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SOME RESULTS ON HARMONIC MAPS

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Abstract

In this note, we will outline the classical results of Eells-Sampson [7] on the harmonic heat flow, Sacks-Uhlenbeck [26] in homotopy classes and Schoen-Uhlenbeck [28] on the partial regularity of minimizing harmonic maps. This note also contains a new proof of the Sacks-Uhlenbeck result [26] by using an estimate of Trudinger [34], which improved the result of Moser in [23].

1. Introduction

The theory of harmonic maps provides a prototype for many complex physical theories including the σ-model, superconductivity, and string theory. The theory of harmonic maps has many important applications to geometry and topology. Motivated by the seminal work of Eells and Sampson [7] on the harmonic map flow, Donaldson [6] established the important Donaldson-Uhlenbeck-Yau theorem by using the Yang-Mills flow, and Hamilton in [14] established many pioneering results on the Ricci flow in order to settle the Poincare conjecture.

One of the important tasks on harmonic maps is to deal with the very challenging **Eells-Sampson question** (e.g. [8]). More precisely, let u_0 be a given smooth map from M to N. **Can u_0 be deformed to a harmonic map in its homotopy class?**

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The Eells-Sampson question is a question of establishing existence of a smooth harmonic map representative in a fixed homotopy class of maps between two manifolds. Main purpose of this note is to discuss three classical results related to this question. We will discuss this question for the case that the target manifolds N have non-positive sectional curvature and some results of minimizing the Dirichlet energy. More precisely, in Section 3, we will outline some key proofs of the classical results of Eells-Sampson [7] on the harmonic heat flow. In Section 4, we will discuss the result of Sacks-Uhlenbeck [26] in homotopy classes in $2D$. In particular, we will present some new proofs on the Sacks-Uhlenbeck result [26] by using an estimate of Trudinger [34], which improved the Moser-Harnack estimate [23]. In Section 5, we will outline some key proofs of Schoen-Uhlenbeck [28] on the partial regularity of minimizing harmonic maps (see also Giaquinta-Giusti [10]). This note was lectured by the author in the Winter School on Geometric Partial Differential Equations at Brisbane, Australia from 2-13 July 2012.

Finally, I would like to dedicate this paper to Professor Neil Trudinger on the occasion of his 70th birthday.

2. Harmonic Maps between Manifolds

Let M be a n-dimensional Riemannian manifold (with or without boundary) with a smooth Riemannian metric g. In a local coordinates around fixed point $p \in M$, g can be represented by

$$g = g_{ij} dx_i \otimes dx_j,$$

where (g_{ij}) is a positive definite symmetric $n \times n$ matrix. Let $(g^{ij}) = (g_{ij})^{-1}$ be the inverse matrix of (g_{ij}) and the volume element of $(M; g)$ is

$$dv_g = \sqrt{|g|} dx,$$

where $|g| = \det\,(g_{ij})$. Let $(N; h)$ be another l-dimensional compact Riemannian manifold without boundary (isometrically embedded into $\mathbb{R}^k$), with a smooth Riemannian metric h.

For a map $u : M \to N$, its Dirichlet energy functional is defined by

$$E(u) = \int_M e(u)\, dv_g,$$

where the density function $e(u)$ is given by

$$e(u)(x) = \frac{1}{2}|\nabla u(x)|^2 = \frac{1}{2}\sum_{\alpha,\beta,i,j} g^{ij}(x)h_{\alpha\beta}(u(x))\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}.$$

A smooth map u from M to N is said to be a harmonic map ([8]) if u is a critical point of the Dirichlet energy functional E; i.e. it satisfies

$$\triangle_M u + A(u)(\nabla u, \nabla u) = 0$$

in M, where $\triangle_M$ is the Laplacian operator with respect to the Riemannian metric of M and A is the second fundamental form of N.

Next, we will give details to get the harmonic map equations.

We recall that a Riemannian manifold M is a smooth manifold which is equipped with a Riemannian metric g; i.e. for each tangent space T_xM, there is an inner product $\langle \cdot, \cdot \rangle$. In local coordinates,

$$g_{ij} := \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle.$$

For $X, Y, Z \in C^\infty(TM)$, the connection ∇ satisfies

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

The connection, which satisfies the above identity, is called Riemannian. In local coordinates, the Christoffel symbols are defined by

$$\nabla_{\frac{\partial}{\partial x^i}}\left(\frac{\partial}{\partial x^j}\right) = \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

More precisely, the Christoffel symbols Γ_{ij}^k can be expressed by

$$\Gamma_{ij}^k := \frac{1}{2}g^{kl}\left(\frac{\partial g_{lj}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

We recall that the curvature tensor of Levi-Civita connection R is given by

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in C^\infty(TM)$. In local coordinates,

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^l} = R^k_{lij}\frac{\partial}{\partial x^k}$$

We set

$$R_{klij} := g_{km}R^m_{lij} = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right\rangle$$

In local coordinates, we have

$$R^k_{lij} = \left(\frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{im}\Gamma^m_{jl} - \Gamma^k_{jm}\Gamma^m_{il}\right). \tag{1}$$

Let N be another compact Riemannian manifold with a metric h.

Let $u = (u^1, \ldots, u^l)$ be a C^1-map from M to N. Intrinsically, the differential du of u is given (see [8] or [18]) by

$$du = \frac{\partial u^\alpha}{\partial x^i}dx^i \otimes \frac{\partial}{\partial u^\alpha}$$

which can be considered as a section of the bundle $T^*M \otimes u^{-1}(TN)$. Then we define the energy density

$$e(u) = \frac{1}{2}\langle du, du\rangle_{T^*M\otimes u^{-1}(TN)} = \frac{1}{2}g^{ij}h_{\alpha\beta}(u)\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}.$$

We define the energy of u as

$$E(u) := \int_M e(u)\, dv_g.$$

Assume that u is a critical point of E. Then for all admissible variation $\varphi \in C^\infty_0(M)$

$$\frac{d}{dt}E(u + t\varphi)\bigg|_{t=0} = 0.$$

It implies that

$$0 = \int_M \left(g^{ij}h_{\alpha\beta}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial \varphi^\beta}{\partial x^j} + \frac{1}{2}g^{ij}h_{\alpha\beta,u^\sigma}\varphi^\sigma\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}\right)\sqrt{|g|}dx$$

$$
\begin{aligned}
&= -\int_M \frac{\partial}{\partial x^j}\left(\sqrt{|g|}g^{ij}\frac{\partial u^\alpha}{\partial x^i}\right) h_{\alpha\beta}\varphi^\beta\, dx - \int_M g^{ij}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\sigma}{\partial x^j} h_{\alpha\beta,u^\sigma}\varphi^\beta\sqrt{|g|}dx \\
&\quad + \int_M \frac{1}{2} g^{ij} h_{\alpha\beta,u^\sigma}\varphi^\sigma \frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}\sqrt{|g|}dx.
\end{aligned}
$$

Put $\eta^\alpha = h_{\alpha\beta}\varphi^\beta$; i.e. $\varphi^\beta = h^{\gamma\beta}\eta^\gamma$. Then

$$
\begin{aligned}
0 &= -\int_M \frac{\partial}{\partial x_j}\left(\sqrt{|g|}g^{ij}\frac{\partial u^\gamma}{\partial x^i}\right)\eta^\gamma\, dx \\
&\quad -\frac{1}{2}\int_M g^{ij}h^{\gamma\sigma}(h_{\alpha\sigma,u^\beta} + h_{\sigma\beta,u^\alpha} - h_{\alpha\beta,u^\sigma})\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}\eta^\gamma\sqrt{|g|}\, dx
\end{aligned}
$$

which implies

$$
\triangle_M u = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial u}{\partial x_i}\right) = -A(u)(\nabla u, \nabla u),
$$

where $A(u) = (A^1, \ldots, A^l)$ is given by

$$
A(u)^\gamma(\nabla u, \nabla u) = g^{ij}\Gamma^\gamma_{\alpha\beta}\frac{\partial u^\alpha}{\partial x_i}\frac{\partial u^\beta}{\partial x_j}.
$$

Let ψ be a vector field along u, i.e. a section of $u^{-1}(TN)$. In local coordinates

$$
\psi = \psi^\alpha(x)\frac{\partial}{\partial u^\alpha}
$$

and

$$
\begin{aligned}
d\psi &= \nabla_{\frac{\partial}{\partial x^i}}\left(\psi^\alpha(x)\frac{\partial}{\partial u^\alpha}\right) dx^i \\
&= \frac{\partial \psi^\alpha}{\partial x^i}\frac{\partial}{\partial u^\alpha}\otimes dx^i + \psi^\alpha\Gamma^\gamma_{\alpha\beta}\frac{\partial u^\beta}{\partial x^i}\frac{\partial}{\partial u^\gamma}\otimes dx^i
\end{aligned}
$$

which is a section of $T^*M\otimes u^{-1}(TN)$. Then ψ induces a variation of u by

$$
u_t(x) = \exp_{u(x)}(t\psi(x)).
$$

We compute

$$
0 = \frac{d}{dt}E(u_t)\bigg|_{t=0} = \int_M \langle du, d\psi\rangle \tag{2}
$$

$$\begin{aligned}
&= \int_M \left\langle du, \nabla_{\frac{\partial}{\partial x^i}} \left(\psi^\alpha(x) \frac{\partial}{\partial u^\alpha} \right) dx^i \right\rangle \\
&= - \int_M \left\langle \nabla_{\frac{\partial}{\partial x^i}} du, \psi^\alpha(x) \frac{\partial}{\partial u^\alpha} dx^i \right\rangle \\
&= - \int_M \langle \mathrm{trace} \nabla du, \psi \rangle
\end{aligned}$$

for all ψ, where ∇ is the covariant derivatives in $T^*M \otimes u^{-1}(TN)$. Note

$$\nabla_{\frac{\partial}{\partial x^j}} dx^i = -{}^M\Gamma^i_{kj} dx^k, \quad \nabla_{\frac{\partial}{\partial u^\beta}} (\frac{\partial}{\partial u^\alpha}) = {}^N\Gamma^\sigma_{\alpha\beta} \frac{\partial}{\partial u^\sigma}.$$

