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NEW COMPARISON THEOREMS IN RIEMANNIAN GEOMETRY

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Abstract

We construct and use solutions, subsolutions, and supersolutions of differential equations as catalysts to link hypotheses on radial *curvature* of a complete n-manifold (M, g) to conclusions on the analysis or geometry of *quadratic forms* and second order *differential operators*. These conclusions are formulated in terms of pointwise estimates on the Hessian and pointwise and weak estimates on the Laplacian of the distance function r from a fixed point x_0 in M. In particular, we prove Hessian Comparison Theorems and Laplacian Comparison Theorems, generalizing the work of Greene and Wu [2]: If the radial curvature K of M satisfies $-\frac{a^2}{c^2+r^2} \le K(r) \le \frac{b^2}{c^2+r^2}$ on $D(x_0)$ where $0 \le a^2, 0 \le b^2 \le \frac{1}{4}, 0 \le c^2$, and $D(x_0) = M\backslash(\mathrm{Cut}(x_0) \cup \{x_0\})$, then

$$\frac{1+\sqrt{1-4b^2}}{2r}\Big(g - dr \otimes dr\Big) \le \mathrm{Hess}(r) \le \frac{1+\sqrt{1+4a^2}}{2r}\Big(g - dr \otimes dr\Big)$$

on $D(x_0)$, in the sense of quadratic forms, and

$$(n-1)\frac{1+\sqrt{1-4b^2}}{2r} \le \Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r}$$

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holds pointwise on $D(x_0)$, and $\Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r}$ holds weakly on M. This is equivalent to that if the radial curvature K on $D(x_0)$ satisfies

$$-\frac{A(A-1)}{r^2} \le K(r) \le \frac{B(1-B)}{r^2}$$

where $1 \le A$, and $\frac{1}{2} \le B \le 1$, then

$$\frac{B}{r}\Big(g - dr\otimes dr\Big) \le \mathrm{Hess}(r) \le \frac{A}{r}\Big(g - dr\otimes dr\Big) \text{ and } (n-1)\frac{B}{r} \le \Delta r \le (n-1)\frac{A}{r}$$

holds pointwise on $D(x_0)$, and $\Delta r \le (n-1)\frac{A}{r}$ holds weakly on M. We also prove and apply Hessian Comparison Theorems via Jacobi Type inequalities, Comparison Theorems on Riccati type inequalities, and Sturm Comparison Theorems. An analog of a theorem of Greene-Wu on negatively pinched manifolds, $-\frac{A}{r^2} \le K(r) \le -\frac{A_1}{r^2} < 0$ for *pointwise* Hessian estimates is given. On positively pinched manifolds, $0 < \frac{b_1^2}{r^2} \le K(r) \le \frac{b^2}{r^2}$, pointwise Hessian estimates are also made. Pointwise Laplacian Comparison Theorems on $D(x_0)$ are then immediately obtained by taking traces in Hessian Comparison Theorems. The corresponding weak upper bound estimates of the Laplacian on all of M are then obtained by Green's Identity and a *double limiting* argument(cf. Lemma 9.1, [4], [6]).

1. Introduction

Robert E. Greene and Hung-Hsi Wu have proved a Hessian Comparison Theorem on a manifold (M,g) with a pole x_0, which can be stated as follows: Let $r(x) = \mathrm{dist}_M(x,x_0)$ be the distance function on M from x_0.

Theorem A ([2, p.38])**.** *If the radial curvature K of M satisfies*

$$-\frac{A}{r^2} \le K(r) \le -\frac{A_1}{r^2} \quad \text{on} \quad M\backslash B_{(a-1)}(x_0), \quad \text{where} \quad 0 < A_1 < A, 1 < a,$$

then

$$\frac{1+\sqrt{1+4A_1}}{2r}\Big(g - dr\otimes dr\Big) \preceq \mathrm{Hess}(r) \preceq \frac{1+\sqrt{1+4A}}{2r}\Big(g - dr\otimes dr\Big).$$

Here we denote for two functions $f_1, f_2 : [a,\infty) \to \mathbb{R}$,

$$f_1 \preceq f_2 \quad \text{if} \quad \lim \frac{f_1}{f_2}(r) \le 1 \quad \text{as} \quad r \to \infty.$$

In this pioneering result, the radial curvature was required to be negatively pinched off a compact set, and the estimates obtained were (off a compact set and) of asymptotical nature near infinity. Since then many efforts have been made to study the behavior of the Hessian under the assumption of radial curvature with a mixed sign, or positively pinched, or $0 \le K(r) \le \frac{B(1-B)}{r^2}$, and to improve the asymptotic estimates to pointwise ones and to weak ones on the entire manifold. There are many statements regarding Hessian comparison theorems in various special cases that are discussed, made, or applied, however, a general yet detailed result with a complete proof is still needed to support some works in the literature.

To this end, we prove in this paper the following fundamental results in a complete n-dimensional Riemannian manifold M: Let $x_0 \in M$ be a fixed point. Denote $\mathrm{Cut}(x_0)$ the cut locus of x_0 in M and let $D(x_0) = M \backslash (\mathrm{Cut}(x_0) \cup \{x_0\})$.

Theorem 1.1 (Hessian Comparison Theorem)**.** *If the radial curvature K of M satisfies*

$$-\frac{a^2}{c^2+r^2} \le K(r) \le \frac{b^2}{c^2+r^2} \tag{1.1}$$

on $D(x_0)$, where $0 \le a^2, 0 \le b^2 \le \frac{1}{4}$, and $0 \le c^2$, then

$$\frac{1+\sqrt{1-4b^2}}{2r}\Big(g - dr \otimes dr\Big) \le \mathrm{Hess}(r) \le \frac{1+\sqrt{1+4a^2}}{2r}\Big(g - dr \otimes dr\Big) \tag{1.2}$$

on $D(x_0)$ in the sense of quadratic forms.

This is equivalent to the following:

Theorem 1.2.

$$\textit{Let } -\frac{a^2}{r^2} \le K(r) \le \frac{b^2}{r^2} \textit{ on } D(x_0) \quad \textit{with } 0 \le a^2 \textit{ and } 0 \le b^2 \le \frac{1}{4}. \tag{1.3}$$

Then (1.2) *holds on $D(x_0)$.*

By taking $A = \frac{1+\sqrt{1+4a^2}}{2}$ and $B = \frac{1+\sqrt{1-4b^2}}{2}$ in Theorem 1.2, one has the following equivalent result:

Theorem 1.3. *If the radial curvature K of M satisfies*

$$-\frac{A(A-1)}{r^2} \le K(r) \le \frac{B(1-B)}{r^2} \text{ on } D(x_0) \textit{ with } 1 \le A \textit{ and } \frac{1}{2} \le B \le 1, \tag{1.4}$$

then

$$\frac{B}{r}\Big(g - dr \otimes dr\Big) \le \mathrm{Hess}(r) \le \frac{A}{r}\Big(g - dr \otimes dr\Big) \ \ on\ D(x_0). \tag{1.5}$$

Theorems 1.1, 1.2 and 1.3 are sharp when $a = b = 0$ or $A = B = 1$, and $M = \mathbb{R}^n$. By setting $c = 1$, Theorem 1.1 takes the following form:

Corollary 1.1. *Suppose the radial curvature K of M satisfies*

$$-\frac{a^2}{1+r^2} \le K(r) \le \frac{b^2}{1+r^2} \tag{1.6}$$

on $D(x_0)$ where $0 \le a^2$ and $0 \le b^2 \le \frac{1}{4}$.

