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K-STABILITY AND CANONICAL METRICS ON TORIC MANIFOLDS

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Dedicated to Professor Neil S. Trudinger on the occasion of his 70th birthday

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Abstract

The K-stability is closely related to the existence of canonical metrics on Kähler manifolds and is an important issue in complex geometry. In this paper we discuss the K-stability on toric Kähler manifolds and present an unstable example of toric Kähler surface with eight $T^2_{\mathbb{C}}$-fixed points.

1. Introduction

The existence of canonical metrics (Kähler-Einstein metrics, constant scalar curvature metrics, and extremal metrics) on Kähler manifolds is a central problem in complex geometry. A well known folklore conjecture by Yau-Tian-Donaldson [43, 38, 14] asserts that a compact complex polarised manifold (M, L) admits canonical metrics in $2\pi c_1(L)$ if and only if the underlying manifold is stable in the sense of geometric invariant theory. Among various notions of stabilities, the K-stability is the most widely studied and significant progress has been made. It is sometimes also called the

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K-polystable when stressing the nontrivial holomorphic vector fields on the manifolds.

The concept of K-stability was first introduced by Tian [37] in the study of Kähler-Einstein metrics. He proved the "only if" part of the conjecture for the first Chern class (if it is positive) on M, i.e, K-stability is a necessary condition for the existence of Kähler-Einstein metrics when the holomorphic automorphism group is trivial. Later, Donaldson extended the K-stability to general polarized varieties [15] and made a conjecture on the relation between the K-stability and the existence of constant scalar curvature Kähler metrics. Stoppa [32] generalized Tian's result to a compact Kähler manifold M with a constant scalar curvature Kähler metric and discrete holomorphic automorphism group. The assumption on the holomorphic automorphism group in these results were later removed by Berman and Mabuchi [5, 25]. Mabuchi also introduced a stronger K-stability to study the conjecture [26, 27, 28].

As a generalization of Kähler-Einstein metrics, the extremal metric in a Kähler class on a compact complex manifold M was introduced by E. Calabi in 1982 [6]. To study the extremal metrics, the definition of K-stability was extended by Szekelyhidi [33] to Kähler classes with nonzero extremal vector fields and was called relative K-stability. Stoppa and Szekelyhidi also proved that the existence of general extremal metrics implies K-stability relative to a maximal torus of automorphisms [36]. However it is still unknown whether it is true that the existence of general extremal metrics implies the K-stability relative to the extremal $\mathbb{C}^*$-action. In the case of toric manifolds, a positive answer was given in [45].

For the "if" part of this conjecture, a remarkable breakthrough has been made very recently. In 2012, the existence of Kähler-Einstein metrics on a Fano manifold is proved to be equivalent to the K-stability as originally expected [39, 8, 9, 10]. The cases for the constant scalar curvature metrics and the more general extremal metrics are more difficult because one has to solve a fourth order elliptic equation. It is a challenge in differential geometry and PDE theory. On some special manifolds, for example, the projective bundles, the conjecture was confirmed, see [3].

With the success on the existence of Kähler-Einstein metrics by Tian and Chen-Donaldson-Sun, the notion of K-stability attracted more attentions. On general complex manifolds, the K-stability is far from being well-understood. However, on toric manifolds, Donaldson [14] set up a strategy for this problem and he proved the conjecture for toric surfaces when the Kähler class admits vanishing Futaki invariant [15, 16, 17].

In this article, we discuss some issues related to Donaldson's strategy on the K-stability on toric manifolds. In Section 2, we recall the definitions of extremal metrics and K-stabilities on general polarised manifolds. Donaldson's reduction of the problem is described in Section 3. Then in Section 4, we focus on the dimension 2 case and state a further simplification of K-stability on toric surfaces. Examples of stable and unstable polytopes associated to toric Kähler surfaces are discussed in the last section. In Section 5, we present a new example of unstable polytope with 8 vertices.

We refer the readers to the expositions [18, 29, 30] for more details on this topic.

2. Extremal metrics and K-stability

In this section, we recall the definition of extremal metrics and K-stabilities.