Then we write $\nabla du = \nabla_{\frac{\partial}{\partial x^j}} (du) dx^j$ with

$$\begin{aligned}
\nabla_{\frac{\partial}{\partial x^j}} (du) &= \nabla_{\frac{\partial}{\partial x^j}} (\frac{\partial u^\alpha}{\partial x^i} dx^i \otimes \frac{\partial}{\partial u^\alpha}) \\
&= \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j} dx^i \otimes \frac{\partial}{\partial u^\alpha} - {}^M\Gamma^i_{lj} \frac{\partial u^\alpha}{\partial x^i} dx^l \otimes \frac{\partial}{\partial u^\alpha} + {}^N\Gamma^\sigma_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} dx^i \otimes \frac{\partial}{\partial u^\sigma}.
\end{aligned}$$

From the above, we have

Lemma 2.1. *The harmonic map equation is*

$$\tau(u) := \mathit{trace} \nabla du = 0,$$

where $\tau(u) = \tau^\sigma(u) \frac{\partial}{\partial u^\sigma}$ *satisfies*

$$\tau^\sigma(u) = g^{ij} \frac{\partial^2 u^\sigma}{\partial x^i \partial x^j} - g^{ij}\ {}^M\Gamma^k_{ij} \frac{\partial u^\sigma}{\partial x^k} + g^{ij}\ {}^N\Gamma^\sigma_{\alpha\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}.$$

Another way to derive the harmonic map equation:

Let $N \subset \mathbb{R}^K$ be an embedded compact manifold in $\mathbb{R}^K$. Then there is a $\delta = \delta(N) > 0$ such that the nearest point project map $\Pi_N : N_\delta \to N$ is smooth, where

$$N_\delta := \left\{ y \in \mathbb{R}^k : d(y, N) = \inf_{x \in N} |y - x| < \delta \right\},$$

and $\Pi_N(y) \in N$ is the projection such that $|y - \Pi_N(y)| = d(y, N)$ for $y \in N_\delta$. The second result is:

Remark 2.2. A smooth map u from M to N is harmonic if and only if it satisfies

$$\triangle_M u \perp T_u N.$$

Proof. Note that $d\Pi_u : \mathbb{R}^k \to T_u N$ is a tangential projection map for any $u \in N$. For any smooth $\phi \in C_0^\infty(M, \mathbb{R}^k)$, set

$$u_t = \Pi(u + t\phi).$$

If u is a critical point of E, then we have

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0} E(u_t) &= \int_M \left\langle \nabla u, \nabla(d\Pi_{u(x)}(\phi(x)))\right\rangle \, dv_g \\
&= -\int_M \left\langle \triangle_M u, d\Pi_{u(x)}(\phi(x))\right\rangle \, dv_g \\
&= -\int_M \left\langle (d\Pi_{u(x)})(\triangle_M u), \phi\right\rangle \, dv_g = 0.
\end{aligned}$$

In fact, one can show that

$$(d\Pi_{u(x)})(\triangle_M u) = \triangle_M u - d^2\Pi_{u(x)}(\nabla u, \nabla u) = \triangle_M u + A(u)(\nabla u, \nabla u). \quad \square$$

3. The Heat Flow Approach

In their pioneering paper [7], Eells and Sampson introduced the harmonic map flow to establish existence of harmonic maps for the case that the sectional curvature of the target manifold is non-positive.

In this section, we consider the following evolution problem:

$$\partial_t u = \triangle_M u + A(u)(\nabla u, \nabla u) \tag{3}$$

with $u(x, 0) = u_0$. We call (3) the heat flow for harmonic maps.

The global estimate is:

Lemma 3.1. *If $u(x, t)$ is a solution to the harmonic map flow (3) in $M \times$*

$[0,T)$ for some T with $0<T\le\infty$, we then have

$$E(u(\cdot,t))+\int_0^t\int_M|\partial_t u|^2\,dvdt=E(u_0)$$

for any $t\in[0,T)$.

Proof. Taking $\psi=\frac{\partial u}{\partial t}$ in (2), we have

$$\begin{aligned}\frac{d}{dt}E(u(\cdot,t))&=\int_M\langle\nabla_{\partial_t}du,du\rangle=\int_M\left\langle d\frac{\partial u}{\partial t},du\right\rangle\\&=-\int_M\left\langle\tau(u),\frac{\partial u}{\partial t}\right\rangle=-\int_M\left|\frac{\partial u}{\partial t}\right|^2.\end{aligned}$$

The result follows from integrating by parts. □

Let R^M and R^N be the Riemannian curvature tensors of M and N respectively.

Let Ric^M denote the Ricci curvature of M and K^N be the sectional curvature of N. Then

Lemma 3.2. *Let $u(x,t)$ be a solution to the harmonic map flow in $M\times[0,T]$. Then we have*

$$\begin{aligned}(\partial_t-\triangle_M)e(u)=&-|\nabla^2u|^2+\left\langle du\cdot Ric^M(e_i),du\cdot e_i\right\rangle\\&-\left\langle R^N(du\cdot e_i,du\cdot e_j)du\cdot e_j,du\cdot e_i\right\rangle,\end{aligned}$$

where $\{e_i\}$ is an orthonormal frame at x If $K^N\le 0$, then

$$(\partial_t-\triangle_M)e(u)\le Ce(u).$$

Proof. This original approach is due to Eells-Sampson [7]. Our proof is essentially due to Jost [18].

We introduce normal coordinates at the points x and $u(x)$ such that $g_{ij}(x)=\delta_{ij}$ and $h_{\alpha\beta}(u(x))=\delta_{\alpha\beta}$ and all first derivatives are zero, so the Christoffel symbols vanish at x and $u(x)$. Since u is a solution of the harmonic map flow,

$$\frac{\partial u^\sigma}{\partial t}=g^{ij}\frac{\partial^2u^\sigma}{\partial x^i\partial x^j}-g^{ij}\ {}^M\Gamma_{ij}^k\frac{\partial u^\sigma}{\partial x^k}+g^{ij}\ {}^N\Gamma_{\alpha\beta}^\sigma\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}.$$

Differentiating this equation at the point x along the direction of x^l, we have

$$\begin{aligned}\frac{\partial^3 u^\sigma}{\partial x^i \partial x^i \partial x^l} =& \frac{\partial u^\sigma_{x^l}}{\partial t} + \frac{1}{2}\left(g_{ik;x^i x^l} + g_{ik;x^i x^l} - g_{ii;x^k x^l}\right)\frac{\partial u^\sigma}{\partial x^k}\\ &- \frac{1}{2}\left(h_{\alpha\sigma;u^\beta u^\gamma} + h_{\sigma\beta;u^\alpha u^\gamma} - h_{\alpha\beta;u^\sigma u^\gamma}\right)\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^l}.\end{aligned}$$

In the coordinates, we have at x

$$g^{ij}_{;x^k x^k} = -g_{ij;x^k x^k}$$

and by the chain rule

$$\triangle_M h_{\alpha\beta}(u(x)) = h_{\alpha\beta;u^\sigma u^\gamma} u^\sigma_{x^k} u^\gamma_{x^k}.$$

Combining the above estimate, we have at x

$$\begin{aligned}&(\triangle_M - \partial_t)\left(\frac{1}{2} g^{ij} h_{\alpha\beta} u^\alpha_{x^i} u^\beta_{x^j}\right)\\ &= u^\alpha_{x^i x^k} u^\alpha_{x^i x^k} + u^\alpha_{x^i}(u^\alpha_{x^i x^k x^k} - \partial_t u^\alpha_{x^i}) + \frac{1}{2}[g^{ij}_{;x^k x^k} u^\alpha_{x^i} u^\alpha_{x^j} + \triangle_M h_{\alpha\beta} u^\alpha_{x^i} u^\beta_{x^i}]\\ &= |\nabla du|^2 - \frac{1}{2}\left(g_{ij;x^k x^k} + g_{kk;x^i x^j} - g_{kj;x^k x^i} - g_{kj;x^k x^i}\right) u^\alpha_{x^i} u^\alpha_{x^j}\\ &\quad + \frac{1}{2}\left(h_{\alpha\beta;u^\sigma u^\gamma} + h_{\sigma\gamma;u^\alpha u^\beta} - h_{\alpha\sigma;u^\beta u^\gamma} - h_{\beta\sigma;u^\alpha u^\gamma}\right) u^\alpha_{x^i} u^\beta_{x^i} u^\sigma_{x^k} u^\gamma_{x^k}\\ &= |\nabla du|^2 + \frac{1}{2} R^M_{ij} u^\alpha_{x^i} u^\alpha_{x^j} - \frac{1}{2} R^N_{\alpha\sigma\beta\gamma} u^\alpha_{x^i} u^\beta_{x^i} u^\sigma_{x^k} u^\gamma_{x^k},\end{aligned}$$

where at x we noted $R^M_{ij} = g^{kl} R^M_{ikjl} = R^M_{ikjk}$ and

$$\begin{aligned}R^M_{klij} &= \frac{1}{2}\left(g_{jk;x^l x^i} + g_{lk;x^i x^j} - g_{jl;x^k x^i} - g_{ik;x^l x^j} - g_{lk;x^i x^j} + g_{il;x^k x^j}\right)\\ &= \frac{1}{2}\left(g_{jk;x^l x^i} + g_{il;x^k x^j} - g_{jl;x^k x^i} - g_{ik;x^l x^j}\right).\end{aligned}$$

