Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **9** (2014), No. 3, pp. 295-315

SOME EXISTENCE AND REGULARITY RESULTS BY NEIL TRUDINGER REVISITED WITHOUT THE WEIGHTED SOBOLEV SPACES FRAMEWORK

LUCIO BOCCARDO[1,a] AND G. RITA CIRMI[2,b]

Dedicated to Neil, "amico" and "maestro"

[1]Dipartimento di Matematica, Sapienza Università di Roma, Piazzale Aldo Moro 5 00185 Roma, Italy.
[a]E-mail: boccardo@mat.uniroma1.it
[2]Dipartimento di Matematica ed Informatica, Università degli Studi di Catania, Viale A. Doria 6, 95125 Catania, Italy.
[b]E-mail: cirmi@dmi.unict.it

Abstract

Following J. Leray and J. L. Lions ([12]), we can say that this paper presents some results by N. Trudinger, concerning linear degenerate elliptic problems, revisited by the methods of [4], [5] (without the use of the weighted Sobolev spaces). Moreover, we study some cases completely new.

1. Introduction

In this paper we are interested in the study of the following boundary value problem

$$\begin{cases} -\text{div}(a(x)Du) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, $a(x)$ ia a non negative measurable function such that

$$a \in L^r(\Omega), \quad r > 1, \tag{2}$$

Received January 10, 2013 and in revised form April 6, 2014.
AMS Subject Classification: 35D30, 35J15, 35J70.
Key words and phrases: Degenerate elliptic equations, $W^{1,0}$-solutions.

$$\frac{1}{a} \in L^s(\Omega), \quad s \geq 1 \tag{3}$$

and the datum f belongs to some Lebesgue spaces, that is

$$f \in L^m(\Omega), \quad m \geq 1. \tag{4}$$

Degenerate problems of this type have been considered by M. K. V. Murthy and G. Stampacchia [13] in the framework of suitable weighted Sobolev spaces $W_0^{1,p}(a,\Omega)$. We recall that, given $p \geq 1 + \frac{1}{s}$, $W^{1,p}(a,\Omega)$ denotes the weighted Sobolev space obtained by completing $C^\infty(\Omega)$ with respect to the norm

$$\|v\|_{W^{1,p}(a,\Omega)} = \left[\int_\Omega (|v(x)|^p + a(x)|Dv(x)|^p)\right]^{\frac{1}{p}},$$

while $W_0^{1,p}(a,\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(a,\Omega)$.

A more general version of problem (1) has been studied by Neil Trudinger during the seventies in the papers [15], [16]. The results he has obtained concern with existence, uniqueness, local and global regularity of solution in weighted Sobolev spaces and under various hypotheses on the datum f. Moreover, the methods introduced have enabled the hypotheses employed in [13] to be considerable relaxed. Concerning the nonlinear case, some existence and regularity results in weighted Sobolev spaces can be found in [8], [9] and [10].

The aim of this paper is twofold.

- We revisit some of these results by choosing as functional setting the usual Sobolev spaces (Theorems 2.1, 2.7). To do this we will approximate problem (1) with some non-degenerate Dirichlet's problems and we will prove some a priori estimate on the solutions of this problems depending on the summability of f. Once this has been accomplished, the linearity of the operator and the summability assumptions on the weight will allow to pass to the limit, thus finding a distributional solution of our problem. We notice that, the solution obtained in [15] by means of weighted Sobolev spaces satisfies our results and, conversely, our solution has the same properties of that obtained in [15].

- Moreover, if $f \in L^1(\Omega)$ we study the existence of solutions of (1) satisfying an entropy condition (see inequality (16) below), without the use of the duality method (Theorem 2.10). Furthermore, if $f \log(1 + |f|)$ belongs to $L^1(\Omega)$ we prove the existence of a distributional solution u of (1) in the borderline case $W_0^{1,\frac{sN}{s(N-1)+N}}(\Omega)$ (Theorem 2.12).

We point out that some of the existence results concern solutions belonging to the nonreflexive space $W_0^{1,1}(\Omega)$.

2. Statement of the Results

The first result concerns the existence of solutions when the datum f has a "good" summability.

Theorem 2.1. *Let hypotheses* (2), (3), (4) *be satisfied and*

$$\frac{1}{s} + \frac{2}{r} \leq 1, \tag{5}$$

$$\frac{1}{m} + \frac{1}{2s} \leq \frac{1}{2} + \frac{1}{N}. \tag{6}$$

Then, there exists a distributional solution $u \in W_0^{1,\frac{2s}{s+1}}(\Omega)$ *of the problem* (1) *such that*

$$\int_\Omega a(x)|Du|^2 \leq \int_\Omega f u\,. \tag{7}$$

Moreover, $u \in L^\infty(\Omega)$ *if*

$$\frac{1}{m} + \frac{1}{s} < \frac{2}{N}, \tag{8}$$

while $u \in L^{\frac{s\,m^{**}}{s+m^{**}}}(\Omega)$ *if*

$$\frac{2}{N} < \frac{1}{m} + \frac{1}{s}. \tag{9}$$

Remark 2.2. Note that the inequality (8) can be written in the form $\frac{1}{m} + \frac{1}{2s} < \frac{1}{2m} + \frac{1}{N}$, which implies (6).

Remark 2.3. We note that the right-hand side of inequality (7) is finite thanks to the assumption (6) and that (7) means that u belongs to the weighted-Sobolev space $W_0^{1,2}(a,\Omega)$.

In the framework of weighted Sobolev spaces inequality (6) implies $f \in (W_0^{1,2}(a,\Omega))'$ and, under this assumption, the existence of a weak solution of problem (1) in the space $W_0^{1,2}(a,\Omega)$ has been studied in [15], Theorem 3.2; moreover, it is easy to prove that this solution belongs to $W_0^{1,\frac{2s}{s+1}}(\Omega)$.

If we assume neither (8) nor (9), previous Theorem and Sobolev immersion imply only $u \in L^{(\frac{2s}{s+1})^*}(\Omega)$. Note that, as a consequence of Theorem 3.2 of [15] and weighted-Sobolev immersion, $u \in L^{\frac{2sN}{s(N-2)+N}}(\Omega)$ and $(\frac{2s}{s+1})^* = \frac{2sN}{s(N-2)+N}$.

