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FUJITA EQUATION WITH BOUNDARY EFFECT

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The paper is dedicated to the 70th anniversary of N. Trudinger

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Abstract

The well-known Fujita analysis for the blow-up and decay of a semilinear heat equation $u_t - \Delta u = u^p$ is studied here for the half-space problem with Robin boundary condition. Our analysis makes use of the explicit construction of Green's functions for the initial-boundary value problem. Our analysis in deriving the adequate expression of the Green's function have provided further development in the explicit construction of the Green's function for a class of partial differential equations.

1. Introduction

The classical Fujita equation, a semilinear heat equation

$$u_t - \Delta u = u^p,\ u = u(\boldsymbol{x}, t) \geq 0,$$

was proposed to study the blow up and decay of the solutions as a consequence of the combined effect of diffusion and nonlinearity. Fujita analysis makes essential use of the explicit expression of the heat kernel, [7]. The

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main purpose of the present study is to consider also the effect of the boundary. For this, we also require the explicit expression of the Green's function for exact computations. For definiteness, we will consider the 3-D case, $\boldsymbol{x} = (x_1, x_2, x_3) = (x, y, z)$, the half-sapce domain $(x, y, z) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}$, and the plane boundary $x = 0$. The boundary condition is the homogeneous Robin condition:

$$\begin{cases} u_t - \Delta u = u^p, \\ (u_x + \alpha u)|_{x=0} = 0, \\ u(x, y, z, 0) = \varepsilon u_0(x, y, z). \end{cases} \tag{1.1}$$

Let $k(x, y, z) = k_3(\boldsymbol{x}) = k_3(x_1, x_2, x_3)$ be the 3-D heat kernel

$$k(x, y, z) = \frac{e^{-\frac{x^2+y^2+z^2}{4t}}}{(4\pi t)^{3/2}}.$$

As in [7], in order to study the combined effect of the diffusion and the nonlinearity, we consider the initial value to be small, i.e. ε small, nonnegative and localized, i.e. dominated by the heat kernel:

$$0 \le u_0(x, y, z) \le k(x, y, z, 1) = \frac{e^{-(x^2+y^2+z^2)/4}}{(4\pi)^{3/2}}. \tag{1.2}$$

Fujita considers the problem in the whole space:

$$\begin{cases} v_t - \Delta v = v^p, \\ v(x, y, z, 0) = \varepsilon u_0(x, y, z) \ge 0. \end{cases} \tag{1.3}$$

He shows that for $p > 5/3$ global solution exists, and for $1 < p < 5/3$, any non-trivial solution blows up. As our purpose is to study the effect of the boundary, we will assume throughout the present study that

$$p > \frac{5}{3}. \tag{1.4}$$

We state the Fujita theorem and gives a short proof to illustrate the necessity of using the heat kernel for the pointwise estimate of the solutions, as in the original analysis of Fujita.

Theorem 1.1 (Fujita). *Suppose that $p > 5/3$ and (1.2) holds. Then for sufficiently small ε, the solution $v(x, y, z, t)$ of (1.3) exists globally in time* *and satisfies*

$$|v(x,y,z,t)| \leq 2\varepsilon k(\boldsymbol{x},t+1) = 2\varepsilon \frac{e^{-\frac{x^2+y^2+z^2}{4(t+1)}}}{(4\pi(t+1))^{3/2}}. \tag{1.5}$$

Proof. By Duhamel's principle,

$$v(\boldsymbol{x},t) = \varepsilon \int_{\mathbb{R}^3} k(\boldsymbol{x}-\boldsymbol{x}_*,t)u_0(\boldsymbol{x}_*)d\boldsymbol{x}_* + \int_0^t \int_{\mathbb{R}^3} k(\boldsymbol{x}-\boldsymbol{x}_*,t-\sigma)|v(\boldsymbol{x}_*,\sigma)|^p d\boldsymbol{x}_* d\sigma. \tag{1.6}$$