Since $e_i = \frac{\partial}{\partial x_i}$ is an orthonormal frame at x, we have

$$\begin{aligned}\triangle_M e(u) - \frac{\partial}{\partial t} e(u) =& |\nabla du|^2 + \frac{1}{2}\left\langle du \cdot Ric^M(e_i), du \cdot e_i\right\rangle\\ &- \frac{1}{2}\left\langle R^N(du \cdot e_i, du \cdot e_j) du \cdot e_j, du \cdot e_i\right\rangle\end{aligned}$$

If $K_N \leq 0$, we have

$$\triangle e(u) - \frac{\partial}{\partial t} e(u) \geq -Ce(u).$$

This proves our claim. $\square$

The following is the well-know Moser-Harnack estimate:

Lemma 3.3. *Let $f \in C^\infty(B_R(x_0) \times [t_0 - R^2, t_0]$ be a nonnegative function satisfying*

$$(\partial_t - \triangle_M) f \leq Cf$$

for a constant $C > 0$. Then there is a constant C such that

$$f(x_0, t_0) \leq CR^{n+2} \int_{t_0 - R^2}^{t_0} \int_{B_R(x_0)} f \, dv_g \, dt.$$

The following theorem is due to Eells-Sampson [7]:

Theorem 3.4. *Let M and N be two compact Riemannian manifolds without boundary. Assume that the sectional curvature K^N is non-positive. Let $u_0 \in C^\infty(M, N)$ be a given map. Then there is a global smooth solution $u \in C^\infty(M \times [0, \infty))$ such that the harmonic map flow with initial value u_0 has a global smooth solution. As $t \to \infty$ suitably, $u(\cdot, t)$ converges smoothly to a harmonic map u_∞.*

Proof. By the local existence, there is a unique smooth solution in $M \times [0, T]$.

Using Lemmas 3.2$-$3.3, there is a constant C such that $|\nabla u|$ is uniformly bounded in $M \times [0, \infty)$. By the L^p-estimates, we can show there is a constant $C = C(p, M, N)$ such that

$$\|u\|_{W^{2,p}(B_R \times (T - R^2, T))} \leq C(p, M, N)$$

for some $R > 0$. By the bootstrap method, u is smooth in $M \times [0, \infty)$. By the energy inequality, we know

$$\int_0^\infty \int_M |\partial_t u|^2 \leq E(u_0) < +\infty.$$

Using the harmonic map heat flow, there is a sequence $t_k \to \infty$ such that $u_t(\cdot, t_k) \to 0$ and $u(\cdot, t_k) \to u_\infty$ smoothly satisfying

$$\triangle_M u_\infty + A(u_\infty)(\nabla u_\infty, \nabla u_\infty) = 0. \qquad \square$$

In fact, u_∞ is unique due to Hartman [15].

Lemma 3.5. *Let $u(x,t,s)$ be a smooth family of solutions of the harmonic map flow with initial values $u(x,0,s) = g(x,s)$ for $0 \le s \le s_0$. Assume again that N has non-positive sectional curvature. For every $s \in [0, s_0]$*

$$\sup_{s\in[0,1]} \sup_{x\in M} \left(h_{\alpha\beta} \frac{\partial u^\alpha}{\partial s} \frac{\partial u^\beta}{\partial s} \right)$$

is non-increasing in t.

Proof. Using normal coordinates we can obtain

$$(\triangle - \frac{\partial}{\partial t}) \left(h_{\alpha\beta} \frac{\partial u^\alpha}{\partial s} \frac{\partial u^\beta}{\partial s} \right) = h_{\alpha\beta} \frac{\partial^2 u^\alpha}{\partial x^k \partial s} \frac{\partial^2 u^\beta}{\partial x^k \partial s} - \frac{1}{2} R^N_{\alpha\beta\sigma\gamma} u^\alpha_s u^\beta_{x^k} u^\sigma_s u^\gamma_{x^k}.$$

Since $K^N \le 0$,

$$(\triangle - \frac{\partial}{\partial t}) \left(h_{\alpha\beta} \frac{\partial u^\alpha}{\partial s} \frac{\partial u^\beta}{\partial s} \right) \ge 0.$$

Then the result follows from the maximum principle for the parabolic equations. $\square$

Assume that u_1 and u_2 are smooth homotopic maps from M to N and $f : M \times [0,1] \to N$ is a smooth homotopy with $f(x,0) = u_1(x)$ and $f(x,1) = u_2(x)$. Then the curve $f(x,\cdot)$ is connecting $u_1(x)$ and $u_2(x)$. Let $g(x,\cdot)$ be the geodesic from $u_1(x)$ and $u_2(x)$, parameterized by the arc length. We define $\tilde{d}(u_1(x), u_2(x))$ to be the arc length of the geodesic arc. Then

Lemma 3.6. *Assume again that N has non-positive sectional curvature. Let $u(x,t,s)$ be a smooth family of solutions of the harmonic map flow with initial values $u(x,0,s) = g(x,s)$ for $0 \le s \le 1$. Then*

$$\sup_{x\in M} \tilde{d}\,(u(x,t,0), u(x,t,1))$$

is non-increasing in $t \in [0, T]$.

Proof. By the construction, at $t = 0$ we have

$$\sup_{x \in M} \left(h_{\alpha\beta} \frac{\partial u^\alpha}{\partial s} \frac{\partial u^\beta}{\partial s} \right) = \sup_{x \in M} |\frac{\partial g}{\partial s}|^2 = \sup_{x \in M} \tilde{d}^2 \left(u(x, 0, 0), u(x, 0, 1) \right)$$

For each $t \in [0, T]$,

$$\tilde{d}^2 \left(u(x, t, 0), u(x, t, 1) \right) \leq \sup_{s \in [0,1]} \sup_{x \in M} \left(h_{\alpha\beta} \frac{\partial u^\alpha}{\partial s} \frac{\partial u^\beta}{\partial s} \right)$$

since $u(x, t, \cdot)$ is a curve joining $u(x, t, 0)$ and $u(x, t, 1)$ in the homotopy class. The claim follow from Lemma 3.5. □

Let u_∞ be the limit of $u(x, t_k)$ as $t_k \to \infty$ and $\tilde{u}_\infty$ be the limit of $u(x, \tilde{t}_k)$ as $\tilde{t}_k \to \infty$. By the above Lemmas,

$$\tilde{d}(u(x, t_k + t), u_\infty) \leq \tilde{d}(u(x, t_k), u_\infty).$$

By choosing a subsequence $\tilde{t}_k$, we show that $u_\infty = \tilde{u}_\infty$.

4. The Sack-Uhlenbeck Functional and Applications

In the two dimensional case, Lemaire [19] and Schoen-Yau [29] established many existence results in each homotopy class under certain topological conditions.

In a well-known paper [26], Sacks and Uhlenbeck established many existence results of minimizing harmonic maps in their homotopy classes by introducing a family of functionals

$$E_\alpha(u) = \int_M (1 + |\nabla u|^2)^\alpha dv$$

for $\alpha > 1$. The α-functional E_α is now called the 'Sacks-Uhlenbeck functional'. For each $\alpha > 1$, there is a minimizer u_α of E_α in the same homotopy class.

Lemma 4.1. *Let $u_0 \in C^\infty(M,N)$ be a given map. For each $\alpha > 1$, there is a minimizer u_α of E_α in the homotopy class $[u_0]$; i.e.*

$$E_\alpha(u_\alpha) = \inf\left\{E_\alpha(v) \,:\, v \in W^{1,2\alpha}(M,N), \quad [v] = [u_0]\right\}.$$

Moreover, u_α satisfies

$$\triangle_M u + (\alpha-1)\frac{\nabla\left|\nabla u\right|^2 \cdot \nabla u}{1+\left|\nabla u\right|^2} + A(u)(\nabla u, \nabla u) = 0. \tag{4}$$

Proof. Set

$$m_\alpha = \inf\left\{E_\alpha(v) \,:\, v \in W^{1,2\alpha}(M,N), \quad [v] = [u_0]\right\}.$$

Then $m_\alpha \le E_\alpha(u_0) \le C$ for a uniform constant $C > 0$ in α. There is a minimizing sequence in $[u_0]$ such that

$$\int_M |\nabla u_i|^{2\alpha} \le 1 + m_\alpha \quad \text{for all } i.$$

By the lower-semi continuity of E_α, we have

$$E_\alpha(u_\alpha) \le \liminf_{i\to\infty} E_\alpha(u_i) = m_\alpha$$

Note that u_i converges to u_α in $W^{1,2\alpha}$ weakly and in $C^\beta(M,N)$ with $\beta = 1-\frac{1}{\alpha}$ by the Sobolev inequality, so $[u_\alpha] = [u_0]$. Therefore, $u_i \to u_\alpha$ in $W^{1,2\alpha}$ strongly. It is easy to check that u_α satisfies (4). $\square$

The following theorem is due to Sacks-Uhlenbeck [26]:

Theorem 4.2. *Let u_α be critical points of E_α and $E_\alpha \le B$ for some constant $B > 0$. As $\alpha \to 1$, u_α weakly sub-converges to a map u in $W^{1,2}(M,N)$. Then there is a finite numbers of points $\{x_1, \ldots, x_L\} \subset M$ such that u_α converges to u in $C^\infty(M\backslash\{x_1,\ldots,x_L\}, N)$. Moreover, u can be extended to a smooth map in M.*

Moreover, a bubbling phenomenon occurs by studying the limits of the critical points of E_α as $\alpha \to 1$ (see section 4 in [26]).