The assumption (6) implies $m \geq 2N/(N+2)$ and then $f \in H^{-1}(\Omega)$; nevertheless $u \notin H_0^1(\Omega)$.

Remark 2.4. Note that if $s \to \infty$, then $\frac{2s}{s+1} \to 2$; while, in the case $s = 1$, previous theorem gives the existence of solutions in the nonreflexive space $W_0^{1,1}(\Omega)$.

Remark 2.5. We point out that, under the hypothesis (8), Theorem 4.1, I of [15] states that problem (1) has a weak solution $u \in W_0^{1,2}(a,\Omega) \cap L^\infty(\Omega)$.

Remark 2.6. Let $\frac{1}{m} + \frac{1}{s} > \frac{2}{N}$.

In this case, the same regularity result stated in Theorem 2.1 has been obtained in Theorem 4.1, of [15], where it is also proved that, if $\frac{1}{m} + \frac{1}{s} = \frac{2}{N}$, the solution of problem (1) belongs to the Orlicz space $L_\phi(\Omega)$, with $\phi(t) = e^{|t|} - 1$ (see also Remark 3.3 below). Moreover, we point out that $\frac{sm^{**}}{s+m^{**}} \geq 1$ iff $\frac{1}{m} + \frac{1}{s} \leq 1 + \frac{2}{N}$ and the last inequality follows by (6).

In the following, given $k > 0$, we set, for every $s \in \mathbb{R}$

$$T_k(s) = \max(-k, \min(s,k)).$$

Next results concern with the case in which inequality (6) does't hold.

Theorem 2.7. *Let hypotheses* (2), (3), (4) *be satisfied,* $m > 1$ *and*

$$\frac{1}{m} + \frac{1}{s} + \frac{1}{r} \leq 1 + \frac{1}{N}, \tag{10}$$

$$\frac{1}{m} + \frac{1}{2s} > \frac{1}{2} + \frac{1}{N}. \tag{11}$$

Then, there exists a distributional solution $u \in W_0^{1,\frac{sm^*}{s+m^*}}(\Omega)$ *of* (1) *such that, for every* $k > 0$

$$T_k(u) \in W_0^{1,\frac{2s}{s+1}}(\Omega)$$

and

$$\int_\Omega a(x)|DT_k(u)|^2 \le \int_\Omega f\, T_k(u)\,. \tag{12}$$

Remark 2.8. Note that $\frac{sm^*}{s+m^*} \ge 1$ if

$$\frac{1}{m} + \frac{1}{s} \le 1 + \frac{1}{N}. \tag{13}$$

and we achieve the existence of a distributional solution in $W_0^{1,1}(\Omega)$ in the particular case $\dfrac{1}{m} + \dfrac{1}{s} = 1 + \dfrac{1}{N}$ and $r = \infty$. Furthemore, by virtue of (11) $\frac{sm^*}{s+m^*} < \frac{2s}{s+1}$.

At least, we point out that here, as in Theorem 2.1, the exponent $\frac{2s}{s+1}$ plays the role of exponent 2 of the non degenerate case.

Remark 2.9. Let the assumptions of Theorem 2.7 be satisfied. Then, in Theorem 4.3 of [15], by a duality method, it is proved that there exists a unique solution u of problem (1) such that

$$\int_\Omega a(x)^{\frac{q_T}{2}}|Du|^{q_T} < +\infty\,, \quad q_T = \frac{2sm^*}{2s+m^*}. \tag{14}$$

Note that such solution has the regularity stated by Theorem 2.7, and that it belongs to $W_0^{1,\frac{sm^*}{s+m^*}}(\Omega)$.

Conversely, we can prove that the solution u given by Theorem 2.7 satisfies condition (14) (see Remark 3.7 below).

Now, we point out that in the previous theorems we cannot take $m = 1$.

In order to handle this last case we recall the following functional setting.

Given $\sigma > 0$ the Marcinkiewicz space $M^\sigma(\Omega)$ is the space of measurable functions v on Ω such that

$$\exists\ C \ge 0 : |\{x \in \Omega : |v(x)| \ge t\}| \le \frac{C}{t^\sigma}\,, \quad \forall t > 0. \tag{15}$$

We recall that the following inclusions hold, if $1 \le p < \sigma < \infty$,

$$L^\sigma(\Omega) \subset M^\sigma(\Omega) \subset L^p(\Omega).$$

Theorem 2.10. *Let hypotheses* (2), (3) *and* (5) *be satisfied. If* $f \in L^1(\Omega)$, *there exists a solution* u *of problem* (1) *such that*

$$u \in M^{\frac{sN}{s(N-2)+N}}(\Omega), \qquad Du \in (M^{\frac{sN}{s(N-1)+N}}(\Omega))^N,$$
$$\log(1+|u|) \in W_0^{1,\frac{2s}{s+1}}(\Omega),$$
$$T_k(u) \in W_0^{1,\frac{2s}{s+1}}(\Omega)$$

and (12) *holds. Moreover* u *is a solution of the elliptic problem* (1) *in the following sense*

$$\int_\Omega a(x)D\varphi DT_k[u-\varphi] \le \int_\Omega f(x)T_k[u-\varphi], \tag{16}$$

$\forall\, k>0,\ \forall\, \varphi \in W_0^{1,(\frac{2s}{s+1})'}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.11. The definition (16) was introduced in [1].

If $s > N$, then $\frac{sN}{s(N-1)+N} > 1$ and assuming only $f \in L^1(\Omega)$ the previous theorem gives the existence of distributional solutions belonging to $W_0^{1,q}(\Omega)$ for every $1 \le q < \frac{sN}{s(N-1)+N}$. Note that $\frac{sN}{s(N-1)+N} = \frac{s1^*}{s+1^*}$.

Theorem 2.12. *Let the hypotheses* (2), (3) *be satisfied,* $s \ge N$, $r = \infty$ *and*

$$f\log(1+|f|) \in L^1(\Omega). \tag{17}$$

Then, there exists $u \in W_0^{1,\frac{sN}{s(N-1)+N}}(\Omega)$, *distributional solution of* (1).

