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EXISTENCE RESULTS FOR THE EINSTEIN-SCALAR FIELD LICHNEROWICZ EQUATIONS ON COMPACT RIEMANNIAN MANIFOLDS IN THE POSITIVE CASE

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Dedicated to Professor Neil Sidney Trudinger on the occasion of his 70th birthday

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Abstract

This is the third and last in our series of papers concerning solution of the Einstein-scalar field Lichnerowicz equations on Riemannian manifolds. Let (M, g) be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$, f, $h > 0$, and $a \geqslant 0$ are smooth functions on M with $\int_M a d\text{vol}_g > 0$. In this article, we prove two major results involving the following partial differential equation arising from the Hamiltonian constraint equation for the Einstein-scalar field system in general relativity

$$\Delta_g u + hu = fu^{2^\star-1} + \frac{a}{u^{2^\star+1}},$$

where $\Delta_g = -\text{div}_g(\nabla_g \cdot)$, $2^\star = \frac{2n}{n-2}$. In the first part of the paper, we prove that if $\int_M a d\text{vol}_g$ is sufficient small, the equation admits one positive smooth solution. In the second part of the paper, we show that the condition for $\int_M a d\text{vol}_g$ can be relaxed if $\sup_M f$ is small. As a by-product of this result, we are able to get a complete characterization of the existence of solutions in the case when $\sup_M f \leqslant 0$. In addition to the two main results above, we should emphasize that we allow a to have zeros in M.

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1. Introduction

This is the third and last in our series of papers concerning solution of the Einstein-scalar field Lichnerowicz equations on compact Riemannian manifolds. Given a smooth compact Riemannian manifold (M,g) without the boundary of dimension $n \geqslant 3$, in this paper, we prove some existence results for the following simple partial differential equation

$$\Delta_g u + hu = fu^{2^\star-1} + au^{-2^\star-1}, \quad u>0, \tag{1.1}$$

where $\Delta_g = -\mathrm{div}_g(\nabla_g \cdot)$ is the Laplace-Beltrami operator, $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent, and $h>0$, f, $a \geqslant 0$ are smooth functions.

The analysis of Eq. (1.1) is motivated by the constraint equations for the initial value problems of general relativity by using the conformal method. Recently, Eq. (1.1) has received much considerable attention due to the nature of their origin. To make the paper self-contained, we briefly recall how the conformal method can be used when we study the Cauchy problem in general relativity and how Eq. (1.1) appears. For interested readers, we refer to [4, Chapter III], see also [4, 5, 6, 7, 16, 17].

Roughly speaking, by a given initial data set $(M,\overline{g},\overline{K})$ we mean an n-dimensional Riemannian manifold $(M,\overline{g})$ and a symmetric $(0,2)$-tensor $\overline{K}$, then the initial value problem asks for a Cauchy development of $(M,\overline{g},\overline{K})$, simply denoted by $(\mathscr{M},\boldsymbol{g})$, which is a Lorentzian manifold of dimension $n+1$. Here the spacetime metric $\boldsymbol{g}$ is required to satisfy the following Einstein equation

$$\mathrm{Ric}_{\boldsymbol{g}} - \frac{1}{2}\mathrm{Scal}_{\boldsymbol{g}}\boldsymbol{g} = \mathscr{T},$$

where $\mathrm{Ric}_{\boldsymbol{g}}$ and $\mathrm{Scal}_{\boldsymbol{g}}$ are the Ricci tensor and the scalar curvature of the spacetime metric $\boldsymbol{g}$. Also, the symmetric $(0,2)$-tensor $\mathscr{T}$ appearing in the Einstein equation is the energy-momentum tensor which is supposed to present the density of all the energies, momenta and stresses of the sources.

In order for $(\mathscr{M},\boldsymbol{g})$ to be a Cauchy development of $(M,\overline{g},\overline{K})$, it is required that $(M,\overline{g},\overline{K})$ must embed isometrically to $(\mathscr{M},\boldsymbol{g})$ as a slice with the second fundamental form $\overline{K}$; and the metric $\overline{g}$ becomes the pullback of the spacetime metric $\boldsymbol{g}$ by the embedding. It turns out that the initial data $(\overline{g},\overline{K})$ cannot be arbitrary, they must satisfy some conditions. As a direct consequence of the Gauss and Codazzi equations, those conditions can be rewritten in a form consisting two equations known as the Hamiltonian and momentum constraints which are defined on $(M, \overline{g})$, namely,

$$\begin{cases} \mathrm{Scal}_{\overline{g}} - |\overline{K}|^2_{\overline{g}} + (\mathrm{trace}_{\overline{g}}\overline{K})^2 - 2\rho = 0, \\ \nabla_{\overline{g}} \cdot \overline{K} - \nabla_{\overline{g}}\mathrm{trace}_{\overline{g}}\overline{K} - \mathscr{J} = 0, \end{cases} \tag{1.2}$$

where all quantities of (1.2) involving a metric are computed with respect to the spacelike metric $\overline{g}$ and $\mathrm{Scal}_{\overline{g}}$ is the scalar curvature of $\overline{g}$. Also in (1.2), ρ is a scalar field on M representing the energy density and $\mathscr{J}$ is a vector field on M representing the momentum density of the nongravitational fields; they are related to the energy-momentum tensor $\mathscr{T}$ as follows

$$\rho = \mathscr{T}(\mathbf{n}, \mathbf{n}), \quad \mathscr{J} = -\mathscr{T}(\mathbf{n}, \cdot),$$

where $\mathbf{n}$ is the unit timelike normal to the slice $M \times \{0\}$, see [4, 6] and [7, Section 5].

By a simple dimension counting argument, it is clear that Eq. (1.2) forms an under-determined system of variable $(\overline{g}, \overline{K})$; thus they are generally hard to solve. However, in literature, the conformal method can be effectively applied in the constant mean curvature setting as remarked in [4], that is to look for

$$\begin{cases} \overline{g} = u^{2^\star - 2} g, \\ \overline{K}_{ij} = \dfrac{\tau}{n} u^{2^\star - 2}\overline{g}_{ij} + u^{-2}(\sigma + \mathscr{L}W)_{ij}, \end{cases} \tag{1.3}$$

where the metric g is fixed, u is a positive (smooth) function to be determined, and W is a 1-form. Note that the operator $\mathscr{L}$ appearing in (1.3) is the conformal Killing operator acting on W which can be given in local coordinates by

$$\mathscr{L}W_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n}(\nabla^k W_k) g_{ij},$$

where ∇ and $\boldsymbol{\nabla}$ are the Levi-Civita connections associated to the metrics g and $\boldsymbol{g}$ respectively. Here $\tau = \overline{g}^{ij}\overline{K}_{ij}$ is the mean curvature of M as a slide of $\mathscr{M}$. The choice for the two-tensor σ is somehow arbitrary.

When the conformal method is applied in this setting, the constraints (1.2) can easily be transformed to a determined system of partial differential equations of variable (u, W) given by

$$\frac{4(n-1)}{n-2}\Delta_g u + \mathrm{Scal}_g u = -\Big(\frac{n-1}{n}\tau^2 - 2\rho\Big)u^{2^\star-1} + |\sigma + \mathscr{L}W|_g^2 u^{-2^\star-1}, \quad (1.4a)$$

$$\mathrm{div}_g(\mathscr{L}W) = \frac{n-1}{n}u^{2^\star} d\tau + u^{\frac{2(n+2)}{n-2}}\mathscr{J}. \quad (1.4b)$$

In the vacuum case and when τ is constant, e.g. $\mathscr{T} \equiv 0$ and hence $\rho \equiv 0$ and $\mathscr{J} \equiv 0$ as well, we know exactly which sets of data lead to solutions and which do not, see [12]. This is because Eq. (1.4b) then only involves W and generically implies $W \equiv 0$ (for example, if M admits no conformal Killing vector field). Therefore, one is left with solving Eq. (1.4a). Clearly, Eq. (1.1) already includes Eq. (1.4a) as a particular case. Throughout this paper, equations of the form (1.1) are called the Einstein-scalar field Lichnerowicz equations.

While, as we have noted, the conformal method can be effectively applied for solving Eq. (1.2) in most cases, it should be pointed out that there are several cases for which either partial result or no result was available, especially in the non-vacuum case, when gravity is coupled to field sources. To see this more precise, we assume the presence of a real scalar field $\boldsymbol{\psi}$ on the space time $(\mathscr{M}, \boldsymbol{g})$ with a potential $\mathscr{U}$ being a function of $\boldsymbol{\psi}$, then Eq. (1.4a) takes the form of (1.1) with

$$h = \frac{n-2}{4(n-1)}\left(\mathrm{Scal}_g - |\nabla\psi|_g^2\right), \quad a = \frac{n-2}{4(n-1)}\left(|\sigma + \mathscr{L}W|_g^2 + \pi^2\right), \quad (1.5)$$

and

$$f = -\frac{n-2}{4(n-1)}\left(\frac{n-1}{n}\tau^2 - 2\mathscr{U}(\psi)\right), \quad (1.6)$$

where π is the transformed normalized time derivative of $\boldsymbol{\psi}$ restricted to M and ψ is the restriction of $\boldsymbol{\psi}$ to M, see [6, 4] for details. Based on the division in [6], one can observe that there are three cases corresponding to either $h < 0$, or $h \equiv 0$, or $h > 0$ with sign-changing f, for which either partial result or no result was achieved.

In the preceding papers [16, 17], we have already proven that, in the case $h \leqslant 0$, a suitable balance between coefficients h, f, a of the Einstein-scalar field Lichnerowicz equations is enough to guarantee the existence of one positive smooth solution. In addition, it was found that under some further conditions we may or we may not have the uniqueness property of solutions of the Einstein-scalar field Lichnerowicz equations. This paper is a continuation of those papers above [16, 17].

In the present paper, we continue our study of the existence of the positive smooth solutions to (1.1) when $h > 0$. We assume hereafter that f and a are smooth functions on M with $a \geqslant 0$. The latter assumption implies no physical restrictions since we always have that $a \geqslant 0$ in the original Einstein-scalar field theory. We also assume $\int_M a d\mathrm{vol}_g > 0$. This assumption prevents us from the study of the prescribing scalar curvature problem in the positive case. Thanks to the conformally covariance property of the Einstein-scalar field Lichnerowicz equations, we can freely choose a background metric g such that manifold M has unit volume.

