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HIGHER ORDER OR FRACTIONAL ORDER HARDY-SOBOLEV TYPE EQUATIONS

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Abstract

In this paper we consider the following higher order or fractional order Hardy-Sobolev type equation

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{u^p(x)}{|y|^t}, \ x = (y, z) \in (R^k \backslash \{0\}) \times R^{n-k}, \tag{1}$$

where $0 < \alpha < n$, $0 < t < \min\{\alpha, k\}$, and $1 < p \leq \tau := \frac{n+\alpha-2t}{n-\alpha}$.

In the case when α is an even number, we first prove that the positive solutions of (1) are super poly-harmonic, i.e.

$$(-\Delta)^i u > 0, \ \ i = 1, \ldots, \frac{\alpha}{2} - 1. \tag{2}$$

Then, based on (2), we establish the equivalence between PDE (1) and the integral equation

$$u(x) = \int_{R^n} G(x, \xi) \frac{u^p(\xi)}{|\eta|^t} d\xi,$$

where $G(x, \xi) = \frac{c_{n,\alpha}}{|x-\xi|^{n-\alpha}}$ is the Green's function of $(-\Delta)^{\frac{\alpha}{2}}$ in R^n.

By the method of moving planes in integral forms, in the critical case, we prove that each nonnegative solution $u(y, z)$ of (1) is radially symmetric and monotone decreasing in y about the origin in R^k and in z about some point z_0 in R^{n-k}. In the subcritical case, we obtain the nonexistence of positive solutions for (1).

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1. Introduction

Let $n \geq 3$, $1 \leq k < n$, and $R^n = R^k \times R^{n-k}$. Write $x = (y, z) \in R^k \times R^{n-k}$. We study the following higher order or fractional order Hardy-Sobolev type equation

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \frac{u^p(x)}{|y|^t}, \quad x \in (R^k \backslash \{0\}) \times R^{n-k}, \tag{3}$$

where $0 < \alpha < n$, $0 < t < \min\{\alpha, k\}$, and $1 < p \leq \tau := \frac{n+\alpha-2t}{n-\alpha}$.

For even values of α, we assume $u \in C^\alpha(R^n)$ and satisfies (3) in the distribution sense

$$\int_{R^n} (-\Delta)^{\alpha/2} u(x)\, \psi(x) dx = \int_{R^n} |y|^{-t} u^p(x) \psi(x) dx, \ \forall \psi \in C_0^\infty(R^n).$$

For other real values of α, we assume

$$u \in \mathcal{D}^{\frac{\alpha}{2},2}(R^n) = \{u \mid \int_{R^n} |\xi|^\alpha |\hat{u}(\xi)|^2 d\xi < \infty\}$$

and satisfies

$$\int_{R^n} (-\Delta)^{\frac{\alpha}{4}} u (-\Delta)^{\frac{\alpha}{4}} \phi dx = \int_{R^n} |y|^{-t} u^p(x) \phi(x) dx, \ \forall \phi \in \mathcal{D}^{\frac{\alpha}{2},2}(R^n). \tag{4}$$

Here,

$$\int_{R^n} (-\Delta)^{\frac{\alpha}{4}} u (-\Delta)^{\frac{\alpha}{4}} \phi dx$$

is defined by the Fourier transform

$$\int_{R^n} |\xi|^\alpha \hat{u}(\xi) \overline{\hat{\phi}(\xi)} d\xi,$$

where $\hat{u}$ and $\hat{\phi}$ are the Fourier transform of u and ϕ respectively.

Equations (3) is closely related to the study of the sharp constants of the Hardy-Sobolev inequality and the Caffarelli-Kohn-Nirenberg inequality (cf. [5, 33, 38] and the references therein). The quantitative and qualitative properties of solutions for these types of equations are also interesting in critical point theory and nonlinear elliptic equations (cf. [1, 2, 37]).

In particular, when $\alpha = 2$, (3) is the Euler-Langrange equation corresponding to the well-known Hardy-Sobolev-Maz'ya inequality (cf. [9, 36]), which has been extensively studied by many authors (cf. [1, 10, 24, 35, 38] and the references therein).

In this special case, equation (3) becomes

$$-\Delta u(x) = \frac{u^p(x)}{|y|^t}, \quad x \in (R^k \backslash \{0\}) \times R^{n-k}. \tag{5}$$

This equation has been systematically studied in [7] by Cao and Li. For the critical exponent $p = \frac{n+2-2t}{n-2}$, and in the special case when $t = 1$, they first established radial symmetry of the positive solutions in y and z respectively, and then they classified all the positive solutions. In addition, they conjectured that the similar results should hold for any $0 < t < \min\{2, k\}$.

For the subcritical exponent $1 < p < \frac{n+2-2t}{n-2}$, Cao and Li proved the nonexistence of positive solutions.

In this paper, by using an entirely different approach–*the method of moving planes in integral forms*, we generalize Cao and Li's results from $\alpha = 2$ to any real value α between 0 and n and for any $0 < t < \min\{\alpha, k\}$.

In order to apply the method in integral forms, we first establish the equivalence between PDE (3) and the integral equation

$$u(x) = \int_{R^n} G(x, \xi) \frac{u^p(\xi)}{|\eta|^t} d\xi \tag{6}$$

where $\xi = (\eta, \zeta)$, and $G(x, \xi) = \frac{c_{n,\alpha}}{|x-\xi|^{n-\alpha}}$ is the Green's function of $(-\Delta)^{\frac{\alpha}{2}}$ in R^n.

Theorem 1.1. *Let α be any even number between 0 and n. If u is a classical positive solution of (3), then u satisfies integral equation (6). If $u \in C^{\alpha}(R^n)$ is a solution of (6), then u satisfies (3).*

Theorem 1.2. *Let α be any real number between 0 and n. If $u \in \mathcal{D}^{\frac{\alpha}{2},2}(R^n)$ is a solution of (3), then u satisfies integral equation (6), and vice versa.*

The proof of Theorem 1.1 is based on the following super poly-harmonic property of solutions.

Theorem 1.3. *If u is a positive solution of* (3)*, then*

$$(-\Delta)^i u(x) > 0, \quad i = 1, \dots, m-1, \quad x \in R^n, \tag{7}$$

where $m = \frac{\alpha}{2}$.

Due to the equivalence between (3) and (6), in order to derive the properties of solutions of (3), we only need to deal with integral equation (6).

By the method of moving planes in integral forms [16], we prove

Theorem 1.4. (i) *For $0 < \alpha < n$, $p = \tau$, $0 < t < \min\{\frac{\alpha}{2}(1 + \frac{k}{n}), k\}$, let $|y|^{-t}u^{\tau-1}(x) \in L^{\frac{n}{\alpha}}_{loc}(R^k \times R^{n-k})$. Assume that $u \in L^q_{loc}(R^n)$ for some $q > \frac{n}{n-\alpha}$. Then each positive solution $u(y, z)$ of* (6) *is radially symmetric and monotone decreasing in y about the origin in R^k and in z about some point z_0 in R^{n-k}; that is, $u = u(|y|, |z - z^0|)$.*
(ii) *For $1 < \alpha < n$, $1 < p < \tau$, let $|y|^{-t}u^{p-1}(x) \in L^{\frac{n}{\alpha}}_{loc}(R^k \times R^{n-k})$. Assume that $u \in L^q_{loc}(R^n)$ for some $q > \frac{n}{n-\alpha}$. If u is a nonnegative solution of* (6)*, then $u \equiv 0$.*

Combining Theorem 1.4 with Theorem 1.1, we conclude, for each even number α between 0 and n,

Corollary 1.1. (i) *For $0 < \alpha < n$, $p = \tau$, $0 < t < \min\{\frac{\alpha}{2}(1 + \frac{k}{n}), k\}$, then each positive solution u of* (3) *is radially symmetric and monotone decreasing in y about the origin in R^k and in z about some point z_0 in R^{n-k}.*
(ii) *For $1 < \alpha < n$, $1 < p < \tau$, if u is a nonnegative solution of* (3)*, then $u \equiv 0$.*

Theorem 1.4 and Theorem 1.2 yield, for each real number α between 0 and n,

Corollary 1.2. (i) *For $0 < \alpha < n$, $p = \tau$, $0 < t < \min\{\frac{\alpha}{2}(1 + \frac{k}{n}), k\}$, then each positive solution u of* (3) *is radially symmetric and monotone decreasing in y about the origin in R^k and in z about some point z_0 in R^{n-k}.*
(ii) *For $1 < \alpha < n$, $1 < p < \tau$, if u is a nonnegative solution of (3), then $u \equiv 0$.*

To apply the method of moving planes in integral forms, one usually needs to assume some global integrability on the solution u. Here by properly using Kelvin transforms, we only need to assume that u is locally integrable.

In the special case $k = n$, when $p = \tau$, Lu and Zhu [33] proved that every positive solution of (6) is radially symmetric and strictly decreasing about the origin; when $0 < p \leq \frac{n-t}{n-\alpha}$, Lei [30] obtained the nonexistence of positive solutions for (3). For more related results, please see [3, 4, 6, 11, 8, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 34, 27, 28, 29, 31, 39, 12] and the references therein.