Remark 2.13. Note that if $s = N$ and, in addition, (17) holds, then we obtain solution in the space $W_0^{1,1}(\Omega)$.

Remark 2.14. Let the assumptions of Theorem 2.10 be satisfied. Then in Theorem 4.3 of [15] the author proved, by duality, that there exists a unique solution u of problem (1) such that

$$\int_\Omega a(x)^{\frac{\beta}{2}}|Du|^\beta < +\infty \quad \forall \beta < q_T, \tag{18}$$

where

$$q_T = \frac{2s1^*}{2s+1^*}.$$

Note that such solution has the regularity stated by Theorem 2.10, that is, its gradient belongs to $M^{\frac{sN}{s(N-1)+N}}(\Omega)$.

Conversely, we can prove that the solution u given by Theorem 2.10 satisfies condition (18) (see Remark 3.9 below).

Remark 2.15. In the paper [7], dedicated to Neil Trudinger on the occasion of his 65th birthday, local versus global properties of solutions u of uniformly elliptic problems with non regular data are studied. Namely, if the right hand side f belongs to $L^1(\Omega)$ and ψ is a positive function belonging to $W^{1,\infty}(\Omega)$, even if u only belongs to $W_0^{1,q}(\Omega)$, $q < \frac{N}{N-1}$, then the function $u\psi^\eta$, for some $\eta > 1$, is more regular.

In the same spirit of this result, it is interesting to study the same property for the solutions u found in the present paper.

3. Approximate Problems and a Priori Bounds

We define

$$a_n(x) = \begin{cases} \dfrac{1}{n} & \text{if } a(x) < \frac{1}{n} \\ a(x) & \text{if } \frac{1}{n} \le a(x) \le n \\ n & \text{if } n < a(x), \end{cases}$$

$$f_n(x) = \frac{f(x)}{1+\frac{1}{n}|f(x)|}$$

and we consider the Dirichlet problems

$$u_n \in W_0^{1,2}(\Omega): \quad -\text{div}(a_n(x)Du_n) = f_n(x). \tag{19}$$

The existence of the solution $u_n \in W_0^{1,2}(\Omega)$ ia a consequence of Lax-Milgram lemma; moreover, for every $n \in \mathbb{N}$, the function u_n is bounded (see [14], [15]).

Remark 3.1. Note that $\{a_n(x)\}$ converges to $a(x)$ a. e. $x \in \Omega$ and $a_n(x) \leq a(x) + 1$ for every $n \in \mathbb{N}$, so that $\{a_n(x)\}$ converges to $a(x)$ in $L^r(\Omega)$; in a similar way $\left\{\frac{1}{a_n(x)}\right\}$ converges to $\frac{1}{a(x)}$ in $L^s(\Omega)$. Moreover, for every $n \in \mathbb{N}$

$$\left\| 1/a_n \right\|_{L^s}^s \leq \left\| 1/a \right\|_{L^s}^s + |\Omega|. \tag{20}$$

In the following, given $k > 0$, let $T_k(s)$ the truncation operator already defined in the previous section and set, for every $s \in \mathbb{R}$

$$G_k(s) = s - T_k(s).$$

3.1. Boundedness of the sequence $\{u_n\}$ in Lebegue's spaces

Let us define

$$q = \frac{2s}{s+1} \tag{21}$$

and note that $q < 2$ and $q = 1$ iff $s = 1$.

Lemma 3.2. *Assume that* (2), (3), (4) *and*

$$\frac{1}{m} + \frac{1}{s} < \frac{2}{N} \tag{22}$$

hold. Then there exists $M > 0$ *such that*

$$\|u_n\|_{L^\infty(\Omega)} \leq M, \ \ \forall\, n \in \mathbb{N}. \tag{23}$$

Proof. We choose $G_k(u_n)$ as test function in (19)

$$\int_\Omega a_n(x)|DG_k(u_n)|^2 \leq \int_\Omega |f(x)||G_k(u_n)|$$

and using the Sobolev and Hölder's inequalities (with exponents $2/q$ and $2/(2-q)$) we obtain

$$\begin{aligned} S_q\left[\int_\Omega |G_k(u_n)|^{q^*}\right]^{\frac{q}{q^*}} &\leq \int_\Omega |DG_k(u_n)|^q = \int_{A_n^k} \frac{(a_n)^{\frac{q}{2}}|DG_k(u_n)|^q}{(a_n)^{\frac{q}{2}}} \\ &\leq \left[\int_\Omega a_n(x)|DG_k(u_n)|^2\right]^{\frac{q}{2}} \left[\int_{A_n^k} \frac{1}{(a_n)^{\frac{q}{2-q}}}\right]^{1-\frac{q}{2}} \end{aligned}$$

$$\leq \left[\int_\Omega |f||G_k(u_n)|\right]^{\frac{q}{2}}\left[\int_\Omega \frac{1}{(a_n)^s}\right]^{1-\frac{q}{2}}$$
$$\leq C_a\Big(\|f\|_{L^m(\Omega)}\|G_k(u_n)\|_{L^{q^*}(\Omega)}\Big|A_n^k\Big|^{1-\frac{1}{q}+\frac{1}{N}-\frac{1}{m}}\Big)^{\frac{q}{2}},$$

where

$$A_n^k = \{x\in\Omega : k \leq |u_n(x)|\}, \quad |A_n^k| = \text{meas}(A_n^k).$$

Thus we proved that

$$\|G_k(u_n)\|_{L^{q^*}(\Omega)} \leq \tilde{C}_{a,f}|A_n^k|^{1-\frac{1}{q^*}-\frac{1}{m}},$$

which implies

$$\int_\Omega |G_k(u_n)| \leq C_{a,f}\,|A_n^k|^{2-\frac{2}{q^*}-\frac{1}{m}}.$$

By standard arguments, last inequality implies

$$|A_n^h| \leq \frac{C_1}{h-k}|A_n^k|^{2-\frac{2}{q^*}-\frac{1}{m}}, \tag{24}$$

for every $h > k > 0$. Note that the assumption (22) gives $2 - \frac{2}{q^*} - \frac{1}{m} > 1$; then, thanks to the Stampacchia's method (see [14], [11]), we conclude that there exists $M > 0$, independent of n, such that $\|u_n\|_{L^\infty(\Omega)} \leq M$, for $n \in \mathbb{N}$. □

Remark 3.3. If we assume

$$\frac{1}{m} + \frac{1}{s} = \frac{2}{N}$$

instead of (22), the inequality (24) becomes

$$|A_n^h| \leq \frac{C_1}{h-k}|A_n^k|,$$

which implies (see [14]) that the sequence $\{e^{\rho|u_n|}\}$ is bounded in $L^1(\Omega)$, for some $\rho > 0$, according to the results by M.K.V. Murty and G. Stamapacchia and by N. Trudinger.