As far as we know, Eq. (1.1) with $h > 0$ was first considered in [10] by using variational methods. In that elegant paper, Hebey–Pacard–Pollack proved, among other things, a fundamental existence result which roughly says that a suitable control of $\int_M a d\mathrm{vol}_g$ from above is enough to guarantee the existence of one positive smooth solution. Their result basically makes use of the fact that the operator $\Delta_g + h$ is coercive. Although the coerciveness property is slightly weaker than the condition $h > 0$, however, this condition is enough to guarantee that the following

$$\|u\|_{H^1_h} = \left(\int_M |\nabla u|_g^2 \, d\mathrm{vol}_g + \int_M h u^2 d\mathrm{vol}_g \right)^{\frac{1}{2}}$$

is an equivalent norm on $H^1(M)$. The advantage of this setting is that the first eigenvalue of the operator $\Delta + h$ is strictly positive, and thus, various goods properties of the theory of weighted Sobolev spaces can be applied. In particular, for all $u \in H^1(M)$, there holds

$$\int_M |\nabla u|^2 d\mathrm{vol}_g + \int_M h u^2 d\mathrm{vol}_g \geqslant \mathcal{S}_h \left(\int_M |u|^{2^\star} d\mathrm{vol}_g \right)^{\frac{2}{2^\star}}, \tag{1.7}$$

where the constant $\mathcal{S}_h$ is called the Sobolev constant and is independent of u. Using our notations, their result can be restated as follows.

Theorem A (see [10]). *Let (M, g) be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Let h, a, and f be smooth* *functions on M for which $\Delta_g + h$ is coercive, $a > 0$ in M, and $\sup_M f > 0$. There exists a constant $C = C(n)$, $C > 0$ depending only on n, such that if*

$$\|\varphi\|_{H_h^1}^{2^\star} \int_M \frac{a}{\varphi^{2^\star}} \,\mathrm{dvol}_g \leqslant \frac{C}{(\mathcal{S}_h \sup_M |f|)^{n-1}} \tag{1.8}$$

and

$$\int_M f\varphi^{2^\star} \,\mathrm{dvol}_g > 0 \tag{1.9}$$

for some smooth positive function $\varphi > 0$ in M, then the Einstein-scalar field Lichnerowicz equation (1.1) *possesses a smooth positive solution.*

As can be seen from Theorem A, the condition $\sup_M f > 0$ is crucial since the condition (1.9) does not hold if $f \leqslant 0$. Moreover, it could be necessary to have $a > 0$ in M in order to get a positive lower bound for smooth solutions of (1.1). Besides, if we denote $f^- = \min(f, 0)$ and $f^+ = \max(f, 0)$, then the condition (1.8) involves not only $\sup_M f^+$ but also $\inf_M f^-$. In other words, for given a, the negative part f^- of f cannot be too negative. This restriction basically reflects the fact that the energy functional has to verify the mountain pass geometry as their solution was found as a mountain pass point. It is worth noticing that an upper bound for $\int_M a d\mathrm{vol}_g$ as in (1.8) is predictable since for given h and f, a cannot be too large, see [10, Section 2].

The present paper was also motivated by a recent paper by Ma–Wei [14]. In their paper, provided $\underline{u}$ is a positive smooth solution, Ma–Wei proved the existence of some mountain pass solution of (1.1) of the form $\underline{u} + v$ for some positive smooth function v. In terms of our notations, we can formulate their result as the following.

Theorem B (see [14])**.** *Assume that a, f, h are positive functions on the compact Riemannian manifold (M, g) of dimension $3 \leqslant n < 6$. Let $\underline{u}$ be a positive smooth solution of* (1.1)*. Assume that the first eigenvalue of*

$$\Delta_g + h - \frac{n+2}{n-2} f \underline{u}^{\frac{4}{n-2}} + \frac{3n-2}{n-2} a \underline{u}^{\frac{4n-4}{n-2}} \tag{1.10}$$

is positive. Then the Einstein-scalar field Lichnerowicz equation (1.1) *possesses a mountain pass, smooth, positive solution.*

It is easy to see that the positivity of the first eigenvalue of the operator given in (1.10) immediately implies that the solution $\underline{u}$ is strictly stable. Therefore, it is natural to seek for positive smooth solutions of (1.1) as local minimizers. Another reason that supports this approach is to look at the profile of the functional associated to (1.1). Due to the presence of the term $au^{-2^\star-1}$, the energy of u is very large when $\max_M u$ is small. Clearly, in the case $f \leqslant 0$, the energy of u is also large when $\max_M u$ is large. Consequently, a local minimizer of the energy functional should exist which could provide a possible solution. Similarly, if one assumes that $\sup_M f > 0$ and that the energy functional admits some mountain pass geometry, a local minimizer of the energy functional again exists.

While searching for positive smooth solutions of Eq. (1.1), we found that the method used in [16, 17] still works in this context. While the non-positive Yamabe-scalar field invariant $h \leqslant 0$ involves more conditions and our analysis of solvability of the Lichnerowicz-scalar field equations strongly depends on the ratio between $\sup_M f$ and $\int_M |f^-| d\mathrm{vol}_g$, the positive Yamabe-scalar field invariant $h > 0$ requires fewer conditions than the non-positive case. In fact, as we shall see later, in the case $\sup_M f > 0$, no condition for f is imposed and we are able to show that if $\int_M a d\mathrm{vol}_g$ is small, then (1.1) possesses at least one smooth positive solutions since the condition for $\sup_M f$ can be absorbed to the condition for $\int_M a d\mathrm{vol}_g$. The first main theorem can be stated as follows.

Theorem 1.1. *Let (M, g) be a smooth compact Riemannian manifold without the boundary of dimension $n \geqslant 3$. Assume that f, $h > 0$, and $a \geqslant 0$ are smooth functions on M such that $\int_M a d\mathrm{vol}_g > 0$ and $\sup_M f > 0$. We assume further that there exists a constant $\tau > \max\{1, (\frac{2}{\mathcal{S}_h}\int_M h d\mathrm{vol}_g)^{\frac{2^\star}{2}}\}$ such that*

$$\int_M a d\mathrm{vol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h}{\tau}\left(\frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f d\mathrm{vol}_g}\right)^{n-1} \tag{1.11}$$

holds. Then (1.1) possesses at least one smooth positive solution.

Observe from (1.11) that τ plays no role but a scaling factor. Therefore, for given $\int_M a d\mathrm{vol}_g$, we could select τ sufficiently large and $\sup_M f$ sufficiently small in such a way that (1.11) is fulfilled. This suggests that under the case when $\sup_M f$ is small, the condition for $\int_M a d\mathrm{vol}_g$ appearing in (1.11) can be relaxed. In the second part of the present paper, we prove this affirmatively. That is the content of the following.

Theorem 1.2. *Let* (M, g) *be a smooth compact Riemannian manifold without the boundary of dimension* $n \geqslant 3$. *Let* f, h, *and* a *be smooth functions on* M *with* $h > 0$, $a \geqslant 0$ *in* M, $\int_M a \mathrm{dvol}_g > 0$, *and* $\sup_M f > 0$. *Then there exists a positive constant* $\mathcal{C}$ *to be specified later such that if* $\sup_M f < \mathcal{C}$, *then Eq.* (1.1) *possesses one positive smooth solution.*

Apparently, Theorem 1.2 provides a slightly stronger result than that of Theorem A as the negative part f^- of the function f could be arbitrarily small. In addition, the condition (1.11) also suggests that if h is large enough, (1.1) always admits at least one positive solution. This is because, as a function of h, $\mathcal{S}_h$ is monotone increasing. It turns out that the size of h really affects the solvability of (1.1). We shall not prove anything about this interesting feature but to summarize the role of h in Table 1 in the last paragraph of the present paper.

In the third part of the present paper, we focus our attention to the case when $f \leqslant 0$. In this context, we are able to get a complete characterization of the existence of solutions of (1.1) in the case when $f \leqslant 0$. Roughly speaking, it should mention that in the statement of Theorem 1.1, $\sup_M f$ is exactly $\sup_M f^+$ where f^+ is the positive part of f. Therefore, without any $\sup_M f$, one can immediately observe that the right hand side of (1.11) goes to $+\infty$ as $\tau \to +\infty$. This suggests that under the condition $f \leqslant 0$, no condition is imposed.

Theorem 1.3. *Let* (M, g) *be a smooth compact Riemannian manifold without the boundary of dimension* $n \geqslant 3$. *Let* f, h, *and* a *be smooth functions on* M *with* $h > 0$, $a \geqslant 0$ *in* M, $\int_M a \mathrm{dvol}_g > 0$, *and* $f \leqslant 0$. *Then Eq.* (1.1) *always possesses one and only one positive smooth solution.*

Concerning Theorem 1.2, it is worth noticing that it generalizes the same result obtained in [6] when the functions f and a take the form (1.5)–(1.6). Loosely speaking, it was proved in [6, Proposition 3] by the method of sub- and super-solutions that (1.1) always possesses one positive solution so long as the functions f and a take the form (1.5)–(1.6) with $f \leqslant 0$ and $a \geqslant 0$. The main ingredient of the proof in [6] is the conformal invariant property of the functions f and a when they take the form (1.5)–(1.6). Apparently, this property is no longer available in our general case.

Besides, we would like to comment that we do not expect that the result in Theorem 1.2 is original and completely new. Recently, it has just come to our attention that other approaches could lead us to the same result, for interested reader, we refer to [11] and [9]. However, more or less, our approach is different from the others.

When written in the form (1.1), one can easily see that the Einstein-scalar field Lichnerowicz equations is closely related to the Yamabe problem and the prescribing scalar curvature problem, which has been studied for years by many great mathematicians, for example, Yamabe [22], Trudinger [21], Aubin [1], Schoen [19], Kazdan–Warner [13], Escobar–Schoen [8], Rauzy [20], Chen–Xu [3] and references therein.

As already used in [16, 17] for the case $h \leqslant 0$, the original idea of our approach was based on Rauzy [20]. However, we found that in the case considered in [20], the assumption of the negative Yamabe invariant $h < 0$ is important; in fact, this approach cannot be applied to the case of the positive Yamabe invariant $h > 0$. Nevertheless, and thanks to the presence of the term with a negative exponent, we can still use the idea of [20] in our case. As always, in the first step to tackle (1.1), we look for positive smooth solutions of the following subcritical problem

$$\Delta_g u + hu = f|u|^{q-2}u + \frac{au}{(u^2+\varepsilon)^{\frac{q}{2}+1}}. \tag{1.12}$$

Our main procedure is to show that the limit exists as first $\varepsilon \to 0$ and then $q \to 2^\star$ under various assumptions. It is worth noticing that in [10], the authors just considered Eq. (1.12) with q replaced by $2^\star$. This difference somehow reflects the fact that we need the compact embedding $H^1(M) \hookrightarrow L^q(M)$ while searching for minimum points.