The paper is organized as follows. In section 2, we obtain the super polyharmonic properties of the positive solutions of partial differential equation by the method of re-centers, and thus prove Theorem 1.3. In section 3, we show the equivalence between the partial differential equation and the integral equation, and thus prove Theorems 1.1 and 1.2. In the last section, when the exponent is critical, by moving planes in R^k and in R^{n-k} separately, we conclude that each nonnegative solution of the integral equation is radially symmetric and monotone decreasing in y about the origin in R^k and in z about some point z_0 in R^{n-k}. When the exponent is subcritical, we move the planes in the whole R^n to derive that each nonnegative solution is radially symmetric and decreasing about any given point $x^0 = (0, z^0) \in R^k \times R^{n-k}$. Hence we conclude that the solution depends only on $|y|$. By exploring a Pohozaev identity in integral forms, we derive $u \equiv 0$, and therefore prove Theorem 1.4.

2. The proof of Theorem 1.3

In this section, we establish super-poly harmonic properties of positive solutions for PDE (3). In the following, C, c, and c_0 denote positive constants whose values may vary from line to line.

Let

$$u_i(x) = (-\Delta)^i u(x), \quad i = 1, \ldots, m-1, \quad x \in R^n.$$

Part I. We first show that

$$u_{m-1}(x) > 0, \quad x \in R^n. \tag{8}$$

Suppose in the contrary, then there are two possible cases.

Case i) There exists $x^1 \in R^n$, such that

$$u_{m-1}(x^1) < 0. \tag{9}$$

Case ii) $u_{m-1}(x) \geq 0$, $\forall\, x \in R^n$, and there is a point $\widetilde{x} \in R^n$, such that

$$u_{m-1}(\widetilde{x}) = 0.$$

In this case, $\widetilde{x}$ is a local minimum of u_{m-1}, and we must have $-\Delta u_{m-1}(\widetilde{x}) \leq 0$. This contradicts with

$$-\Delta u_{m-1} = \frac{u^p(x)}{|y|^t} > 0, \quad x \in (R^k \backslash \{0\}) \times R^{n-k}.$$

Therefore we only need to deal with Case i).

Step 1. In this step, we will show that m must be even. If not, we assume that m is odd. Let

$$\bar{u}(r) = \frac{1}{|\partial B_r(x^1)|} \int_{\partial B_r(x^1)} u(x) d\sigma \tag{10}$$

be the spherical average of u. Then by the well-known property that

$$\overline{\Delta u} = \Delta \overline{u},$$

we have

$$\begin{cases} -\Delta \overline{u}_{m-1} = \overline{\frac{u^p(x)}{|y|^t}}, \\ -\Delta \overline{u}_{m-2} = \overline{u}_{m-1}, \\ \cdots \\ -\Delta \overline{u} = \overline{u}_1. \end{cases} \tag{11}$$

From the first equation in (11), by Jensen's inequality, we have

$$\begin{aligned} -\Delta \overline{u}_{m-1} &= \frac{1}{|\partial B_r(x^1)|} \int_{\partial B_r(x^1)} \frac{u^p(x)}{|y|^t} d\sigma \\ &\geq (r + |x^1|)^{-t} \frac{1}{|\partial B_r(x^1)|} \int_{\partial B_r(x^1)} u^p(x) d\sigma \end{aligned}$$

$$\begin{aligned}
&\geq\ (r+|x^1|)^{-t}\left(\frac{1}{|\partial B_r(x^1)|}\int_{\partial B_r(x^1)}u(x)d\sigma\right)^p\\
&=\ (r+|x^1|)^{-t}\bar{u}^p(x)>0,\ \text{for any } r>0. \qquad (12)
\end{aligned}$$

Then integrating both sides from 0 to r yields

$$\overline{u}'_{m-1}(r)<0,\ \text{and}\ \overline{u}_{m-1}(r)<\overline{u}_{m-1}(0)=u_{m-1}(x^1):=-c_0<0,\ r>0. \qquad (13)$$

By the second equation in (11), we deduce

$$-\frac{1}{r^{n-1}}\left(r^{n-1}\overline{u}'_{m-2}\right)'=\overline{u}_{m-1}(r)<-c_0,\quad \forall r>0.$$

That is

$$\left(r^{n-1}\overline{u}'_{m-2}\right)'>r^{n-1}c_0,\quad \forall r>0.$$

Integrating yields

$$\overline{u}'_{m-2}(r)>\frac{c_0}{n}r,\quad \text{and}\quad \overline{u}_{m-2}(r)\geq\overline{u}_{m-2}(0)+\frac{c_0}{2n}r^2,\quad \forall r>0. \qquad (14)$$

Hence, $\exists\ r_1>0$ such that

$$\bar{u}_{m-2}(r_1)>0.$$

Making average at a new center x^2 with $|x^1-x^2|=r_1$, i.e,

$$\bar{\bar{u}}(r)=\frac{1}{|\partial B_r(x^2)|}\int_{\partial B_r(x^2)}\bar{u}(x)d\sigma,$$

we have

$$\bar{\bar{u}}_{m-2}(0)=\bar{u}_{m-2}(x^2)>0. \qquad (15)$$

Then by (12), $(\bar{\bar{u}},\bar{\bar{u}}_1,\ldots,\bar{\bar{u}}_{m-1})$ satisfies

$$\begin{cases}
-\Delta\overline{\overline{u}}_{m-1}\geq\overline{(|x-x^1|+|x^1|)^{-t}\bar{u}^p(x)},\\
-\Delta\overline{\overline{u}}_{m-2}=\overline{\overline{u}}_{m-1},\\
\cdots\\
-\Delta\overline{\overline{u}}=\overline{\overline{u}}_1.
\end{cases} \qquad (16)$$

By (16) and Jensen's inequality, we obtain

$$-\Delta\overline{\overline{u}}_{m-1}(r) \geq (r+|x^2-x^1|+|x^1|)^{-t}\overline{\overline{u}}^p(x) > 0, \text{ for any } r \geq 0.$$

By the same arguments as in deriving (14), we conclude

$$\overline{\overline{u}}_{m-2}(r) \geq \overline{\overline{u}}_{m-2}(0) + \frac{c_0}{2n}r^2, \quad \forall r \geq 0. \tag{17}$$

By (13), (15), and (17), we have

$$\overline{\overline{u}}_{m-1}(r) < 0, \ \overline{\overline{u}}_{m-2}(r) > 0, \text{ for any } r \geq 0.$$

Continuing this way, after $m-1$ steps of re-centers (denotes the results by $\tilde{u}$), we conclude, for any $r \geq 0$,

$$-\Delta\tilde{u}_{m-1}(r) \geq (r+|x^{m-1}-x^{m-2}|+\cdots+|x^2-x^1|+|x^1|)^{-t}\tilde{u}^p(r) > 0, \tag{18}$$

and

$$(-1)^i\tilde{u}_{m-i}(r) > 0, \ i=1,\ldots,m-1, \text{ for any } r \geq 0. \tag{19}$$

Since m is odd, (19) implies

$$\tilde{u}_2(r) < 0, \text{ for any } r \geq 0.$$

And then we derive

$$\tilde{u}_1'(r) > 0 \text{ and } \tilde{u}_1(r) \geq \tilde{u}_1(0) := c > 0, \text{ for any } r \geq 0.$$

From the last equation in (16), we deduce

$$\begin{aligned} \tilde{u}(r) &\leq \tilde{u}(0) - \frac{c}{2n}r^2 \\ &\to -\infty, \text{ as } r \to \infty \end{aligned}$$

which contradicts with the positiveness of u. Hence m must be even.

Step 2. Let

$$u_\lambda(x) = \lambda^{\frac{2m-t}{p-1}}u(\lambda x)$$

be the rescaling of u. It still satisfies equation (3) for any $\lambda > 0$. By (19), we derive

$$\tilde{u}(r) \geq \tilde{u}(0) > 0, \text{ for any } r \geq 0.$$

Then we choose a sufficiently large λ such that for any $a_0 > 0$,

$$\tilde{u}(r) \geq a_0 \geq a_0 r^{\sigma_0}, \ \forall\, r \in [0,1], \tag{20}$$

where $\sigma_0 > 1$ and $\sigma_0 p \geq 2m+n$. By (18) and (20), we have

$$\begin{aligned} -\Delta \tilde{u}_{m-1}(r) &\geq (r + |x^{m-1} - x^{m-2}| + \cdots + |x^2 - x^1| + |x^1|)^{-t} \tilde{u}^p(x) \\ &\geq (1 + |x^{m-1} - x^{m-2}| + \cdots + |x^2 - x^1| + |x^1|)^{-t} a_0^p r^{p\sigma_0} \\ &:= c_0 a_0^p r^{p\sigma_0}, \ 0 < c_0 < 1. \end{aligned}$$

It follows that

$$\tilde{u}_{m-1}(r) \leq \tilde{u}_{m-1}(0) - \frac{c_0 a_0^p r^{\sigma_0 p + 2}}{(\sigma_0 p + n)(\sigma_0 p + 2)}.$$

Since m is even, by (19), we obtain

$$\tilde{u}_{m-1}(r) \leq -\frac{c_0 a_0^p r^{\sigma_0 p + 2}}{(\sigma_0 p + n)(\sigma_0 p + 2)} \leq -\frac{c_0 a_0^p r^{\sigma_0 p + 2}}{(2\sigma_0 p)^2}, \ \forall\, r \in [0,1]. \tag{21}$$

Similar to (21), by the second equation in (16), (19), and (21) , we deduce

$$\tilde{u}_{m-2}(r) \geq \frac{c_0 a_0^p r^{\sigma_0 p + 4}}{(2\sigma_0 p)^4}, \ \forall\, r \in [0,1]. \tag{22}$$