Then (1.2) holds *on $D(x_0)$.*

By setting $A = 1$, Theorem 1.3 takes the following form:

Corollary 1.2. *If K of M satisfies*

$$0 \le K(r) \le \frac{B(1-B)}{r^2} \tag{1.7}$$

on $D(x_0)$ where $\frac{1}{2} \le B \le 1$, then

$$\frac{B}{r}\Big(g - dr \otimes dr\Big) \le \mathrm{Hess}(r) \le \frac{1}{r}\Big(g - dr \otimes dr\Big) \tag{1.8}$$

on $D(x_0)$.

When $B = 1$, and $D(x_0) = M\backslash\{x_0\}$, Theorem 1.3 takes the following form:

Corollary 1.3. *If the radial curvature K of M satisfies*

$$-\frac{A(A-1)}{r^2} \le K(r) \le 0 \tag{1.9}$$

on $M\backslash\{x_0\}$ where $1 \le A$, then

$$\frac{1}{r}\Big(g - dr \otimes dr\Big) \le \mathrm{Hess}(r) \le \frac{A}{r}\Big(g - dr \otimes dr\Big) \tag{1.10}$$

on $M\backslash\{x_0\}$ in the sense of quadratic forms.

Corollary 1.4 (cf. [2]). *If M is a manifold with a pole and*

$$0 \le K(r) \quad \Big(resp. \quad K(r) \le 0\Big), \tag{1.11}$$

then

$$\operatorname{Hess} r \le \frac{1}{r}\big(g - dr \otimes dr\big) \quad \Big(resp. \quad \frac{1}{r}\big(g - dr \otimes dr\big) \le \operatorname{Hess}(r)\Big) \text{ on } M\backslash\{x_0\}. \tag{1.12}$$

In particular,

$$\text{if} \quad K(r) \equiv 0 \quad \text{then} \quad \operatorname{Hess}(r) = \frac{1}{r}\big(g - dr \otimes dr\big).$$

The technique we employed is to construct and use solutions, subsolutions, and supersolutions of differential equations as catalysts to link hypotheses on radial *curvature* of a complete manifold M to conclusions on the analysis or geometry of *quadratic forms* and second order *differential operators*. These conclusions are formulated in terms of pointwise estimates on the Hessian and pointwise and weak estimates on the Laplacian of the distance function r from a fixed point x_0 in M. Comparison theorems in differential equations lead naturally to comparison theorems in differential geometry and the second order linear Jabobi equations are transformed to the first order nonlinear Riccati equations. More specifically, to obtain Hessian comparison theorems, we construct and use solutions, supersolutions and subsolutions of the Jabobi equation in Sect. 3, and apply Hessian Comparison Theorems via Jacobi type inequalities (in Sect. 4), Comparison Theorems on Riccati type inequalities (in Sect. 5) and the Sturm Comparison Theorem (in Sect. 6). For more discussion, background, or insight of geometric analytic approach, we refer the reader to Stefano Pigola, Marco Rigoli, and Alberto G. Setti's book ([4]), Peter Petersen's book ([3]), and recent articles [1], [5], etc.

The above technique of constructing and using solutions, subsolutions, and supersolutions of the Jacobi equation as catalysts to link radial curvature with the Hessian can be employed to more general settings. In Sect. 7, we prove an analog of Theorem A for *pointwise* estimates of the Hessian as follows:

Theorem 7.1(An extension of a theorem of Greene-Wu)**.** *Let the radial curvature K of M satisfy*

$$-\frac{A}{r^2} \le K(r) \le -\frac{A_1}{r^2} \quad \text{on} \quad M\backslash\{x_0\}, \quad \text{where} \quad 0 < A_1 < A.$$

Then

$$\frac{1+\sqrt{1+4A_1}}{2r}\Big(g-dr\otimes dr\Big)\le \mathrm{Hess}(r)\le \frac{1+\sqrt{1+4A}}{2r}\Big(g-dr\otimes dr\Big) \text{ on } M\backslash\{x_0\}.$$

By constructing different comparison functions for solutions of Jacobi type equations in consideration, we have the following:

Theorem 7.2. *Let the radial curvature K of M satisfy*

$$-\frac{a^2}{c^2+r^2} \le K(r) \le -\frac{a_1^2}{c^2+r^2} \text{ on } M\backslash\{x_0\}, \text{ where } 0<a_1^2<a^2, 0\le c.$$

Then

$$\frac{1+\sqrt{1+4a_1^2}}{2(r+c)}\Big(g-dr\otimes dr\Big)\le \mathrm{Hess}(r)\le \frac{1+\sqrt{1+4a^2}}{2r}\Big(g-dr\otimes dr\Big) \text{ on } M\backslash\{x_0\}.$$

We then turn to the study on positively pinched manifolds in Sect 8 and obtain

Theorem 8.1(An analog of Theorem 7.1)**.** *Let the radial curvature K of M satisfy*

$$\frac{b_1^2}{r^2} \le K(r) \le \frac{b^2}{r^2} \quad \text{on} \quad D(x_0), \quad \text{where} \quad 0 < b_1^2 < b^2 \le \frac{1}{4}. \tag{8.1}$$

Then

$$\frac{1+\sqrt{1-4b^2}}{2r}\Big(g-dr\otimes dr\Big)\le \mathrm{Hess}(r)\le \frac{1+\sqrt{1-4b_1^2}}{2r}\Big(g-dr\otimes dr\Big) \text{ on } D(x_0).$$

In Sect 9, we obtain immediate pointwise Laplacian Comparison Theorems by taking traces in Hessian Comparison Theorems. The corresponding weak upper bound estimates of the Laplacian on all of M (c.f. Theorems 9.1, 9.2, 9.3, 9.4, 9.5 and 9.6) are then obtained by an exhaustion method, Green's Identity, and a *double limiting* argument(cf. Lemma 9.1, [4], [6]). In particular, we prove

Theorem 9.6 *Under the radial curvature assumption* (8.1) *on* $D(x_0)$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{1+\sqrt{1-4b^2}}{2r} \leq \Delta r \leq (n-1)\frac{1+\sqrt{1-4b_1^2}}{2r} \quad \text{pointwise on } D(x_0)\,, \tag{9.6}$$
$$\text{and} \quad \Delta r \leq (n-1)\frac{1+\sqrt{1-4b_1^2}}{2r} \quad \text{weakly on } M.$$

We end this paper in Sect 10 by discussing the equivalence of Hessian Comparison Theorems and the equivalence of the Laplacian Comparison Theorems with their immediate consequences (cf. Theorems 10.1 and 10.2).