Now we assume that the datum f is less regular and we study the boundedness of the sequence $\{u_n\}$ in some Lebesgue space.

Lemma 3.4. *We assume*

$$\frac{2}{N} < \frac{1}{s} + \frac{1}{m} \le 1 + \frac{2}{N}, \quad m > 1. \tag{25}$$

Then there exists $C > 0$ such that

$$\|u_n\|_{L^{\frac{s\,m^{**}}{s+m^{**}}}(\Omega)} \le C\ ,\ \forall\, n \in \mathbb{N}. \tag{26}$$

Proof. Define

$$\gamma = \frac{m'}{2m' - q^*} \tag{27}$$

and note that $\gamma > \frac{1}{2}$ and $q^*\gamma = (2\gamma - 1)m'$.

Given $\epsilon > 0$ we use $[(\epsilon + |u_n|)^{2\gamma-1} - \epsilon^{2\gamma-1}]sign(u_n)$ as test function in (19) and we get

$$(2\gamma - 1)\int_\Omega a_n(x)|Du_n|^2(\epsilon + |u_n|)^{2\gamma-2} \le \int_\Omega |f(x)|(\epsilon + |u_n|)^{2\gamma-1} \tag{28}$$

which implies

$$C_\gamma \int_\Omega a_n(x)|D[(\epsilon + |u_n|)^\gamma - \epsilon^\gamma]|^2 \le \|f\|_{L^m(\Omega)}\Big[\int_\Omega [(\epsilon + |u_n|)^\gamma]^{\frac{(2\gamma-1)m'}{\gamma}}\Big]^{\frac{1}{m'}}.$$

Recall that $s = \frac{q}{2-q}$. Then

$$\begin{aligned} &S_q\Big[\int_\Omega [(\epsilon + |u_n|)^\gamma - \epsilon^\gamma]^{q*}\Big]^{\frac{q}{q^*}} \\ &\le \int_\Omega |D|u_n|^\gamma|^q = \int_\Omega \frac{(a_n)^{\frac{q}{2}}|D|u_n|^\gamma|^q}{(a_n)^{\frac{q}{2}}} \\ &\le \Big[\int_\Omega a_n(x)|D[(\epsilon + |u_n|)^\gamma - \epsilon^\gamma]|^2\Big]^{\frac{q}{2}}\Big[\int_\Omega \frac{1}{(a_n)^s}\Big]^{1-\frac{q}{2}} \\ &\le C_a\|f\|^{\frac{q}{2}}_{L^m(\Omega)}\Big[\int_\Omega [(\epsilon + |u_n|)^\gamma]^{\frac{(2\gamma-1)m'}{\gamma}}\Big]^{\frac{q}{2m'}}. \end{aligned}$$

The limit as $\epsilon \to 0$ implies

$$S_q\Big[\int_\Omega [|u_n|^\gamma]^{q*}\Big]^{\frac{q}{q^*}} \le C_{a,f}\Big[\int_\Omega [|u_n|^\gamma]^{\frac{(2\gamma-1)m'}{\gamma}}\Big]^{\frac{q}{2m'}}.$$

Now the assumption $\frac{1}{s}+\frac{1}{m} > \frac{2}{N}$ gets $\frac{q}{q^*} > \frac{q}{2m'}$ (recall that $q^*\gamma = (2\gamma-1)m' = \frac{s\,m^{**}}{s+m^{**}}$) and then the estimate (26) follows. □

3.2. Boundedness of the sequence $\{u_n\}$ in Sobolev's spaces

Lemma 3.5. *Assume that* (2), (3), (4) *and (6) hold. Then, up to subsequences, the sequence* $\{u_n\}$ *weakly converges in* $W_0^{1,\frac{2s}{s+1}}(\Omega)$.

Proof. First of all, we note that the assumption $\frac{1}{m}+\frac{1}{2s} \le \frac{1}{2}+\frac{1}{N}$ implies that $m \ge \frac{2N}{N+2}$.

Choosing u_n as test function in (19) we obtain

$$\int_\Omega a_n(x)|Du_n|^2 \le \int_\Omega |f(x)||u_n(x)|\,. \tag{29}$$

Using Sobolev and Hölder's inequalities (with exponents $2/q$ and $2/(2-q)$) and working as in the proof of Lemma 3.2, we give

$$\begin{aligned} S_q\|u_n\|^q_{L^{q^*}} &\le \int_\Omega |Du_n|^q = \int_\Omega \frac{(a_n)^{\frac{q}{2}}|Du_n|^q}{(a_n)^{\frac{q}{2}}} \\ &\le \left[\int_\Omega a_n(x)|Du_n|^2\right]^{\frac{q}{2}}\left[\int_\Omega \frac{1}{(a_n)^{\frac{q}{2-q}}}\right]^{1-\frac{q}{2}} \\ &\le \left[\int_\Omega |f||u_n|\right]^{\frac{q}{2}}\left[\int_\Omega \frac{1}{(a_n)^s}\right]^{1-\frac{q}{2}} \le C_a\|f\|^{\frac{q}{2}}_{L^{(q^*)'}(\Omega)}\|u_n\|^{\frac{q}{2}}_{L^{q^*}(\Omega)}. \end{aligned}$$

Now we note that $(q^*)' \le m$ since (6) holds and by previous inequality it follows that the sequence $\{u_n\}$ is bounded in $W_0^{1,\frac{2s}{s+1}}(\Omega)$.