One of key steps is to derive a Bochner type formula. Let (g_{ij}) be a Riemannian metric on M. Then

Lemma 4.3. *(Bochner's type formula) Let $u(x)$ be a smooth solution to the α-equation (4) and set $e(u) := |\nabla u|^2$. Then, for $\alpha - 1$ sufficiently small, we have*

$$\left(g^{ij} + \frac{(\alpha-1)}{1+|\nabla u|^2} g^{ik}\frac{\partial u^\beta}{\partial x_k} g^{jl}\frac{\partial u^\beta}{\partial x_l}\right)\frac{\partial^2 e(u)}{\partial x_i \partial x_j} \geq -Ce(u)(e(u)+1), \tag{5}$$

where the constant C does not depend on α and u.

Proof. In a neighborhood of each point $x \in M$, we can choose an orthonormal frame $\{e_1, e_2\}$. We denote by ∇_i the first covariant derivative with respect to e_i and by u_{ji} the second covariant derivatives of u and so on. In a local frame, we have

$$\nabla_j e(u) = 2u_k^\gamma u_{kj}^\gamma, \quad |\nabla^2 u|^2 = \sum_{k,i,\gamma} \left|u_{ki}^\gamma\right|^2.$$

The Ricci identity is

$$u_{iki} = u_{iik} + R_{ik}u_i,$$

where R_{ik} is the Ricci curvature. Then we have

$$\begin{aligned}
&\nabla_i\left((\delta_{ij} + 2(\alpha-1)\frac{u_i^\beta u_j^\beta}{1+|\nabla u|^2})\nabla_j e(u)\right)\\
&= 2\nabla_i\left(u_k^\gamma u_{ki}^\gamma + 2(\alpha-1)\frac{u_i^\beta u_j^\beta u_k^\gamma u_{kj}^\gamma}{1+|\nabla u|^2}\right)\\
&= 2|\nabla^2 u|^2 + 2u_k^\gamma u_{iik}^\gamma + 4(\alpha-1)\nabla_k\left(\frac{u_k^\gamma u_j^\gamma u_i^\beta u_{ij}^\beta}{1+|\nabla u|^2}\right)\\
&\geq |\nabla^2 u|^2 + 2u_k^\gamma \nabla_k\left(u_{ii}^\gamma + 2(\alpha-1)\frac{u_j^\gamma u_i^\beta u_{ij}^\beta}{1+|\nabla u|^2}\right) - Ce(u)
\end{aligned}$$

for $\alpha-1$ sufficiently small, where we used the Ricci identity twice for switching third order derivatives. Using the α-equation (4) and the Young's inequality, we have

$$-\left(\delta_{ij} + 2(\alpha-1)\frac{u_i^\beta u_j^\beta}{1+|\nabla u|^2}\right)\nabla_{ij}^2 e(u)$$

$$\begin{aligned}&\le -\frac{1}{2}|\nabla^2 u|^2 - 2u_k^\gamma \nabla_k \left(A^\gamma(u)(\nabla u, \nabla u)\right) + Ce(u)\\&\le Ce(u)(e(u)+1)\end{aligned}$$

for $\alpha - 1$ sufficiently small. This proves our claim. □

The following local Harnack inequality is taken from Theorem 9.20 of Gilberg-Trudinger's book [13].

Lemma 4.4. *Let $v(x) \in W^{2,n}(\Omega)$ and let*

$$a_{ij} D_{ij} v + Cv \ge 0,$$

where a_{ij} are measurable functions in $\Omega \subset \mathbb{R}^n$ satisfying

$$\lambda|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

for any two positive constants λ and Λ. Then for any $p > 0$ and $R > 0$ with $B_R(x) \subset \Omega$, we have

$$|v(x)| \le C \left(\frac{1}{R^n}\int_{B_R(x)} (v^+)^p\right)^{1/p}.$$

Remark. Moser [23] proved the inequality for $p > 1$ and Trudinger [34] proved the inequality for all $p > 0$. In fact, we need the case for $p = 1$.

The following ε-regularity estimate is essentially due to Schoen in [27]:

Lemma 4.5. *Let $u(x)$ be a solution of the α-equation (4). There is a small constant $\varepsilon_0 > 0$ such that if*

$$\int_{B_R} |\nabla u(x)|^2\, dx \le \varepsilon_0$$

for a ball B_R with some $R > 0$, then

$$|\nabla u(x)|^2 \le \frac{C}{R^2}\int_{B_R} |\nabla u|^2\, dv_g \quad \forall x \in B_{R/2},$$

where the constant C depends not on x and α.

Proof. We choose $\sigma_0 \in [0, R]$ such that

$$(R-\sigma_0)^2 \sup_{B_{\sigma_0}} e(u) = \max_{\sigma\in[0,R]} \left\{ (R-\sigma)^2 \sup_{B_\sigma} e(u) \right\}.$$

Let x_0 be the point in $\bar{B}_{\sigma_0}$ such that

$$e_0 =: e(u)(x_0) = \sup_{B_{\sigma_0}} e(u).$$

Set $\rho_0 = \frac{1}{2}(R-\sigma_0)$, which implies $R-(\sigma_0+\rho_0) = \rho_0$. Then

$$\sup_{B_{\rho_0}(x_0)} e(u) \le \sup_{B_{\sigma_0+\rho_0}} e(u) \le 4e_0.$$

We claim

$$r_0 = (e_0)^{1/2}\rho_0 \le 1.$$

Otherwise, we may assume that $r_0 > 1$; i.e. $e_0(R-\sigma_0)^2 > 4$. We define a new map $v \in C^2(B_{r_0}(x_0))$ by

$$v(x) = u(x_0 + \frac{x}{e_0^{1/2}})$$

for $x \in B_{r_0}(0)$. Then v satisfies the scaled α-equation

$$\frac{\operatorname{div}\left((e_0^{-1}+|\nabla v|^2)^{\alpha-1}\nabla v\right)}{(e_0^{-1}+|\nabla v|^2)^{\alpha-1}} + A(v)(\nabla v, \nabla v) = 0$$

and

$$e(v)(0) = 1, \quad \sup_{B_{r_0}} e(v) \le 4. \tag{6}$$

By Lemma 4.3 and (6), we have

$$-a_{ij}(v)\nabla^2_{ij} e(v) \le Ce(v),$$

where

$$a_{ij}(v) = \delta_{ij} + 2(\alpha-1)\frac{v_i^\beta v_j^\beta}{e_0^{-1}+|\nabla v|^2}.$$

The symmetric matrix $(a_{ij}(v))$ has positive eigenvalues satisfying the uniform elliptic condition. By the Moser-Trudinger estimate (Lemma 4.4), we have

$$1 = e(v)(0) \le C \int_{B_1(0)} e(v) \le C\varepsilon_0,$$

which is impossible if we choose ε_0 small, where we note

$$\int_{B_{r_0}(0)} e(v) = \int_{B_{\rho_0}(x_0)} e(u) \le \varepsilon_0. \tag{7}$$

This proves that $r_0 \le 1$.

Using the Moser-Trudinger estimate again, we have

$$1 = e(v)(0) \le C r_0^{-2} \int_{B_{r_0}} e(v) = C \frac{1}{e_0 \rho_0^2} \int_{B_{\rho_0}(x_0)} e(u)$$

which implies

$$\left(\frac{R}{2}\right)^2 |\nabla u(x)|^2 \le 4 e_0 \rho_0^2 \le C \int_{B_R} |\nabla u|^2 \, dv_g \quad \forall x \in B_{R/2}. \qquad \square$$

Using above two Lemmas, We prove Theorem 4.2:

Proof. We can see that there is a constant C such that

$$\int_M |\nabla u_\alpha|^2 \le C.$$

Then there are finite singular points of the set

$$\Sigma = \{x_1, \cdots, x_l\}$$

such that for every point $x_0 \in M \backslash \Sigma$, there is $r_0 > 0$ satisfying

$$\int_{B_{r_0}(x_0)} |\nabla u_\alpha|^2 \le \varepsilon_0$$

By Lemma 4.5, we have

$$\|u_\alpha\|_{C^k(B_{r_0/2}(x_0))} \le C(k, x_0), \quad \forall k \ge 1.$$

Then there exists a subsequence such that $u_{\alpha_i} \to u$ in $C^k_{\mathrm{loc}}(M \backslash \Sigma, N)$ for all $k \ge 1$ and $u \in C^\infty(M \backslash \Sigma, N)$ is harmonic map. By the removable singularity theorem (see [26] and also below Theorem 4.8), $u \in C^\infty(M, N)$. $\square$

Theorem 4.6. *If $dim(M) = 2$ and $\pi_2(N) = \emptyset$, then any smooth map u_0 is homotopic to a smooth harmonic map.*

Proof.Let $u_i = u_{\alpha_i}$ be the above minimizers of E_{α_i} in the same homotopy class $[u_0]$. Using Theorem 4.2, there exist finitely many points $x_1, \ldots, x_l$ such that u_i converges to u smoothly in M away from these points. By the well-known removable singularity theorem on harmonic maps (see below), u can be extended to a smooth map on M.