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2. Preliminaries

The radial vector field ∂ on $D(x_0)$ is the unit vector field such that for any $x \in D(x_0)$, $\partial\,(x)$ is the unit vector tangent to the unique geodesic joining x_0 to x and pointing away from x_0. A radial plane is a plane π which contains $\partial(x)$ in the tangent space T_xM. By the radial curvature K of a manifold, we mean the restriction of the sectional curvature function to all the radial planes. We define $K(t)$ to be the radial curvature of M at x such that $r(x) = t$. Let a tensor $g - dr \otimes dr = 0$ on the radial direction, and be just the metric tensor g on the orthogonal complement $\partial^\perp$. At $x \in M$, the Hessian of r, denoted by Hess(r) is a quadratic form on T_xM given by $\mathrm{Hess}(r)(v,w) = (\nabla_v dr)w = g(\nabla_v \nabla r, w)$ for $v, w \in T_xM$. Here ∇r is the gradient vector field of r, and hence is dual to the differential dr of r. Thus, $\mathrm{Hess}(r)(\nabla r, \nabla r) = 0$. The Laplacian of r, is defined to be $\Delta r = \mathrm{trace}\big(\mathrm{Hess}(r)\big)$. M is said to be a manifold with a pole x_0, if $D(x_0) = M\backslash\{x_0\}$. We recall the following:

Theorem B (cf. [2] [1])**.** *Let (M, g) be a complete Riemannian manifold with a pole x_0, i.e. $D(x_0) = M\backslash\{x_0\}$.*

(i) *If $-\alpha^2 \le K(r) \le -\beta^2$ with $\alpha > 0$, $\beta > 0$, then*

$$\beta\coth(\beta r)\big(g - dr\otimes dr\big) \le \mathrm{Hess}(r) \le \alpha\coth(\alpha r)\big(g - dr\otimes dr\big)$$

(ii) *If $K(r) = 0$, then*

$$\frac{1}{r}\big(g - dr\otimes dr\big) = \mathrm{Hess}(r)$$

(iii) *If $-\frac{A}{(1+r^2)^{1+\epsilon}} \le K(r) \le \frac{B}{(1+r^2)^{1+\epsilon}}$ with $\epsilon > 0, A \ge 0$, and $0 \le B < 2\epsilon$, then*

$$\frac{1-\frac{B}{2\epsilon}}{r}\big(g - dr\otimes dr\big) \le \mathrm{Hess}(r) \le \frac{e^{\frac{A}{2\epsilon}}}{r}\big(g - dr\otimes dr\big)$$

(iv) *If $-Ar^{2q} \le K(r) \le -Br^{2q}$ with $A \ge B > 0$ and $q > 0$, then*

$$B_0 r^q\big(g - dr\otimes dr\big) \le \mathrm{Hess}(r) \le (\sqrt{A}\coth\sqrt{A})r^q\big(g - dr\otimes dr\big)$$

for $r \ge 1$, where $B_0 = \min\{1, -\frac{q+1}{2} + (B + (\frac{q+1}{2})^2)^{1/2}\}$.

Proof. (i), (ii), and (iv) are treated in Section 2 of [2].

(iii) is treated in [1], for completeness, we include a proof here. Since for every $\epsilon > 0$,

$$\frac{d}{ds}\Big(-\frac{1}{2\epsilon}(1+s^2)^{-\epsilon}\Big) = \frac{s}{(1+s^2)^{1+\epsilon}},$$

we have

$$\int_0^\infty s\frac{A}{(1+s^2)^{1+\epsilon}}ds = \frac{A}{2\epsilon} < \infty \quad \text{and} \quad \int_0^\infty s\frac{B}{(1+s^2)^{1+\epsilon}}ds = \frac{B}{2\epsilon} < 1.$$

Now the assertion is an immediate consequence of Quasi-isometry Theorem due to Greene-Wu [2, p.57] in which $1 \le \eta \le e^{\frac{A}{2\epsilon}}$ and $1 - \frac{B}{2\epsilon} \le \mu \le 1$. □

3. Proof of Theorem 1.1

Let $\phi_1 = r^\alpha$, where $\alpha = \frac{1+\sqrt{1+4a^2}}{2}$.

Then $\phi_1' = \alpha r^{\alpha-1}$, and $(2\alpha-1)^2 = 1 + 4a^2$, i.e. $\alpha(\alpha-1) = a^2$.

Hence, for $r > 0$, $\phi_1'' = \alpha(\alpha-1)r^{\alpha-2}$, i.e.

$$\phi_1'' + \frac{-a^2}{r^2}\phi_1 = 0. \tag{3.1}$$

Furthermore,

$$\frac{\phi_1'}{\phi_1} = \frac{\alpha}{r} = \frac{1+\sqrt{1+4a^2}}{2r} \tag{3.2}$$

Let h_1 satisfy

$$\begin{cases} h_1'' + G_1 h_1 = 0, \\ h_1(0) = 0, h_1'(0) = 1 \end{cases} \tag{3.3}$$

where $G_1 = -\frac{a^2}{c^2+r^2}$, and let

$$r_1 = \sup\{r : h_1 > 0 \quad \text{on} \quad (0,r), \quad \text{where} \quad h_1 \quad \text{satisfies} \quad (3.3)\} \tag{3.4}$$

We note $r_1 = \infty$. This can be seen by comparing the solution h_1 of (3.3) with the solution $\tilde{h}(r) = r$ of the following

$$\begin{cases} \tilde{h}_1'' + 0 \cdot \tilde{h}_1 = 0, \\ \tilde{h}_1(0) = 0, \tilde{h}_1'(0) = 1 \end{cases}$$

and applying a standard Sturm Comparison Theorem. Furthermore, $(\phi_1' h_1 - h_1'\phi_1)(0) = 0$, and in view of (3.1),(3.3) and (3.4), for $r \in (0,\infty)$

$$\begin{aligned} (\phi_1' h_1 - h_1'\phi_1)' &= \phi_1'' h_1 - h_1''\phi_1 \\ &= h_1\phi_1\big(\frac{a^2}{r^2} - \frac{a^2}{c^2+r^2}\big) \\ &\geq 0 \end{aligned}$$

The monotonicity then implies that $\phi_1' h_1 \geq h_1'\phi_1$ on $(0,\infty)$ which in turn via (3.2) yields

$$\frac{h_1'}{h_1} \leq \frac{\phi_1'}{\phi_1} = \frac{1+\sqrt{1+4a^2}}{2r} \quad \text{on} \quad (0,\infty). \tag{3.5}$$

Similarly, suppose $\beta = \frac{1+\sqrt{1-4b^2}}{2}$, with $b^2 \leq \frac{1}{4}$, and $\phi_2 = r^\beta$.

Then $(2\beta-1)^2 = 1-4b^2$, i.e. $\beta(\beta-1) = -b^2$, and $\phi_2' = \beta r^{\beta-1}$ for $r > 0$.