Before closing this section, we briefly mention the organization of the paper and highlight some techniques used. Section 2 mainly concerns basic properties of positive solutions of (1.1) such as point-wise estimate and regularity. In Section 3, a careful analysis of the energy functional is presented by proving the various properties involving the asymptotic behavior of the energy functional that is needed in later parts. Having these preparation, we spend Sections 4, 5 and 6 to prove Theorems 1.1, 1.2, and 1.3. Some comments and remarks will appear in Section 7. Part of this paper is a revision of Chapter 6 of the first author's doctoral thesis [15] submitted to the National University of Singapore under the supervision of the second author.

2. Preliminary

2.1. Notations

As usual, let $H^p(M)$ be the standard Sobolev space equipped with the standard norm. We also denote by $2^\flat$ the average of 2 and $2^\star$, that is, $2^\flat = \frac{2n-2}{n-2}$. Observe that $\tau > 1$ and therefore $\tau \sup_M f > \int_M f\, d\text{vol}_g$. Having this, we then introduce

$$k_{1,q} = \left(\frac{q+2}{4q} \frac{\mathcal{S}_h \tau^{\frac{2}{q}}}{\tau \sup_M f - \int_M f\, d\text{vol}_g} \right)^{\frac{q}{q-2}}, \quad k_{2,q} = \tau k_{1,q}. \tag{2.1}$$

One can observe that $k_{1,q} < k_{2,q}$. Moreover, one can easily bound $k_{1,q}$ from below and $k_{2,q}$ from above, that is, there exists two positive numbers $\underline{k} < 1$ and $\overline{k} > 1$ independent of q and ε such that $\underline{k} \leqslant k_{1,q} < k_{2,q} \leqslant \overline{k}$. In order to find such bounds, one first observes that

$$\tau k_{1,q} = \left(\frac{q+2}{4q} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f\, d\text{vol}_g} \right)^{\frac{q}{q-2}}.$$

Therefore, we can choose

$$\underline{k} \;=\; \frac{1}{\tau} \min \left\{ \left(\frac{1}{4} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f\, d\text{vol}_g} \right)^{\frac{2^\flat}{2^\flat - 2}}, 1 \right\} \tag{2.2}$$

and

$$\overline{k} \;=\; \max \left\{ \left(\frac{1}{2} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f\, d\text{vol}_g} \right)^{\frac{2^\flat}{2^\flat - 2}}, 1 \right\}. \tag{2.3}$$

2.2. Basic properties for positive solutions

This section is devoted to proving several properties of positive solutions of (1.12). We first derive a lower bound for positive C^2 solutions of (1.12).

It is worth noticing that such a result was already proved in [10], here we just derive a precise lower bound for positive C^2 solutions of (1.12). Then we recall a regularity result for weak solutions of (1.12).

Lemma 2.1. *Let u be a positive C^2 solution of* (1.12) *with $h > 0$. Then it holds*

$$\min_M u \geqslant \min\left\{\left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{1}{2^\flat+2}}, 1\right\} \tag{2.4}$$

for any $q \in [2^\flat, 2^\star)$ and any

$$\varepsilon < \min\left\{\left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{2}{2^\flat+2}}, 1\right\}. \tag{2.5}$$

Proof. Following [10], we let $\delta > 0$ be the unique positive solution of the following algebraic equation

$$\delta^{q+2}\left(\sup_M h + (\sup_M |f|)\delta^{q-2}\right) = \frac{1}{2^{2^\flat}}\inf_M a. \tag{2.6}$$

Since δ depends on q, we shall prove that for $q \in (2^\flat, 2^\star)$, δ has a strictly positive lower bound. We have the following two cases.

Case 1. Suppose

$$\sup_M h + \sup_M |f| \geqslant \frac{1}{2^{2^\flat}}\inf_M a.$$

In this case, there holds $\delta \leqslant 1$. Consequently, we can estimate

$$\frac{1}{2^{2^\flat}}\inf_M a \leqslant \delta^{q+2}\left(\sup_M h + \sup_M |f|\right)$$

which immediately gives us

$$\delta \geqslant \left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{1}{q+2}} \geqslant \left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{1}{2^\flat+2}}.$$

Case 2. Suppose

$$\sup_M h + \sup_M |f| < \frac{1}{2^{2^\flat}}\inf_M a.$$

In this case, there holds $\delta \geqslant 1$ which immediately gives us a lower bound for δ.

Combining two cases above, we conclude that

$$\delta \geqslant \min\left\{\left(\frac{1}{2^{2^\flat}}\frac{\inf_M a}{\sup_M h + \sup_M |f|}\right)^{\frac{1}{2^\flat+2}}, 1\right\}.$$

Suppose that u is a positive C^2 solution of (1.12) with $\varepsilon > 0$ satisfying the condition (2.5) above, that is,

$$\frac{\Delta_g u}{u} + h = f u^{q-2} + \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}+1}}.$$

Let us assume that u achieves its minimum value at x_0, then we have

$$h(x_0) + (-f(x_0))u(x_0)^{q-2} \geqslant \frac{a(x_0)}{\left(u(x_0)^2+\varepsilon\right)^{\frac{q}{2}+1}}. \tag{2.7}$$

We assume $u(x_0) < \delta$. From the choice of δ, one can verify that

$$\sup_M |h| + (\sup_M |f|)\delta^{q-2} \geqslant h(x_0) + (-f(x_0))u(x_0)^{q-2}. \tag{2.8}$$

Since $\varepsilon < \delta^2$ and $u(x_0) < \delta$, it is easy to see that

$$\frac{a(x_0)}{\left(u(x_0)^2+\varepsilon\right)^{\frac{q}{2}+1}} \geqslant \frac{\inf_M a}{(\sqrt{2}\delta)^{q+2}} > \frac{1}{2^{2^\flat}}\frac{\inf_M a}{\delta^{q+2}}. \tag{2.9}$$

Using (2.7), (2.8), and (2.9), we easily get a contradiction, thus proving that $u(x_0) \geqslant \delta$. In particular, there holds

$$u \geqslant \delta \quad \text{in } M.$$

This proves our lemma. □

From (2.4), our lower bound for $\min_M u$ clearly depends on $\inf_M a$. As mentioned in Introduction, it could be necessary to have $\inf_M a > 0$ in order to guarantee that $\min_M u$ stays away from 0 for any positive solution u. We now quote the following regularity result whose proof can be mimicked from a similar result proved in [16].

Lemma 2.2. *Assume that $u \in H^1(M)$ is an almost everywhere non-negative weak solution of Eq.* (1.12)*. Then*

(a) *If $\varepsilon > 0$, then $u \in C^\infty(M)$. In particular, $u \geqslant 0$ in M.*

(b) *If $\varepsilon = 0$ and $u^{-1} \in L^p(M)$ for all $p \geqslant 1$, then $u \in C^\infty(M)$.*

3. The analysis of the energy functionals when $\sup_M f > 0$

As indicated in the title of this section, throughout this section, we mainly consider the energy functional associated to (1.12) in the case when $\sup_M f > 0$. As such, unless otherwise stated, we always assume that $\sup_M f > 0$ and $\inf_M a > 0$.

3.1. Functional setting

For each $q \in (2, 2^\star)$ and $k > 0$, we introduce $\mathscr{B}_{k,q}$ a hyper-surface of $H^1(M)$ which is defined by

$$\mathscr{B}_{k,q} = \left\{ u \in H^1(M) : \|u\|_{L^q} = k^{\frac{1}{q}} \right\}. \tag{3.1}$$

Notice that for any $k > 0$, our set $\mathscr{B}_{k,q}$ is non-empty since it contains $k^{\frac{1}{q}}$. Now we construct the energy functional associated to problem (1.12). For each $\varepsilon > 0$ small satisfying (2.5), consider the functional $\mathcal{F}_q^\varepsilon : H^1(M) \to \mathbb{R}$ defined by

$$\mathcal{F}_q^\varepsilon(u) = \frac{1}{2}\int_M \left(|\nabla u|^2 + hu^2\right) d\mathrm{vol}_g - \frac{1}{q}\int_M f|u|^q d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} d\mathrm{vol}_g.$$

By a standard argument, $\mathcal{F}_q^\varepsilon$ is differentiable on $H^1(M)$. Since $\mathcal{F}_q^\varepsilon\big|_{\mathscr{B}_{k,q}}$ is bounded from below by $-k|\sup_M f|$, we can define

$$\mu_{k,q}^\varepsilon = \inf_{u \in \mathscr{B}_{k,q}} \mathcal{F}_q^\varepsilon(u).$$

Since critical points of $\mathcal{F}_q^\varepsilon$ are weak solutions of (1.12), we wish to find critical points of the functional $\mathcal{F}_q^\varepsilon$. It was proved in [16] that $\mu_{k,q}^\varepsilon$ is achieved by some smooth positive function, say u_ε. The proof is standard and we refer the reader to [16] for the details of the proof.

3.2. Asymptotic behavior of $\mu_{k,q}^\varepsilon$ in the case $\sup_M f > 0$

In this subsection, we investigate the behavior of $\mu_{k,q}^\varepsilon$ when both k and ε vary. We first study the behavior of $\mu_{k,q}^\varepsilon$ as $k \to +\infty$. Using the idea developed in [16], we can easily prove the following lemma.

Lemma 3.1. *$\mu_{k,q}^\varepsilon \to -\infty$ as $k \to +\infty$ if $\sup_M f > 0$.*

We are going to show that $\mu_{k_{1,q},q}^\varepsilon < \mu_{k_{2,q},q}^\varepsilon$ where $k_{1,q}$ and $k_{2,q}$ are given in (2.1). To this purpose, we first need a rough estimate for $\mu_{k_{1,q},q}^\varepsilon$.

Lemma 3.2. *There holds*

$$\mu_{k_{1,q},q}^\varepsilon \leqslant \frac{1}{2} k_{1,q}^{\frac{2}{q}} \int_M h\,d\mathrm{vol}_g - \frac{k_{1,q}}{q} \int_M f\,d\mathrm{vol}_g + \frac{1}{q k_{1,q}} \int_M a\,d\mathrm{vol}_g \tag{3.2}$$

where $k_{1,q}$ is given in (2.1).