Continuing this way, we derive

$$\tilde{u}(r) \geq \frac{c_0 a_0^p r^{\sigma_0 p + 2m}}{(2\sigma_0 p)^{2m}} \geq \frac{c_0 a_0^p r^{2\sigma_0 p}}{(2\sigma_0 p)^{2m}}, \ \forall\, r \in [0,1].$$

Set

$$\sigma_1 = 2\sigma_0 p, \ \sigma_k = 2\sigma_{k-1} p, \ k = 2, \ldots,$$

$$a_1 = \frac{c_0 a_0^p}{(2\sigma_0 p)^{2m}}, \ a_k = \frac{c_0 a_{k-1}^p}{(2\sigma_{k-1} p)^{2m}}, \ k = 2, \ldots.$$

Repeating the above arguments, by induction, one can prove

$$\tilde{u}(r) \geq a_k r^{\sigma_k}, \ \forall \, r \in [0,1]. \tag{23}$$

Through elementary calculations, we have

$$\begin{aligned} a_k &= \frac{c_0^{\frac{p^k-1}{p-1}} a_0^{p^k}}{(2p)^{2m(k+(k-1)p+(k-2)p^2+\cdots+p^{k-1})} \sigma_0^{\frac{2m(p^k-1)}{p-1}}} \\ &\geq \frac{c_0^{\frac{p^k-1}{p-1}} a_0^{p^k}}{(2p)^{2m\frac{p^{k+1}-p}{(p-1)^2}} \sigma_0^{\frac{2m(p^k-1)}{p-1}}} \\ &\geq c_0^{\frac{-1}{p-1}} \left(\frac{c_0^{\frac{1}{p-1}} a_0}{(2p)^{\frac{2mp}{(p-1)^2}} \sigma_0^{\frac{2m}{p-1}}} \right)^{p^k}, \ k = 1, \ldots . \end{aligned}$$

We take

$$a_0 = 2c_0^{-\frac{1}{p-1}} (2p)^{\frac{2mp}{(p-1)^2}} \sigma_0^{\frac{2m}{p-1}}.$$

Then by (23), we deduce

$$\tilde{u}(1) \geq 2^{p^k} \to \infty, \ \text{as } k \to \infty.$$

This is impossible. Hence (8) must hold.

Part II. Now we show that all other $u_k(x)$, $k = 1, \ldots, m-2$, $x \in R^n$, must be positive. On the contrary, suppose for some i, $2 \leq i \leq m-1$, $\exists \, x^0 \in R^n$, such that

$$u_{m-1}(x) > 0, \ \ u_{m-2}(x) > 0, \ \ \cdots, \ \ u_{m-i+1}(x) > 0, \ \ x \in R^n, \tag{24}$$

$$u_{m-i}(x^0) < 0. \tag{25}$$

Repeating the similar arguments as in *Step* 1 of **Part I**, after a few steps of re-centers, the signs of $\tilde{u}_{m-j}(r)$, $j = i, \ldots, m-1$, are alternating, and by the positiveness of u, we must have

$$\tilde{u}_1(r) < 0, \ \text{for any } r \geq 0.$$

Therefore,

$$\tilde{u}(r) \geq \tilde{u}(0) := c > 0, \ \text{for any } r \geq 0.$$

By (18), we obtain

$$-\Delta\tilde{u}_{m-1}(r)\geq(1+|x^{m-1}-x^{m-2}|+\cdots+|x^2-x^1|+|x^1|)^{-t}\tilde{u}^p(r)\geq c_0c^p:=C>0.$$

Integrating both sides from 0 to r yields

$$\tilde{u}_{m-1}(r)\leq\tilde{u}_{m-1}(0)-\frac{Cr^2}{2n}\to-\infty,\ \text{as } r\to\infty.$$

This contradicts with (8), and therefore (7) must be true. This completes the proof of Theorem 1.3.

3. Equivalence between PDE and IE

3.1. The proof of Theorem 1.1

In this subsection, we consider the positive classical solution u of higher order equation

$$(-\Delta)^m u(x)=\frac{u^p(x)}{|y|^t},\ x=(y,z)\in(R^k\backslash\{0\})\times R^{n-k}, \tag{26}$$

where m is a positive integer and $2m<n$.

We show that u is also a solution of the integral equation, and vice versa.

Let $\delta(x-\xi)$ be the Dirac Delta function. For fixed $x=(y,z)\in R^k\times R^{n-k}$, $G_r(x,\xi)$ is the Green's function:

$$\begin{cases}(-\Delta)^m G_r(x,\xi)=\delta(x-\xi), & \text{in } B_r(x),\\ G_r(x,\xi)=\Delta G_r(x,\xi)=\cdots=\Delta^{m-1}G_r(x,\xi)=0, & \text{on } \partial B_r(x).\end{cases} \tag{27}$$

By the Maximum principle, one can easily verify that the outward normal derivative

$$\frac{\partial}{\partial\nu_\xi}[(-\Delta)^i G_r(x,\xi)]\leq 0,\ i=0,\ldots,m-1,\ \text{on } \partial B_r(x). \tag{28}$$

Multiply both sides of (26) by $G_r(x,\xi)$ and integrate on $B_r(x)$. After integrating by parts, and due to Theorem 1.3 and (28), we arrive at

$$\int_{B_r(x)}G_r(x,\xi)\frac{u^p(\xi)}{|\eta|^t}d\xi$$

$$= \; u(x) + \sum_{i=0}^{m-1} \int_{\partial B_r(x)} [(-\Delta)^i u] \frac{\partial}{\partial \nu_\xi} [(-\Delta)^{m-1-i} G_r(x,\xi)] d\sigma \tag{29}$$

$$\leq \; u(x). \tag{30}$$

Solving equations (27) directly and letting $r \to \infty$, we have

$$G_r(x,\xi) \;\to\; \frac{C}{|x-\xi|^{n-2m}}, \tag{31}$$

$$(-\Delta)^i G_r(x,\xi) \;\to\; \frac{C}{|x-\xi|^{n+2i-2m}},\ i = 1,\ldots,m-1, \tag{32}$$

and

$$|\frac{\partial}{\partial \nu_\xi}[(-\Delta)^{m-1-i} G_r(x,\xi)]| \;\leq\; \frac{C}{|x-\xi|^{n-2i-1}},\ i = 0,\ldots,m-1. \tag{33}$$

It follows from (30) that

$$\int_{R^n} \frac{1}{|x-\xi|^{n-2m}} \frac{u^p(\xi)}{|\eta|^t} d\xi < \infty. \tag{34}$$

By (34), there exists $r_k \to \infty$, such that

$$0 < \frac{1}{r_k^{n-2m-1}(|x|+r_k)^t} \int_{\partial B_{r_k}(x)} u^p(\xi) d\sigma \leq \frac{1}{r_k^{n-2m-1}} \int_{\partial B_{r_k}(x)} \frac{u^p(\xi)}{|\eta|^t} d\sigma \to 0.$$

We further deduce that

$$\frac{1}{r_k^{n+t-2m-1}} \int_{\partial B_{r_k}(x)} u^p(\xi) d\sigma \to 0, \ \text{ as } \ r_k \to \infty.$$

Then by Jensen's inequality, we have

$$\frac{1}{r_k^{n-1-\frac{2m-t}{p}}} \int_{\partial B_{r_k}(x)} u(\xi) d\sigma \to 0, \ \text{ as } \ r_k \to \infty. \tag{35}$$

Since $t < 2m$, it is easy to see

$$\frac{1}{r_k^{n-1}} \int_{\partial B_{r_k}(x)} u(\xi) d\sigma \to 0, \ \text{ as } \ r_k \to \infty. \tag{36}$$

Set

$$(-\Delta)^i u = u_i,\ i = 1,\ldots,m-1, \tag{37}$$

$$(-\Delta)^i G_r(x,\xi) = G_i(x,\xi),\ i = 1,\ldots,m-1.$$

Multiply both sides of (37) by $G_{m-i}(x,\xi)$ and integrate on $B_r(x)$. After integrating by parts, by Theorem 1.3 and (28) again, we deduce

$$\begin{aligned}
&\int_{B_r(x)} u_i(\xi)G_{m-i}(x,\xi)d\xi \\
&= u(x)+\sum_{j=0}^{i-1}\int_{\partial B_r(x)}[(-\Delta)^j u]\frac{\partial}{\partial\nu_\xi}[(-\Delta)^{m-1-j}G_r(x,\xi)]d\sigma \\
&\le u(x),\ i=1,\ldots,m-1. \qquad (38)
\end{aligned}$$

(32) and (38) imply

$$\int_{R^n}\frac{u_i(\xi)}{|x-\xi|^{n-2i}}d\xi<\infty,\ i=1,\ldots,m-1.$$

Then there exists $r_k\to\infty$, such that

$$\frac{1}{r_k^{n-2i-1}}\int_{\partial B_{r_k}(x)}u_i(\xi)d\sigma\to 0,\quad i=1,\ldots,m-1. \qquad (39)$$

From equation (29), by (33), (34), (36), and (39), letting $r_k\to\infty$, we arrive at

$$u(x)=C\int_{R^n}\frac{1}{|x-\xi|^{n-2m}}\frac{u^p(\xi)}{|\eta|^t}d\xi. \qquad (40)$$

Now assume that $u\in C^{2m}(R^n)$ is a solution of integral equation (6), then

$$\begin{aligned}
(-\Delta)^m u(x) &= \int_{R^n}(-\Delta)^m G(x,\xi))\frac{u^p(\xi)}{|\eta|^t}d\xi \\
&= \int_{R^n}\delta(x-\xi)\frac{u^p(\xi)}{|\eta|^t}d\xi=\frac{u^p(x)}{|y|^t}.
\end{aligned}$$

This completes the proof of Theorem 1.1.