By (1.2), one has that

$$\left|\varepsilon \int_{\mathbb{R}^3} k(\boldsymbol{x}-\boldsymbol{x}_*,t)u_0(\boldsymbol{x}_*)d\boldsymbol{x}_*\right| \leq \varepsilon \int_{\mathbb{R}^3} k(\boldsymbol{x}-\boldsymbol{x}_*,t)k(\boldsymbol{x}_*,1)d\boldsymbol{x}_* = \varepsilon k(\boldsymbol{x},t+1), \tag{1.7}$$

where the last equality follows from the semigroup property of the heat kernel. Thus we make the ansatz assumption for the solution $v(\boldsymbol{x},t)$:

$$|v(\boldsymbol{x},t)| \leq 4\varepsilon k(\boldsymbol{x},t+1). \tag{1.8}$$

Substituting the ansatz (1.8) into (1.6) and keeping in mind that $p > 5/3$, we obtain through direct calculations that

$$\begin{aligned}
|v(\boldsymbol{x},t)| &\leq \varepsilon k(\boldsymbol{x},t+1)| + 4^p\varepsilon^p \int_0^t \int_{\mathbb{R}^3} k(\boldsymbol{x}-\boldsymbol{x}_*,t-\sigma)k(\boldsymbol{x}_*,\sigma+1)^p d\boldsymbol{x}_* d\sigma \\
&\leq \varepsilon k(\boldsymbol{x},t+1)| + 4^p\varepsilon^p k(\boldsymbol{x},t+1) \int_0^t \frac{1}{(4\pi\sigma)^{3(p-1)/2}} d\sigma \\
&= \varepsilon \left(1 + \frac{2\ 4^p \varepsilon^{p-1}}{(3p-5)(4\pi)^{3(p-1)/2}}\right) k(\boldsymbol{x},t+1) \\
&< 2\varepsilon k(\boldsymbol{x},t+1) \text{ when } \varepsilon \ll 1.
\end{aligned}$$

Thus, the ansatz (1.8) holds when $\varepsilon \ll 1$ and (1.5) is proved. □

Our main theorem is that, under the stability hypothesis (1.4) for the whole space, the boundary condition can make a non-trivial solution to blow up in finite time.

Theorem 1.2. *Suppose that $\alpha > 0$ and that the initial value $u(x,y,z,0)$ is nonzero. Then there is a solution $u(x,y,z,t)$ of* (1.1) *blows up in finite time.*

Remark 1.1. In the unstable case $\alpha > 0$, the linear heat equation is not well-posed and does not satisfy the Maximum Principle. In the above theorem, we consider the solution constructed through the fundamental solution. We study the blow-up of only such a solution. The authors would like to thank the referee for pointing this out to us.

Theorem 1.3. *For $\alpha < 0$ and $\varepsilon \ll 1$, the solution of the problem* (1.1) *exists globally in time and satisfies*

$$|u(x,y,z,t)| \leq O(1)\varepsilon \frac{e^{-\frac{x^2+y^2+z^2}{4(t+1)}}}{(4\pi(t+1))^{3/2}}. \tag{1.9}$$

Remark 1.2. When $\alpha = 0$ the solution $u(x,y,z,t)$ of (1.1) is time-asymptotically stable. This follows immediately from Theorem 1.1 by making an even extension of the solution $u(x,y,z,t)$ to the whole space domain, $v(x,y,z,t) \equiv u(|x|,y,z,t)$.

Remark 1.3. The pointwise analysis for the proof of Theorem 1.2 turns out to be quite different from that for Theorem 1.3. Exact expression of the Green's function for the initial-boundary value problem is needed for the analysis in both cases.

There have been extensive studies on the formation of singularities for the Fujita equation with boundary. Energy methods and maximum principles are the commonly used methods. We list here some survey articles and recent papers on the related subject, [1, 6, 8, 2, 14, 10, 4, 5]. The present study uses the explicit expression of the Green's function, (2.18), constructed in the next section. In this, we are following the original approach of Fujita, [7]; though there are some subtleties in our process of deriving adequate expressions of the Green's function.

2. Green's Function

Consider the the linearized version of (1.1):

$$\begin{cases} (\partial_t - \Delta)U(x,y,z,t) = 0, \ x,t > 0, y,z \in \mathbb{R}, \\ (U_x + \alpha U)|_{x=0} = 0, \\ U|_{t=0} = U_0. \end{cases} \tag{2.1}$$

Let $G(x,y,z,t;x_*)$ be the Green's function:

$$\begin{cases}(\partial_t-\Delta_{\boldsymbol{x}})G=0,\ x,t>0,y,z\in\mathbb{R},\\ (G_x+\alpha G)|_{x=0}=0,\\ G|_{t=0}=\delta(x-x_*)\delta(y)\delta(z).\end{cases} \tag{2.2}$$