Without loss of generality, we assume that $l = 1$. Let $\eta(r)$ be a smooth cutoff function in $\mathbb{R}$ with the property that $\eta \equiv 1$ for $r \geq 1$ and $\eta \equiv 0$ for $r \leq 1/2$. For some $\rho > 0$, we define a new sequence of maps $v_i : M \to N$ such that v_i is the same as u_i outside $B_\rho(x_1)$, and for $x \in B_\rho(x_1)$,

$$v_i(x) = \exp_{u(x)}\left(\eta(\frac{|x|}{\rho})\exp^{-1}_{u(x)} \circ u_i(x)\right),$$

where exp is the exponential map on N.

We claim that

$$\|v_i - u\|_{W^{1,2}(M)} \to 0 \tag{8}$$

as $i \to \infty$.

To see this, it suffices to consider $B_\rho(x_1) \setminus B_{\rho/2}(x_1)$ because $v_i \equiv u$ on $B_{\rho/2}(x_1)$ and $v_i \equiv u_i$ outside $B_\rho(x_1)$. On the other hand, u_i converges to u on $B_\rho(x_1) \setminus B_{\rho/2}(x_1)$ strongly in $W^{1,2}$ and C^β for some $\beta > 0$. Hence for large i, $v_i(B_{\rho(x_1)} \setminus B_{\rho/2}(x_1))$ lies in a small neighborhood of $u(x_1)$, where $\exp^{-1}_{u(x)}$ is a well defined smooth map (if ρ is small). Since $F(y) = \exp_{u(x)}\left(\eta(\frac{|x|}{\rho})\exp^{-1}_{u(x)} y\right)$ is a smooth map from a neighborhood of $u(x_1)$ into itself, we have

$$\begin{aligned}\sup_{B_\rho \setminus B_{\rho/2}(x_1)} |\nabla(v_i - u)| &= \sup_{B_\rho \setminus B_{\rho/2}(x_1)} |\nabla(F \circ u_i - F \circ u)| \\ &\leq C \sup_{B_\rho \setminus B_{\rho/2}(x_1)} |\nabla(u_i - u)| \to 0 \quad \text{as } i \to \infty.\end{aligned}$$

The claim (8) is proved.

Since $\pi_2(N)$ is trivial, v_i is in the same homotopy class as u_i. Since u_i is a minimizer of E_{α_i} and u_i converges weakly to u in $W^{1,2}$, we have

$$\begin{aligned} E(u)+|M| &\le \liminf_{i\to\infty} E(u_i)+|M| \\ &\le \limsup_{i\to\infty} E_{\alpha_i}(u_i) \le \limsup_{i\to\infty} E_{\alpha_i}(v_i) \\ &= E(u)+|M|, \end{aligned}$$

which implies

$$E(u)=\lim_{i\to\infty} E(u_i).$$

Now, u_i converges to u strongly in $W^{1,2}(M,N)$, which means that there is no energy concentration, and Theorem 4.2 in turn shows that the convergence is in C^β for some $\beta>0$ and hence also in $C^\infty(M,N)$. □

In order to establish the removable singularity theorem of Sack-Uhlenbeck [26], we need

Lemma 4.7. *Let $u \in C^\infty(\bar{B}\backslash\{0\},N)$ be a smooth harmonic map with $E(u;B)\le +\infty$, where $B=B_1$. Then for any $0<r\le 1$,*

$$\int_0^{2\pi}\left|\frac{\partial u}{\partial r}\right|^2(r,\theta)d\theta = r^{-2}\int_0^{2\pi}\left|\frac{\partial u}{\partial \theta}\right|^2(r,\theta)d\theta$$

Proof. The result is a consequence of the Pohozaev identity (see [22]). However, 0 is a singular point. We need to use a test function to cut off the singularity. For a very small $\varepsilon>0$, let $\phi(x)=\phi_\varepsilon(r)\in C^\infty(B)$ with $r=|x|$ be a cut-off function such that $\phi=0$ in B_ε and $\phi=1$ in $B\backslash B_{2\varepsilon}$, $0\le\phi\le 1$ and $|\nabla\phi|\le 2/\varepsilon$.

Multiplying the harmonic map equation by $\phi x\cdot\nabla u$ and then integrating by parts, we have

$$\begin{aligned} 0 &= \int_B \triangle u\cdot(\phi x\cdot\nabla u)\,dx \\ &= \int_{\partial B}\left(|\partial_r u|^2-\frac{1}{2}|\nabla u|^2\right)d\theta + \int_B |x|\phi'(|x|)\left(\frac{1}{2}|\nabla u|^2-|\partial_r u|^2\right)dx. \end{aligned}$$

Since $E(u;B)$ is finite, the claim follows from taking $\varepsilon\to 0$ in the above identity. □

Theorem 4.8. *If $u \in C^\infty(\bar{B}\backslash\{0\}) \to N$ is a harmonic map and $E(u;B) < +\infty$, then $u \in C^\infty(B,N)$.*

Proof. This proof is due to Sack-Uhlenbeck in [26]. Since E is conformal invariant, we assume that $\int_{B_2} |\nabla u|^2 \le \varepsilon_0^2$, where ε_0 is a small constant. For any nonzero point $x \in B$, we have $E(u, B_{|x|}) \le \varepsilon_0^2$. Then Lemma 4.5 (with $\alpha = 1$) yields

$$|x||\nabla u|(x) \le C\|\nabla u\|_{L^2(B)} \le C\varepsilon_0.$$

For any integer $m \ge 1$, set

$$A_m = \{x \in B :\ 2^{-m} \le |x| \le 2^{-m+1}\}.$$

There exists a radial symmetric harmonic function $q(x) = q(r)$ in A_m to solve the harmonic equation

$$\triangle q = 0 \quad \text{ in } A_m$$

with boundary conditions $q(2^{-m}) = \frac{1}{2\pi}\int_0^{2\pi} u(2^{-m},\theta)\, d\theta$ and $q(2^{-m+1}) = \frac{1}{2\pi}\int_0^{2\pi} u(2^{-m+1},\theta)\, d\theta$. By the maximum principle, we have

$$\begin{aligned}|q(x) - u(x)| = |q(r) - u(r,\theta)| &\le 2 \max_{x,y\in A_m} \{|u(x) - u(y)|\}\\ &\le 2^{-m+3} \max_{x\in A_m} |\nabla u(x)| \le C\left(\int_{|x|\le 2^{-m+2}} |\nabla u|^2\right)^{1/2} \le C\varepsilon_0.\end{aligned}$$

Multiplying the harmonic map equation by $u - q$ and then integrating by parts yields that

$$\begin{aligned}&\int_B |\nabla(q(x) - u(x))|^2 = \sum_{m=1}^\infty \int_{A_m} |\nabla(q(x) - u(x))|^2\\ &= \sum_{m=1}^\infty r\int_0^{2\pi} (q(r) - u(r,\theta)) \cdot (u_r(r,\theta) - q'(r))\, d\theta\Big|_{r=2^{-m}}^{r=2^{-m+1}} + \int_B \triangle u \cdot (u - q).\end{aligned}$$

Note for any $m \ge 1$

$$\int_0^{2\pi} (q(r) - u(r,\theta)) \cdot q'(r)\, d\theta\Big|_{r=2^{-m}}$$

$$= \left(q(2^{-m})2\pi - \int_0^{2\pi} u(2^{-m},\theta)d\theta \right) \cdot q'(r) \; = 0.$$

Since u, q and u_r are continuous, the boundary terms with u_r cancel for any finite m; i.e.

$$\begin{aligned}
&\sum_{m=1}^{\infty} r \int_0^{2\pi} (q(r) - u(r,\theta)) \cdot u_r(r,\theta)\, d\theta \Big|_{r=2^{-m}}^{r=2^{-m+1}} \\
&= \int_0^{2\pi} (q(1) - u(1,\theta)) \cdot u_r(1,\theta)\, d\theta \\
&- \lim_{m\to\infty} 2^{-m} \int_0^{2\pi} (q(2^{-m}) - u(2^{-m},\theta)) \cdot u_r(2^{-m},\theta)\, d\theta \\
&= \int_0^{2\pi} (q(1) - u(1,\theta)) \cdot u_r(1,\theta)\, d\theta.
\end{aligned}$$

Since $|A(u)(\nabla u, \nabla u)| \le C|\nabla u|^2$, we have

$$\left| \int_B \triangle u \cdot (u-q) \right| \le C\|u-q\|_{L^\infty(B)} \int_B |\nabla u|^2\, dx \le C\varepsilon_0 \|\nabla u\|^2_{L^2(B)}.$$

Therefore

$$\begin{aligned}
\int_B |\nabla(u-q)|^2 \le &\left(\int_0^{2\pi} |q(1) - u(1,\theta)|^2\, d\theta \right)^{1/2} \left(\int_0^{2\pi} |u_r(1,\theta)|^2\, d\theta \right)^{1/2} \\
&+ C\varepsilon_0 \|\nabla u\|^2_{L^2(B)}.
\end{aligned}$$

Since q does not depend on θ, it follows from Lemma 4.7 that

$$\frac{1}{2}\int_B |\nabla u|^2 = \frac{1}{2}\int_0^1 \int_0^{2\pi} |u_r|^2 + \frac{1}{r^2}|u_\theta|^2 d\theta r\, dr \le \int_B |\nabla(u-q)|^2.$$