3.2. The proof of Theorem 1.2

In this subsection, we consider fractional order PDE (3) and and show that its weak solutions satisfy IE (6); and vice versa.

Set $f(x,u)=\frac{u^p(x)}{|y|^t}$, $x=(y,z)\in R^k\times R^{n-k}$.

(i) We first assume $u \in \mathcal{D}^{\frac{\alpha}{2},2}(R^n)$ is a solution of (3). Then we have

$$\int_{R^n} |\xi|^\alpha \widehat{u}(\xi)\overline{\widehat{\varphi}(\xi)}d\xi = \int_{R^n} f(x,u(x))\varphi(x)dx, \ \ \forall\, \varphi \in \mathcal{D}^{\frac{\alpha}{2},2}(R^n). \tag{41}$$

For any $\psi \in C_0^\infty(R^n)$, let

$$\varphi(x) = \int_{R^n} \frac{C}{|x-\xi|^{n-\alpha}}\psi(\xi)d\xi.$$

Choose an appropriate constant C such that $(-\Delta)^{\frac{\alpha}{2}}\varphi = \psi$. Consequently,

$$\varphi \in \mathcal{D}^{\alpha,2}(R^n) \subset \mathcal{D}^{\frac{\alpha}{2},2}(R^n) \ \text{ and } \ \widehat{\varphi}(\xi) = \frac{1}{|\xi|^\alpha}\widehat{\psi}(\xi).$$

Combining this with (41), we deduce

$$\int_{R^n} \widehat{u}(\xi)\overline{\widehat{\psi}(\xi)}d\xi = \int_{R^n}\left(f * \frac{C}{|x|^{n-\alpha}}\right)\psi(x)dx, \ \ \forall\, \psi \in C_0^\infty(R^n).$$

Then

$$\int_{R^n} u(x)\psi(x)dx = \int_{R^n}\left(f * \frac{C}{|x|^{n-\alpha}}\right)\psi(x)dx, \ \ \forall\, \psi \in C_0^\infty(R^n).$$

Therefore, we obtain

$$u(x) = f(x,u) * \frac{C}{|x|^{n-\alpha}} = \int_{R^n} \frac{C}{|x-\xi|^{n-\alpha}} f(\xi,u(\xi))d\xi = \int_{R^n} \frac{C}{|x-\xi|^{n-\alpha}}\frac{u^\tau(\xi)}{|\eta|^t}d\xi.$$

(ii) If u is a soution of (6), differentiating under the integral sign, one can show that u satisfies (3). This completes the proof of Theorem 1.2.

4. The proof of Theorem 1.4

4.1. The critical exponent $p = \tau$

Because there is no global integrability assumptions on the solution u, one is not able to carry on the method of moving planes directly on u. To circumvent this difficulty, we resort to Kelvin type transforms. Let

$$v(x) = \frac{1}{|x|^{n-\alpha}}u\left(\frac{x}{|x|^2}\right) \tag{42}$$

be the Kelvin transform of u. If $u(x)$ is a solution of

$$u(x)=\int_{R^n} G(x,\xi)\frac{u^\tau(\xi)}{|\eta|^t}d\xi, \tag{43}$$

then v is also a solution of (43). Since $|y|^{-t}u^{\tau-1}(x)\in L^{\frac{n}{\alpha}}_{loc}(R^k\times R^{n-k})$, it is easy to see that v has no singularity at infinity; i.e, for any domain Ω that is a positive distance away from the origin

$$\int_\Omega\left(\frac{v^{\tau-1}(\xi)}{|\eta|^t}\right)^{\frac{n}{\alpha}}d\xi<\infty. \tag{44}$$

We divide the proof into two parts. In Part I, we show that each positive solution u of (43) is radially symmetric about the origin in R^k. In Part II, we prove that u is radially symmetric about some point z_0 in R^{n-k}. Then we derive the monotonicity of u with respect to y and z.

4.1.1. Part I: Move the Plane in R^k

Let λ be a real number and let the moving plane be

$$T_\lambda \;=\; \{x=(y,z)\in R^k\times R^{n-k}\mid y_1=\lambda\}.$$

We denote

$$\Sigma_\lambda \;=\; \{x\in R^k\times R^{N-k}\mid y_1<\lambda\}.$$

Let

$$x^\lambda \;=\; (y^\lambda,z)=(2\lambda-y_1,\ldots,y_k,z)$$

be the reflection of the point $x=(y_1,\ldots,y_k,z)$ about the plane T_λ, and

$$v_\lambda(x):=v(x^\lambda)\ \text{ and }\ w_\lambda(x):=v_\lambda(x)-v(x).$$

For any $x,\xi\in\Sigma_\lambda$, $x\neq\xi$, it is easy to check

$$G(x,\xi)=G(x^\lambda,\xi^\lambda)>G(x,\xi^\lambda)=G(x^\lambda,\xi). \tag{45}$$

The following lemma is the key ingredient in integral estimates.

Lemma 4.1. *For any $x \in \Sigma_\lambda$, it holds*

$$v(x) - v_\lambda(x) = \int_{\Sigma_\lambda} \Big[G(x,\xi) - G(x^\lambda,\xi)\Big] \left[\frac{v^\tau(\xi)}{|\eta|^t} - \frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t}\right] d\xi.$$

Proof. Since

$$v(x) = \int_{\Sigma_\lambda} G(x,\xi)\frac{v^\tau(\xi)}{|\eta|^t} d\xi + \int_{\Sigma_\lambda} G(x,\xi^\lambda)\frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t} d\xi,$$

$$v(x^\lambda) = \int_{\Sigma_\lambda} G(x^\lambda,\xi)\frac{v^\tau(\xi)}{|\eta|^t} d\xi + \int_{\Sigma_\lambda} G(x^\lambda,\xi^\lambda)\frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t} d\xi,$$

by (45), we have

$$\begin{aligned} v(x) - v(x^\lambda) &= \int_{\Sigma_\lambda} \Big[G(x,\xi) - G(x^\lambda,\xi)\Big] \frac{v^\tau(\xi)}{|\eta|^t} d\xi \\ &\quad + \int_{\Sigma_\lambda} \Big[G(x,\xi^\lambda) - G(x^\lambda,\xi^\lambda)\Big] \frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t} d\xi \\ &= \int_{\Sigma_\lambda} \Big[G(x,\xi) - G(x^\lambda,\xi)\Big] \left[\frac{v^\tau(\xi)}{|\eta|^t} - \frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t}\right] d\xi. \end{aligned}$$

We also need the following inequality.

Lemma 4.2 ([12])**.** *(An Equivalent Form of the Hardy-Littlewood-Sobolev Inequality) Let $g \in L^{\frac{np}{n+\alpha p}}(R^n)$ for $\frac{n}{n-\alpha} < p < \infty$. Define*

$$Tg(x) = \int_{R^n} \frac{1}{|x-y|^{n-\alpha}} g(y) dy.$$

Then

$$\|Tg\|_{L^p} \leq C(n,p,\alpha)\|g\|_{L^{\frac{np}{n+\alpha p}}}. \tag{46}$$

In Part I, the proof of Theorem 1.4 (i) consists of two steps. In the first step, we will show that for sufficiently negative value of λ,

$$w_\lambda(x) \geq 0, \ \ \forall\, x \in \Sigma_\lambda \backslash \{0^\lambda\}. \tag{47}$$

In the second step, we will move the plane $T_\lambda = \{x = (y,z) \in R^k \times R^{N-k} \mid y_1 = \lambda\}$ along the positive direction of y_1-axis as long as inequality

(47) holds up to the limiting position, and show that v is symmetric about the limiting plane.

Step 1. *Start Moving the Plane from near* $y_1 = -\infty$.