From this we can easily construct the Green's function $G(\boldsymbol{x},\ \boldsymbol{x}_*,t-\sigma)$, $\boldsymbol{x}=(x,y,z)$, $\boldsymbol{x}_*=(x_*,y_*,z_*)$, for the initial-boundary value problem (2.1) with more general initial value

$$\begin{aligned}&G=G(x,y,z,t,x_*,y_*z_*)\equiv G(x,y-y_*,z-z_*,t,x_*),\\ &(\partial_t-\Delta_{\boldsymbol{x}})G=0,\ x,t>0,y,z\in\mathbb{R},\\ &(G_x+\alpha G)|_{x=0}=0,\\ &G|_{t=0}=\delta(x-x_*)\delta(y-y_*)\delta(z-z_*).\end{aligned}$$

Let $k_n(x_1,\ldots,x_n,t)$ be the n-dimensional heat kernel:

$$k_n(x_1,\ldots,x_n,t)=\frac{e^{-\frac{x_1^2+x_2^2+\cdots+x_n^2}{4t}}}{(4\pi t)^{n/2}}.$$

One has

$$k_n(x_1,\ldots,x_n,t)=\prod_{j=1}^{n}k_1(x_j,t), \tag{2.3}$$

the semigroup property

$$k_1(x,t)=\int_{\mathbb{R}}k_1(x-x_*,t-\sigma)k_1(x_*,\sigma)dx_*\text{ for }\sigma\in(0,t), \tag{2.4}$$

and, for $0<\sigma<\tau$,

$$k_1(x,\sigma)<\frac{\sqrt{\tau}}{\sqrt{\sigma}}k_1(x,\tau). \tag{2.5}$$

The main goal of the present article is to demonstrate the importance of the explicit construction of the Green's function, even for study of the nonlinear problems. Although for the heat and wave equations, the Green's function for a class of the initial-boundary value problem, such as that given in (2.18), have been found, for instance, by the image method, [9], we construct the Green's function by the systematic LY algorithm. It uses the

Fourier-Laplace transforms:

$$\begin{cases} v(x,\boldsymbol{\eta},t)=\mathscr{F}[u](x,\boldsymbol{\eta},t)\equiv\int_{\mathbb{R}^2}u(x,y,z,t)e^{-i(y,z)\cdot\boldsymbol{\eta}}dydz, \\ \qquad\qquad\qquad\qquad\qquad\text{(Fourier transform in } (y,z)\text{)}, \\ V(x,\boldsymbol{\eta},s)=\mathbb{L}[v](x,\boldsymbol{\eta},s)\equiv\int_0^\infty e^{-st}v(x,\boldsymbol{\eta},t)dt, \\ \qquad\qquad\qquad\qquad\qquad\text{(Laplace transformation in } t\text{)}. \end{cases} \tag{2.6}$$

The LY algorithm assume that the fundamental solution, here the heat kernel $k(x,y,z,t)=k_3(x,y,z,t)$, is known. We start with the transforms of the heat kernel

$$\begin{cases} h(x,\boldsymbol{\eta},t)=h(x,\eta^1,\eta^2,t)=\mathscr{F}_{\boldsymbol{y}}[k](x,\boldsymbol{\eta},t)=\int_{\mathbb{R}^2}\frac{e^{-\frac{x^2+y^2+z^2}{4t}}}{4\pi t}.e^{-i(y,z)\cdot\boldsymbol{\eta}}dydz, \\ \qquad\qquad\qquad\qquad\qquad\text{(Fourier transform in } (y,z)\text{)}, \\ H(x,\boldsymbol{\eta},s)=\mathbb{L}_t[h](x,\boldsymbol{\eta},s)\equiv\int_0^\infty e^{-st}h(x,\boldsymbol{\eta},t)dt, \\ \qquad\qquad\qquad\qquad\qquad\text{(Laplace transformation in } t\text{)}. \end{cases} \tag{2.7}$$

The defining equations for heat kernel are

$$\begin{cases} (\partial_t-\Delta_{\boldsymbol{x}})k=0,\ (x,y,z)\in\mathbb{R}^3,\ t>0, \\ k|_{t=0}=\delta(\boldsymbol{x})=\delta(x)\delta(y)\delta(z). \end{cases} \tag{2.8}$$

Take first the Fourier transform in (y,z) to obtain

$$(\partial_t+|\boldsymbol{\eta}|^2-\partial_x^2)h=0.$$

Then take the Laplace transform in t and use the initial value to yield

$$(s+|\boldsymbol{\eta}|^2-\partial_x^2)H=\delta(x).$$

This is viewed as an ordinary differential equation in x and has general continuous solutions

$$H(x,\boldsymbol{\eta},s)=Ae^{-\lambda|x|},$$

where

$$\lambda=\lambda(\boldsymbol{\eta},s)=\sqrt{s+|\boldsymbol{\eta}|^2}. \tag{2.9}$$