By the Poincare inequality on S^1, we have

$$\int_{r=1} |u-q|^2\, d\theta \le \int_{r=1} |u_\theta|^2\, d\theta = \frac{1}{2}\int_{r=1} |\nabla u|^2\, d\theta.$$

Choosing ε_0 sufficiently small with $\delta_0 = C\varepsilon_0 < 1$, we obtain

$$(1-\delta_0)\int_B |\nabla u|^2 \le \int_{\partial B} |\nabla u|^2.$$

By scaling in r, we can obtain

$$(1-\delta_0)\int_{B_r}|\nabla u|^2 \le r\int_{\partial B_r}|\nabla u|^2 = r\frac{d}{dr}\left(\int_{B_r}|\nabla u|^2\right)$$

for all r with $0 < r \le 1$. This implies

$$\int_{B_r}|\nabla u|^2 \le r^{1-\delta_0}\int_B |\nabla u|^2$$

Using the ε-regularity, we have

$$|x|^2|\nabla u|^2(x) \le C\int_{B_{2|x|}}|\nabla u|^2 \le C|x|^{1-\delta_0}\int_B|\nabla u|^2, \quad \forall 0<r<\frac{1}{2}.$$

This implies $\nabla u \in L^p(B)$ for some $p>2$ and $u \in C^\alpha(B)$ for some $0<\alpha<1$. By using the elliptic theory of partial differential equations, $u \in C^\infty(B,N)$. $\square$

In fact, we can improve the above result as follows. Let u_i be a sequence of smooth maps minimizing $E(u)=\int_M|\nabla u|^2\,dv$ in a fixed homotopy class of maps. Since u_i is bounded in $W^{1,2}$, there is a weak limit u in $W^{1,2}(M,N)$. In general, u may not be in the same homotopy class, but we can show:

Remark 4.9. Let u be the weak limit of the above minimizing sequence $\{u_i\}$. Then it is a harmonic map from M to N and there exist harmonic maps $\omega_k : S^2 \to N$ with $k=1,\dots,l$ such that

$$\lim_{i\to\infty}E(u_i)=E(u)+\sum_{k=1}^{l}E(\omega_k). \tag{9}$$

Moreover, if $\pi_2(N)$ is trivial, then u_i converges strongly to u in $W^{1,2}(M,N)$ and u is a minimizer in the homotopy class of u_i. (see [17], also [22]).

5. The Partial Regularity of Minimizing Harmonic Maps

The study of partial regularity of various classes of weakly harmonic maps has been of great interest for a number of years. Schoen-Uhlenbeck [28] and Giaquinta-Giusti [10] established that an energy minimizing map $u: M\to N$ between Riemannian manifolds is smooth in M away from a singular set Σ that has Hausdorff dimension $\le n-3$, where n is the dimension of M. Bethuel [1] proved that a weak stationary harmonic map $u : M \to N$ is smooth away from a singular set of vanishing $(n-2)$-dimensional Hausdorff measure. Lin [L] proved an important result that if there is no non-constant harmonic map from S^2 to N, then the singular set of any stationary harmonic map into N has to be $(n-4)$-rectifiable.

Let n and k be positive integers with $n \geq 3$. Let Ω be a bounded smooth domain in n-dimensional space $\mathbb{R}^n$ and let $N \subset \mathbb{R}^l$ be a compact k-dimensional Riemannian manifold without boundary for some integer l.

For a map $u \in W^{1,2}(\Omega, N) := \{v \in W^{1,2}(\Omega, \mathbb{R}^l) | \, v \in N\}$, its Dirichlet energy is given by

$$E(u, \Omega) = \int_\Omega |\nabla u|^2 \, dx,$$

where ∇u is the gradient of u.

A map $u \in W^{1,2}(\Omega, N)$ is said to be a (weakly) harmonic map if u belongs to $W^{1,2}(\Omega, N)$ and satisfies

$$\int_\Omega \langle \nabla u, \nabla \phi \rangle + A(u)(\nabla u, \nabla u) \cdot \phi \, dv_g = 0,$$

for all $\phi \in C^\infty(\Omega, \mathbb{R}^l)$.

Without any assumption on weak harmonic maps, Rivière in [24] gave an counterexample that weakly harmonic maps may have singularities. In this section, we will prove partial regularity of the classic result of minimizing harmonic maps by Schoen-Uhlenbeck.

Definition 5.1. For $0 \leq s \leq n$, the s-dimensional Hausdorff measure $\mathcal{H}^s$ on $\mathbb{R}^n$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(A), \quad A \subset \mathbb{R}^n,$$

with

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_i r_i^s : \, A \subset \bigcup_i B_{r_i}, \, r_i \leq \delta \right\}.$$

The Hausdorff dimension of $A \subset \mathbb{R}^n$ is defined by

$$dim_{\mathcal{H}}(A) := \inf \{s : \mathcal{H}^s(A) = 0\} = \sup \left\{s : \mathcal{H}^t(A) = \infty \right\}.$$

The main result of this section is:

Theorem 5.2. *Let $u \in W^{1,2}(\Omega; N)$ be a minimizer of $E(u)$ in $W^{1,2}(\Omega; N)$. Then, u is smooth in $M \backslash \Sigma$, where Σ is the singular set of u and is defined by*

$$\Sigma := \{x \in \Omega : u \textit{ is discontinuous at } x.\}$$

Moreover, the Hausdorff dimension of Σ is less or equal to $n-3$.

Lemma 5.3 (Monotonicity)**.** *For $n \geq 3$, let $u \in W^{1,2}(\Omega, N)$ be a minimizing harmonic map. Then for any $x_0 \in \Omega$ and for any two r and R, we have*

$$R^{2-n} \int_{B_R(x_0)} |\nabla u|^2 - s^{2-n} \int_{B_s(x_0)} |\nabla u|^2 \geq \int_{B_R(x_0) \backslash B_s(x_0)} r^{2-n} \left| \frac{\partial u}{\partial r} \right|^2 \tag{10}$$

with $r = |x - x_0|$.

Proof. The hint is to use that $u_r(x) = u(\frac{rx}{|x|})$ for $x \in B_r$ with $r > 0$. Using the minimality of u, we have

$$\int_{B_r} |\nabla u|^2 \leq \int_{B_r} |\nabla u_r|^2 = \frac{r}{n-2} \int_{\partial B_r} \left(|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2 \right) d\mathcal{H}^{n-1}.$$

We omit all details. (see [28]). □

In fact, the inequality also holds for stationary harmonic maps.

Assume that $u : B_1 \to N$ is a minimizing harmonic map satisfying

$$E(u; B_1) = \int_{B_1} |\nabla u|^2 \, dv_g \leq \varepsilon.$$

Let $\phi \in C^\infty(\mathbb{R}^n, \mathbb{R}^+)$ be a radial mollifying function so that supp $\phi \subset B_1$ and $\int_{\mathbb{R}^n} \phi = 1$ (see Chapter 7 of the book of Gilberg-Trudinger [13]).

Let $h \in (0, \frac{1}{4}]$, set

$$u^h(x) = \int_{B_1} \phi^h(x-y) u(y) dy, \quad \forall x \in B_{1/2},$$

where $\phi^h(x) = h^{-n}\phi(\frac{x}{h})$. Then

$$\mathrm{dist}^2(u^h(x), N) \leq \frac{1}{|B_h|}\int_{B_h} |u(y)-u^h(x)|^2\, dy \leq Ch^{2-n}\int_{B_h} |\nabla u(y)|^2\, dy \leq C\varepsilon,$$

where we used a variant of Poincare's inequality

$$\int_{B_1} |u(x) - \int_{B_1} \phi(y)u(y)dy|^2\, dx \leq C\int_{B_1} |\nabla u|^2.$$

For a sufficiently small, $u^{\bar{h}}(B_{1/2}) \subset N_{\delta_0}$ and we can define

$$u_{\bar{h}} := \Pi_N(u^{\bar{h}}) : B_{1/2} \to N.$$

Lemma 5.4. *For $\bar{h} = \varepsilon^{1/4}$, we have*

$$\int_{B_{1/2}} |\nabla u^{\bar{h}}|^2 \;\leq\; C\int_{B_1} |\nabla u|^2, \tag{11}$$

$$\sup_{x\in B_{1/2}} |u^{\bar{h}}(x) - u^{\bar{h}}(0)|^2 \;\leq\; C\varepsilon^{1/2} \tag{12}$$

where the constant C does not depend on α and u.

Proof. For any $x \in B_{1/2}$, we have

$$\begin{aligned}
|\nabla u^{\bar{h}}|^2(x) &= \left|\int_{B_1} \phi^{\bar{h}}(x-y)\nabla u(y)\, dy\right|^2 \\
&\leq \int_{B_1} \phi^{\bar{h}}(x-y)|\nabla u(y)|^2\, dy \\
&\leq C\frac{1}{\bar{h}^n}\int_{B_{\bar{h}}(x)} |\nabla u(y)|^2\, dy \leq C\frac{\varepsilon}{\bar{h}^2} = C\varepsilon^{1/2}.
\end{aligned}$$

The inequality (12) also follows. $\square$

Let $\bar{h} = \varepsilon^{1/4}$, $\tau = \varepsilon^{1/8}$. We choose $h(x) = h(r)$, $r = |x|$ to be a non-increasing smooth function of r such that

$$h(x) = h(r) = \bar{h},\ \text{for } r \leq \theta, \quad h(\theta+\tau) = 0, \quad |h'(r)| \leq 2\varepsilon^{1/8}.$$

Then we set

$$u^{h(x)}(x) = \int_{B_1} \phi^{h(x)}(x-y)u(y)\,dy$$

We know

$$u_{h(x)} := \Pi \circ u^{h(x)}(x) \in N.$$

Then we have

Lemma 5.5. *For $\theta \in (\tau, \frac{1}{4}]$, the above map $u_{h(x)}$ satisfies $u_h = u$ on $B_{1/2} \backslash B_{\theta+\tau}$ and*

$$\int_{B_{\theta+\tau} \backslash B_\theta} |\nabla u_h|^2 dx \le C \int_{B_{\theta+2\tau} \backslash B_{\theta-\tau}} |\nabla u|^2 dx$$

where the constant C does not depend on θ and u.