If $s > 1$ then $\frac{2s}{s+1} > 1$ and, up to a subsequence still denoted by $\{u_n\}$, $\{u_n\}$ converges to some function u weakly in $W_0^{1,\frac{2s}{s+1}}(\Omega)$, strongly in $L^1(\Omega)$ and almost everywhere in Ω.

In the case $s = 1$ (which implies $q = 1$), since the a priori estimate is not enough to pass to the limit, we need something more in order to prove the weak compactness of the sequence $\{u_n\}$ in $W_0^{1,1}(\Omega)$ and we follow some techniques already used in [2], [3], [6].

Note that (6) with $s = 1$ gives $m \ge N$. Let E be a measurable subset of Ω, and let i be in $\{1, \ldots, N\}$. Then we adapt the above inequalities and we have

$$\int_E |\partial_i u_n| \le \int_E |Du_n| = \int_E \frac{(a_n)^{\frac12}|Du_n|}{(a_n)^{\frac12}} \le \Big[\int_\Omega a_n(x)|Du_n|^2\Big]^{\frac12}\Big[\int_E \frac{1}{a_n}\Big]^{\frac12}$$
$$\le \big(\|f\|_{L^N(\Omega)}\|u_n\|_{L^{1^*}(\Omega)}\big)^{\frac12}\Big[\int_E \frac{1}{a_n}\Big]^{\frac12} \le C_1\Big[\int_E \frac{1}{a_n}\Big]^{\frac12}.$$

Since the sequence $\{\frac{1}{a_n}\}$ is compact in $L^1(\Omega)$, we can use the Vitali theorem on the last term; thus, we can say that the first term $\{\partial_i u_n\}$ is equiintegrable. By Dunford-Pettis theorem, and up to subsequences, there exists Y_i in $L^1(\Omega)$ such that $\{\partial_i u_n\}$ weakly converges to Y_i in $L^1(\Omega)$. Since $\partial_i u_n$ is the distributional derivative of u_n, we have, for every n in $\mathbb{N}$,

$$\int_\Omega \partial_i u_n\,\varphi = -\int_\Omega u_n\,\partial_i\varphi\,, \quad \forall\varphi \in C_c^\infty(\Omega)\,.$$

We now pass to the limit in the above identities, using that $\{\partial_i u_n\}$ weakly converges to Y_i in $L^1(\Omega)$, and that $\{u_n\}$ strongly converges to u in $L^\mu(\Omega)$, $1<\mu<\frac{N}{N-1}$; we obtain

$$\int_\Omega Y_i\,\varphi = -\int_\Omega u\,\partial_i\varphi\,, \quad \forall\varphi \in C_c^\infty(\Omega)\,,$$

which implies that $Y_i = \partial_i u$, and this result is true for every i. Since Y_i belongs to $L^1(\Omega)$ for every i, u belongs to $W_0^{1,1}(\Omega)$. □

The next results concern with the case in which m doesn't satisfy inequality (6)

Lemma 3.6. *Let hypotheses* (2), (3), (4) *be satisfied and (11) and (13) hold. Then, up to subsequences, the sequence* $\{u_n\}$ *weakly converges in* $W_0^{1,\frac{sm^*}{s+m^*}}(\Omega)$.

Proof. In the first part of the proof we assume $s>1$. First of all, we note that assumption (11) implies $\frac1s+\frac1m>\frac2N$; thus the sequence $\{u_n\}$ is bounded in $L^{\frac{sm^{**}}{s+m^{**}}}(\Omega)$, by virtue of Lemma 3.4. Moreover, if $\gamma>\frac12$ is the number defined in the proof of Lemma 3.4, the inequality (28) can be rewritten as follows, with $\epsilon=1$,

$$(2\gamma-1)\int_\Omega \frac{a_n(x)|Du_n|^2}{(1+|u_n|)^{2(1-\gamma)}} \le \int_\Omega |f(x)|(1+|u_n|)^{2\gamma-1}.$$

We point out that here the assumption (11) implies $\gamma < 1$ and that the right hand side of the above inequality is bounded (with respect to n), since $(2\gamma - 1)m' = \frac{sm^{**}}{s+m^{**}}$. Then

$$\int_\Omega \frac{a_n(x)|Du_n|^2}{(1+|u_n|)^{2(1-\gamma)}} \le C_0, \quad \forall\, n \in \mathbb{N}. \tag{30}$$

Let us define

$$\overline{q} = \frac{s\,m^*}{s+m^*} \tag{31}$$

and $\overline{p}$ such that

$$\overline{p}\,\overline{q}\,(1-\gamma) = \frac{s\,m^{**}}{s+m^{**}}. \tag{32}$$

Note that $\overline{q} > 1$, since $\frac{1}{s} + \frac{1}{m} < 1 + \frac{1}{N}$, $\overline{q} < 2$ and easy calculations show that

$$\frac{\overline{q}}{2} + \frac{1}{\overline{p}} + \frac{\overline{q}}{2s} = 1.$$

Then in the equality

$$\int_\Omega |Du_n|^{\overline{q}} = \int_\Omega \frac{a_n(x)^{\frac{\overline{q}}{2}}|Du_n|^{\overline{q}}}{(1+|u_n|)^{\overline{q}(1-\gamma)}}(1+|u_n|)^{\overline{q}(1-\gamma)}\frac{1}{a_n(x)^{\frac{\overline{q}}{2}}} \tag{33}$$

we can use Hölder's inequality with exponents $\frac{2}{\overline{q}}$, $\overline{p}$ and $\frac{2s}{\overline{q}}$. At least, thanks to the choice of $\overline{p}$ and using the inequalities (30) and (20) we prove that the sequence $\{u_n\}$ is bounded in $W_0^{1,\overline{q}}(\Omega)$.

If $\frac{1}{m}+\frac{1}{s} < 1+\frac{1}{N}$ (which implies $\overline{q} > 1$), up to a subsequence still denoted by $\{u_n\}$, $\{u_n\}$ converges to some function u weakly in $W_0^{1,\overline{q}}(\Omega)$, strongly in $L^1(\Omega)$ and almost everywhere in Ω.