The constant A is determined to be $A = 1/(2\lambda)$ so as to yield the the $\delta(x)$ from $\partial_x^2 H$ and so

$$H(x, \boldsymbol{\eta}, s) = \frac{1}{2\lambda} e^{-\lambda|x|}.$$

From this the Fourier-Laplace transform of the differentials of the heat kernel is

$$\mathscr{F}_{\boldsymbol{y}}[\mathbb{L}_t[\partial_x^m k]](x, \boldsymbol{\eta}, s) = \partial_x^m H(x, \boldsymbol{\eta}, s) = \frac{(-1)^m}{2} \lambda^{m-1} e^{-\lambda x}, \tag{2.10}$$

$$x > 0,\ m = 0, 1, 2, \ldots. \tag{2.11}$$

We now construct the Green's function. The first step is to make the initial value zero by considering the function $G - k$:

$$\begin{cases} (\partial_t - \Delta_{\boldsymbol{x}})u = 0,\ x, t > 0, y, z \in \mathbb{R}, \\ u|_{t=0} = 0, \\ (u_x + \alpha u)|_{x=0} = [-k_x - \alpha k](x - x_*, y, z, t)|_{x=0} = [k_x - \alpha k](x_*, y, z, t), \\ u(x, y, z, t) \equiv G(x, y, z, t, x_*) - k(x - x_*, y, z, t). \end{cases} \tag{2.12}$$

Here we have noticed that k is even in x and k_x is odd in x.

Remark 2.4. This first step of making the initial value zero is crucial, as the homogeneous initial value yields a simple expression for the Laplace transform of the solution in the time variable t. The boundary condition for the new function u is now non-homogeneous.

From (2.12) the transformed variable V, (2.6), satisfies

$$\begin{cases} (s + |\boldsymbol{\eta}|^2)V - V_{xx} = 0, \\ V_x^0 + \alpha V^0 = [H_x - \alpha H](x_*, y, z, t) = (-\frac{1}{2} - \frac{\alpha}{2\lambda})e^{-\lambda x_*}, \\ V^0 \equiv V|_{x=0},\ V_x^0 \equiv V_x|_{x=0}, \end{cases} \tag{2.13}$$

where we have used the expression (2.11) for the transformed heat kernel. The first equation in (2.13) has general solutions

$$\begin{aligned} &V(x, \boldsymbol{\eta}, s) = Ae^{-\lambda x} + Be^{\lambda x},\ \lambda = \sqrt{s + |\boldsymbol{\eta}|^2}, \\ &A + B = V^0,\ -\lambda A + \lambda B = V_x^0. \end{aligned}$$

The stability criterion demands that the solution does not contain the exponentially growing term $e^{\lambda x}$, or B=0 and so

$$\begin{cases} V(x,\boldsymbol{\eta},s) = V^0 e^{-\lambda x}, \\ -\lambda V^0 = V_x^0, \ \text{Master relationship.} \end{cases} \tag{2.14}$$

From the Master relationship and the boundary condition in (2.13), we have

$$V^0 = \frac{1}{\alpha-\lambda}(-\frac{1}{2} - \frac{\alpha}{2\lambda})e^{-\lambda x_*},$$

and so the first equation in (2.13) yields the solution in the transformed variable

$$\begin{aligned} V &= -\frac{1}{2\lambda}\frac{\alpha+\lambda}{\alpha-\lambda}e^{-\lambda(x+x_*)} = -\frac{1}{2\lambda}\frac{(\alpha+\lambda)^2}{\alpha^2-\lambda^2}e^{-\lambda(x+x_*)} \\ &= \frac{1}{s+|\boldsymbol{\eta}|^2-\alpha^2}(\alpha+\lambda)^2\frac{e^{-\lambda(x+x_*)}}{2\lambda}. \end{aligned} \tag{2.15}$$

The first factor can be inverted easily first in the Laplace and then the Fourier transform:

$$\mathbf{F}^{-1}\mathbf{L}^{-1}[\frac{1}{s+|\boldsymbol{\eta}|^2-\alpha^2}] = \boldsymbol{F}^{-1}[e^{\alpha^2 t-|\boldsymbol{\eta}|^2 t}] = e^{\alpha^2 t}\frac{1}{4\pi t}e^{-\frac{y^2+z^2}{4t}}.$$

The second factor has the following inverse transform by (2.11):

$$\mathbf{F}^{-1}\mathbf{L}^{-1}[(\alpha+\lambda)^2 e^{-\lambda(x+x_*)}] = (\alpha-\partial_x)^2 k(x+x_*,y,z,t).$$