Hence, for $r > 0$, $\phi_2'' = \beta(\beta-1) r^{\beta-2}$, i.e.

$$\phi_2'' + \frac{b^2}{r^2}\phi_2 = 0. \tag{3.6}$$

and

$$\frac{\phi_2'}{\phi_2} = \frac{1+\sqrt{1-4b^2}}{2r} \quad \text{for} \quad r > 0. \tag{3.7}$$

Let h_2 satisfy

$$\begin{cases} h_2'' + G_2 h_2 = 0, \\ h_2(0) = 0, h_2'(0) = 1 \end{cases} \tag{3.8}$$

where $G_2 = \frac{b^2}{c^2+r^2}$, and let

$$r_2 = \sup\{r : h_2 > 0 \quad \text{on} \quad (0,r), \quad \text{where} \quad h_2 \quad \text{satisfies} \quad (3.8)\} \tag{3.9}$$

Then by l'Hospital's Rule,

$$\lim_{r\to 0^+} (\phi_2' h_2 - h_2'\phi_2)(r) = \lim_{r\to 0^+} -\frac{\beta}{\beta-1} h_2'(r)\, r^{\beta} = 0.$$

Furthermore, in view of (3.6), (3.8) and (3.9) for $r \in (0, r_2)$

$$\begin{aligned} (\phi_2' h_2 - h_2'\phi_2)' &= \phi_2'' h_2 - h_2''\phi_2 \\ &= \phi_2 h_2 \Big(\frac{-b^2}{r^2} + \frac{b^2}{c^2+r^2}\Big) \\ &\le 0 \end{aligned}$$

Integrating the above inequality on $[\epsilon_1, r_2-\epsilon_2] \subset (0, r_2)$, where $\epsilon_1, \epsilon_2 > 0$, and passing $\epsilon_1, \epsilon_2 \to 0$ we have $\phi_2' h_2 \le h_2'\phi_2$ on $(0, r_2)$. This in turn via (3.7) implies that

$$\frac{1+\sqrt{1-4b^2}}{2r} = \frac{\phi_2'}{\phi_2} \le \frac{h_2'}{h_2} \quad \text{on} \quad (0, r_2). \tag{3.10}$$

Integrating (3.10) on $[\epsilon, r] \subset (0, r_2)$, we have

$$0 < C(\epsilon) r^{\beta} \leq h_2 \quad \text{on} \quad (\epsilon, r) \quad \text{for} \quad \text{every} \quad 0 < \epsilon < r < r_2, \tag{3.11}$$

where $C(\epsilon) > 0$ is a constant depending on ϵ. Thus $h_2 > 0$ on $(0, r_2)$. We claim $r_2 = \infty$. Otherwise there would exist a $\delta > 0$ such that $r_2 + \delta < \infty$ at which $h_2 > 0$ by the continuity, and would lead to, via (3.9) a contradiction

$$r_2 < r_2 + \delta \leq r_2.$$

Applying (3.5) and (3.10) to the following Comparison Theorem 4.1, we obtain the desired (1.2) on $D(x_0)$.

4. Hessian Comparison Theorems via Jacobi Type Inequalities

Theorem 4.1. *Let radial curvature K of a complete n-manifold M satisfying*

$$G_1 \leq K \quad \text{on} \quad D(x_0) \quad \Big(\textit{resp. } K \leq \tilde{G}_2 \quad \text{on} \quad D(x_0)\Big) \tag{4.1}$$

where G_i (resp. $\tilde{G}_i$) is a continuous function on $\mathbb{R}^+ \cup \{0\}$, and $(0, r_i) \subset (0, \infty)$ is the maximal interval in which $h_i, i = 1, 2$ is a positive solution of the following

$$\begin{cases} h_1'' + G_2 h_1 \geq 0, \\ h_1(0) = 0, h_1'(0) = 1 \end{cases} \qquad \left(\textit{resp.} \quad \begin{cases} h_2'' + \tilde{G}_1 h_2 \leq 0, \\ h_2(0) = 0, h_2'(0) = 1 \end{cases} \right) \tag{4.2}$$

Assume

$$G_2 \leq G_1 \quad \Big(\textit{resp.} \quad \widetilde{G_2} \leq \widetilde{G_1}\Big).$$

Then

$$\operatorname{Hess}(r) \leq \frac{h_1'}{h_1}(g - dr \otimes dr) \text{ on } B_{r_1}(x_0) \cap D(x_0), \tag{4.3}$$

$$\left(\textit{resp.} \, \frac{h_2'}{h_2}\big(g - dr \otimes dr\big) \leq \operatorname{Hess}(r) \text{ on } B_{r_2}(x_0) \cap D(x_0)\right)$$

in the sense of quadratic forms.

For the case $G_1 = G_2 = \tilde{G_1} = \tilde{G_2}$, $h_1 = h_2$ and G_1 is a smooth function on $\mathbb{R}$, we refer the reader to Stefano Pigola, Marco Rigoli, and Alberto G. Setti's book [4].

Proof. Let γ be the unit speed geodesic curve joining $x_0 = \gamma(0)$ to $x = \gamma(t_0)$, and V be a parallel vector field along $\gamma(t)$, for $0 \le t \le t_0$. In view of Gauss lemma $\gamma'(t) = \nabla r(\gamma(t))$. By the definitions of curvature tensor R, geodesic, parallel vector field V and zero torsion of the Riemannian connection ∇, one has at x

$$\begin{aligned}\nabla_{\nabla r}\nabla_V \nabla r &= \nabla_V \nabla_{\nabla r}\nabla r + \nabla_{[\nabla r, V]}\nabla r - R(V, \nabla r)\nabla r \\ &= \nabla_{\nabla_{\nabla r} V}\nabla r - \nabla_{\nabla_V \nabla r}\nabla r - R(V, \nabla r)\nabla r \\ &= -\nabla_{\nabla_V \nabla r}\nabla r - R(V, \nabla r)\nabla r\end{aligned}$$

Taking the inner product with V,

$$\langle \nabla_{\nabla r}\nabla_V \nabla r, V\rangle + \langle \nabla_{\nabla_V \nabla r}\nabla r, V\rangle = -\langle R(V, \nabla r)\nabla r, V\rangle \tag{4.4}$$

We note the above second term on the left hand

$$\langle \nabla_{\nabla_V \nabla r}\nabla r, V\rangle = \sum_{i=1}^{n}\langle \nabla_V \nabla r, e_i\rangle\langle \nabla_{e_i}\nabla r, V\rangle = \langle \nabla_V \nabla r, \nabla_V \nabla r\rangle$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field, and the last step follows from the symmetry of the Hessian of r.

Since V is parallel, it follows from (4.4) and (4.1) that

$$\frac{d}{dt}\langle \nabla_V \nabla r, V\rangle + \langle \nabla_V \nabla r, \nabla_V \nabla r\rangle = -\langle R(V, \nabla r)\nabla r, V\rangle \le -G_1 \tag{4.5}$$

Define

$$\lambda_{\max}(x) = \max_{\{v \in T_x(M)\setminus\{0\}, v \perp \nabla r(x)\}} \frac{\operatorname{Hess} r(v,v)}{\langle v, v\rangle}$$

Select a unit vector v at $x = \gamma(t_0)$ such that

$$\langle \nabla_v \nabla r, v\rangle := \operatorname{Hess} r(v,v) = \lambda_{\max} \circ \gamma(t_0)$$

Then

$$\langle \nabla_v \nabla r, \nabla_v \nabla r \rangle = \lambda_{\max}^2 \circ \gamma(t_0)$$

Let the parallel vector field V along γ satisfying $V(t_0) = v$. Then the function $\operatorname{Hess} r(V,V) - \lambda_{\max} \circ \gamma(t) \le 0$, attains its maximum value 0 at $t = t_0$, and if at this point $\lambda_{\max} \circ \gamma$ is differentiable,

$$\frac{d}{dt}\bigg|_{t=t_0} \operatorname{Hess} r(V,V) = \frac{d}{dt}\bigg|_{t=t_0} \lambda_{\max} \circ \gamma(t) \tag{4.6}$$