Define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \backslash \{0^\lambda\} \mid w_\lambda(x) < 0\}.$$

where 0^λ is the reflection of 0 about the plane T_λ. We show that for λ sufficiently negative, Σ_λ^- must be measure zero, and thus (47) holds. In fact, for any $x \in \Sigma_\lambda^-$, by the Mean Value Theorem and Lemma 4.1, we obtain

$$\begin{aligned}
0 &< v(x) - v_\lambda(x) \\
&= \int_{\Sigma_\lambda} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{v^\tau(\xi)}{|\eta|^t} - \frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t}\right] d\xi \\
&= \int_{\Sigma_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{v^\tau(\xi)}{|\eta|^t} - \frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t}\right] d\xi \\
&\quad + \int_{\Sigma_\lambda \backslash \Sigma_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{v^\tau(\xi)}{|\eta|^t} - \frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t}\right] d\xi \\
&\le \int_{\Sigma_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{v^\tau(\xi)}{|\eta|^t} - \frac{v_\lambda^\tau(\xi)}{|\eta^\lambda|^t}\right] d\xi \\
&\le \int_{\Sigma_\lambda^-} G(x,\xi) \frac{1}{|\eta|^t} \left[v^\tau(\xi) - v_\lambda^\tau(\xi)\right] d\xi \\
&= \tau \int_{\Sigma_\lambda^-} G(x,\xi) \frac{1}{|\eta|^t} \psi_\lambda^{\tau-1}(\xi)[v(\xi) - v_\lambda(\xi)] d\xi \\
&\le \tau \int_{\Sigma_\lambda^-} G(x,\xi) \frac{1}{|\eta|^t} v^{\tau-1}(\xi)[v(\xi) - v_\lambda(\xi)] d\xi, \qquad (48)
\end{aligned}$$

where $\psi_\lambda(\xi)$ is valued between $v(\xi)$ and $v_\lambda(\xi)$, and

$$0 \le v_\lambda(\xi) \le \psi_\lambda(\xi) \le v(\xi),\ \xi \in \Sigma_\lambda^-.$$

We apply Hardy-Littlewood-Sobolev inequality (46) and Hölder inequality to (48) to obtain, for any $q > \frac{n}{n-\alpha}$,

$$\begin{aligned}
\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} &\le C \||\eta|^{-t} v^{\tau-1} w_\lambda\|_{L^{\frac{nq}{n+\alpha q}}(\Sigma_\lambda^-)} \\
&\le C \||\eta|^{-t} v^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \qquad (49)
\end{aligned}$$

By (44), we can choose N sufficiently large, such that $\lambda \leq -N$, we have

$$C\||\eta|^{-t}v^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} \leq \frac{1}{2}.$$

Now inequality (49) implies

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0,$$

and therefore Σ_λ^- must be measure zero.

Step 2. Move the Plane to the Origin to Derive Symmetry.

Inequality (47) provides a starting point to move the plane

$$T_\lambda = \{x = (y, z) \in R^k \times R^{N-k} | \; y_1 = \lambda\}.$$

Now we start from the neighborhood of $y_1 = -\infty$ and move the plane to the right as long as (47) holds. We show that by moving this way, the plane will not stop before hitting the origin in R^k.

Define

$$\lambda_0 = \sup\{\lambda \leq 0 \mid w_\rho(x) \geq 0, \; \rho \leq \lambda, \; \forall \, x \in \Sigma_\rho \backslash \{0^\lambda\}\}.$$

Now, we prove

$$\lambda_0 = 0. \tag{50}$$

First, we show that $v(x)$ is symmetric about the plane T_{λ_0}, i.e.

$$w_{\lambda_0} \equiv 0, \;\; a.e. \;\; \forall \, x \in \Sigma_{\lambda_0} \backslash \{0^\lambda\}. \tag{51}$$

Suppose in the contrary, then we have $w_{\lambda_0} \geq 0$, but $w_{\lambda_0} \not\equiv 0$ *a.e.* on $\Sigma_{\lambda_0} \backslash \{0^\lambda\}$. We will show that the plane can be moved further. More precisely, there exists an $\epsilon > 0$ such that for all λ in $[\lambda_0, \lambda_0 + \epsilon)$

$$w_\lambda(x) \;\; a.e \;\; \text{on} \;\; \Sigma_\lambda \backslash \{0^\lambda\}.$$

In fact, by inequality (49), we have

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} \leq C\||\eta|^{-t}v^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \tag{52}$$

Again by (44), we choose ϵ sufficiently small so that for all λ in $[\lambda_0, \lambda_0+\epsilon)$,

$$C\||\eta|^{-t}v^{\tau-1}\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} \leq \frac{1}{2}. \tag{53}$$

We postpone the proof of (53) for a moment. Now by (52) and (53), we have $\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0$, and therefore Σ_λ^- must be measure zero. Hence, for these values of $\lambda > \lambda_0$, we have

$$w_\lambda(x) \geq 0, \ \ a.e. \ \forall\, x \in \Sigma_\lambda \backslash \{0^\lambda\}.$$

This contradicts with the definition of λ_0. Therefore (51) must hold.

Next, we show that (50) is true. Otherwise, if the plane stops at $y_1 = \lambda_0 < 0$, then

$$v_{\lambda_0}(x) = v(x), \ \ \forall\, x \in \Sigma_{\lambda_0} \backslash \{0^\lambda\}.$$

By Lemma 4.1, we have, for any $x \in \Sigma_{\lambda_0}$, $|\eta^{\lambda_0}| < |\eta|$,

$$0 = v(x) - v_{\lambda_0}(x) = \int_{\Sigma_{\lambda_0}} \Big[G(x,\xi) - G(x^{\lambda_0},\xi)\Big] \left[\frac{v^\tau(\xi)}{|\eta|^t} - \frac{v_{\lambda_0}^\tau(\xi)}{|\eta^{\lambda_0}|^t}\right] d\xi > 0.$$

This is obviously a contradiction. Therefore λ_0 must be zero, and hence

$$w_0(x) \geq 0, \ \ a.e. \ \forall\, x \in \Sigma_0.$$

If we move the plane from the positive infinity to the left and carry on the same procedure as done above in *Steps 1* and *2*, we can also prove that

$$w_0(x) \leq 0, \ \ a.e. \ \forall\, x \in \Sigma_0.$$

This implies that $v(x)$ is symmetric about the plane T_0.

Since the direction of y_1 in R^k can be chosen arbitrarily, we deduce that $v(x)$ must be radially symmetric in $y \in R^k$ about $y = 0$ and decreasing about the origin in R^k. By expression (42), we conclude that $u(x)$ must be radially symmetric in $y \in R^k$ about $y = 0$.

Now what left is to derive inequality (53). For any small $\eta > 0$, we can choose R sufficiently large so that

$$\||\eta|^{-t}v^{\tau-1}\|_{L^{\frac{n}{\alpha}}(R^n\backslash\{0\}\backslash B_R(0))} < \eta. \tag{54}$$

We fix this R and then show that the measure of $\Sigma_\lambda^- \cap B_R(0)$ is sufficiently small for λ close to λ_0. By Lemma 4.1, it is easy to see

$$w_{\lambda_0}(x) > 0 \tag{55}$$

in the interior of $\Sigma_{\lambda_0} \backslash \{0^\lambda\}$.

For any $\gamma > 0$, let

$$E_\gamma \; = \; \{x \in (\Sigma_{\lambda_0} \backslash \{0^\lambda\}) \cap B_R(0) | \; w_{\lambda_0}(x) > \gamma\}$$

and

$$F_\gamma \; = \; \Big((\Sigma_{\lambda_0} \backslash \{0^\lambda\}) \cap B_R(0) \Big) \backslash E_\gamma .$$

It is obviously

$$\lim_{\gamma \to 0} \mu(F_\gamma) = 0.$$

For $\lambda > \lambda_0$, let

$$D_\lambda = \Big((\Sigma_\lambda \backslash \{0^\lambda\}) \backslash \Sigma_{\lambda_0} \Big) \cap B_R(0).$$

Then it is easy to see that

$$(\Sigma_\lambda^- \cap B_R(0)) \subset (\Sigma_\lambda^- \cap E_\gamma) \cup F_\gamma \cup D_\lambda . \tag{56}$$

Apparently, the measure of D_λ is small for λ close to λ_0. We show that the measure of $\Sigma_\lambda^- \cap E_\gamma$ can also be sufficiently small as λ close to λ_0. In fact, for any $x \in \Sigma_\lambda^- \cap E_\gamma$, we have

$$w_\lambda(x) = v_\lambda(x) - v(x) \; = \; v_\lambda(x) - v_{\lambda_0}(x) + v_{\lambda_0}(x) - v(x) < 0.$$

Hence

$$v_{\lambda_0}(x) - v_\lambda(x) \; > \; w_{\lambda_0}(x) > \gamma .$$

It follows that

$$(\Sigma_\lambda^- \cap E_\gamma) \subset G_\gamma \equiv \{x \in B_R(0) | \;\; v_{\lambda_0}(x) - v_\lambda(x) > \gamma\}. \tag{57}$$

By the well-known Chebyshev inequality, we have

$$\mu(G_\gamma) \; \leq \; \frac{1}{\gamma^{p+1}} \int_{G_\gamma} |v_{\lambda_0}(x) - v_\lambda(x)|^{p+1} dx$$

$$\leq \frac{1}{\gamma^{p+1}} \int_{B_R(0)} |v_{\lambda_0}(x) - v_\lambda(x)|^{p+1} dx.$$

For each fixed γ, as λ close to λ_0, the right hand side of the above inequality can be made as small as we wish. Therefore by (56) and (57), the measure of $\Sigma_\lambda^- \cap B_R(0)$ can also be made sufficiently small. Combining this with (54), we obtain (53).

4.1.2. Part II: Move the Plane in R^{n-k}

For each real number λ, let the moving plane be

$$\hat{T}_\lambda = \{x = (y, z) \in R^k \times R^{n-k} \mid z_1 = \lambda\}.$$

We denote

$$\hat{\Sigma}_\lambda = \{x = (y, z_1, \ldots, z_{n-k}) \in R^k \times R^{N-k} \mid z_1 < \lambda\}.$$

Let

$$x^\lambda = (y, z^\lambda) = (y, 2\lambda - z_1, \ldots, z_{n-k})$$

be the reflection of the point $x = (y, z_1, \ldots, z_{n-k})$ about the plane $\hat{T}_\lambda$.