From these we conclude from above and (2.15) that u is the convolution in $(\boldsymbol{y},t)$:

$$u = \mathbf{F}^{-1}\mathbf{L}^{-1}[V] = (\alpha-\partial_x)^2[e^{\alpha^2 t}\frac{1}{4\pi t}e^{-\frac{y^2+z^2}{4t}} \star_{\boldsymbol{yt}} k(x+x_*,y,z,t)],$$

or

$$\begin{aligned} u &= (\alpha-\partial_x)^2\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[e^{\alpha^2(t-\sigma)}\frac{1}{4\pi(t-\sigma)}e^{-\frac{(y-\bar{y})^2+(z-\bar{z})^2}{4(t-\sigma)}} \\ &\quad \times k(x+x_*,y-\bar{y},z-\bar{z},\sigma)\Big]d\bar{y}d\bar{z}d\sigma. \end{aligned}$$

The integration in $(\bar{y},\bar{z})$ is straightforward and follows directly from the semi-group property of the heat kernel:

$$u = k_1(y,t)k_1(z,t)(\alpha-\partial_x)^2\int_0^t e^{\alpha^2(t-\sigma)}k_1(x+x_*,\sigma)]d\sigma. \tag{2.16}$$

By $\partial_x^2 k_1(x+x_*,\sigma)=\partial_\sigma k_1(x+x_*,\sigma)$, this can be rewritten as

$$\begin{aligned} u &= k_1(y,t)k_1(z,t)[(\alpha^2-2\alpha\partial_x)\int_0^t e^{\alpha^2(t-\sigma)}k_1(x+x_*,\sigma)]d\sigma \\ &\quad +\int_0^t e^{\alpha^2(t-\sigma)}\partial_\sigma k_1(x+x_*,\sigma)]d\sigma]. \end{aligned}$$

By integration by parts and noticing that the three heat kernels $k_1(y,t)$, $k_1(z,t)$ and $k_1(x+x_*,t)$ have disjoint support at $t=0$, we obtain

$$u = k_1(y,t)k_1(z,t)[k_1(x+x_*,t)+2\alpha(\alpha-\partial_x)\int_0^t e^{\alpha^2(t-\sigma)}k_1(x+x_*,\sigma)]d\sigma]. \tag{2.17}$$

Finally, from the definition of the function u in the last identity of (2.12), we have the explicit expression of the Green's function:

$$\begin{aligned} G(\boldsymbol{x},\boldsymbol{x}_*,t) &= k_1(y-y_*,t)k_1(z-z_*,t)\Big(k_1(x-x_*,t)+k_1(x+x_*,t) \\ &\quad +2\alpha(\alpha-\partial_x)\int_0^t e^{\alpha^2(t-\sigma)}k_1(x+x_*,\sigma)d\sigma\Big). \end{aligned} \tag{2.18}$$

We have thus follow the LY algorithm, [15, 16], to obtain the Green's function through a systematic approach. The approach requires that there is an explicit construction of the fundamental solutions for the corresponding initial value problems. The explicit construction of the fundamental solutions of the heat equation and the wave equation are classical; they are the heat kernel and the Kirchhoff-Hadamard formulas. For these equations, the image method, [9], applies for the construction of the Green's function for certain class of the initial-boundary value problem. The LY algorithm would apply to a general class of initial-boundary value problems for heat and wave equations. For a class of hyperbolic-parabolic partial differential equations in the continuum physics, such as the compressible Navier-Stokes equations, the explicit construction of the fundamental solution for the whole space has been studied, [19, 18, 11, 12, 13]. For such a system, the fundamental solution contains variable scalings, no simple image method is available and the LY algorithm can be applied for the explicit construction of the Green's functions for the corresponding initial-boundary value problem, c.f. [17]. In the following two sections we will carry out our analysis through exact computations using the explicit expression of the Green's function (2.18).