It follows from (4.5), (4.6), and the fact $\operatorname{Hess}(r) = \frac{1}{r}\big(g - dr \otimes dr\big) + o(1)$ as $t \to 0^+$, that $\lambda_{\max} \circ \gamma$ satisfies

$$\begin{cases} \frac{d}{dt}\lambda_{\max} \circ \gamma + \lambda_{\max}^2 \circ \gamma + G_1 \le 0, \ \text{for a.e.} \quad t > 0, \ \text{where} \ \gamma([0,t]) \subset D(x_0) \\ \lambda_{\max} \circ \gamma = \frac{1}{t} + O(1) \quad \text{as} \quad t \to 0^+ \end{cases}$$

On the other hand, one can transform Jacobi type inequalities (4.2) into the following Ricatti type inequalities by setting $\phi_1 = \frac{h_1'}{h_1}$:

$$\begin{cases} \phi_1' + \phi_1^2 = \frac{h_1''}{h_1} \ge -G_2 \quad \text{on} \quad (0, r_1) \\ \phi_1(t) = \frac{1}{t} + O(1) \quad \text{as} \quad t \to 0^+ \end{cases} \tag{4.7}$$

Indeed, $h_1(t) = t + O(t^2)$, $h_1' = 1 + O(t)$, and

$$\begin{aligned} \phi_1(t) &= \frac{h_1'(t)}{h_1(t)} = \frac{1+O(t)}{t+O(t^2)} = \frac{1}{t}\frac{1+O(t)}{1+O(t)} \\ &= \frac{1}{t}\left(\frac{1}{1+O(t)} + O(t)\right) = \frac{1}{t}(1+O(t)) = \frac{1}{t} + O(1), \ \text{as} \ t \to 0^+ \end{aligned} \tag{4.8}$$

Now the first part of result (4.3) follows from Comparison Theorem 5.1 in which $k_1 = \lambda_{\max} \circ \gamma$ is a supersolution of a generalized Riccati equation and $k_2 = \phi_1$ is a subsolution of the other equation in (5.1).

Similarly, define

$$\lambda_{\min}(x) = \min_{\{v \in T_x(M)\backslash\{0\}, v \perp \nabla r(x)\}} \frac{\operatorname{Hess} r(v,v)}{\langle v, v \rangle}$$

Arguing in the same way by setting $\phi_2 = \frac{h_2'}{h_2}$ and using (4.8), one compares

$$\begin{cases} \frac{d}{dt}\lambda_{\min}\circ\gamma+\lambda_{\min}^2\circ\gamma+\tilde{G}_2\geq 0 \ \text{ for a.e. } t>0, \text{ where } \gamma([0,t])\subset D(x_0) \\ \lambda_{\min}\circ\gamma = \frac{1}{t}+O(1) \quad \text{as} \quad t\to 0^+ \end{cases}$$

with

$$\begin{cases} \phi_2'+\phi_2^2 = \frac{h_2''}{h_2} \leq -\widetilde{G_1} \quad \text{on} \quad (0,r_2) \\ \phi_2(t) = \frac{1}{t}+O(1) \quad \text{as} \quad t\to 0^+ \end{cases} \tag{4.9}$$

Now the counter-part of the results follows from Comparison Theorem 5.1 where supersolution $k_1 = \phi_2$ and subsolution $k_2 = \lambda_{\min}\circ\gamma$ in (5.1), $\kappa = 1$ and $G_2 = \tilde{G}_2 \leq \tilde{G}_1 = G_1$. □

5. Comparison Theorems on Riccati Type Inequalities

Theorem 5.1 (Comparison Theorem for Subsolutions and Supersolutions of Riccati type equations). *Let G_1, G_2 be continuous functions on $[0,\infty)$ with $G_2 \leq G_1$. For $i = 1,2$, let $k_i \in AC(0,t_i)$ be solutions of*

$$k_1{}' + \frac{k_1{}^2}{\kappa} + \kappa G_1 \leq 0 \qquad\qquad k_2{}' + \frac{k_2{}^2}{\kappa} + \kappa G_2 \geq 0 \tag{5.1}$$

a.e. in $(0,t_i)$ satisfying the asymptotic condition

$$k_i(t) = \frac{\kappa}{t} + O(1) \quad \text{as} \quad t\to 0^+,$$

for some constant $\kappa > 0$. Then $t_1 \leq t_2$ and $k_1 \leq k_2$ on $(0,t_1)$.

This is treated in [4], where $G_1 = G_2$.

Proof. Without loss of generality we may assume (5.1) with $\kappa = 1$, since we can rescale k_i so that $\frac{k_i}{\kappa}, i = 1,2$ satisfies (5.1) with $\kappa = 1$. Now observe $k_i(s) - \frac{1}{s}$ is bounded and locally integrable in a neighborhood of 0. Let

$$\phi_i(t) = t\exp\left(\int_0^t k_i(s) - \frac{1}{s}\, ds\right) \tag{5.2}$$

Then $\phi_i \in C^1(0,t_i), \phi_i(0) = 0$ and $\phi_i > 0$ on $(0,t_i)$.

Differentiating (5.2) with respect to t gives

$$\phi_i'(t) = k_i(t)\phi_i(t) \tag{5.3}$$

Hence, $\phi_i'(0) = 1$ and $\phi_i' \in AC(0, t_i)$. Differentiating (5.3) and applying (5.1), one has

$$\phi_1'' + G_1\phi_1 \leq 0 \quad \text{on} \quad (0, t_1) \qquad \phi_2'' + G_2\phi_2 \geq 0 \quad \text{on} \qquad (0, t_2)$$

It follows from the following Sturm Comparison Theorem 6.1 that

$$t_1 \leq t_2 \quad \text{and} \quad k_1 = \frac{\phi_1{}'}{\phi_1} \leq \frac{\phi_2{}'}{\phi_2} = k_2. \qquad \square$$

6. Sturm Comparison Theorems

Theorem 6.1. *Let $G_1, G_2 \in C([0, \infty))$ with $G_2 \leq G_1$, and let $\psi_1, \psi_2 \in C^1([0, \infty))$ with $\psi_1', \psi_2' \in AC([0, \infty))$ be solutions of the problems*

$$\begin{cases} \psi_1'' + G_1\psi_1 \leq 0, & \text{a.e. in}\,(0, \infty), \\ \psi_1(0) = 0, \end{cases} \qquad \begin{cases} \psi_2'' + G_2\psi_2 \geq 0, & \text{a.e. in}\,(0, \infty), \\ \psi_2(0) = 0, \psi_2'(0) > 0 \end{cases}$$

If $\psi_1(r) > 0$ for $r \in (0, t_1)$ $\psi_1'(0) \leq \psi_2'(0)$, and $(0, t_i)$ in the maximum domain in which $\psi_i > 0, i = 1, 2,$ then

$$t_1 \leq t_2$$

and

$$\psi_1'(0) > 0, \quad \frac{\psi_1'}{\psi_1} \leq \frac{\psi_2'}{\psi_2} \quad \text{and} \quad \psi_1 \leq \psi_2 \quad \text{in} \quad (0, t_1).$$

For the case $G_1 = G_2$, please see [4].