If $\frac{1}{m} + \frac{1}{s} = 1 + \frac{1}{N}$ (which implies $\overline{q} = 1$), we work as in the proof of previous lemma. Let E be a measurable subset of Ω and $i \in \{1, \ldots, N\}$; by adapting (33) and using Hölder's inequality with exponents 2, $\overline{p}$ and $2s$, we obtain

$$\int_E |\partial_i u_n| \;\le\; \int_E |Du_n| = \int_E \frac{a_n(x)^{\frac{1}{2}}|Du_n|}{(1+|u_n|)^{(1-\gamma)}}(\epsilon+|u_n|)^{(1-\gamma)}\frac{1}{a_n(x)^{\frac{1}{2}}}$$

$$\le\ C_1\Big[\int_E \frac{1}{a_n(x)^s}\Big]^{\frac{1}{2s}}.$$

Then we prove that the sequence $\{u_n\}$ converges weakly in $W_0^{1,1}(\Omega)$, up to subsequences, to some function u. As a matter of fact, we can repeat the last part of the proof of Lemma 3.5, since in the framework of this case the choice $s=1$ implies $m=N$.

Remark 3.7. Let $q_T=\frac{2sm^*}{2s+m^*}$, γ and $\overline{q}$ as in the previous lemma. Then, there exists a constant $c>0$, independent on n such that

$$\int_\Omega a_n(x)^{\frac{q_T}{2}}|Du_n|^{q_T}\le c \quad \forall n\in\mathbb{N}. \tag{34}$$

As a matter of fact, by Holder's inequality we have

$$\int_\Omega a_n(x)^{\frac{q_T}{2}}|Du_n|^{q_T}\le\Big[\int_\Omega a_n(x)\frac{|Du_n|^2}{(1+|u_n|)^{2(1-\gamma)}}\Big]^{\frac{q_T}{2}}\Big[\int_\Omega (1+|u_n|)^{\frac{q_T(1-\gamma)}{2-q_T}}\Big]^{1-\frac{q_T}{2}}.$$

Since

$$\frac{q_T(1-\gamma)}{2-q_T}=\overline{q}^*$$

using the estimate (30) we conclude that the right hand side of previous inequality is bounded.

We note that estimate (34) says that $\{u_n\}$ is bounded in the weighted Sobolev space $W_0^{1,q_T}(\Omega)$, where q_T is the summability exponent obtained by N. Trudinger in Theorem 4.3 of [15].

3.3. The case $m=1$

Here we study the case $f\in L^1(\Omega)$, since, if $m=1$, in the previous inequalities it is not possible to use m'.

Lemma 3.8. *Let the hypotheses* (2), (3), (4) *be satisfied and* $m=1$. *Then*

$$\{T_k(u_n)\} \text{ is bounded in } W_0^{1,\frac{2s}{s+1}}(\Omega),\quad \forall k>0, \tag{35}$$

$$\{\log(1+|u_n|)\} \text{ is bounded in } W_0^{1,\frac{2s}{s+1}}(\Omega), \tag{36}$$

$$\{u_n\} \text{ is bounded in } M^{\frac{sN}{s(N-2)+N}}(\Omega), \tag{37}$$

$$\textit{the sequence } \{Du_n\} \textit{ is bounded in } \left(M^{\frac{sN}{s(N-1)+N}}(\Omega)\right)^N. \tag{38}$$

Proof. Let $k > 0$; if we choose $T_k(u_n)$ as test function in (19), we obtain

$$\int_\Omega a_n(x)|DT_k(u_n)|^2 = \int_\Omega f_n(x)T_k(u_n) \le k\|f\|_{L^1(\Omega)}, \ \ \forall\, n \in \mathbb{N}. \tag{39}$$

Let $q = \frac{2s}{s+1}$ (recall that $q = 1$ if $s = 1$). Working as in the proof of Lemma 3.5 and using the previous inequality we get

$$\int_\Omega |DT_k(u_n)|^q \le C_0\, k^{\frac{q}{2}}. \tag{40}$$

Now we follow the proof of Lemma 4.1 in [1]. Indeed (40) and the Sobolev inequality give

$$k^{q^*}\mathrm{meas}\{k < |u_n|\} = \int_{k<|u_n|} |T_k(u_n)|^{q^*} \le C_1\, k^{\frac{q^*}{2}},$$

which implies that

$$\mathrm{meas}\{k < |u_n|\} \le \frac{C_1}{k^{\frac{q^*}{2}}},$$

that is the estimate stated in (37).

Moreover (40) also implies that

$$\lambda^q \,\mathrm{meas}\{|u_n| \le k,\, \lambda \le |Du_n|\} \le \int_{|u_n|\le k,\, \lambda\le|Du_n|} |Du_n|^q \le C_0\, k^{\frac{q}{2}}.$$

Then

$$\mathrm{meas}\{\lambda \le |Du_n|\} = \mathrm{meas}\{|u_n| \le k,\, \lambda \le |Du_n|\} + \mathrm{meas}\{k < |u_n|\}$$

$$\le C_0 \frac{k^{\frac{q}{2}}}{\lambda^q} + \frac{C_1}{k^{\frac{q^*}{2}}}.$$

The choice $k = \lambda^{\frac{2(N-q)}{2N-q}}$ gives the estimate stated in (38).

In order to prove (36), we use in (19) as test function $\frac{u_n}{1+|u_n|}$ and we have

$$\int_\Omega \frac{|Du_n|^2}{(1+|u_n|)^2} \le \int_\Omega |f|,$$

which implies (once more we use the Hölder inequality)

$$\int_\Omega \frac{|Du_n|^{\frac{2s}{s+1}}}{(1+|u_n|)^{\frac{2s}{s+1}}} = \int_\Omega a_n(x)^{\frac{s}{s+1}} \frac{|Du_n|^{\frac{2s}{s+1}}}{(1+|u_n|)^{\frac{2s}{s+1}}} \frac{1}{a_n(x)^{\frac{s}{s+1}}}$$
$$\le \left[\int_\Omega |f|\right]^{\frac{s}{s+1}} \left[\int_\Omega \frac{1}{a_n(x)^s}\right]^{\frac{1}{s+1}}. \qquad \square$$

Remark 3.9. Let the assumptions of previous lemma be satisfied. Then, there exists a positive constant c, independent of n, such that, for every $n \in \mathbb{N}$ the following estimate holds

$$\int_\Omega a_n(x)^{\frac{\beta}{2}} |Du_n|^\beta < c, \quad \forall \beta < q_{_T}, \tag{41}$$

where $q_{_T} = \frac{2s1^*}{2s+1^*}$ is the number introduced in Remark 2.14.