Proof. Since Π is smooth, it suffices to prove this lemma for u^h instead of u_h. Note that

$$u^h = \int_{B_1} \phi(y) u(x - h(x)y)\,dy.$$

We compute

$$\frac{\partial u^h}{\partial x^\alpha} = \int_{B_1} \phi(y) \left[\frac{\partial u}{\partial x^\alpha}(x - hy) - \frac{\partial h}{\partial x^\alpha} \cdot \nabla u(x - hy) \right] dy.$$

Then

$$\begin{aligned} \int_{B_{\theta+\tau} \backslash B_\theta} |\nabla u^h|^2 &\le C \int_{B_{\theta+\tau} \backslash B_\theta} \int_{B_1} \phi(y) |\nabla u|^2 (x - hy)\,dydx \\ &\le C \int_{B_{\theta+2\tau} \backslash B_{\theta-\tau}} |\nabla u|^2\,dx. \end{aligned}$$ □

Lemma 5.6. *(Energy decay estimate) For $n \ge 3$, there are a small constant $\varepsilon = \varepsilon(n, M)$ and another constant $\theta \in (0, \frac{1}{4})$ such that if $u : B_1 \to N$ is a minimizing harmonic map satisfying*

$$E(u; B_1) = \int_{B_1} |\nabla u|^2\,dv_g \le \varepsilon,$$

then

$$\theta^{2-n}\int_{B_\theta}|\nabla u|^2 \le \frac{1}{2}\int_{B_1}|\nabla u|^2. \tag{13}$$

Proof. The proof is divided into three parts.

Claim 1: For any $\theta \in (0, \frac{1}{4}]$,

$$\theta^{2-n}\int_{B_\theta}|\nabla u_{\bar{h}}|^2 \le C(\theta^{2-n}\varepsilon^{1/4}+\theta^2)\int_{B_1}|\nabla u|^2 \tag{14}$$

Claim 2: There is a $\theta \in [\bar{\theta}, 2\bar{\theta}]$ with $\bar{\theta} = \varepsilon^{\gamma_n}$ with $\gamma_n = \min\{\frac{1}{32(n-2)}, \frac{1}{64}\}$ and $\tau = \varepsilon^{1/8}$ such that

$$\int_{B_{\theta+\tau}\setminus B_\theta}|\nabla u^{h(x)}|^2 \le C\varepsilon^{\frac{1}{16}}\int_{B_1}|\nabla u|^2.$$

Claim 3: Since u is minimizing,

$$\int_{B_{\theta+\tau}}|\nabla u|^2 \le C\int_{B_{\theta+\tau}}|\nabla u^{h(x)}|^2.$$

Using Claims 1–3 and noting $\theta \in [\bar{\theta}, 2\bar{\theta}]$, we obtain

$$\begin{aligned}\theta^{2-n}\int_{B_\theta}|\nabla u|^2 &\le \theta^{2-n}\int_{B_{\theta+\tau}}|\nabla u|^2\\ &\le C\theta^{2-n}\Big(\int_{B_\theta}|\nabla u^h|^2+\int_{B_{\theta+\tau}\setminus B_\theta}|\nabla u^h|^2\Big)\\ &\le C(\theta^{2-n}\varepsilon^{\frac{1}{4}}+\theta^2+\varepsilon^{\frac{1}{16}})\int_{B_1}|\nabla u|^2 \le C\varepsilon^{2\gamma_n}\int_{B_1}|\nabla u|^2.\end{aligned}$$

Choosing ε sufficiently small, the required result follows.

Claim 3 follows from Lemma 5.5. Next, we are going to prove **Claims 1–2**.

To prove Claim 1, let v be the solution of

$$\begin{aligned}\triangle v &= 0 \quad \text{in } B_{1/2}\\ v &= u^{\bar{h}} \quad \text{on } \partial B_{1/2}.\end{aligned}$$

By the maximal principle, we have

$$\sup_{B_{1/2}} |v - u^{\bar{h}}| \le C\varepsilon^{1/4}.$$

By the mean value inequality ($\triangle|\nabla v|^2 \ge 0$), we have

$$\sup_{B_{1/4}} |\nabla v|^2 \le C\int_{B_{1/2}} |\nabla v|^2 \le C\int_{B_{1/2}} |\nabla u^{\bar{h}}|^2 \le C\int_{B_1} |\nabla u|^2.$$

Hence for any $\theta \in (0, \frac{1}{4}]$,

$$\theta^{2-n}\int_{B_\theta} |\nabla u_{\bar{h}}|^2 \le 2\theta^{2-n}\int_{B_\theta} |\nabla (u_{\bar{h}} - v)|^2 + 2\theta^{2-n}\int_{B_\theta} |\nabla v|^2 \tag{15}$$

$$\le 2\theta^{2-n}\int_{B_\theta} |\nabla (u_{\bar{h}} - v)|^2 + C\theta^2\int_{B_1} |\nabla u|^2. \tag{16}$$

Note

$$\begin{aligned}
\triangle u^h &= \int_{\mathbb{R}^n} [\triangle_x \phi^{\bar{h}}(x-y)]u(y)dy \\
&= \int_{\mathbb{R}^n} [\triangle_y \phi^{\bar{h}}(x-y)]u(y)dy] = \int_{\mathbb{R}^n} \phi^{\bar{h}}(x-y)\triangle_y u(y)dy \\
&= \int_{\mathbb{R}^n} \phi^{\bar{h}}(x-y)]A(u)(\nabla u, \nabla u)(y)dy
\end{aligned}$$

which implies

$$\int_{B_{1/2}} |\triangle u^{\bar{h}}| \le C\int_{B_1} |\nabla u|^2.$$

Then

$$\int_{B_{1/2}} |\nabla (u^{\bar{h}} - v)|^2 = -\int_{B_{1/2}} \triangle u^{\bar{h}} \cdot (u^{\bar{h}} - v) \le C\varepsilon^{1/4}\int_{B_1} |\nabla u|^2. \tag{17}$$

Claim 1 follows from (15)–(17).

Now we are going to prove Claim 2.

We recall $\bar{\theta} = \varepsilon^{\gamma n}$ with $\gamma \le \frac{1}{16}$. Let $l = \left[\frac{\bar{\theta}}{3\tau}\right]$ $(\ge \frac{1}{3}\varepsilon^{-\frac{1}{16}} - 1)$ be the integer of $\frac{\theta}{3\tau}$ and write

$$[\bar{\theta}, \bar{\theta} + 3\tau l] = \cup_{1\le i\le l} I_i, \quad |I_i| = 3\tau,$$

where each I_i is a closed interval of length 3τ. Since $\gamma_n \leq \frac{1}{16}$, $l \geq \frac{1}{3}\varepsilon^{-\frac{1}{16}}$. Then

$$\int_{B_{\bar{\theta}+3l\tau}\setminus B_{\bar{\theta}}} |\nabla u|^2 dx = \sum_{1\leq i\leq l} \int_{|x|\in I_i} |\nabla u|^2 dx \leq \int_{B_1} |\nabla u|^2.$$

There is at least one interval I_j with $1 \leq j \leq l$ such that

$$\int_{|x|\in I_j} |\nabla u|^2 \, dx \leq l^{-1} \int_{B_1} |\nabla u|^2 \leq C\varepsilon^{\frac{1}{16}} \int_{B_1} |\nabla u|^2.$$

Let θ be the number such that $I_j = [\theta - \tau, \theta + 2\tau] \subset [\bar{\theta}, 2\bar{\theta}]$, and let $h = h(x)$ be as in Lemma 5.3. Then $u_h \in W^{1,2}(B_{1/2}, N)$ and $u_h = u$ for any $|x| \geq \theta+\tau$, and

$$\int_{B_{\theta+\tau}\setminus B_\theta} |\nabla u_h|^2 \leq C\varepsilon^{\frac{1}{16}} \int_{B_1} |\nabla u|^2.$$

This proves **Claim 2**. □

As a consequence of this lemma, we can prove:

Theorem 5.7. *Let $u \in W^{1,2}(M;N)$ be a minimizer of $E(u)$ in $W^{1,2}(M;N)$. Then, u is smooth in $M\setminus\Sigma$, where Σ is the singular set defined by*

$$\Sigma = \{x \in M : \lim_{r\to 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2 \geq \varepsilon_0^2\} \tag{18}$$

and $\mathcal{H}^{n-2}(\Sigma) = 0$.