Proof. This is trivial since $\mu_{k_{1,q},q}^\varepsilon \leqslant \mathcal{F}_q^\varepsilon(k_{1,q}^{\frac{1}{q}})$. The proof follows. □

As a consequence of Lemma 3.2 and thanks to the fact that $\underline{k} < 1$ and $\overline{k} \geqslant 1$, we can bound $\mu_{k_{1,q},q}^\varepsilon$ with the bound independent of q and ε as follows

$$\mu_{k_{1,q},q}^\varepsilon \leqslant \frac{\overline{k}^{\frac{2}{2^\flat}}}{2} \int_M h\,d\mathrm{vol}_g + \frac{\overline{k}}{2} \sup_M |f| + \frac{1}{2\underline{k}} \int_M a\,d\mathrm{vol}_g.$$

As can be seen, the right hand side of (3.2) is always positive. In order to make $\mu_{k_{2,q},q}^\varepsilon > \mu_{k_{1,q},q}^\varepsilon$ with $k_{2,q} > k_{1,q}$, we need $\sup_M f$ to be small. We now study the asymptotic behavior of $\mu_{k,q}^\varepsilon$ as $k \to 0$. This result together with Lemmas 3.1 and 3.6 give us a full description of the asymptotic behavior of $\mu_{k,q}^\varepsilon$.

Lemma 3.3. *There holds $\lim_{k\to 0+} \mu_{k,q}^{k^{\frac{2}{q}}} = +\infty$. In particular, there is some $k_\star$ sufficiently small and independent of both q and ε such that*

$$\mu_{k_\star,q}^\varepsilon \geqslant \overline{k}^{\frac{2}{2^\flat}} \int_M h\,d\mathrm{vol}_g + \overline{k} \sup_M |f| + \frac{1}{\underline{k}} \int_M a\,d\mathrm{vol}_g$$

for any $\varepsilon \leqslant k_\star$. In particular, there holds $\mu_{k_\star,q}^\varepsilon > \mu_{k_{1,q},q}^\varepsilon$.

Proof. The way that ε comes and plays immediately shows us that $\mu_{k,q}^{\varepsilon}$ is strictly monotone decreasing in ε for fixed k and q. Following [16, Lemma 3.1], for any $\varepsilon \leqslant k^{\frac{2}{q}}$, any $1 < q/2 < 2^\star/2$, and any $u \in \mathscr{B}_{k,q}$, we have

$$\int_M \sqrt{a}\, d\mathrm{vol}_g \leqslant 2^{\frac{q}{4}} \sqrt{k} \left(\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} d\mathrm{vol}_g \right)^{\frac{1}{2}}. \tag{3.3}$$

By squaring (3.3) and using $q < 2^\star$, we get that

$$\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}} d\mathrm{vol}_g \geqslant \frac{1}{2^{\frac{2^\star}{2}} k} \left(\int_M \sqrt{a}\, d\mathrm{vol}_g \right)^2.$$

This helps us to conclude

$$\mathcal{F}_q^{\varepsilon}(u) \geqslant -\frac{k}{q} \sup_M f + \frac{1}{2^{\frac{2^\star}{2}} qk} \left(\int_M \sqrt{a}\, d\mathrm{vol}_g \right)^2,$$

which proves that $\mu_{k,q}^{k^{\frac{2}{q}}} \to +\infty$ as $k \to 0$. Since the right hand side of the preceding inequality is independent of u, in order to get the desired estimate, it suffices to find some small $k_\star < 1$ independent of both q and ε such that the following inequality

$$\begin{aligned} -\frac{k_\star}{q} \sup_M f + \frac{1}{2^{\frac{2^\star}{2}} qk_\star} \left(\int_M \sqrt{a}\, d\mathrm{vol}_g \right)^2 & \\ \geqslant \overline{k}^{\frac{2}{2^\flat}} \int_M h\, d\mathrm{vol}_g + \overline{k} \sup_M |f| + \frac{1}{\underline{k}} \int_M a\, d\mathrm{vol}_g & \end{aligned} \tag{3.4}$$

holds. In order to find such a $k_\star$, we first let $k_\star < 1$. Since $q > 2$, it suffices to select $k_\star$ in such a way that

$$\frac{1}{2^{\frac{2^\star}{2}} 2^\star k_\star} \left(\int_M \sqrt{a}\, d\mathrm{vol}_g \right)^2 \geqslant \overline{k}^{\frac{2}{2^\flat}} \int_M h\, d\mathrm{vol}_g + \overline{k} \sup_M |f| + \frac{1}{\underline{k}} \int_M a\, d\mathrm{vol}_g + \frac{1}{2} \sup_M f$$

which is equivalent to

$$\begin{aligned} k_\star \leqslant \frac{1}{2^{\frac{2^\star}{2}} 2^\star} & \left(\int_M \sqrt{a}\, d\mathrm{vol}_g \right)^2 \\ & \left(\overline{k}^{\frac{2}{2^\flat}} \int_M h\, d\mathrm{vol}_g + \overline{k} \sup_M |f| + \frac{1}{\underline{k}} \int_M a\, d\mathrm{vol}_g + \frac{1}{2} \sup_M f \right)^{-1}. \end{aligned}$$

Hence, one can choose $k_\star$ as

$$\begin{aligned} k_\star \;=\; \min\Big\{ & \frac{1}{2^{\frac{2^\star}{2}} 2^\star} \Big(\int_M \sqrt{a} d\mathrm{vol}_g \Big)^2 \\ & \Big(\overline{k}^{\frac{2}{2^\flat}} \int_M h d\mathrm{vol}_g + \big(\overline{k} + \frac{1}{2}\big) \sup_M |f| + \frac{1}{\underline{k}} \int_M a d\mathrm{vol}_g \Big)^{-1}, \underline{k}, 1 \Big\}. \qquad (3.5) \end{aligned}$$

Since $k_\star \leqslant 1$, we always have $k_\star < k_\star^{\frac{2}{q}}$. By Lemma 3.2, we can check that $\mu^\varepsilon_{k_\star,q} > \mu^\varepsilon_{k_{1,q},q}$, thus concluding the lemma with $\varepsilon \leqslant k_\star$. Notice that, we have used $\underline{k}$ in (3.5). The reason is that we wish to ensure that $k_\star < k_{1,q}$ in any case. The proof now follows easily. $\square$

Our next result concerns the continuity of the function $\mu^\varepsilon_{k,q}$ with respect to k for each $\varepsilon > 0$ and $q \in (2^\flat, 2^\star)$ fixed. Since a similar result has been proved in [16], we omit its proof here and refer the reader to [16, Proposition 3.9].

Proposition 3.4. *For $\varepsilon > 0$ and $q \in (2^\flat, 2^\star)$ fixed, the function $\mu^\varepsilon_{k,q}$ is continuous with respect to k.*

In the rest of this section, our aim here is to study $\mu^\varepsilon_{k,q}$ when $k \geqslant k_{1,q}$. It is found that $\mu^\varepsilon_{k_{1,q},q} < \mu^\varepsilon_{k_{2,q},q}$ provided $\sup_M f$ is sufficiently small. To this end, we need to estimate $\mu^\varepsilon_{k,q}$ for $k \geqslant k_{1,q}$. A similar result was studied in [20, Proposition 2], [16, Proposition 3.14], or [17, Proposition 4.5]. Recall that $\mathcal{S}_h$ is the Sobolev constant appearing in (1.7).

Proposition 3.5. *For any $u \in \mathscr{B}_{k,q}$ with $k \geqslant k_{2,q}$, any $q \in [2^\flat, 2^\star)$, and any $\varepsilon > 0$, there holds*

$$\mathcal{F}^\varepsilon_q(u) \geqslant \frac{1}{2} \mathcal{S}_h k^{\frac{2}{q}} - \frac{k}{q} \sup_M f.$$

In particular,

$$\mu^\varepsilon_{k,q} \geqslant \frac{1}{2} \mathcal{S}_h k^{\frac{2}{q}} - \frac{k}{q} \sup_M f$$

for any $k \geqslant k_{2,q}$.

Proof. Suppose $u \in \mathscr{B}_{k,q}$ where k is arbitrary. We now estimate $\mathcal{F}^\varepsilon_q(u)$ from below. In view of (1.7) and the Hölder inequality, we obviously have

$$\int_M |\nabla u|^2 d\mathrm{vol}_g + \int_M h u^2 d\mathrm{vol}_g \geqslant \mathcal{S}_h k^{\frac{2}{q}}$$

Using this, we then easily have

$$\mathcal{F}_q^\varepsilon(u) \geqslant \frac{1}{2}\left(\int_M |\nabla u|^2 \, d\mathrm{vol}_g + \int_M hu^2 \, d\mathrm{vol}_g\right) - \frac{1}{q}\int_M f^+|u|^q \, d\mathrm{vol}_g.$$

In particular, there holds

$$\mathcal{F}_q^\varepsilon(u) \geqslant \frac{1}{2}\mathcal{S}_h k^{\frac{2}{q}} - \frac{k}{q}\sup_M f.$$

Thus, we can conclude the lemma by taking the infimum with respect to $u \in \mathscr{B}_{k,q}$. □

In order to prove the existence of a local minimum point, the following lemma plays an important role in our analysis.

Lemma 3.6. *Assume that, for some $\tau > 1$, the total integral of a satisfies*

$$\int_M a \, d\mathrm{vol}_g < \frac{q-2}{4q}\frac{\mathcal{S}_h}{\tau}\left(\frac{q+2}{4q}\frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g}\right)^{\frac{q+2}{q-2}}. \qquad (3.6)$$

Then there holds

$$\mu^\varepsilon_{k_{1,q},q} < \min\{\mu^\varepsilon_{k_\star,q}, \mu^\varepsilon_{k_{2,q},q}\}$$

for any $q \in [q_{\eta_0}, 2^\star)$ and any $\varepsilon \in (0, k_\star)$.