As a matter of fact, let us take as test function in (19) the function $\dfrac{1-(1+|G_k(u_n)|)^{1-2\delta}}{2\delta-1}\text{sign}(u_n)$ where $\delta > \frac{1}{2}$ will be choosen later on and we obtain

$$\int_\Omega a_n(x) \frac{|DG_k(u_n)|^2}{(1+|G_k(u_n)|)^{2\delta}} \le \|f\|_{L^1(\Omega)}. \tag{42}$$

Let us fix $\beta < q_{_T}$. We have

$$\int_\Omega a_n(x)^{\frac{\beta}{2}} |Du_n|^\beta = \int_\Omega a_n(x)^{\frac{\beta}{2}} |DT_k(u_n)|^\beta + \int_\Omega a_n(x)^{\frac{\beta}{2}} |DG_k(u_n)|^\beta.$$

The first integral in the right hand side of above equality is bounded by virtue of (39), while the second one can be treated as follows

$$\int_\Omega a_n(x)^{\frac{\beta}{2}} |DG_k(u_n)|^\beta$$
$$= \int_\Omega a_n(x)^{\frac{\beta}{2}} \frac{|DG_k(u_n)|^\beta}{(1+|G_k(u_n)|)^{\delta\beta}} (1+|G_k(u_n)|)^{\delta\beta}$$

$$\leq \left[\int_\Omega a_n(x)^{\frac{\beta}{2}} \frac{|DG_k(u_n)|^2}{(1+|G_k(u_n)|)^{2\delta}} \right]^{\frac{\beta}{2}} \left[\int_\Omega \Big(1+|G_k(u_n)|\Big)^{\frac{2\delta\beta}{2-\beta}} \right]^{1-\frac{\beta}{2}}.$$

Using the estimate (42) and the boundedness of $\{u_n\}$ (and consequently of $\{G_k(u_n)\}$) in the space $M^{\frac{sN}{s(N-2)+N}}(\Omega)$, we conclude that the second member of previous inequality is bounded if we can take $\delta > \frac{1}{2}$ such that

$$\frac{2\delta\beta}{2-\beta} < \frac{sN}{s(N-1)+N}$$

and this choice is possible, since $\beta < q_T$.

Lemma 3.10. *Let* $s \geq N$ *and* $f \log(1+|f|)$ *be a function belonging to* $L^1(\Omega)$. *Then*

$$\{u_n\} \textit{ is bounded in } L^{\frac{sN}{s(N-2)+N}}(\Omega), \tag{43}$$

$$\{Du_n\} \textit{ is bounded in } L^{\frac{sN}{s(N-1)+N}}(\Omega). \tag{44}$$

Moreover if $s = N$ *then*

$$\{Du_n\} \textit{ is weakly compact in } (L^1(\Omega))^N. \tag{45}$$

Proof. We use $\log(1+|u_n|)\text{sign}(u_n)$ as test function in (19) and we get

$$\int_\Omega a_n(x) \frac{|Du_n|^2}{1+|u_n|} \leq \int_\Omega |f| \log(1+|u_n|).$$

We recall now the following inequality (for positive real numbers z, t)

$$z\,t \leq z\log(1+z) + \mathrm{e}^t - 1\,,$$

so that we have

$$\int_\Omega a_n(x) \frac{|Du_n|^2}{1+|u_n|} \leq \int_\Omega |f| \log(1+|f|) + \int_\Omega |u_n|.$$

Let $\tilde{q} = \frac{sN}{s(N-1)+N}$; from

$$\int_\Omega |Du_n|^{\tilde{q}} = \int_\Omega a_n(x)^{\frac{\tilde{q}}{2}} \left[\frac{|Du_n|}{\sqrt{1+|u_n|}} \right]^{\tilde{q}} \Big[1+|u_n| \Big]^{\frac{\tilde{q}}{2}} \frac{1}{a_n(x)^{\frac{\tilde{q}}{2}}}$$

we deduce, thanks to the Hölder inequality with exponents $\frac{2}{\tilde q}$, $\frac{2\tilde q^*}{\tilde q}$ and $\frac{2s}{\tilde q}$

$$\int_\Omega |Du_n|^{\tilde q} \le \left[\int_\Omega |f|\log(1+|f|)+\int_\Omega |u_n|\right]^{\frac{\tilde q}{2}} \left[\int_\Omega (1+|u_n|)^{\tilde q^*}\right]^{\frac{\tilde q}{2\tilde q^*}} \left[\int_\Omega \frac{1}{a_n(x)^s}\right]^{\frac{\tilde q}{2s}}.$$

Here we can use (37), with $s \ge N$, since now $\frac{sN}{s(N-2)+N}$ is strictly grater than 1. Thus we have

$$\int_\Omega |u_n| \le C_1$$

and

$$\mathcal{S}\|u_n\|^{\tilde q}_{L^{\tilde q^*}(\Omega)} \le \int_\Omega |Du_n|^{\tilde q} \le C_2\left[\int_\Omega |f|\log(1+|f|)+C_1\right]^{\frac{\tilde q}{2}} \|1+|u_n|\|^{\frac{\tilde q}{2}}_{L^{\tilde q^*}(\Omega)},$$

which implies (43) (note that $\tilde q^* = \frac{sN}{s(N-2)+N}$) and then (44).