Proof. Let $\tau = \sup\{s \in (0, t_1) : \psi_2 > 0 \text{ in } (0, s]\}$. Then $\psi_1, \psi_2 > 0$ on $(0, \tau)$, $\tau \leq t_2$, $(\psi_2'\psi_1 - \psi_1'\psi_2)(0) = 0$, and

$$(\psi_2'\psi_1 - \psi_1'\psi_2)' = \psi_2''\psi_1 - \psi_1''\psi_2 \geq 0 \quad \text{a.e.} \quad \text{in} \quad (0, \tau)$$

Whence $\psi_2'\psi_1 - \psi_1'\psi_2 \geq 0$, and

$$\frac{\psi_1'}{\psi_1} \leq \frac{\psi_2'}{\psi_2}$$

Integrating from $\epsilon(>0)$ to $r(<\tau)$, and passing ϵ to 0 from the right, one has

$$\psi_1(r) = \lim_{\epsilon\to 0^+} \psi_1(r) \leq \lim_{\epsilon\to 0^+} \frac{\psi_1(\epsilon)}{\psi_2(\epsilon)}\psi_2(r) = \frac{\psi_1'(0)}{\psi_2'(0)}\psi_2(r) \leq \psi_2(r) \text{ in } [0,\tau)$$

Thus, $\psi_1'(0) > 0$. Or $\psi_1(r) \leq 0$ on some interval $(0,\delta)$, a contradiction. Furthermore, $t_1 = \tau \leq t_2$. Otherwise, $\tau < t_1$ would lead to, by the continuity

$$0 < \psi_1(\tau) \leq \psi_2(\tau) \quad \text{and hence } \psi_2 > 0 \text{ in } (0, \tau+\delta],$$

for some $0 < \delta < t_1 - \tau$, contradicting the definition of τ. □

7. An extension of A Theorem of Greene-Wu

In contrast to Theorem A, where asymptotic estimates are given, we have the following:

Theorem 7.1. *If*

$$-\frac{A}{r^2} \leq K(r) \leq -\frac{A_1}{r^2} \quad \text{on} \quad M\backslash\{x_0\}, \quad \text{where} \quad 0 < A_1 < A, \tag{7.1}$$

then

$$\frac{1+\sqrt{1+4A_1}}{2r}\Big(g - dr\otimes dr\Big) \leq \mathrm{Hess}(r) \leq \frac{1+\sqrt{1+4A}}{2r}\Big(g - dr\otimes dr\Big) \text{ on } M\backslash\{x_0\}. \tag{7.2}$$

Proof. Arguing as in the proof of Theorem 1.1, Sect 3: We choose $\phi_1 = r^\alpha$, where $\alpha = \frac{1+\sqrt{1+4A}}{2}$. and choose h_1 as in (3.3) where $G_1 = -\frac{A}{r^2}$, Then $(\phi_1' h_1 - h_1'\phi_1)(0) = 0$, and for $r \in (0,\infty)$,

$$\begin{aligned}(\phi_1' h_1 - h_1'\phi_1)' &= \phi_1'' h_1 - h_1''\phi_1 \\ &= h_1\phi_1\big(\frac{A}{r^2} - \frac{A}{r^2}\big) \\ &= 0.\end{aligned}$$

Thus, corresponding to (3.5) one has

$$\frac{h_1'}{h_1} = \frac{\phi_1'}{\phi_1} = \frac{1+\sqrt{1+4A}}{2r} \quad \text{on} \quad (0,\infty).$$

Similarly, choose $\phi_2 = r^\beta$, where $\beta = \frac{1+\sqrt{1+4A_1}}{2}$, with $A_1 > 0$. Choose h_2 as in (3.8) where $G_2 = -\frac{A_1}{r^2}$, Then

$$\lim_{r\to 0^+} (\phi_2' h_2 - h_2'\phi_2)(r) = 0.$$

Furthermore, for $r \in (0,\infty)$

$$\begin{aligned}(\phi_2' h_2 - h_2'\phi_2)' &= \phi_2'' h_2 - h_2''\phi_2 \\ &= \phi_2 h_2 \big(\frac{A_1}{r^2} + \frac{-A_1}{r^2}\big) \\ &= 0\end{aligned}$$

The corresponding (3.10) becomes

$$\frac{1+\sqrt{1+4A_1}}{2r} = \frac{\phi_2'}{\phi_2} = \frac{h_2'}{h_2} \quad \text{on} \quad (0,\infty).$$

The assertions follow from Theorem 4.1. □

The following Theorem recaptures Theorem 7.1, when $c = 0$.

Theorem 7.2. *If*

$$-\frac{a^2}{c^2+r^2} \le K(r) \le -\frac{a_1^2}{c^2+r^2} \ \text{ on } M\backslash\{x_0\}, \text{ where } 0 < a_1^2 < a^2, 0 \le c, \tag{7.3}$$

then

$$\frac{1+\sqrt{1+4a_1^2}}{2(r+c)}\Big(g - dr\otimes dr\Big) \le \text{Hess}(r) \le \frac{1+\sqrt{1+4a^2}}{2r}\Big(g - dr\otimes dr\Big). \tag{7.4}$$

on $M\backslash\{x_0\}$.

Proof. We choose $\phi_2 = (c+r)^\beta$, where $\beta = \frac{1+\sqrt{1+4a_1^2}}{2}$ and let h_2 satisfy

(3.8), where $G_2 = -\frac{a_1^2}{c^2+r^2}$. Then on $(0,\infty)$,

$$\begin{aligned}(\phi_2' h_2 - h_2'\phi_2)' &= \phi_2'' h_2 - h_2''\phi_2 \\ &= \phi_2 h_2\big(\frac{a_1^2}{(c+r)^2} - \frac{a_1^2}{c^2+r^2}\big) \\ &\le 0.\end{aligned}$$

Since $(\phi_2' h_2 - h_2'\phi_2)(0) = -c^\beta \le 0$, $\phi_2' h_2 \le h_2'\phi_2$ on $[0,\infty)$. Hence

$$\frac{1+\sqrt{1+4a_1^2}}{2(c+r)} = \frac{\phi_2'}{\phi_2} \le \frac{h_2'}{h_2} \quad \text{on} \quad (0, r_2).$$

Arguing in the same way as in the proof of Theorem 1.1 completes the proof. □

8. Hessian Comparison Theorems on Positively Pinched Manifolds

In contrast to Corollary 10.1 in which $c = 0$, we have the following:

Theorem 8.1. *Let the radial curvature K of M satisfy*

$$\frac{b_1^2}{r^2} \le K(r) \le \frac{b^2}{r^2} \quad \text{on} \quad D(x_0), \quad \text{where} \quad 0 < b_1^2 < b^2 \le \frac{1}{4}. \tag{8.1}$$

Then

$$\frac{1+\sqrt{1-4b^2}}{2r}\Big(g - dr\otimes dr\Big) \le \operatorname{Hess}(r) \le \frac{1+\sqrt{1-4b_1^2}}{2r}\Big(g - dr\otimes dr\Big) \text{ on } D(x_0). \tag{8.2}$$