3. Proof of Theorem 1.2

We consider nonzero, nonnegative initial value. It is clear from the local theory that, for such a fixed initial value, the solution at any positive, small time, is bounded below by a multiple of the heat kernel. For the case of $\alpha > 0$ considered here, it is easy to see by direct computation that the Green's function (2.18) is positive. As a consequence, there is a monotonicity property of the solution operator for

$$u_t - \Delta u = S,\ (u_x + \alpha u)|_{x=0} = 0$$

in that the solution increases as the source S or the initial value $u(\boldsymbol{x}, 0)$ increases. Moreover, the source $u^p,\ u \geq 0$, in (1.1) is a monotone function of the solution u. Thus, for the blow up property we intend to verify, it is sufficient to consider

$$u_0(x, y, z) = H_3(x, y, z, 1). \tag{3.1}$$

From the formula (2.18) with the property $\alpha > 0$ and $x, x_* > 0$, by (2.5) one has the estimate on the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_*, t)$:

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{x}_*, t) &\geq H_1(y - y_*, t) H_1(z - z_*, t) \Big(H_1(x - x_*, t) + H_1(x + x_*, t) \\ &\quad + 2\alpha^2 \int_0^t e^{\alpha^2 (t-\sigma)} H_1(x + x_*, \sigma) d\sigma \Big) \\ &\geq 2\alpha^2 H_1(y - y_*, t) H_1(z - z_*, t) H_1(x + x_*, 1) e^{\alpha^2 t} K(t), \end{aligned} \tag{3.2}$$

$$K(t) \equiv \begin{cases} 0 \text{ for } t < 1, \\ \displaystyle\int_1^t \frac{e^{-\alpha^2 \sigma}}{\sqrt{\sigma}} d\sigma \text{ for } t > 1. \end{cases} \tag{3.3}$$

By the Duhamel's principle, the initial data (3.1), and the estimate of the Green's function (3.2), one has the following estimate of the solution

$u(x,y,z,t)$ of (1.1):

$$\begin{aligned} u(x,y,z,t) \ \geq\ & 2\varepsilon\alpha^2 H_1(y,t+1)H_1(z,t+1)e^{\alpha^2 t}K(t)\sqrt{2}H_1(x,1) \\ & + \int_0^t \int_{\mathbb{R}^3_+} 2\alpha^2 e^{\alpha^2(t-\sigma)}K(t-\sigma)H_1(x+x_*,1)H_1(y-y_*,t-\sigma) \\ & H_1(z-z_*,t-\sigma)|u(x_*,y_*,z_*,\sigma)|^p dx_* dy_* dz_* d\sigma. \end{aligned} \tag{3.4}$$

We now show that there is $T_0(\varepsilon,\alpha)>0$ such that, for $t>T_0(\varepsilon,\epsilon)$,

$$u(0,0,0,t) \geq \infty.$$

This is done by iterations. Define

$$u_1(x,y,z,t) \equiv 2\varepsilon\alpha^2 H_1(y,t+1)H_1(z,t+1)e^{\alpha^2 t}K(t)\sqrt{2}H_1(x,1)>0,$$

and for $n\geq 2$,

$$\begin{aligned} u_n(x,y,z,t) \ \equiv\ & \int_0^t \int_{\mathbb{R}^3_+} 2\alpha^2 e^{\alpha^2(t-\sigma)}K(t-\sigma)H_1(x+x_*,1)H_1(y-y_*,t-\sigma) \\ & \cdot H_1(z-z_*,t-\sigma)|u_{n-1}(x_*,y_*,z_*,\sigma)|^p dx_* dy_* dz_* d\sigma > 0. \end{aligned} \tag{3.5}$$

From (3.4) we have

$$u(x,y,z,t) \geq u_1(x,y,z,t) + \sum_{n=2}^{N} u_n(x,y,z,t) \text{ for any } N\geq 2, \tag{3.6}$$

and from (2.4) and (2.5)

$$\begin{aligned} u_2(x,y,z,t) \ \geq\ & \varepsilon H_1(y,t+1)H_1(z,t+1)H_1(x,1) \\ & \times \int_0^t \frac{2^{p+1}\varepsilon^{p-1}\alpha^{2+2p}(t+1)e^{p\alpha^2\sigma}k(\sigma)^p}{(t-\sigma+p(\sigma+1))(\sigma+1)^{p-1}} e^{\alpha^2(t-\sigma)}K(t-\sigma)d\sigma \\ \geq\ & \varepsilon^p \alpha^{p+2} 2^{p+1} e^{\alpha^2 t} H_1(y,t+1)H_1(z,t+1)H_1(x,1) \\ & \times \int_0^t \frac{k(\sigma)^p K(t-\sigma)}{(t-\sigma+p(\sigma+1))(\sigma+1)^{p-1}} e^{(p-1)\alpha^2\sigma} d\sigma. \end{aligned} \tag{3.7}$$