Proof. First, using Lemma 3.3, it suffices to verify $\mu^\varepsilon_{k_{1,q},q} < \mu^\varepsilon_{k_{2,q},q}$ for all $q \in [q_{\eta_0}, 2^\star)$. By Lemma 3.2 and Proposition 3.5, the following facts have already proved

$$\mu^\varepsilon_{k_{1,q},q} < \frac{k_{1,q}^{\frac{2}{q}}}{2}\int_M h \, d\mathrm{vol}_g - \frac{k_{1,q}}{2}\int_M f \, d\mathrm{vol}_g + \frac{1}{2k_{1,q}}\int_M a \, d\mathrm{vol}_g$$

and

$$\frac{1}{2}\mathcal{S}_h k_{2,q}^{\frac{2}{q}} - \frac{k_{2,q}}{q}\sup_M f \leqslant \mu^\varepsilon_{k_{2,q},q}.$$

Therefore, it suffices to prove that

$$k_{1,q}^{\frac{2}{q}}\int_M h \, d\mathrm{vol}_g - k_{1,q}\int_M f \, d\mathrm{vol}_g + \frac{1}{k_{1,q}}\int_M a \, d\mathrm{vol}_g \leqslant \mathcal{S}_h k_{2,q}^{\frac{2}{q}} - k_{2,q}\sup_M f,$$

for any $q \in [q_{\eta_0}, 2^\star)$. Notice that, from the choice of τ, we can verify that $\mathcal{S}_h \tau^{\frac{2}{q}} \geqslant 2 \int_M h d\mathrm{vol}_g$. This amounts to saying that

$$k_{1,q}^{\frac{2}{q}} \int_M h d\mathrm{vol}_g \leqslant \frac{1}{2} \mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2}{q}} = \frac{1}{2} \mathcal{S}_h k_{2,q}^{\frac{2}{q}}.$$

Therefore, it suffices to show that

$$-k_{1,q} \int_M f \, d\mathrm{vol}_g + \frac{1}{k_{1,q}} \int_M a d\mathrm{vol}_g \leqslant \frac{1}{2} \mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2}{q}} - \tau k_{1,q} \sup_M f$$

or equivalently,

$$\begin{aligned} \int_M a d\mathrm{vol}_g &\leqslant \frac{1}{2} \mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{1+\frac{2}{q}} - k_{1,q}^2 \left(\tau \sup_M f - \int_M f \, d\mathrm{vol}_g \right) \\ &= k_{1,q}^2 \left(\frac{1}{2} \mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2-q}{q}} - \left(\tau \sup_M f - \int_M f \, d\mathrm{vol}_g \right) \right), \end{aligned} \tag{3.7}$$

for any $q \in [q_{\eta_0}, 2^\star)$. Again, from the choice of $k_{1,q}$, it is clear to see that

$$\begin{aligned} \tau^{\frac{2}{q}} k_{1,q}^{\frac{2-q}{q}} &= \tau^{\frac{2}{q}} \left(\frac{q+2}{4q} \frac{\mathcal{S}_h \tau^{\frac{2}{q}}}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g} \right)^{-1} \\ &= \frac{4q}{q+2} \frac{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g}{\mathcal{S}_h}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{2} \mathcal{S}_h \tau^{\frac{2}{q}} k_{1,q}^{\frac{2-q}{q}} - \left(\tau \sup_M f - \int_M f \, d\mathrm{vol}_g \right) \\ &= \frac{q-2}{q+2} \left(\tau \sup_M f - \int_M f \, d\mathrm{vol}_g \right) \\ &= \frac{q-2}{4q} \mathcal{S}_h \tau^{\frac{2}{q}} \left(\frac{q+2}{4q} \frac{\mathcal{S}_h \tau^{\frac{2}{q}}}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g} \right)^{-1}. \end{aligned}$$

By using this, (3.7) is equivalent to

$$\int_M a d\mathrm{vol}_g \leqslant \frac{q-2}{4q} \mathcal{S}_h \tau^{\frac{2}{q}} \left(\frac{q+2}{4q} \frac{\mathcal{S}_h \tau^{\frac{2}{q}}}{\tau \sup_M f - \int_M f \, d\mathrm{vol}_g} \right)^{\frac{q+2}{q-2}}$$

$$= \frac{q-2}{4q}\frac{s_h}{\tau}\left(\frac{q+2}{4q}\frac{s_h\tau}{\tau\sup_M f - \int_M f\,d\mathrm{vol}_g}\right)^{\frac{q+2}{q-2}}. \tag{3.8}$$

The proof follows easily by comparing (3.6) and (3.8). □

4. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. The proof that we provide here consists of two steps. First, in view of Lemma 2.1 we need to make use of the condition $\inf_M a > 0$ in order to guarantee the existence of one solution. Second, by using a simple sub- and super- solutions argument, we prove that Eq. (1.1) still admits one positive smooth solution even that $\inf_M a = 0$.

4.1. The case $\inf_M a > 0$

In this subsection, we obtain the existence of one solution of (1.1) under the assumption $\inf_M a > 0$. For the sake of clarity, we divide the proof into several claims.

Claim 1. There holds

$$\mu^{\varepsilon}_{k_{1,q},q} < \min\{\mu^{\varepsilon}_{k_\star,q}, \mu^{\varepsilon}_{k_{2,q},q}\}$$

for all $q \in (q_{\eta_0}, 2^\star)$ and for all $\varepsilon \in (0, k_\star)$ satisfying (2.5).

Proof of Claim 1. This is a consequence of Lemma 3.6. In order to apply Lemma 3.6, we have to derive (3.6) for suitable q close enough to $2^\star$. Observe that

$$\lim_{q\to 2^\star}\frac{q+2}{q-2} = n-1, \quad \lim_{q\to 2^\star}\frac{q-2}{4q}\left(\frac{q+2}{4q}\right)^{\frac{q+2}{q-2}} = \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}.$$

Hence, we can choose $q_{\eta_0} \in [2^\flat, 2^\star)$ sufficiently close to $2^\star$ such that the condition $s_h\tau^{\frac{2}{q}} \geqslant 2$ and the following inequality

$$\int_M a\,d\mathrm{vol}_g < \frac{q-2}{4q}\frac{s_h}{\tau}\left(\frac{q+2}{4q}\frac{s_h\tau}{\tau\sup_M f - \int_M f\,d\mathrm{vol}_g}\right)^{\frac{q+2}{q-2}} \tag{4.1}$$

holds for any $q \in [q_{\eta_0}, 2^\star)$. Thanks to (4.1), we can now use Lemma 3.6 to finish the proof of this claim. Note that the condition $\inf_M a > 0$ is crucial since this makes the right hand side of (2.5) strictly positive.

It is important to note that q_{η_0} is independent of q and ε. Thus, from now on, we only consider $q \in [q_{\eta_0}, 2^\star)$.

Claim 2. Eq. (1.12) with ε replaced by 0 has a positive solution, say $u_{1,q}$, that is, $u_{1,q}$ solves the following subcritical equation

$$\Delta_g u_{1,q} + h u_{1,q} = f(u_{1,q})^{q-1} + \frac{a}{(u_{1,q})^{q+1}}, \tag{4.2}$$

for any $q \in [q_{\eta_0}, 2^\star)$.

Proof of Claim 2. Again for the sake of clarity, we divide our proof into two steps.

Step 1. *The existence of $u^\varepsilon_{1,q}$ with energy μ^ε_q.* We now define

$$\mu^\varepsilon_q = \inf_{u \in \mathscr{D}_q} \mathcal{F}^\varepsilon_q(u)$$

where the set $\mathscr{D}_q$ is given by

$$\mathscr{D}_q = \left\{ u \in H^1(M) : k_\star \leqslant \|u\|^q_{L^q} \leqslant k_{2,q} \right\}.$$

In term of $\mathscr{B}_{k,q}$, we can rewrite $\mathscr{D}_q$ as follows $\mathscr{D}_q = \bigcup_{k_\star \leqslant k \leqslant k_{2,q}} \mathscr{B}_{k,q}$. It follows from $k_{1,q} \in (k_\star, k_{2,q})$ and Lemma 3.2 that

$$\mu^\varepsilon_q \leqslant \mu^\varepsilon_{k_{1,q},q} \leqslant \overline{k}^{\frac{2}{2^\flat}} \int_M h \, d\mathrm{vol}_g + \overline{k} \sup_M |f| + \frac{1}{\underline{k}} \int_M a \, d\mathrm{vol}_g.$$

In other words, we have proved that μ^ε_q is bounded. By a standard argument and the Ekeland Variational Principle, one can show that there exists a H^1-bounded minimizing sequence for μ^ε_q in $\mathscr{D}_q$. A standard argument shows that μ^ε_q is achieved by some positive function $u^\varepsilon_{1,q} \in \mathscr{D}_q$. Notice that one can claim $u^\varepsilon_{1,q} \in \mathscr{D}_q$ since $q < 2^\star$, furthermore, by Claim 1, $u^\varepsilon_{1,q}$ does not lie on the boundary of $\mathscr{D}_q$; hence $u^\varepsilon_{1,q}$ is a weak solution of (1.12). Thus, the regularity result, Lemma 2.2(a), developed in Section 2 can be applied to (1.12) to conclude that $u^\varepsilon_{1,q} \in C^\infty(M)$. Finally, with Lemma 2.1 and the Strong Minimum Principle in hand, in order to see why $u^\varepsilon_{1,q} > 0$, it is necessary to rule out the case $u^{\varepsilon}_{1,q} \equiv 0$. To this purpose, we observe that $\|u^{\varepsilon}_{1,q}\|^q_{L^q} > k_\star > 0$.

Step 2. *The existence of $u_{1,q}$ with energy $\mu_{k_{1,q}}$.* Next, in order to send $\varepsilon \to 0$, we need a uniform bound for $u^{\varepsilon}_{1,q}$ in $H^1(M)$. Using the Hölder inequality and the fact that $\|u^{\varepsilon}_{1,q}\|_{L^2} \leqslant \|u^{\varepsilon}_{1,q}\|_{L^q}$, it is not hard to prove that $\|u^{\varepsilon}_{1,q}\|_{H^1}$ is bounded from above with the bound independent of q and ε. In what follows, we let $\{\varepsilon_j\}_j$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$. For each j, let $u^{\varepsilon_j}_{1,q}$ be a smooth positive function in M solving

$$\Delta_g u^{\varepsilon_j}_{1,q} + h u^{\varepsilon_j}_{1,q} = f(u^{\varepsilon_j}_{1,q})^{q-1} + \frac{a u^{\varepsilon_j}_{1,q}}{((u^{\varepsilon_j}_{1,q})^2 + \varepsilon_j)^{\frac{q}{2}+1}}. \tag{4.3}$$

Being a bounded sequence in $H^1(M)$, there exists $u_{1,q} \in H^1(M)$ such that, up to subsequences, as $j \to \infty$,

- $u^{\varepsilon_j}_{1,q} \rightharpoonup u_{1,q}$ in $H^1(M)$;
- $u^{\varepsilon_j}_{1,q} \to u_{1,q}$ strongly in $L^2(M)$; and
- $u^{\varepsilon_j}_{1,q} \to u_{1,q}$ almost everywhere in M.

Using Lemma 2.1, the Lebesgue Dominated Convergence Theorem can be applied to conclude that $\int_M (u_{1,q})^{-p} d\mathrm{vol}_g$ is finite for all p. Now sending $j \to \infty$ in (4.3), we get that $u_{1,q}$ is a weak solution of the subcritical equation (4.2). Thus Lemma 2.2(b) can be applied to (4.2). It follows that $u_{1,q} \in C^\infty(M)$. Using the strong convergence in $L^p(M)$ and the fact that $k^{\varepsilon}_1 \geqslant k_\star$, one can see that $\|u_{1,q}\|^q_{L^q} \geqslant k_\star > 0$, thus proving $u_{1,q} \not\equiv 0$. Again with Lemma 2.1 and the Strong Minimum Principle in hand, it is easy to prove that $u_{1,q}$ is strictly positive. Keep in mind that $\|u_{1,q}\|^q_{L^q} \leqslant k_{2,q}$ since we still have a strong convergence. This settles Claim 2.