Proof. We modify the first part of the proof of Theorem 1.1 by choosing $\alpha = \frac{1+\sqrt{1-4b_1^2}}{2}$, and $G_1 = \frac{b_1^2}{r^2}$. Then by l'Hospital's Rule, $\lim_{r\to 0^+}(\phi_1' h_1 - h_1'\phi_1)(r) = \lim_{r\to 0^+}\frac{\alpha}{1-\alpha} h_1'(r)\, r^\alpha = 0$, and for $r \in (0, r_1)$, r_1 is as in (3.4)

$$\begin{aligned}(\phi_1' h_1 - h_1'\phi_1)' &= \phi_1'' h_1 - h_1''\phi_1 \\ &= h_1\phi_1\big(-\frac{b_1^2}{r^2} + G_1\big) \\ &= 0\end{aligned}$$

Thus, corresponding to (3.5) one has

$$\frac{h_1'}{h_1} = \frac{\phi_1'}{\phi_1} = \frac{1+\sqrt{1-4b_1^2}}{2r} \quad \text{on} \quad (0, r_1)$$

which implies that $r_1 = \infty$. Similarly, choose $\beta = \frac{1+\sqrt{1-4b^2}}{2}$ and $G_2 = \frac{b^2}{r^2}$. Then $\lim\limits_{r\to 0^+}(\phi_2' h_2 - h_2'\phi_2)(r) = \lim\limits_{r\to 0^+}\frac{\beta}{1-\beta} h_2'(r)\, r^\beta = 0$ and for $r \in (0, r_2)$, r_2 is as in (3.9)

$$\begin{aligned}
(\phi_2' h_2 - h_2'\phi_2)' &= \phi_2'' h_2 - h_2''\phi_2 \\
&= h_2\phi_2\big(-\frac{b^2}{r^2} + G_2\big) \\
&= 0
\end{aligned}$$

Thus corresponding to (3.10) one has

$$\frac{h_2'}{h_2} = \frac{\phi_2'}{\phi_2} = \frac{1+\sqrt{1-4b^2}}{2r} \quad \text{on} \quad (0, r_2).$$

which implies that $r_2 = \infty$. Applying Theorem 4.1 completes the proof. $\square$

9. Laplacian Comparison Theorems

Taking traces in Theorem 1.1, Theorem 1.2, and Corollary 1.1, we immediately obtain the pointwise estimates for Δr on $D(x_0)$. The corresponding weak estimates on M follow from the following Lemma by a *double limiting* argument (cf. [4], [6]):

Lemma 9.1. *If $\Delta r \le f(r)$ holds pointwise in $D(x_0)$, where $f \in C^0(0,\infty)$, then $\Delta r \le f(r)$ holds weakly on M. That is, for every $0 \le \varphi(r) \in C_0^\infty(M)$,*

$$\int_M \varphi(r)\Delta r\, dv \le \int_M \varphi(r) f(r)\, dv$$

Proof. Let $\Omega = \exp_{x_0}(E)$, where E is the maximal star shaped domain $(\subset T_{x_0}M)$ on which $\exp_{x_0} : E \to \Omega$ is a diffeomorphism. Then $\mathrm{Cut}(x_0) = \partial(\exp_{x_0}(E))$ has measure 0, and $M = \Omega \cup \mathrm{Cut}(x_0)$, and Ω is star-shaped. We can exhaust Ω by a family $\{\Omega_n\}_{n=1}^\infty$ of relatively compact and star-shaped domains with smooth boundaries such that

$$\overline{\Omega_n} \subset \Omega_{n+1} \quad \text{and} \quad \bigcup_{n=1}^{\infty} \Omega_n = \Omega$$

Since $\mathrm{Cut}(x_0)$ has measure 0, for every $0 \le \varphi(r) \in C_0^\infty(M)$,

$$\int_M r\Delta\varphi\, dv = \int_\Omega r\Delta\varphi\, dv = \lim_{n\to\infty} \int_{\Omega_n} r\Delta\varphi\, dv$$

Let ν_n be the unit outer normal to $\partial\Omega_n \cup \partial B_\delta(x_0)$. Then it follows from Green's Identity and $\frac{\partial r}{\partial \nu_n} > 0$ on the boundary of star-shape Ω_n that

$$\begin{aligned}
&\int_{\Omega_n} r\Delta\varphi\, dv \\
&= \int_{\Omega_n \setminus B_\delta(x_0)} r\Delta\varphi\, dv + \int_{B_\delta(x_0)} r\Delta\varphi\, dv \\
&= \int_{\Omega_n \setminus B_\delta(x_0)} \varphi\Delta r\, dv - \int_{\partial\Omega_n \cup \partial B_\delta(x_0)} \big(\varphi \frac{\partial r}{\partial \nu_n} - r\frac{\partial \varphi}{\partial \nu_n}\big)\, dS + \int_{B_\delta(x_0)} r\Delta\varphi\, dv \\
&= \int_{\Omega_n \setminus B_\delta(x_0)} \varphi\Delta r\, dv - \int_{\partial\Omega_n} \varphi \frac{\partial r}{\partial \nu_n}\, dS + \big(\int_{\partial\Omega_n} r\frac{\partial \varphi}{\partial \nu_n}\, dS\big) \\
&\qquad + \big(\int_{B_\delta(x_0)} r\Delta\varphi\, dv - \int_{\partial B_\delta(x_0)} \varphi \frac{\partial r}{\partial \nu_n} - r\frac{\partial \varphi}{\partial \nu_n}\, dS\big) \\
&:= \int_{\Omega_n \setminus B_\delta(x_0)} \varphi\Delta r\, dv - \int_{\partial\Omega_n} \varphi \frac{\partial r}{\partial \nu_n}\, dS + I_n + I_\delta \\
&\le \int_{\Omega_n \setminus B_\delta(x_0)} \varphi f(r)\, dv + 0 + I_n + I_\delta \\
&\to \int_M \varphi f(r)\, dv \quad \text{as} \quad \delta \to 0 \quad \text{and} \quad n \to \infty
\end{aligned}$$

Combining the above identity and the inequality gives the desired. $\square$

Theorem 9.1. *Under the radial curvature assumption* (1.1) *or* (1.3) *or* (1.6) *on* $D(x_0)$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{1+\sqrt{1-4b^2}}{2r} \le \Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r} \quad \text{pointwise on } D(x_0), \tag{9.1}$$
$$\text{and } \Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r} \quad \text{weakly on} \quad M.$$

As an immediate consequence of Theorem 1.3 and Lemma 9.1,

Theorem 9.2. *Under the radial curvature assumption* (1.4) *on* $D(x_0)$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{B}{r} \le \Delta r \le (n-1)\frac{A}{r} \quad \text{pointwise on } D(x_0), \tag{9.2}$$
$$\text{and} \quad \Delta r \le (n-1)\frac{A}{r} \quad \text{weakly on } M.$$

Theorem 4.1 and Lemma 9.1 imply immediately

Theorem 9.3. *Under the curvature assumption* (4.1) *on* $D(x_0)$, *the assumption* (4.2) *in which* $(0, r_i) \subset (0, \infty)$ *is the maximal interval in which* $h_i, i = 1, 2$ *is a positive solution, with* $G_2 \le G_1$(*resp.* $\widetilde{G_2} \le \widetilde{G_1}$), *the Laplacian of the distance function satisfies:*

$$\Delta r \le (n-1)\frac{h_1'}{h_1} \text{ on } B_{r_1}(x_0) \cap D(x_0), \text{and weakly on } B_{r_1}(x_0), \tag{9.3}$$
$$\Big(\textit{resp.}\, (n-1)\frac{h_2'}{h_2} \le \Delta r \text{ on } B_{r_2}(x_0) \cap D(x_0)\Big).$$