By direct computations we can show that there exists $C_0>0$ such that, for

$p > 1$

$$\int_0^t \frac{k(\sigma)^p K(t-\sigma)}{2(t-\sigma+p(\sigma+1))(\sigma+1)^{p-1}} e^{(p-1)\alpha^2\sigma} d\sigma > C_0 e^{(p-1)\alpha^2 t/2} \text{ for } t \geq 4, \tag{3.8}$$

and so

$$\begin{aligned} u_2(x,y,z,t) \ &\geq \ e^{\alpha^2 t} k_1(y,t+1) k_1(z,t+1) k_1(x,1) \\ &\qquad \text{for } t > T_0 \equiv 4\left(1 + \left|\frac{\log\left(\varepsilon^p \alpha^{p+2} 2^{p+1}/C_0\right)}{(p-1)\alpha^2}\right|\right). \end{aligned}$$

In general we have, for all $n \geq 1$,

$$u_n(x,y,z,t) \geq e^{\alpha^2 t} k_1(y,t+1) k_1(z,t) k_1(x,1) \geq 1 \text{ for } t > T_0.$$

This and (3.6) show that for $t > T_0$

$$u(x,y,z,t) = \infty.$$

This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.3

The boundary condition in this case is stable, as easily seen from the energy estimate. We are considering the Fujita stable case of $p > \frac{5}{3}$, (1.4), and so the stability result in Theorem 1.3 is expected. As in the original Fujita analysis, the energy method alone is not sufficient for the stability analysis, and the explicit expression of the Green's function is needed. However, in the formula (2.18), the Green's function $G(\boldsymbol{x}, \boldsymbol{x}_*, t)$ contains the exponentially growing factor $e^{\alpha^2(t-\sigma)}$. Thus to use the Green's function, there is a need to see some cancellation so as to eliminate this growing factor in order to assert the strong pointwise stability (1.9). We start with the identity

$$\int_0^\infty k_1(x,\sigma) e^{-\alpha^2\sigma} d\sigma = \frac{e^{-|\alpha x|}}{2|\alpha|} \tag{4.1}$$

to assert that for $k < 0$ and $x > 0$

$$(\alpha - \partial_x) \int_0^\infty k_1(x,\sigma) e^{-\alpha^2\sigma} d\sigma = 0. \tag{4.2}$$

It follows from (4.2) that

$$\begin{aligned}
&\left|(\alpha-\partial_x)\int_0^t k_1(x+x_*,\sigma)e^{\alpha^2(t-\sigma)}d\sigma\right| \\
&\quad= \left|(\alpha-\partial_x)\left(\int_0^\infty-\int_t^\infty\right)k_1(x+x_*,\sigma)e^{\alpha^2(t-\sigma)}d\sigma\right| \\
&\quad= \left|\Big(e^{\alpha^2 t}(\alpha-\partial_x)\int_0^\infty k_1(x+x_*,\sigma)e^{-\alpha^2\sigma}d\sigma\right. \\
&\qquad \left.-(\alpha-\partial_x)\int_t^\infty k_1(x+x_*,\sigma)e^{\alpha^2(t-\sigma)}d\sigma\Big)\right| \\
&\quad= \left|(\alpha-\partial_x)\int_t^\infty k_1(x+x_*,\sigma)e^{\alpha^2(t-\sigma)}d\sigma\right| \\
&\quad\le \frac{1}{\alpha^2}\left(|\alpha|k_1(x+x_*,t)+|\partial_x k_1(x+x_*,t)|\right). \qquad (4.3)
\end{aligned}$$

This and (2.18) give the estimate fo the Green's function without the exponentially growing factor $e^{\alpha^2(t-\sigma)}$:

$$\begin{aligned}
|G(\boldsymbol{x},\boldsymbol{x}_*,t)| \le\ & k_1(y-y_*,t)k_1(z-z_*,t)\Big(k_1(x-x_*,t)+6k_1(x+x_*,t) \\
&+\frac{6}{\alpha}|\partial_x k_1(x+x_*,t)|\Big). \qquad (4.4)
\end{aligned}$$

With the estimate (4.4) of the Green's function, one can apply similar arguments as in the proof of Theorem 1.1 to establish Theorem 1.3. Details are omitted.