Claim 3. Eq. (1.1) has at least one positive solution.

Proof of Claim 3. Let us denote by $\mu_{k_{1,q}}$ the energy of $u_{1,q}$ found in Claim 2, i.e.,

$$\begin{aligned}\mu_{k_{1,q}} = &\frac{1}{2}\int_M |\nabla u_{1,q}|^2 d\mathrm{vol}_g + \frac{1}{2}\int_M h(u_{1,q})^2 d\mathrm{vol}_g \\ &- \frac{1}{q}\int_M f(u_{1,q})^q d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{(u_{1,q})^q} d\mathrm{vol}_g.\end{aligned}$$

Here by k_1 we mean $\|u_{1,q}\|_{L^q}^q = k_1$. Since $q < 2^\star$, by strong convergences, we have

$$\begin{aligned}
\mu_{k_1,q} &= \limsup_{j\to\infty} \mu_q^{\varepsilon_j} \\
&\leqslant \limsup_{j\to\infty} \mu_{k_{1,q},q}^{\varepsilon_j} \\
&\leqslant \frac{\overline{k}^{\frac{2}{2^\flat}}}{2}\int_M h\, d\mathrm{vol}_g + \frac{\overline{k}}{2}\sup_M |f| + \frac{1}{2\underline{k}}\int_M a\, d\mathrm{vol}_g. \qquad (4.4)
\end{aligned}$$

We now estimate the H^1-norm of the sequence $\{u_{1,q}\}_q$. Clearly, since $h > 0$ and $a \geqslant 0$, we get that

$$\begin{aligned}
\frac{1}{2}\int_M |\nabla u_{1,q}|^2 d\mathrm{vol}_g &= \mu_{k_1,q} - \frac{1}{2}\int_M h(u_{1,q})^2 d\mathrm{vol}_g \\
&\quad + \frac{1}{q}\int_M f(u_{1,q})^q d\mathrm{vol}_g - \frac{1}{q}\int_M \frac{a}{(u_{1,q})^q} d\mathrm{vol}_g \\
&\leqslant \mu_{k_1,q} + \frac{1}{q}\int_M f(u_{1,q})^q d\mathrm{vol}_g \\
&\leqslant \mu_{k_1,q} + \frac{k_1}{2}\sup_M |f|.
\end{aligned}$$

Since $k_1 \in [k_\star, k_{2,q}]$, we then easily obtain

$$\frac{1}{2}\int_M |\nabla u_{1,q}|^2 d\mathrm{vol}_g \leqslant \frac{\overline{k}^{\frac{2}{2^\flat}}}{2}\int_M h\, d\mathrm{vol}_g + \overline{k}\sup_M |f| + \frac{1}{2\underline{k}}\int_M a\, d\mathrm{vol}_g.$$

This and the fact that $\|u_{1,q}\|_{L^2}^2 \leqslant (\overline{k})^{\frac{2}{2^\flat}}$ imply that the sequence $\{u_{1,q}\}_q$ remains bounded in $H^1(M)$. Thus, up to subsequences, there exists $u_1 \in H^1(M)$ such that, as $q \to 2^\star$,

- $u_{1,q} \rightharpoonup u_1$ in $H^1(M)$;
- $u_{1,q} \to u_1$ strongly in $L^2(M)$; and
- $u_{1,q} \to u_1$ almost everywhere in M.

Our aim is now to prove that u_1 is the desired solution. Notice that $u_{1,q}$ verifies

$$\begin{aligned}\int_M \nabla u_{1,q}\cdot\nabla v\,d\mathrm{vol}_g &+ \int_M h u_{1,q} v\,d\mathrm{vol}_g \\ &- \int_M f(u_{1,q})^{q-1} v\,d\mathrm{vol}_g - \int_M \frac{a}{(u_{1,q})^{q+1}} v\,d\mathrm{vol}_g = 0\end{aligned} \tag{4.5}$$

for any $v \in H^1(M)$. As a standard routine, all we need to do is to take the limit in (4.5) as $q \to 2^\star$. First, thanks to $\nabla u_{1,q} \rightharpoonup \nabla u_1$, there holds

$$\int_M (\nabla u_{1,q} - \nabla u_1)\cdot \nabla v\,d\mathrm{vol}_g \to 0$$

as $q \to 2^\star$. Since $u_{1,q} \to u_1$ strongly in $L^2(M)$, it is not hard to see

$$\int_M (u_{1,q} - u_1)\, v\,d\mathrm{vol}_g \to 0$$

as $q \to 2^\star$. Lemma 2.1 and the dominated convergence theorem imply that

$$\int_M \frac{av}{(u_{1,q})^{q+1}}\,d\mathrm{vol}_g \to \int_M \frac{av}{(u_1)^{2^\star+1}}\,d\mathrm{vol}_g,$$

as $q \to 2^\star$. So far, we can pass to the limit every terms on the left hand side of (4.5) except the term involving f. By the Hölder inequality, one obtains

$$\left\|(u_{1,q})^{q-1}\right\|_{L^{\frac{2^\star}{2^\star-1}}} \leqslant \left(\left(\int_M (u_{1,q})^{2^\star}\,d\mathrm{vol}_g\right)^{\frac{q-1}{2^\star-1}}\right)^{\frac{2^\star-1}{2^\star}} = \|u_{1,q}\|_{L^{2^\star}}^{q-1}. \tag{4.6}$$

Making use of the Sobolev inequality and (4.6), we can prove the boundedness of $(u_{1,q})^{q-1}$ in $L^{\frac{2^\star}{2^\star-1}}(M)$. In addition, since $u_{1,q} \to u_1$ almost everywhere, $(u_{1,q})^{q-1} \to (u_1)^{2^\star-1}$ almost everywhere. According to [2, Theorem 3.45], we conclude that $(u_{1,q})^{q-1} \rightharpoonup (u_1)^{\frac{2^\star}{2^\star-1}}$ weakly in $L^{\frac{2^\star}{2^\star-1}}(M)$. Therefore, by definition of weak convergence and the smoothness of f, one has

$$\int_M f(u_{1,q})^{q-1} v\,d\mathrm{vol}_g \to \int_M f(u_1)^{2^\star-1} v\,d\mathrm{vol}_g \tag{4.7}$$

as $q \to 2^\star$. Using (4.7), one can see, by sending $q \to 2^\star$ in (4.5), that u_1 are weak solutions to (1.1). Using Lemma 2.2(b) we conclude that $u_1 \in C^\infty(M)$

and $u_1 > 0$ in M.

4.2. The case $\inf_M a = 0$

Under this context, making use of the method of sub- and super-solutions is the key argument. As far as we know, this idea was first introduced in [9, Pages 43–44][1]. However, it is worth mentioning that our construction of sub-solutions is different from that of [9]. We let $\varepsilon_0 > 0$ sufficiently small and then fix it so that the following inequality

$$\int_M a\,d\mathrm{vol}_g + \varepsilon_0 < \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h}{\tau}\left(\frac{\mathcal{S}_h\tau}{\tau\sup_M f - \int_M f\,d\mathrm{vol}_g}\right)^{n-1} \tag{4.8}$$

still holds. Since the manifold M has unit volume, we can conclude that from (4.8), the function $a+\varepsilon_0$ verifies all assumptions in the previous subsection, thus showing that there exists a positive smooth function $\overline{u}$ solving the following equation

$$\Delta_g \overline{u} + h\overline{u} = f\overline{u}^{2^\star-1} + \frac{a+\varepsilon_0}{\overline{u}^{2^\star+1}}.$$

Obviously, $\overline{u}$ is a super-solution to (1.1), that is

$$\Delta_g \overline{u} + h\overline{u} \geqslant f\overline{u}^{2^\star-1} + \frac{a}{\overline{u}^{2^\star+1}}.$$

Our aim is to find a sub-solution to (1.1). In this context, we consider the following equation

$$\Delta_g u + (h - f^-)u = a. \tag{4.9}$$

Since $h - f^- > 0$, $a \geqslant 0$, $a \not\equiv 0$, and the manifold M is compact without the boundary, the standard argument shows that (4.9) always admits a weak solution, say u_0. By a standard regularity result, one can easily deduce that u_0 is at least continuous. Thus, by the Maximum Principle, we conclude $u_0 > 0$.

As before, we now find the sub-solution $\underline{u}$ of the form εu_0 for small $\varepsilon > 0$

[1] Some other unpublished results can also be found in [9].

to be determined. To this purpose, we first write

$$\Delta_g \underline{u} + h\underline{u} = \varepsilon a + f^- \underline{u}. \tag{4.10}$$

Since $\max_M u_0 < +\infty$, it is easy to see that, for any $0 < \varepsilon \leqslant (\max_M u_0)^{-\frac{2^\star+1}{2^\star+2}}$, there holds

$$\varepsilon a \leqslant \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}. \tag{4.11}$$

Besides, since $f^- \leqslant 0$ and $2^\star > 2$, it is not difficult to see that the following inequality

$$\varepsilon u_0 f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f^-$$

holds provided $1 \leqslant \max_M u_0$. In particular, the following

$$\varepsilon u_0 f^- \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f \tag{4.12}$$

holds provided $1 \leqslant \max_M u_0$. Combining all estimates (4.10), (4.11), and (4.12) above, we conclude that for small ε, there holds

$$\Delta_g \underline{u} + h\underline{u} \leqslant \varepsilon^{2^\star-1} u_0^{2^\star-1} f + \frac{a}{\varepsilon^{2^\star+1} u_0^{2^\star+1}}.$$

In other words, we have shown that $\underline{u}$ is a sub-solution of (1.1). Finally, since $\overline{u}$ has a strictly positive lower bound, we can choose $\varepsilon > 0$ sufficiently small such that $\underline{u} \leqslant \overline{u}$. Using the sub- and super-solutions method, see [13, Lemma 2.6], we can conclude the existence of a positive solution u to (1.1). By a regularity result developed in [13], we know that u is smooth.

5. Proof of Theorem 1.2

In order to prove Theorem 1.2, we need to show that the condition (1.11) is fulfilled. Although we have not assumed that $\int_M a d\mathrm{vol}_g$ is bounded from above, we are able to show that we can recover the condition (1.11) provided $\sup_M f$ is bounded from above by some small constant $\mathcal{C}$ depending not only on f^- but also on a and h. Here we only consider the existence of such a $\mathcal{C}$, the dependence of $\mathcal{C}$ in a and h will be considered in the last section of the paper.