If $s = N$, then (44) says that $\{Du_n\}$ is bounded in $L^1(\Omega)$ and we need something more in order to prove (45). Let E be a measurable subset of Ω; since

$$\int_E |Du_n| = \int_E a_n(x)^{\frac{1}{2}} \frac{|Du_n|}{\sqrt{1+|u_n|}} \sqrt{1+|u_n|} \frac{1}{a_n(x)^{\frac{1}{2}}}$$

due to the Hölder inequality with exponents 2, $\frac{2N}{N-1}$ and $2N$, we deduce

$$\int_E |Du_n| \le \left[\int_\Omega |f|\log(1+|f|)+\int_\Omega |u_n|\right]^{\frac{1}{2}} \left[\int_\Omega (1+|u_n|)^{\frac{N}{N-1}}\right]^{\frac{N-1}{2N}} \left[\int_E \frac{1}{a_n(x)^N}\right]^{\frac{1}{2N}}.$$

Here we can use (43), thus we have

$$\int_E |Du_n| \le C_3\left[\int_E \frac{1}{a_n(x)^N}\right]^{\frac{1}{2N}}.$$

Since the sequence $\{\frac{1}{a_n}\}$ is compact in $L^1(\Omega)$, the sequence $\{\partial_i u_n\}$ is equi-integrable. Thus, by Dunford-Pettis theorem, as in the proof of Lemma 3.5, we prove (45).

4. Proof of Existence Theorems

4.1. Proof of Theorem 2.1

Lemma 3.5 says that, up to subsequences, the sequence $\{u_n\}$ weakly converges to a function u in $W_0^{1,\frac{2s}{s+1}}(\Omega)$. Then, thanks to (5), it is easy to pass to the limit in the weak formulation of (19)

$$\int_\Omega a_n(x)Du_nDv = \int_\Omega f_n(x)v, \quad \forall\, v \in W_0^{1,(\frac{2s}{s+1})'}(\Omega).$$

Moreover, the summability (boundedness) of u is a consequence of the boundedness of the sequence $\{u_n\}$ in Lebegue's spaces proved in Subsection 3.1. □

4.2. Proof of Theorem 2.7

Here we use Lemma 3.6 instead of Lemma 3.5.

4.3. Proof of Theorem 2.12

Here we use Lemma 3.10 instead of Lemma 3.5.

4.4. Proof of Theorem 2.10

As a consequence of (36), there exists a subsequence (not relabelled) such that

$$\{\log(1+|u_n|)\mathrm{sign}(u_n)\} \text{ converges weakly in } W_0^{1,\frac{2s}{s+1}}(\Omega) \text{ and a. e. in } \Omega \tag{46}$$

Then, $\{u_n(x)\}$ converges a. e. in Ω to a measurable function $u(x)$ such that $\log(1+|u|) \in W_0^{1,\frac{2s}{s+1}}(\Omega)$. Moreover, as a consequence of (35), for every $k>0$, the sequence $\{T_k(u_n)\}$ converges weakly in $W_0^{1,\frac{2s}{s+1}}(\Omega)$ to $T_k(u)$.

Thus, if we take $T_k[u_n-\varphi]$ as test function in the weak formulation of problem (19) , we have, $\forall\, k>0$ and $\forall \varphi \in W_0^{1,(\frac{2s}{s+1})'}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a_n(x)Du_nD\,T_k[u_n-\varphi] = \int_\Omega f(x)T_k[u_n-\varphi],$$

which implies

$$\int_\Omega a_n(x) D\varphi D\, T_k[u_n - \varphi] \le \int_\Omega f(x) T_k[u_n - \varphi].$$

Here it is easy to pass to the limit, due to (5) and the weak convergence in $W_0^{1,\frac{2s}{s+1}}(\Omega)$ of $\{T_k(u_n)\}$ to $T_k(u)$, and we obtain (16). □

Acknowledgments

A first and partial version of the result of the present paper (consequence of a period of staying in Canberra, february 1997) was presentend during a series of lectures at Sevilla University, April 1997.

References

1. P. Bénilan, L. Boccardo, T. Gallouet, R. Gariepy, M. Pierre and J. L. Vazquez, An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Annali Sc. Norm. Sup. Pisa*, **22** (1995), 241-273.

2. L. Boccardo, G. Croce and L. Orsina, A semilinear problem with a $W_0^{1,1}$ solution, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat. Rend. Lincei* (9) *Mat. Appl.*, **23** (2012), no. 2, 97–103.

3. L. Boccardo, G. Croce and L. Orsina, Nonlinear degenerate elliptic problems with $W_0^{1,1}$ solutions, *Manuscripta Math.*, **137** (2012), 419-439.

4. L. Boccardo and T. Gallouet, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.*, **87** (1989), 149-169.

5. L. Boccardo and T. Gallouet, Nonlinear elliptic equations with right hand side measures, *Comm. P.D.E.*, **17** (1992), 641-655.

6. L. Boccardo and T. Gallouet, $W_0^{1,1}$-solutions in some borderline cases of Calderon-Zygmund theory, *J. Differential-Equations*, **253** (2012), 2698-2714.

7. L. Boccardo and T. Leonori, Local properties of solutions of elliptic equations depending on local properties of the data, *Methods and applications of analysis*, **15** (2008), 53-64.

8. G. R. Cirmi, On the existence of solutions to nonlinear degenerate elliptic equations with measure data, *Ricerche Mat.*, **42**, (1993), 315-329.

9. G. R. Cirmi, Some regularity results for nonlinear degenerate elliptic equations, *Differential and Integral Equations*, **8** (1995), 131-140.

10. G. R. Cirmi and M. M. Porzio, L^∞- solutions for some nonlinear degenerate elliptic and parabolic equations, *Ann. Mat. Pura Appl.*, **169** (1995), 67-86.

11. P. Hartman and G. Stampacchia, On some nonlinear elliptic differential–functional equations, *Acta Math.*, **115** (1966), 271-310.

12. J. Leray and J.-L. Lions, Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France*, **93** (1965), 97-107.

13. M. Murthy and G. Stampacchia, Boundary value problems for some degenerate-elliptic operators, *Ann. Mat. Pura Appl.*, **80** (1968), 1-122.

14. G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, *Ann. Inst. Fourier* (*Grenoble*), **15** (1965), 189–258.

15. N. S. Trudinger, Linear elliptic operator with measurable coefficients, *Annali Sc. Norm. Sup. Pisa*, **27** (1973), 65-308.

16. N. S. Trudinger, Maximum principles for linear, non-uniformly elliptic operators with measurable coefficients, *Math. Z.*, **156** (1977), 291-301.