The algebraic manipulations (4.1) and (4.2) to achieve the cancellation (4.3), (4.4) do not seem to be obvious. In fact, the present study have provided impetus for further development of the LY approach. In [3] the cancellation is seen as a natural consequence of the algebraic manipulations in Fourier-Laplace variables. We now adopt the procedure in [3] for the heat equation by starting from the equation (2.15):

$$\begin{aligned}
V &= \frac{1}{s+|\boldsymbol{\eta}|^2-\alpha^2}(\alpha+\lambda)^2\frac{e^{-\lambda(x+x_*)}}{2\lambda} = (\partial_x-\alpha)^2\Big[\frac{1}{s+|\boldsymbol{\eta}|^2-\alpha^2}\frac{e^{-\lambda(x+x_*)}}{2\lambda}\Big] \\
&= (\partial_x-\alpha)^2\Big[\frac{1}{s+|\boldsymbol{\eta}|^2-\alpha^2}\mathscr{F}[\mathbb{L}[k]](x+x_*,\boldsymbol{\eta},s)\Big],
\end{aligned}$$

where we have used (2.11) in the last step. It follows that the inverse transform is of the form

$$\begin{cases} u = [\mathscr{F}_{\boldsymbol{y}}]^{-1}[\mathbb{L}_t]^{-1}V = (\partial_x - \alpha)^2 w, \ \mathscr{F}[\mathbb{L}[w]] = W, \\ (s + |\boldsymbol{\eta}|^2 - \alpha^2)W = \mathscr{F}[\mathbb{L}[k]]. \end{cases} \tag{4.5}$$

The second identity above can easily be inverted to yield

$$(\partial_t - \Delta_{y,z} - \alpha^2)w = k(x + x_*, y, z, t).$$

Because of the presence of the heat kernel $k(x + x_*, y, z, t)$, we also have

$$(\partial_t - \Delta_{x,y,z})w = 0.$$

The last two identities yield ODE

$$(\partial_x^2 - \alpha^2)w = k(x + x_*, y, z, t). \tag{4.6}$$

We thus rewrite (4.5) as

$$u = (\partial_x - \alpha)h, \ (\partial_x + \alpha)h = k(x + x_*, y, z, t). \tag{4.7}$$

Note that in the ODE (4.6) the sign of α does not matter, while the reformulation (4.7) makes use of (4.5) and the sign of α becomes important for the function h. In order to yield solution vanishing at $x = \infty$ we now have two cases. For the unstable case $\alpha > 0$ we solve the ODE (4.7) from $x = 0$:

$$h(x, y, z, t) = e^{-\alpha x} h(0, y, z, t) + \int_0^x e^{-\alpha(x-\bar{x})} k(\bar{x} + x_*, y, z, t) d\bar{x}. \tag{4.8}$$

For the stable case $\alpha < 0$ we solve the ODE (4.7) from $x = \infty$:

$$h(x, y, z, t) = -\int_x^\infty e^{-\alpha(x-\bar{x})} k(\bar{x} + x_*, y, z, t) d\bar{x}. \tag{4.9}$$

For the stable case, we have from (4.7) and (4.9) the solution representation with no growing mode

$$u(x, y, z, t) = k(x + x_*, y, z, t) + 2\alpha \int_x^\infty e^{-\alpha(x-\bar{x})} k(\bar{x} + x_*, y, z, t) d\bar{x}. \tag{4.10}$$

When comparing this with (2.17), it is to equate two expressions, the first with growing mode $e^{\alpha^2(t-\sigma)} = e^{\alpha^2|t-\sigma|}$ and the second with non-growing mode $e^{-\alpha(x-\bar{x})}=e^{-|\alpha(x-\bar{x})|}$:

$$l=m,\ l\equiv(\alpha-\partial_x)\int_0^t e^{\alpha^2(t-\sigma)}k_1(x+x_*,\sigma)]d\sigma,\ m\equiv\int_x^\infty e^{-\alpha(x-\bar{x})}k_1(\bar{x}+x_*,t)d\bar{x}.$$

This follows directly from the observation that both $n=l,m$ have zero initial value for $x,x_*>0$ and satisfy the same equation

$$n_t=-k_x+\alpha k+\alpha^2 n.$$

Note in the above algebraic manipulations that the bi-directional operator $\partial_x^2-\alpha^2$ in (4.6) is strategically separated in (4.7) so that the flow of information from the boundary for the unstable case, (4.8), and from the interior for the stable case, (4.9), can be effectively registered. By adopting the approach of [3] we obtain the alternate expression of the Green's function from (4.10):

$$\begin{aligned} G(\boldsymbol{x},\boldsymbol{x}_*,t) &= k_1(y-y_*,t)k_1(z-z_*,t)\Big(k_1(x-x_*,t)+k_1(x+x_*,t) \\ &\quad +2\alpha\int_x^\infty e^{-\alpha(x-\bar{x})}k_1(\bar{x}+x_*,t)d\bar{x}\Big),\ \alpha<0. \end{aligned} \tag{4.11}$$

We can also use this expression of the Green's function for the proof of Theorem 1.3.

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