As usual, we first assume $\inf_M a > 0$. Depending on the sign of $\int_M f\,d\mathrm{vol}_g$, we have two cases.

Case 1. Suppose $\int_M f\,d\mathrm{vol}_g \geqslant 0$. In this context, we can easily verify that

$$\frac{\mathcal{S}_h}{\sup_M f} \leqslant \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f\,d\mathrm{vol}_g}.$$

Therefore, it suffices to show that

$$\int_M a\,d\mathrm{vol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \frac{\mathcal{S}_h}{\tau} \left(\frac{\mathcal{S}_h}{\sup_M f} \right)^{n-1},$$

which is equivalent to

$$\sup_M f < \left(\frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \frac{\mathcal{S}_h^n}{\tau \int_M a\,d\mathrm{vol}_g} \right)^{\frac{1}{n-1}}.$$

Case 2. Suppose $\int_M f\,d\mathrm{vol}_g < 0$. In this context, we assume for a moment that $\sup_M f > 0$ is small in such a way that we can select

$$\tau = \frac{1}{\sup_M f} \geqslant \max \left\{ 1, \Big(\frac{2}{S_h} \int_M h\,d\mathrm{vol}_g \Big)^{\frac{2^\star}{2}} \right\}.$$

Then, we have

$$\begin{aligned} \frac{\mathcal{S}_h \tau}{\tau \sup_M f - \int_M f\,d\mathrm{vol}_g} &= \frac{1}{\sup_M f} \frac{\mathcal{S}_h}{1 - \int_M f\,d\mathrm{vol}_g} \\ &\geqslant \frac{1}{\sup_M f} \frac{\mathcal{S}_h}{1 + \int_M |f^-|\,d\mathrm{vol}_g}. \end{aligned}$$

Therefore, it suffices to show that

$$\int_M a\,d\mathrm{vol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \mathcal{S}_h \sup_M f \left(\frac{1}{\sup_M f} \frac{\mathcal{S}_h}{1 + \int_M |f^-|\,d\mathrm{vol}_g} \right)^{n-1},$$

which is equivalent to

$$\sup_M f < \left(\frac{(2n-1)^{n-1}}{2^{2n-1} n^n} \frac{\mathcal{S}_h^n}{(1 + \int_M |f^-|\,d\mathrm{vol}_g) \int_M a\,d\mathrm{vol}_g} \right)^{\frac{1}{n-2}}.$$

From calculation above, we conclude that there exists some positive constant $\mathcal{C} > 0$ depending only on a, h, and f^- such that if $0 < \sup_M f < \mathcal{C}$, our equation (1.1) always admits at least one positive smooth solution.

It remains to consider the case $\inf_M a = 0$. However, since the size of a plays no role in the above calculation, we can freely add a small constant ε_0 to a as in the second stage of the proof of Theorem 1.1. This procedure ensures that we get a super-solution of (1.1) with a strictly positive lower bound and this is enough since a suitable positive sub-solution always exists.

6. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. Since we are in the case $\sup_M f \leqslant 0$, we generally have two cases. First we observe that if $f \equiv 0$, then it is easy to see that (1.1) always admits a positive solution since a small constant and a large constant are sub- and super-solutions. Therefore, it suffices to study the case $f \not\equiv 0$. This and the condition $\sup_M f \leqslant 0$ immediately imply that $\int_M f \, d\mathrm{vol}_g < 0$.

To prove Theorem 1.3, we use the same approach as in the proof of Theorem 1.1. However, unlike the case when $\sup_M f > 0$ that forces $\mu^\varepsilon_{k,q} \to -\infty$ as $k \to \infty$, in the case $\sup_M f \leqslant 0$, we always have $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to \infty$ and this is enough to guarantee the existence of at least solution.

Before doing so and since $f^+ \equiv 0$, throughout this section, let us denote

$$k_{1,q} = \left(\frac{q+2}{4q} \frac{\mathcal{S}_h \tau^{\frac{2}{q}}}{-\int_M f \, d\mathrm{vol}_g} \right)^{\frac{q}{q-2}}, \quad k_{2,q} = \tau k_{1,q}, \tag{6.1}$$

where $\tau > 1$ is a scaling constant to be determined later. Thanks to $\int_M f \, d\mathrm{vol}_g < 0$, $k_{1,q}$ and then $k_{2,q}$ are clearly well-defined.

6.1. Asymptotic behavior of $\mu^\varepsilon_{k,q}$ in the case $\sup_M f \leqslant 0$

As we have already seen that the behavior of $\mu^\varepsilon_{k,q}$ for small k and small ε depends strongly on the term involving a. Despite the fact that we are under the case $\sup_M f \leqslant 0$, we can still go through Lemma 3.3 without any difficulty, that is, for small ε, $\mu^\varepsilon_{k,q} \to +\infty$ as $k \to 0$. It is worth noticing that we always assume $\inf_M a > 0$. We now study the behavior of $\mu^{\varepsilon}_{k,q}$ for $k \to +\infty$ when $\sup_M f \leqslant 0$.

Proposition 6.1. *Suppose* $\sup_M f \leqslant 0$, *then* $\mu^{\varepsilon}_{k,q} \to +\infty$ *as* $k \to +\infty$ *for any* $\varepsilon > 0$ *and any* $q \in [2^\flat, 2^\star)$ *but all are fixed.*

Proof. By using (1.7) and the Hölder inequality, for any $u \in \mathscr{B}_{k,q}$, any $q \in (q_{\eta_0}, 2^\star)$, and any $\varepsilon > 0$, there holds

$$\begin{aligned}\mathcal{F}^{\varepsilon}_q(u) =&\frac{1}{2}\int_M (|\nabla u|^2 + hu^2)\,d\mathrm{vol}_g - \frac{1}{q}\int_M f|u|^q\,d\mathrm{vol}_g + \frac{1}{q}\int_M \frac{a}{(u^2+\varepsilon)^{\frac{q}{2}}}\,d\mathrm{vol}_g\\ \geqslant&\frac{1}{2}\mathcal{S}_h k^{\frac{2}{q}},\end{aligned}$$

which immediately implies that $\mu^{\varepsilon}_{k,q} \geqslant \frac{1}{2}\mathcal{S}_h k^{\frac{2}{q}}$. Thus, we have shown that $\mu^{\varepsilon}_{k,q} \to +\infty$ as $k \to +\infty$. □

Our next lemma gives a full description for $\mu^{\varepsilon}_{k,q}$ similarly to that proved in Section 3.

Lemma 6.2. *There holds*

$$\mu^{\varepsilon}_{k_{1,q},q} < \min\{\mu^{\varepsilon}_{k_\star,q}, \mu^{\varepsilon}_{k_{2,q},q}\}$$

for any $\varepsilon \in (0, k_\star)$ *and any* $q \in (q_{\eta_0}, 2^\star)$.

Proof. As in the proof of Lemma 3.6, the proof is similar and straightforward. To see this, for new $k_{1,q}$ and $k_{2,q}$, we can also bound by $\underline{k}$ and $\overline{k}$ given in (2.2) and (2.3) by dropping $\sup_M f$. Therefore, we can define $k_\star$ as in (3.5). Having such a $k_\star$, the estimate $\mu^{\varepsilon}_{k_{1,q},q} \leqslant \mu^{\varepsilon}_{k_\star,q}$ still holds; therefore, it suffices to prove $\mu^{\varepsilon}_{k_{1,q},q} \leqslant \mu^{\varepsilon}_{k_{2,q},q}$ by choosing a suitable $\tau \gg 1$. Equivalently, we need to prove that

$$k^{\frac{2}{q}}_{1,q}\int_M h\,d\mathrm{vol}_g - k_{1,q}\int_M f\,d\mathrm{vol}_g + \frac{1}{k_{1,q}}\int_M a\,d\mathrm{vol}_g \leqslant \mathcal{S}_h k^{\frac{2}{q}}_{2,q},$$

for any $q \in [q_{\eta_0}, 2^\star)$. From the choice of τ, we only need to prove that

$$\int_M a\,d\mathrm{vol}_g \leqslant k^2_{1,q}\left(\frac{1}{2}\mathcal{S}_h \tau^{\frac{2}{q}} k^{\frac{2-q}{q}}_{1,q} + \int_M f\,d\mathrm{vol}_g\right). \tag{6.2}$$

A simple calculation shows that (6.2) is equivalent to

$$\int_M a\,d\mathrm{vol}_g \leqslant \tau^{\frac{4}{q-2}} \frac{q-2}{4q} \mathcal{S}_h \left(\frac{q+2}{4q} \frac{\mathcal{S}_h}{-\int_M f\,d\mathrm{vol}_g} \right)^{\frac{q+2}{q-2}}.$$

Hence, by choosing τ sufficiently large, one easily gets the desired result. $\square$

6.2. Proof of Theorem 1.3: The existence

The proof of the existence part of Theorem 1.3 consists of two parts.

Case 1. In the first stage of the proof, we assume that $\inf_M a > 0$ and $\varepsilon \in (0, k_\star)$ satisfying (2.5). With information that we have already proved in Lemma 6.2, we can define

$$\mu_q^\varepsilon = \inf_{u \in \mathscr{D}_q} \mathcal{F}_q^\varepsilon(u)$$

where

$$\mathscr{D}_q = \left\{ u \in H^1(M) : k_\star \leqslant \|u\|_{L^q}^q \leqslant k_{2,q} \right\}.$$

Then by an usual routine as we have already used before, we can easily prove the existence of at least one positive smooth solution to (1.1) that conclude the proof of Theorem 1.3.

Case 2. In the second stage of the proof, we assume $\inf_M a = 0$. Since we have no control on $\int_M a\,d\mathrm{vol}_g$, we can freely add small $\varepsilon_0 > 0$ to a as in the proof of Theorem 1.1. Since the trick that was used in the proof of Theorem 1.1 still works in our context, a sub- and super-solutions argument as used before concludes that (1.1) has at least one positive smooth solution for any $q \in [2^\flat, 2^\star)$.

6.3. Proof of Theorem 1.3: The uniqueness

The uniqueness of positive solutions of (1.1) follows from the fact that the following functions $t \mapsto -t^{2^\star-1}$ and $t \mapsto t^{-2^\star-1}$ are monotone decreasing. We note that this type of argument is standard and was used once in the proof of [17, Theorem 1.2].