For any $x, \xi \in \hat{\Sigma}_\lambda$, $x \neq \xi$, it is easy to see

$$G(x, \xi) = G(x^\lambda, \xi^\lambda) > G(x, \xi^\lambda) = G(x^\lambda, \xi), \tag{58}$$

and then

$$v(x) - v_\lambda(x) = \int_{\hat{\Sigma}_\lambda} \left[G(x, \xi) - G(x^\lambda, \xi)\right] \frac{1}{|\eta|^t} \left[v^\tau(\xi) - v_\lambda^\tau(\xi)\right] d\xi.$$

The proof is similar to that in Part I.

Step 1. Define

$$\hat{\Sigma}_\lambda^- = \{x \in \hat{\Sigma}_\lambda \backslash \{0^\lambda\} \mid w_\lambda(x) < 0\},$$

where 0^λ is the reflection of 0 about the plane $\hat{T}_\lambda$. We show that for λ sufficiently negative, Σ_λ^- must be measure zero. We only need to check, for $x \in \hat{\Sigma}_\lambda^-$, we have

$$\begin{aligned} 0 &< v(x) - v_\lambda(x) \\ &= \int_{\hat{\Sigma}_\lambda} \left[G(x, \xi) - G(x^\lambda, \xi)\right] \frac{1}{|\eta|^t} \left[v^\tau(\xi) - v_\lambda^\tau(\xi)\right] d\xi \end{aligned}$$

$$
\begin{aligned}
&= \int_{\hat{\Sigma}_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \frac{1}{|\eta|^t} \left[v^\tau(\xi) - v_\lambda^\tau(\xi)\right] d\xi \\
&\quad + \int_{\hat{\Sigma}_\lambda \setminus \hat{\Sigma}_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \frac{1}{|\eta|^t} \left[v^\tau(\xi) - v_\lambda^\tau(\xi)\right] d\xi \\
&\le \int_{\hat{\Sigma}_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \frac{1}{|\eta|^t} \left[v^\tau(\xi) - v_\lambda^\tau(\xi)\right] d\xi \\
&\le \int_{\hat{\Sigma}_\lambda^-} G(x,\xi) \frac{1}{|\eta|^t} \left[v^\tau(\xi) - v_\lambda^\tau(\xi)\right] d\xi \\
&= \tau \int_{\hat{\Sigma}_\lambda^-} G(x,\xi) \frac{1}{|\eta|^t} \psi_\lambda^{\tau-1}(\xi)[v(\xi) - v_\lambda(\xi)] d\xi \\
&\le \tau \int_{\hat{\Sigma}_\lambda^-} G(x,\xi) \frac{1}{|\eta|^t} u^{\tau-1}(\xi)[v(\xi) - v_\lambda(\xi)] d\xi. \qquad (59)
\end{aligned}
$$

The rest is similar to *Step* 1. in Part I.

Step 2. Define

$$\lambda_0 = \sup\{\lambda \le 0 \mid w_\rho(x) \ge 0,\ \rho \le \lambda,\ \forall x \in \hat{\Sigma}_\rho \setminus \{0^\lambda\}\}.$$

Through a similar argument as in *Step* 2. in Part I, we can show that $v(x)$ is symmetric about the plane $\hat{T}_{\lambda_0}$, i.e.

$$w_{\lambda_0} \equiv 0,\ \ a.e.\ \ \forall x \in \hat{\Sigma}_{\lambda_0} \setminus \{0^\lambda\}.$$

If the plane stops at $z_1 = \lambda_0 < 0$, then $v(x)$ must be symmetric and monotone about the plane $z_1 = \lambda_0$. This implies that $v(x)$ has no singularity at the origin in R^{n-k}. Then we have

$$u(x) = O(\frac{1}{|x|^{n-\alpha}}).$$

Combining this with $0 < t < \min\{\frac{\alpha}{2}(1+\frac{k}{n}), k\}$ and $|y|^{-t}u^{\tau-1}(x) \in L_{loc}^{\frac{n}{\alpha}}(R^k \times R^{n-k})$, we derive

$$\int_{R^n} \left(\frac{u^{\tau-1}(\xi)}{|\eta|^t}\right)^{\frac{n}{\alpha}} d\xi < \infty.$$

In this case, we can carry on the moving of planes on $u(x)$ directly to obtain the radial symmetry and monotonicity of u in z about some point $z^0 \in R^{n-k}$.

Otherwise, we can move the plane all the way to $z_1 = 0$. Since the direction of z_1 can be chosen arbitrary in R^{n-k}, we deduce that $v(x)$ must be radially symmetric in $z \in R^{n-k}$ about $z = 0$ and decreasing about the origin in R^{n-k}. Similar to the case when $p = \tau$, one can deduce that $u(x)$ must be radially symmetric in $z \in R^{n-k}$ about $z = 0$.

Now we prove the monotonicity of u. Without loss of generality, we may assume

$$u(x) = u(|y|, |z|) := u(r, s) = u(re_1, se_2),$$

where e_1 and e_2 are unit vectors in R^k and R^{n-k} respectively.

Since

$$\begin{aligned}
&\int_{R^n} \frac{u^p(\xi)\eta \cdot e_1}{[(re_1 - \eta)^2 + (se_2 - \varsigma)^2]^{\frac{n-\alpha+2}{2}} |\eta|^t} d\xi \\
= &\int_{R^{n-k}} \int_{R^k} \frac{u^p(|\eta|, |\varsigma|)\eta \cdot e_1}{[(re_1 - \eta)^2 + (se_2 - \varsigma)^2]^{\frac{n-\alpha+2}{2}} |\eta|^t} d\eta d\varsigma \\
= &\int_{R^{n-k}} \int_0^\infty \int_{\partial B_\tau(0)} \frac{u^p(|\eta|, |\varsigma|)\eta \cdot e_1}{[(re_1 - \eta)^2 + (se_2 - \varsigma)^2]^{\frac{n-\alpha+2}{2}} |\eta|^t} d\sigma_\eta d\tau d\varsigma \\
= &\int_{R^{n-k}} \int_0^\infty \frac{u^p(\tau, |\varsigma|)}{\tau^t} \int_{\partial B_\tau(0)} \frac{\eta \cdot e_1}{[(re_1 - \eta)^2 + (se_2 - \varsigma)^2]^{\frac{n-\alpha+2}{2}}} d\sigma_\eta d\tau d\varsigma \\
= &\ 0,
\end{aligned}$$

we calculate

$$\begin{aligned}
\frac{\partial}{\partial r} u(re_1, se_2) &= \frac{\partial}{\partial r} \int_{R^n} \frac{u^p(\xi)}{[(re_1 - \eta)^2 + (se_2 - \varsigma)^2]^{\frac{n-\alpha}{2}} |\eta|^t} d\xi \\
&= (\alpha - n) \int_{R^n} \frac{u^p(\xi)(re_1 - \eta) \cdot e_1}{[(re_1 - \eta)^2 + (se_2 - \varsigma)^2]^{\frac{n-\alpha+2}{2}} |\eta|^t} d\xi \\
&= (\alpha - n) r \int_{R^n} \frac{u^p(\xi)}{[(re_1 - \eta)^2 + (se_2 - \varsigma)^2]^{\frac{n-\alpha+2}{2}} |\eta|^t} d\xi \\
&< 0, \text{ for any } r > 0. \qquad (60)
\end{aligned}$$

Similarly, we have

$$\frac{\partial}{\partial s} u(re_1, se_2) < 0, \text{ for any } s > 0.$$

This completes the proof of Theorem 1.4 (i).

4.2. The Subcritical Case $\frac{n}{n-\alpha} < p < \tau$

For any given $x^0 = (0, z^0) = (\underbrace{0, \ldots, 0}_{k}, \underbrace{z_1^0, \ldots, z_{n-k}^0}_{n-k}) \in R^k \times R^{n-k}$, let

$$v(x) = \frac{1}{|x - x^0|^{n-\alpha}} u(\frac{x - x^0}{|x - x^0|^2} + x^0) \tag{61}$$

be the Kelvin transform of u centered at x^0. We calculate

$$\begin{aligned} v(x) &= \frac{1}{|x - x^0|^{n-\alpha}} u(\frac{x - x^0}{|x - x^0|^2} + x^0) \\ &= \frac{1}{|x - x^0|^{n-\alpha}} \int_{R^n} G(\frac{x - x^0}{|x - x^0|^2} + x^0, \xi) \frac{u^p(\xi)}{|\eta|^t} d\xi \\ &= \frac{1}{|x - x^0|^{n-\alpha}} \int_{R^n} \frac{G(\frac{x-x^0}{|x-x^0|^2} + x^0, \frac{\tilde{\xi}-x^0}{|\tilde{\xi}-x^0|^2} + x^0) u^p(\frac{\tilde{\xi}-x^0}{|\tilde{\xi}-x^0|^2} + x^0)}{|\frac{\tilde{\eta}}{|\tilde{\xi}-x^0|^2}|^t |\tilde{\xi} - x^0|^{2n}} d\tilde{\xi} \\ &= \int_{R^n} \frac{G(\frac{x-x^0}{|x-x^0|^2} + x^0, \frac{\tilde{\xi}-x^0}{|\tilde{\xi}-x^0|^2} + x^0)}{|x - x^0|^{n-\alpha} |\tilde{\xi} - x^0|^{n-\alpha} |\tilde{\eta}|^t} \left[\frac{u(\frac{\tilde{\xi}-x^0}{|\tilde{\xi}-x^0|^2} + x^0)}{|\tilde{\xi} - x^0|^{n-\alpha}} \right]^p \frac{1}{|\tilde{\xi} - x^0|^\beta} d\tilde{\xi} \\ &= \int_{R^n} G(x, \xi) \frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} d\xi \end{aligned} \tag{62}$$

where $1 < p \le \tau$, and $\beta = (n - \alpha)(\tau - p) > 0$.

