi^{(2)}$ is not a polynomial, Lemma 14 shows that there exists n such that $\deg D_{\mathrm{nc}}^{(n)} \geq 3$, and when $\phi^{(2)}$ is a polynomial, $\deg D_{\mathrm{nc}}^{(n)}$ is always 1. It would be worthwhile to create various maps for which $\sup_n \deg D_{\mathrm{nc}}^{(n)} \leq N+1$, so that we have a better idea of when integral points in orbits are Zariski-dense.

Another desired complementary result would be a generalization of Proposition 15. In particular, we would like to have similar examples in higher-dimensions, such as $\phi_1 = [F_0 : F_1(X_1, X_2) : F_2(X_1, X_2) : X_2 X_3^{d-1}]$ discussed above. Because of the insufficiency of the normal-crossings part, results such as Theorem 7 do not apply to these examples. The maps satisfying $\sup_n \deg D_{\mathrm{nc}}^{(n)} \leq N+1$ and the examples generalizing Proposition 15 should help us obtain a condition closer to a necessary condition for Question 2.

As for extensions, we would like to strengthen Theorems 10 and 12. Both of them require knowing the heights of orbit points, so it would be much better if we can instead come up with a condition that relies solely on the arithmetic degree, regardless of how the limsup (or the limit, conjecturally) is attained. Even if we can only conclude smaller σ, we would like to have a hypothesis that is more easily verifiable and that only uses notions that do not depend on the arithmetic of the initial point. This would enable us to obtain a potential candidate for Question 2 (1) for rational maps.

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