Suppose that there exists two positive smooth solutions u_1 and u_2 of (1.1). By setting $w(x) = u_1(x) - v(x)$ with $x \in M$, we arrive at

$$w\Delta_g w + hw^2 = f\big(u_1^{2^\star-1} - v^{2^\star-1}\big)(u_1-u_2) + a\big(u_1^{-2^\star-1} - v^{-2^\star-1}\big)(u_1-u_2).$$

Thanks to $h > 0$, simply integrating both sides over M gives

$$\begin{aligned}
0 &\leqslant \int_M |\nabla w|^2 d\mathrm{vol}_g + \int_M h|w|^2 d\mathrm{vol}_g \\
&= \int_M f(u_1^{2^\star-1} - u_2^{2^\star-1})(u_1-u_2)d\mathrm{vol}_g \\
&\quad + \int_M a(u_1^{-2^\star-1} - u_2^{-2^\star-1})(u_1-u_2)d\mathrm{vol}_g \\
&\leqslant 0.
\end{aligned}$$

Thanks to $f \leqslant 0$ and $a \geqslant 0$ with $a \not\equiv 0$, the only possibility for which the preceding inequality holds is that w vanishes in M, thus proving the uniqueness of positive smooth solution of Eq. (1.1).

7. Some Remarks and Discussion

7.1. The constant $\mathcal{C}$ in Theorem 1.2

As can be seen from the proof of Theorem 1.2, our choice for $\mathcal{C}$ basically depends on both f^-, h, and a. Then one can ask whether or not there is some constant $\mathcal{C}$ independent of h and a such that the result in Theorem 1.2 still holds for any f with $\sup_M f < \mathcal{C}$. Here we prove that such a constant $\mathcal{C}$ never exists. Indeed, the key argument is a non-existence result due to Hebey–Pacard–Pollack [10, Theorem 2.1]. In our context, their result claims that (1.1) has no positive solution if $f > 0$ and the following inequality holds

$$\left(\frac{n^n}{(n-1)^{n-1}}\right)^{\frac{n+2}{4n}} \int_M a^{\frac{n+2}{4n}} f^{\frac{3n-2}{4n}} d\mathrm{vol}_g > \int_M h^{\frac{n+2}{4}} f^{\frac{2-n}{4}} d\mathrm{vol}_g. \tag{7.1}$$

Therefore, by contradiction, if such a constant $\mathcal{C}$ existed, we would have an existence result for (1.1) for any given $a \geqslant 0$ and $h > 0$. However, for any f fixed with $\sup_M f < \mathcal{C}$, this is impossible as one can construct counter-examples by either enlarging a or reducing h in such a way that the condition (7.1) holds.

Note that the dependence of $\mathcal{C}$ in a and h could be seen from the proof of Theorem 1.2, for example, the larger $\int_M a d\text{vol}_g$ is, the smaller $\sup_M f$ is.

7.2. Translation of f and a turning point

In view of Theorem 1.3, Eq. (1.1) always admits at least one positive smooth solution provided $\sup_M f \leqslant 0$. In particular, the condition $\int_M f d\text{vol}_g < 0$ holds. We now assume at the beginning that the smooth function f verifies the condition $\sup_M f \leqslant 0$. For each $\lambda \in \mathbb{R}$, we construct a new family of functions, say f_λ, given as follows

$$f_\lambda(x) = f(x) + \lambda, \quad x \in M.$$

Let us now consider the following equation

$$\Delta_g u + hu = f_\lambda u^{2^\star-1} + \frac{a}{u^{2^\star+1}}, \quad u > 0. \tag{$(7.2)_\lambda$}$$

Obviously, Eq. $(7.2)_\lambda$ always admits one positive smooth solution provided $\lambda \leqslant 0$. We are now interested in the case $\lambda > 0$. By using Theorem 1.2, we can prove the following theorem.

Theorem 7.1. *There exists a constant $\lambda^\star > 0$ such that*

(i) *Problem $(7.2)_\lambda$ has no positive smooth solution if $\lambda > \lambda^\star$.*

(ii) *Problem $(7.2)_\lambda$ has at least one positive smooth solution if $\lambda < \lambda^\star$.*

We now sketch a proof of this theorem.

Proof. In order to prove this theorem, let us observe from Theorem 1.2 that Eq. $(7.2)_\lambda$ has at least one positive smooth solution for some small $\lambda > 0$ since f_λ depends continuously on λ. In order to see this, let us observe that in this context, $\sup_M f_\lambda = \lambda$. Since $\int_M f d\text{vol}_g < 0$, we can select $\lambda > 0$ small such that $\int_M f_\lambda d\text{vol}_g < 0$. As usual, let us first suppose $\inf_M a > 0$. Now we show that there exists some

$$\tau > \max\left\{1, \left(\frac{2}{\mathcal{S}_h} \int_M h d\text{vol}_g\right)^{\frac{2^\star}{2}}\right\}$$

and some $\lambda \in (0,1)$ small enough such that

$$\int_M a\,d\mathrm{vol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{S_h}{\tau}\Big(\frac{S_h\tau}{\tau\sup_M f_\lambda - \int_M f_\lambda\,d\mathrm{vol}_g}\Big)^{n-1}.$$

Indeed, we can start with small λ having

$$\frac{1}{\lambda} > \max\Big\{1, \Big(\frac{2}{\mathcal{S}_h}\int_M h\,d\mathrm{vol}_g\Big)^{\frac{2^\star}{2}}\Big\}$$

and $\int_M f_\lambda d\mathrm{vol}_g < 0$. In particular, we can choose $\tau = \frac{1}{\lambda}$ and observe that

$$0 < 1 - \int_M f_\lambda d\mathrm{vol}_g = 1 - \lambda - \int_M f\,d\mathrm{vol}_g < 1 - \int_M f\,d\mathrm{vol}_g.$$

Therefore, a simple calculation shows that it suffices to show that

$$\int_M a\,d\mathrm{vol}_g < \frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h^n}{\lambda^{n-2}}\Big(\frac{1}{1-\int_M f\,d\mathrm{vol}_g}\Big)^{n-1},$$

or equivalently,

$$\lambda < \left(\frac{(2n-1)^{n-1}}{2^{2n-1}n^n}\frac{\mathcal{S}_h^n}{\int_M a\,d\mathrm{vol}_g}\right)^{\frac{1}{n-2}}\Big(\frac{1}{1-\int_M f\,d\mathrm{vol}_g}\Big)^{\frac{n-1}{n-2}},$$

which proves the existence of some small λ as claimed in the case $\inf_M a > 0$. In the case $\inf_M a = 0$, as in the second stage of the proof of Theorem 1.1, we simply replace a by $a + \varepsilon_0$ for some small $\varepsilon_0 > 0$ and repeat the above procedure to obtain a super-solution. Since a sub-solution always exists, the existence result for small λ follows.

Therefore, we can define

$$\lambda^\star = \sup_{\lambda\in\mathbb{R}}\{\ (7.2)_\lambda \text{ has at least one positive smooth solution }\}.$$

We now prove the following comparison: if $0 < \lambda_1 < \lambda_2 < \lambda^\star$ such that $(7.2)_{\lambda_2}$ has at least one positive smooth solution, then $(7.2)_{\lambda_1}$ also has at least one positive smooth solution. Indeed, suppose that u_2 is a positive smooth solution of $(7.2)_{\lambda_2}$, we then see that u_2 is a super-solution of $(7.2)_{\lambda_1}$ since $f_{\lambda_2} > f_{\lambda_1}$ pointwise. Having such an u_2, one can easily construct a sub-solution u_1 of $(7.2)_{\lambda_1}$ with $u_1 < u_2$. By the method of sub- and super-solutions, one can prove the existence of at least one positive smooth solution of $(7.2)_{\lambda_1}$.

In order to see why should we have $\lambda^\star < +\infty$, we make use of [10, Theorem 2.1]. Indeed, for sufficiently large λ, we obviously have $f_\lambda > 0$. Moreover, the following estimate

$$\left(\frac{n^n}{(n-1)^{n-1}}\right)^{\frac{n+2}{4n}} \int_M a^{\frac{n+2}{4n}} f_\lambda^{\frac{3n-2}{4n}} \, d\mathrm{vol}_g > \int_M h^{\frac{n+2}{4}} f_\lambda^{\frac{2-n}{4}} \, d\mathrm{vol}_g$$

holds, which immediately proves the finiteness of $\lambda^\star$ since $n \geqslant 3$. $\square$

Clearly, the theorem above does not cover the critical case $\lambda = \lambda^\star$. In fact, it would be interesting if one can answer whether or not we have the solvability in the critical case above. Since we do not have any good control for solutions when λ is near $\lambda^\star$, we cannot say anything about this critical case.

7.3. Interaction between coefficients

Finally, before closing the present paper, we would like to mention the interaction between the coefficients of the Einstein-scalar field Lichnerowicz equations (1.1) for any sign of h. Using our previous results for the negative case in [16] and for the null case in [17] together with Theorems 1.1, 1.2, and 1.3 in the present paper, one can obtain the following table which shows us how the coefficients in (1.1) depend on each other in order to get the existence of solutions.

Table 1: Interaction between the coefficients of (1.1) for any h.

h	$-\lambda_f < h < 0$	$h = 0$	$h = 0$	$h > 0$	$h > 0$	$h \gg 0$
f	$\sup f^+ < C(f^-)$	$\sup f^+ < C(f^-)$			$\sup f^+ < C(f^-, a)$	
a	$\int a < C(f^-)$	$\int a < C(f^-)$	$\sup a < C(f)$	$\int a < C(f)$		

The second column basically says that h cannot be too negative as it must satisfy $h > -\lambda_f$ for some positive constant λ_f given in [16, Eq. (2.1)]. Under this condition, we guarantee an existence result for (1.1) provided

$\sup_M f^+$ and $\int_M a d\mathrm{vol}_g$ are bounded in terms of f^-. This result still holds for the case $h = 0$; however, the boundedness of $\sup_M f^+$ can be relaxed if we replace $\int_M a d\mathrm{vol}_g$ by $\sup_M a$ as shown in the third column. The fourth column shows that in the case $h > 0$, the boundedness of $\sup_M a$ can be weakened by using $\int_M a d\mathrm{vol}_g$. For the fifth column, it shows that no condition is required if $\sup_M f^+$ is small in terms of f^- and a. In the last column, it shows that (1.1) is always solvable if $\inf_M h$ is sufficiently large, for example, if h satisfies

$$h \geqslant \sup_M f + \sup_M a$$

in M. (See also the paragraph right after the statement of Theorem 1.2 in Introduction.) This is because in this case the constant 1 is a super-solution for (1.1) and this is enough since a sub-solution for (1.1) which is less than 1 always exists.

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