Theorem 9.4. *Under the radial curvature assumption* (7.1) *on* $M\backslash\{x_0\}$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{1+\sqrt{1+4A_1}}{2r} \le \Delta r \le (n-1)\frac{1+\sqrt{1+4A}}{2r} \text{ pointwise on } M\backslash\{x_0\}, \tag{9.4}$$
$$\text{and} \quad \Delta r \le (n-1)\frac{1+\sqrt{1+4A}}{2r} \quad \text{weakly on } M.$$

Theorem 9.5. *Under the radial curvature assumption* (7.3) *on* $M\backslash\{x_0\}$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{1+\sqrt{1+4a_1^2}}{2(c+r)} \le \Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r} \text{ pointwise on } M\backslash\{x_0\}, \tag{9.5}$$
$$\text{and} \quad \Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r} \quad \text{weakly on } M.$$

Theorem 9.6. *Under the radial curvature assumption* (8.1) *on* $D(x_0)$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{1+\sqrt{1-4b^2}}{2r} \le \Delta r \le (n-1)\frac{1+\sqrt{1-4b_1^2}}{2r} \text{ pointwise on } D(x_0), \tag{9.6}$$

$$\text{and} \quad \Delta r \le (n-1)\frac{1+\sqrt{1-4b_1^2}}{2r} \quad \text{weakly on } M.$$

Corollary 9.1. *Under the radial curvature assumption* (10.1) *on* $D(x_0)$, *the Laplacian of the distance function satisfies*

$$(n-1)\frac{1+\sqrt{1-4b^2}}{2r} \le \Delta r \le (n-1)\frac{1}{r} \quad \text{pointwise on } D(x_0), \tag{9.7}$$
$$\text{and} \quad \Delta r \le (n-1)\frac{1}{r} \quad \text{weakly on } M.$$

Corollary 9.2. *Under the radial curvature assumption* (10.3) *on* $M\backslash\{x_0\}$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{1}{r} \le \Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r} \quad \text{on} \quad M\backslash\{x_0\}. \tag{9.8}$$
$$\text{and} \quad \Delta r \le (n-1)\frac{1+\sqrt{1+4a^2}}{2r} \quad \text{weakly on } M.$$

As an immediate consequence of Corollary 1.2,

Corollary 9.3. *Under the radial curvature assumption* (1.7) *on* $D(x_0)$, *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{B}{r} \le \Delta r \le (n-1)\frac{1}{r} \quad \text{pointwise on } D(x_0), \tag{9.9}$$
$$\text{and} \quad \Delta r \le \frac{n-1}{r} \quad \text{weakly on } M.$$

Corollary 1.3 implies at once the following:

Corollary 9.4. *Under the assumption of* (1.9), *the Laplacian of the distance function satisfies:*

$$(n-1)\frac{1}{r} \le \Delta r \le (n-1)\frac{A}{r} \quad \text{pointwise on} \quad M\backslash\{x_0\}, \tag{9.10}$$
$$\text{and} \quad \Delta r \le (n-1)\frac{A}{r} \quad \text{weakly on } M.$$

Corollary 9.5 (cf. [2])**.** *If* (1.11) *holds, then*

$$\Delta r \le (n-1)\frac{1}{r} \quad \Big(\text{resp.} \quad (n-1)\frac{1}{r} \le \Delta r.\Big) \tag{9.11}$$

In particular,

$$\text{if} \quad K(r) \equiv 0 \quad \text{then} \quad \Delta r = \frac{n-1}{r}.$$

10. The equivalence of Hessian Comparison Theorems and The equivalence of Laplacian Comparison Theorems

Proposition 10.1. *Three Hessian Comparison Theorems* 1.1, 1.2, *and* 1.3 *are equivalent.*

Proof. (i) Theorem 1.1 $\Longleftrightarrow$ Theorem 1.2: If (1.1) holds then (1.3) holds by choosing $c = 0$ in (1.1). Conversely if (1.3) holds, then (1.1) holds, since $[-\frac{a^2}{c^2+r^2}, \frac{b^2}{c^2+r^2}] \subset [-\frac{a^2}{r^2}, \frac{b^2}{r^2}]$ (ii) Theorem 1.2 $\Longleftrightarrow$ Theorem 1.3: This is due to the fact that $A = \frac{1+\sqrt{1+4a^2}}{2}$ and $B = \frac{1+\sqrt{1-4b^2}}{2}$ if and only if $a^2 = A(A-1)$ and $b^2 = B(1-B)$. $\square$

Proposition 10.2. *Three implications that state Laplacian Comparison Theorems in Theorem 9.1, i.e.* (1.1) $\Longrightarrow$ (9.1), (1.3) $\Longrightarrow$ (9.1), (1.6) $\Longrightarrow$ (9.1) *are equivalent.*

By setting $a = 0$, Theorem 1.1 takes the following form:

Corollary 10.1.

$$0 \le K(r) \le \frac{b^2}{c^2+r^2} \tag{10.1}$$

on $D(x_0)$ *where* $0 \le b^2 \le \frac{1}{4}$, *and* $0 \le c^2$, *then*

$$\frac{1+\sqrt{1-4b^2}}{2r}\Big(g - dr \otimes dr\Big) \le \operatorname{Hess}(r) \le \frac{1}{r}\Big(g - dr \otimes dr\Big) \tag{10.2}$$

on $D(x_0)$ *in the sense of quadratic forms. Furthermore, the pointwise and weak Laplacian estimates* (9.7) *hold.*

By setting $b = 0$, Theorem 1.1 takes the following form:

Corollary 10.2. *If*

$$-\frac{a^2}{c^2+r^2} \le K(r) \le 0 \tag{10.3}$$

on $M\backslash\{x_0\}$ *where* $0 \le a^2$, *and* $0 \le c^2$, *then*

$$\frac{1}{r}\Big(g - dr \otimes dr\Big) \le \mathrm{Hess}(r) \le \frac{1+\sqrt{1+4a^2}}{2r}\Big(g - dr \otimes dr\Big) \quad \text{on} \quad M\backslash\{x_0\}. \tag{10.4}$$

Furthermore, the pointwise and weak Laplacian estimates (9.8) *hold.*

Combining Theorem 8.1 and Corollary 10.1 in which $c = 0$, one has

Theorem 10.1. *Let the radial curvature* K *of* M *satisfy*

$$\frac{b_1^2}{r^2} \le K(r) \le \frac{b^2}{r^2} \quad \text{on} \quad D(x_0), \quad \text{where} \quad 0 \le b_1^2 < b^2 \le \frac{1}{4}. \tag{8.1}$$

Then the Hessian estimates (8.2) *and the pointwise and weak Laplacian estimates* (9.6) *hold.*

Analogously, combining Theorem 7.2 and Corollary 10.2 in which $c = 0$, one has

Theorem 10.2. *Let*

$$-\frac{a^2}{r^2} \le K(r) \le -\frac{a_1^2}{r^2} \quad \text{on} \quad M\backslash\{x_0\}, \quad \text{where} \quad 0 \le a_1^2 < a^2. \tag{10.5}$$

Then the Hessian estimates (7.4) *and the pointwise and weak Laplacian estimates* (9.5) *hold.*

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