We apply the method of moving planes on $v(x)$. Since $|y|^t u^{p-1} \in L^{\frac{n}{\alpha}}_{loc}(R^n)$, for any domain Ω that is a positive distance away from x^0, we have

$$\int_\Omega [\frac{v^{p-1}(\xi)}{|\eta|^t |\xi - x^0|^\beta}]^{\frac{n}{\alpha}} d\xi < \infty. \tag{63}$$

Since

$$\begin{aligned} v(x) &= \int_{\Sigma_\lambda} G(x, \xi) \frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} d\xi + \int_{\Sigma_\lambda} G(x, \xi^\lambda) \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta} dy, \\ v(x^\lambda) &= \int_{\Sigma_\lambda} G(x^\lambda, \xi) \frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} d\xi + \int_{\Sigma_\lambda} G(x^\lambda, \xi^\lambda) \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta} dy, \end{aligned}$$

we have

$$v(x) - v_\lambda(x)$$

$$
\begin{aligned}
&= \int_{\Sigma_\lambda} \Big[G(x,\xi) - G(x^\lambda,\xi)\Big] \frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} d\xi \\
&\quad + \int_{\Sigma_\lambda} \Big[G(x,\xi^\lambda) - G(x^\lambda,\xi^\lambda)\Big] \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta} d\xi \\
&= \int_{\Sigma_\lambda} \Big[G(x,\xi) - G(x^\lambda,\xi)\Big] \left[\frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} - \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta}\right] d\xi. \quad (64)
\end{aligned}
$$

We now outline the ideas of the proof of Theorem 1.4 (ii). First, by the method of moving plane in integral forms, we prove that $v(x)$ must be radially symmetric and decreasing about x^0. Take any line passing through x^0, and call it x_1 axis. We move the plane

$$T_\lambda = \{x \mid x_1 = \lambda\}$$

along the direction of x_1 axis. The proof consists of two steps. In Step 1, we show that for any sufficiently negative λ,

$$v(x) \le v(x^\lambda), \text{ in } \Sigma_\lambda \backslash \{(x^0)^\lambda\}, \tag{65}$$

where $\Sigma_\lambda = \{x \in R^n \mid x_1 < \lambda\}$, and $(x^0)^\lambda$ is the reflection of x^0 about the plane T_λ.

In Step 2, we move the plane T_λ along the x_1 direction continuously from near negative infinity to the right as long as (65) holds. We show that the plane can be moved all the way up to the point x^0.

Since the direction of x_1 can be chosen arbitrarily, we deduce that $v(x)$ is radially symmetric and decreasing about x^0, hence $u(x)$ is also radially symmetric about x^0. Since $x^0 = (0, z^0) \in R^k \times R^{n-k}$ is any given point, we further derive that $u(x) = u(y,z)$ is independent of z. We analyze the integral itself to obtain Pohozaev identity. Then we deduce that $u \equiv 0$ in the case of subcritical exponent.

Step 1. Define

$$\Sigma_\lambda^- = \{x \in \Sigma_\lambda \backslash \{(x^0)^\lambda\} \mid w_\lambda(x) < 0\}.$$

We show that for λ sufficiently negative, Σ_λ^- must be measure zero.

By the Mean Value Theorem, we have, for sufficiently negative values of λ, and for $x \in \Sigma_\lambda^-$,

$$\begin{aligned}
0 &< v(x) - v_\lambda(x) \\
&= \int_{\Sigma_\lambda} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} - \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta}\right] d\xi \\
&= \int_{\Sigma_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} - \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta}\right] d\xi \\
&\quad + \int_{\Sigma_\lambda \setminus \Sigma_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} - \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta}\right] d\xi \\
&\leq \int_{\Sigma_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{1}{|\eta|^t} \frac{v^p(\xi)}{|\xi - x^0|^\beta} - \frac{1}{|\eta^\lambda|^t} \frac{v_\lambda^p(\xi)}{|\xi^\lambda - x^0|^\beta}\right] d\xi \\
&\leq \int_{\Sigma_\lambda^-} \left[G(x,\xi) - G(x^\lambda,\xi)\right] \left[\frac{v^p(\xi) - v_\lambda^p(\xi)}{|\eta|^t |\xi - x^0|^\beta}\right] d\xi \\
&\leq p \int_{\Sigma_\lambda^-} G(x,\xi) \frac{v^{p-1}(\xi)}{|\eta|^t |\xi - x^0|^\beta} [v(\xi) - v_\lambda(\xi)] d\xi \\
&\leq \int_{\Sigma_\lambda^-} \frac{C}{|x - \xi|^{n-\alpha}} \left|\frac{v^{p-1}(\xi)}{|\eta|^t |\xi - x^0|^\beta}\right| |v(\xi) - v_\lambda(\xi)| d\xi. \qquad (66)
\end{aligned}$$

Here we have used the fact that $|\eta^\lambda| < |\eta|$ and $|\xi^\lambda - x^0| < |\xi - x^0|$.

We apply Hardy-Littlewood-Sobolev inequality (46) and Hölder inequality to (66) to obtain, for any $q > \frac{n}{n-\alpha}$,

$$\begin{aligned}
\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} &\leq C \left\| \frac{v^{p-1}}{|\eta|^t |\xi - x^0|^\beta} w_\lambda \right\|_{L^{\frac{nq}{n+\alpha q}}(\Sigma_\lambda^-)} \\
&\leq C \left\| \frac{v^{p-1}}{|\eta|^t |\xi - x^0|^\beta} \right\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} \|w_\lambda\|_{L^q(\Sigma_\lambda^-)}. \qquad (67)
\end{aligned}$$

By (63), we can choose N sufficiently large, such that for $\lambda \leq -N$,

$$C \left\| \frac{v^{p-1}}{|\eta|^t |\xi - x^0|^\beta} \right\|_{L^{\frac{n}{\alpha}}(\Sigma_\lambda^-)} \leq \frac{1}{2}.$$

Now inequality (67) implies

$$\|w_\lambda\|_{L^q(\Sigma_\lambda^-)} = 0,$$

and therefore Σ_λ^- must be measure zero. Then we get

$$w_\lambda(x) \geq 0, \quad a.e.\ x \in \Sigma_\lambda. \tag{68}$$

Step 2. (Move the plane to the limiting position to derive symmetry.)

Inequality (68) provides a starting point to move the plane T_λ. Now we start from the neighborhood of $x_1 = -\infty$ and move the plane to the right as long as (68) holds to the limiting position. Define

$$\lambda_0 = \sup\{\lambda \leq 0 \mid w_\rho(x) \geq 0,\ \rho \leq \lambda,\ \forall x \in \Sigma_\rho \backslash \{(x^0)^\lambda\}\}.$$

The rest is similar to the case when $p = \tau$. We only need to use $\int [\frac{v^{p-1}(y)}{|\eta|^t|\xi - x^0|^\beta}]^{\frac{n}{\alpha}} d\xi$ instead of $\int [\frac{v^{\tau-1}(y)}{|\eta|^t}]^{\frac{n}{\alpha}} d\xi$. We also conclude

$$w_{\lambda_0}(x) \equiv 0, \quad a.e.\ \forall\ x \in \Sigma_{\lambda_0} \backslash \{(x^0)^\lambda\}.$$

Now, we show that the plane can not stop before hitting the point x^0. If not, by symmetry, since v is singular at x^0, v must also be singular at $(x^0)^\lambda$. This is impossible.

Since the direction of x_1 can be chosen arbitrary in R^n, we deduce that $v(x)$ must be radially symmetric and decreasing about the point x^0 in R^n.

For any $x^1,\ x^2 \in R^n$, choose x^0 as the mid point such that $|x^1 - x^0| = |x^2 - x^0|$. Set

$$X^1 = \frac{x^1 - x^0}{|x^1 - x^0|^2} + x^0,\ X^2 = \frac{x^2 - x^0}{|x^2 - x^0|^2} + x^0.$$

Then

$$|X^1 - x^0| = |X^2 - x^0|.$$

Therefore,

$$v(X^1) = v(X^2).$$

By the relation between $v(x)$ and $u(x)$, we have

$$u(x^1) = u(x^2)$$

which implies that $u(x)$ is radially symmetric about the point x^0 in R^n.