Proof. If $x_0 \notin \Sigma$, then there is a $r_0 > 0$ such that

$$r_0^{2-n} \int_{B_{r_0}(x_0)} |\nabla u|^2 \leq \varepsilon_0^2$$

implying

$$(\frac{r_0}{2})^{2-n} \int_{B_{r_0}(x)} |\nabla u|^2 \leq 2^{n-2}\varepsilon_0^2, \quad \forall x \in B_{\frac{r_0}{2}}(x_0).$$

By the monotonicity, we have

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq 2^{n-2}\varepsilon_0^2 \leq \varepsilon, \quad \forall x \in B_{\frac{r_0}{2}}(x_0) \text{ and } 0 < r \leq \frac{r_0}{2}$$

for ε_0 sufficiently small, where ε is the constant in Lemma 5.6. By the above lemma, there is a $\theta \in (0,1)$ such that

$$\theta^{2-n} \int_{B_\theta} |\nabla u|^2 \leq \frac{1}{2} \int_{B_1} |\nabla u|^2.$$

We consider a rescaling map $u_\theta(x) = u(\theta x)$. Then

$$\int_{B_1} |\nabla u_\theta|^2 = \theta^{2-n} \int_\theta |\nabla u|^2 \leq \varepsilon.$$

Using again Lemma, we have

$$(\theta^2)^{2-n} \int_{B_{\theta^2}} |\nabla u|^2 = (\theta)^{2-n} \int_{B_\theta} |\nabla u_\theta|^2 \leq \frac{1}{2} \int_{B_1} |\nabla u_\theta|^2 = \left(\frac{1}{2}\right)^2 \int_{B_1} |\nabla u|^2.$$

By the induction argument, we have

$$(\theta^i)^{2-n} \int_{B_{\theta^i}} |\nabla u|^2 \leq \left(\frac{1}{2}\right)^i \int_{B_1} |\nabla u|^2.$$

For any $r \in (0,1)$, there is integer i so that $r \in [\theta^{i+1}, \theta^i]$. Then

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq C(\theta^i)^{2\alpha} \int_{B_1} |\nabla u|^2 \leq C r^{2\alpha} \int_{B_1} |\nabla u|^2$$

for some $\alpha = \log 2/(2\log\theta^{-1}) > 0$. Repeating the above arguments, we can obtain

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq C r^{2\alpha}, \quad \forall x \in B_{r_0/2}(x_0) \text{ and } 0 < r \leq \frac{r_0}{2}$$

for some $\alpha \in (0,1)$ depending on ε_0, M and N. By Morrey's Lemma, $u \in C^\alpha(B_{r_0/2}(x_0), N)$.

Next we will show $\mathcal{H}^{n-2}(\Sigma) = 0$.

Since M is compact and Σ is relatively closed, by Vitali's covering lemma (see Giaquinta's book [9]), there are disjoint balls $\{B_{r_i}(x_i)\}_{i\in I}$ such that

$$\Sigma \subset \cup_i B_{3r_i}(x_i), \quad r_i \leq \delta$$

so

$$\mathcal{H}^{n-2}_{5\delta}(\Sigma) \leq \sum_{i\in I}(5r_i)^{n-2} \leq \frac{5^{n-2}}{\varepsilon_0^2}\int_{\cup_{i\in I}B_{r_i}(x_i)} |\nabla u|^2\,dx \leq C\int_M |\nabla u|^2 < +\infty.$$

Note

$$\text{meas}|\cup_{i\in I} B_{r_i}(x_i)| \leq C\delta^2$$

Hence $\mathcal{H}^{n-2}(\Sigma) = 0$ by letting $\delta \to 0$. $\square$

Lemma 5.8. *Let $u_i \in W^{1,2}(\Omega, N)$ be a sequence of minimizing harmonic maps. If $u_i \to u$ weakly in $W^{1,2}(\Omega, N)$, then $u_i \to u$ strongly in $W^{1,2}_{loc}(\Omega, N)$ and u is a minimizing harmonic map.*

The proof is based on the application of Luckhaus's Lemma. (We omit details and refer to see Leon Simon's book [31] or Lin-Wang's book [22])

Next we will prove the Hausdorff dimension of the singular set is $n-3$.

Following [28], we define

$$\varphi^p(E) = \inf\left\{\sum_i r_i^s : E \subset \cup_i B_{r_i}(x_i)\right\}$$

Then

$$\varphi^p(E) = 0 \text{ if and only if } \mathcal{H}^p(E) = 0.$$

Moreover, if $\varphi^s(E) > 0$, then the following density result of Federer holds:

$$\limsup_{\lambda\to 0} \lambda^{-s}\varphi^s(E\cap B_\lambda) \geq c > 0$$

for φ^s a. e. $x \in E$ (see [28]).

Lemma 5.9. *Suppose u_i is a sequence of minimizing maps in $W^{1,2}(M, N)$, which converges weakly to u in $W^{1,2}$. Let Σ_i be the singular set of u_i and Σ denotes the singular set of u. Then we have*

$$\varphi^s(\Sigma\cap B_1) \geq \limsup_{i\to 0}\varphi^s(\Sigma_i \cap B_1)$$

for any $s \geq 0$

We complete the proof of Theorem 5.2.

Proof. Suppose $u \in W^{1,2}(M,N)$ is a minimizing harmonic map with the singular set $\Sigma \subset \text{int M}$. Let $0 \leq s < n-2$ be such that $\varphi^s(\Sigma) > 0$. Then by the density result, we can choose $x_0 \in \Sigma$ such that

$$\lim_{\lambda_i \to 0} \lambda_i^{-s} \varphi^s(\Sigma \cap B_{\lambda_i}(x_0)) > 0$$

for a sequence of $\lambda_i \to 0$. Then we consider the scaled maps $u_\lambda(x) = u(\lambda x)$. By the monotonicity formula (Lemma 5.3) and Theorem 5.9, u_{λ_i} converges to a minimizing harmonic map u_0 weakly in $W^{1,2}(B_2, N)$ and strongly in $W^{1,2}(B_1)$. Note that $\varphi^s(\Sigma_\lambda \cap B_1) = \lambda^{-s}\varphi^s(\Sigma \cap B_1)$. The density result implies

$$\lim_{\lambda_i \to 0} \varphi^s(\Sigma_{\lambda_i} \cap B_1) > 0$$

By Lemma 5.9, we obtain

$$\varphi^s(\Sigma \cap B_1) > 0.$$

Since $\frac{\partial u_0}{\partial r} = 0$, we have $\lambda\Sigma_0 \subset \Sigma_0$ for any $\lambda > 0$.

There are two cases: either $s \leq 0$ or there is a point $x_1 \in \Sigma_0 \cap \partial B_1$ such that

$$\limsup_{\lambda \to 0} \lambda^{-s} \varphi^s(\Sigma_0 \cap B_\lambda(x_1)) > 0.$$

Then repeating the above argument at x_1, there is a radially symmetric minimizing harmonic map u_1 with $\phi^s(\Sigma_1 \cap B_1) > 0$, where Σ_1 is the singular set of Σ_1. If $s - 1 \leq 0$, we stop. Otherwise, we repeat the above argument so that there is a point $x_2 \in \Sigma_1 \cap \partial B_1$. If we repeat this procedure m times, we get minimizing harmonic maps $u_j \in W^{1,2}(\mathbb{R}^n, N)$ for $j = 1, \ldots, m$ such that $\frac{\partial u_j}{\partial x^k} = 0$ for $k = 1, \ldots, j$. By the construction u_m, it must have that $s - m + 1 > 0$ and $s \leq m$. Since $s < n-2$ and m is an integer, then $m \leq n-2$. If $m = n-2$, then we have

$$\Sigma_m \supset \mathbb{R}^{n-2} = \{(x^1, \ldots, x^{n-2}, 0, 0)\}$$

which contradicts with the fact that $\mathcal{H}^{n-2}(\Sigma_m) = 0$. Therefore, we have $m \leq n-3$. Hence $\varphi^t(\Sigma \cap B_1) = 0$ for all $t \geq n-3$. This implies that $\dim \Sigma \leq n-3$. $\square$

In fact, Leon Simon [31] (also also [22]) presented another beautiful proof based on the ideas of Almgren.

6. Further Developments

In this section, we would like to make a few remarks about harmonic maps and related topics.

1. Minimizing harmonic maps:

Leon Simon [30] proved the rectifiablity of the singular set of minimizing harmonic maps.

2. Partial regularity result on Stationary harmonic maps:

Partial regularity result on Stationary harmonic maps have been established by Bethuel [1]. A new approach was presented by Riviere-Struwe [25]. Lin [21] established a result on the structure of the singular set.

3. Heat flow for Harmonic maps:

Struwe [32] proved the global existence of the weak solution to the harmonic map flow and that the solution to the flow converges to a harmonic map as $t \to \infty$. Chang, Ding and Ye [3] constructed an example where the harmonic map flow blows up at finite time. Chen-Struwe [5] (also [22]) used the Ginzburg-Landau approximation to establish the global existence and partial regularity of the harmonic map flow. Recently, Hong-Yin [17] introduced the Sack-Uhlenbeck flow in 2D to establish new existence of the harmonic map flow.

4. Relaxed energy for harmonic maps and a new approximation approach:

Bethuel-Brisis-Coron [2] introduced the relaxed energy functional. Giaquinta-Modica-Soucek [12] proved the partial regularity using Cartesian currents. Giaquinta-Hong-Yin [11] proposed a new approximation method to prove the partial regularity.

5. Biharmonic maps between manifolds:

Partial regularity of stationary bi-harmonic maps was established by Chang-Wang-Yang [4], Wang [36] and Struwe [33]. Recently, Hong and Yin [16] solved a problem on the relaxed energy for biharmonic maps.

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