For any $x^1 \in (y, z^1)$, $x^2 \in (y, z^2) \in R^k \times R^{n-k}$, set $z^0 = \frac{z^1+z^2}{2}$. Let $x^0 = (0, z^0)$ be the projection of $\bar{x} = (y, z^0)$. Repeat the above arguments, we conclude that $u(x)$ is radially symmetric about the point x^0 in R^n. Due to $|x^1 - x^0| = |x^2 - x^0|$, we have $u(x^1) = u(x^2)$. Since $x^1 \in (y, z^1)$, $x^2 \in (y, z^2) \in R^k \times R^{n-k}$ are arbitrary, u is independent of z variables, i.e,

$$u(x) = u(y, z) = u(y).$$

We calculate

$$\begin{aligned}
u(y,z) &= \int_{R^n} \frac{u^p(\xi)}{|x-\xi|^{n-\alpha}|\eta|^t} d\xi \\
&= \int_{R^k} \frac{u^p(\eta)}{|\eta|^t} d\eta \int_{R^{n-k}} \frac{d\zeta}{(|y-\eta|^2 + |z-\zeta|^2)^{\frac{n-\alpha}{2}}} \\
&= c_n \int_{R^k} \frac{u^p(\eta)}{|\eta|^t} d\eta \int_0^\infty \int_{\partial B_r(z)} \frac{r^{n-k-1} d\sigma dr}{(|y-\eta|^2 + r^2)^{\frac{n-\alpha}{2}}} \\
&= c_n \int_{R^k} \frac{u^p(\eta)}{|\eta|^t |y-\eta|^{k-\alpha}} d\eta \int_0^\infty \int_{\partial B_{|y-\eta|\tau}(z)} \frac{\tau^{n-k-1} d\sigma d\tau}{(1+\tau^2)^{\frac{n-\alpha}{2}}} \\
&= c_n \int_{R^k} \frac{u^p(\eta)}{|\eta|^t |y-\eta|^{k-\alpha}} d\eta.
\end{aligned}$$

Now, we consider

$$u(x) = \int_{R^k} \frac{u^p(y)}{|x-y|^{k-\alpha}|y|^t} dy, \ x \in R^k. \tag{69}$$

By (69), we have

$$u(\mu x) = \int_{R^k} \frac{u^p(y)}{|\mu x - y|^{k-\alpha}|y|^t} dy. \tag{70}$$

Since

$$\begin{aligned}
&\frac{d}{d\mu}(|\mu x - y|^{\alpha-k}) \\
&= \frac{d}{d\mu}(|\mu x - y|^2)^{\frac{\alpha-k}{2}} \\
&= \frac{\alpha-k}{2} \cdot (|\mu x - y|^2)^{\frac{\alpha-k}{2}-1} \frac{d}{d\mu}\left[(\mu x_1 - y_1)^2 + \cdots + (\mu x_k - y_k)^2\right] \\
&= \frac{\alpha-k}{2} \cdot (|\mu x - y|^2)^{\frac{\alpha-k}{2}-1} \left(2(\mu x_1 - y_1)x_1 + \cdots + 2(\mu x_k - y_k)x_k\right)
\end{aligned}$$

$$= (\alpha-k)|\mu x-y|^{\alpha-k-2}\sum_{i=1}^{k}(\mu x_i-y_i)x_i = (\alpha-k)|\mu x-y|^{\alpha-k-2}x\cdot(\mu x-y),$$

we differentiate (70) with respect to μ:

$$x\cdot\nabla u(\mu x) = (\alpha-k)\int_{R^k}\frac{x\cdot(\mu x-y)u^p(y)}{|\mu x-y|^{k-\alpha+2}|y|^t}dy,\ x\neq 0.$$

Letting $\mu=1$ yields

$$x\cdot\nabla u(x) = (\alpha-k)\int_{R^k}\frac{x\cdot(x-y)u^p(y)}{|x-y|^{k-\alpha+2}|y|^t}dy,\ x\neq 0. \tag{71}$$

Multiplying both sides of (71) by $\frac{u^p(x)}{|x|^t}$ and integrating on $B_r\backslash B_\epsilon := B_r(0)\backslash$

$B_\epsilon(0)$, we integrate by parts to obtain

$$\begin{aligned}
\text{Left} &= \int_{B_r\backslash B_\epsilon}\frac{u^p(x)}{|x|^t}(x\cdot\nabla u(x))dx\\
&= \frac{1}{p+1}\int_{\partial B_r}\frac{u^{p+1}(x)}{|x|^t}(x\cdot\frac{x}{|x|})d\sigma+\frac{1}{p+1}\int_{\partial B_\epsilon}\frac{u^{p+1}(x)}{|x|^t}(x\cdot\frac{x}{|x|})d\sigma\\
&\quad -\frac{k-t}{p+1}\int_{B_r\backslash B_\epsilon}\frac{u^{p+1}(x)}{|x|^t}dx.\\
&= \frac{1}{p+1}\int_{\partial B_r}r^{1-t}u^{p+1}(x)d\sigma+\frac{1}{p+1}\int_{\partial B_\epsilon}\epsilon^{1-t}u^{p+1}(x)d\sigma\\
&\quad -\frac{k-t}{p+1}\int_{B_r\backslash B_\epsilon}\frac{u^{p+1}(x)}{|x|^t}dx, \qquad (72)\\
\text{Right} &= (\alpha-k)\int_{B_r\backslash B_\epsilon}\int_{R^k}\frac{x\cdot(x-y)u^p(y)u^p(x)}{|x-y|^{k-\alpha+2}|y|^t|x|^t}dydx. \qquad (73)
\end{aligned}$$

Similar to (60), we derive

$$\frac{d}{dr}u(r)<0,\ \text{for any } r>0.$$

Then we have

$$\begin{aligned}
u(r)=u(re) &= \int_{R^k}\frac{u^p(y)}{|re-y|^{k-\alpha}|y|^t}dy\\
&\geq \int_0^r\int_{\partial B_s}\frac{u^p(y)}{|re-y|^{k-\alpha}|y|^t}d\sigma ds
\end{aligned}$$

$$\begin{aligned} &\geq c\int_0^r \int_{\partial B_1} \frac{u^p(r)}{|re-s\omega|^{k-\alpha}} s^{k-1-t} d\omega ds \\ &= c\frac{u^p(r)}{r^{k-\alpha}} \int_0^r \int_{\partial B_1} \frac{s^{k-1-t}}{|e-\frac{s}{r}\omega|^{k-\alpha}} d\omega ds \\ &=: c\frac{u^p(r)}{r^{k-\alpha}} \int_0^r s^{k-1-t} f(s) ds. \end{aligned} \tag{74}$$

Obviously, for each fixed $0 \leq s \leq r$, $f(s) > 0$, set $t = \frac{s}{r}$, then $0 \leq t \leq 1$, $g(t) := f(s)$. Since $[0,1]$ is a compact set, $g(t)$ is continuous in t, we must have $g(t) \geq c_0 > 0$. Then by (74), we deduce

$$u(r) \ \geq \ cu^p(r) r^{\alpha-t}.$$

This implies

$$u(r) \ \leq \ \frac{c}{r^{\frac{\alpha-t}{p-1}}} \ \text{ as } \ r \to \infty. \tag{75}$$

Since $p < \tau$, by (75), we deduce

$$\int_{R^k} \frac{u^{p+1}(y)}{|y|^t} dy < \infty. \tag{76}$$

Then there exists a sequence $r_j \to \infty$ as $j \to \infty$ such that

$$r_j^{1-t} \int_{\partial B_{r_j}} u^{p+1}(x) d\sigma \to 0. \tag{77}$$

Since $x \cdot (x-y) + y \cdot (y-x) = |x-y|^2$, by symmetry, we have

$$\begin{aligned} &\frac{\alpha-k}{2} \int_{R^k} \frac{u^{p+1}(x)}{|x|^t} dx \\ &= \frac{\alpha-k}{2} \int_{R^k} \int_{R^k} \frac{u^p(y)u^p(x)}{|x-y|^{k-\alpha}|y|^t|x|^t} dxdy \\ &= \frac{\alpha-k}{2} \int_{R^k} \int_{R^k} \frac{x\cdot(x-y)u^p(y)u^p(x)}{|x-y|^{k-\alpha+2}|y|^t|x|^t} dydx \\ &\quad + \frac{\alpha-k}{2} \int_{R^k} \int_{R^k} \frac{y\cdot(y-x)u^p(y)u^p(x)}{|x-y|^{k-\alpha+2}|y|^t|x|^t} dxdy \\ &= (\alpha-k) \int_{R^k} \int_{R^k} \frac{x\cdot(x-y)u^p(y)u^p(x)}{|x-y|^{k-\alpha+2}|y|^t|x|^t} dydx. \end{aligned} \tag{78}$$

By (76) and (78), let $\epsilon \to 0$ in (72) and (73), we derive

$$\begin{aligned}&\frac{1}{p+1}\int_{\partial B_r} r^{1-t}u^{p+1}(x)d\sigma - \frac{k-t}{p+1}\int_{B_r}\frac{u^{p+1}(x)}{|x|^t}dx\\ &= (\alpha-k)\int_{B_r}\int_{R^k}\frac{x\cdot(x-y)u^p(y)u^p(x)}{|x-y|^{k-\alpha+2}|y|^t|x|^t}dydx.\end{aligned}$$

Combining this with (77) and (78), we arrive at

$$-\frac{k-t}{p+1}\int_{B_r}\frac{u^{p+1}(x)}{|x|^t}dx = \frac{\alpha-k}{2}\int_{R^k}\frac{u^{p+1}(x)}{|x|^t}dx. \tag{79}$$

If $\alpha \geq k$, (79) yields

$$u \equiv 0 \ \text{ in } \ R^n.$$

If $\alpha < k$, since

$$p+1 < \frac{2(n-t)}{n-\alpha} = 2 + \frac{2(\alpha-t)}{n-\alpha} < 2 + \frac{2(\alpha-t)}{k-\alpha} = \frac{2(k-t)}{k-\alpha},$$

(79) implies

$$u \equiv 0 \ \text{ in } \ R^n.$$

This completes the proof of Theorem 1.4 (ii).

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