2.1. Extremal metrics

Let (M^n, ω_g) be an n-dimensional compact Kähler manifold, where g is a Kähler metric and ω_g is its Kähler form. In [6, 7], Calabi proposed to study critical points of the energy functional

$$\frac{1}{V(M)}\int_M S(\omega)^2\frac{\omega^n}{n!}$$

in the fixed Kähler class $[\omega_g]$. Here $\omega \in [\omega_g]$, $S(\omega)$ is the scalar curvature of ω, and $V(M)$ is the volume of M. A Kähler metric in $[\omega_g]$ is called *extremal* if it is a critical point of Calabi's energy.

In [23], it is shown that for a given Kähler class $[\omega_g]$, there exists an extremal holomorphic vector field X, no matter whether the extremal metric exists. An equivalent definition of extremal metrics can be given as follows: A Kähler metric ω in the Kähler class $[\omega_g]$ is extremal iff

$$S(\omega) = \bar{S} + \theta_X(\omega), \tag{2.1}$$

where $\bar{S}$ is the average of the scalar curvature of ω_g and $\theta_X(\omega)$ is the normalized potential of X with respect to ω such that

$$i_X\omega = \sqrt{-1}\,\bar{\partial}\theta_X(\omega), \quad \text{and} \int_M \theta_X(\omega)\frac{\omega^n}{n!} = 0. \tag{2.2}$$

In particular, if X vanishes (i.e., Fukaki invariant vanishes), the extremal metric, if exists, is a constant scalar curvature metric. Furthermore, if we choose the Kähler class to be a multiple of the first Chern class and X vanishes, then the extremal metric, if exists, is a Kähler-Einstein metric.

2.2. Donaldson-Futaki invariant

In this subsection, we recall the definition of Donaldson-Futaki invariant. Futaki invariant was an holomorphic invariant first constructed by Futaki and Calabi on any Kähler manifold. It is an obstruction to the existence of constant scalar curvature metric [7, 22]. This definition was extended to the case of Fano normal varieties in [20]. Later, Tian defined the notion of K-stability of a Fano manifold M using this invariant and some degenerations of M. In [14], Donaldson defined the general Futaki invariant for polarized scheme in an algebraic way. Here we state this definition as follows.

Let (M, L) be a polarized scheme, where L is an ample line bundle. Let α be a $\mathbb{C}^*$-action on (M, L). Then for any positive integer k, α induces a $\mathbb{C}^*$-action on the vector space $H_k = H^0(M, L^k)$. Denote by d_k the dimension of the vector space H_k and $w_k(\alpha)$ the weight of the induced action on the highest exterior power H_k. Then d_k and w_k are given by polynomials of k as

$$\begin{aligned} d_k &= \alpha_0 k^n + \alpha_1 k^{n-1} + \cdots, \\ w_k(\alpha) &= \beta_0 k^{n+1} + \beta_1 k^n + \cdots. \end{aligned}$$

The *Donaldon-Futaki invariant* of α on (M, L) is defined to be

$$\mathcal{F}(\alpha) = \frac{\alpha_1\beta_0 - \alpha_0\beta_1}{\alpha_0}.$$

Donaldson also proved that when M is a smooth manifold and the $\mathbb{C}^*$-action is induced by a holomorphic vector field X, this definition coincides with Futaki's original result [22]: let ω be a Kähler metric in $2\pi c_1(L)$, then

$$\mathcal{F}(\alpha) = -\frac{1}{4V(M)}\int_M X(h)\frac{\omega^n}{n!}, \tag{2.3}$$

where $h = G(S(\omega) - \bar{S})$, G is the Green's operator, $\bar{S}$ is the average of scalar curvature. Note that the integral in (2.3) is the original Futaki invariant. Hence, when M is a manifold, Donaldson-Futaki invariant is the original Futaki invariant multiplied by a constant.

As was pointed out, Futaki invariant is an obstruction to the existence of constant scalar curvature metric. When Futaki invariant does not vanish, we consider general extremal metrics. Hence, a modification of Futaki invariant is needed.

To define the modified Donaldson-Futaki invariant, we first need an inner product for the $\mathbb{C}^*$-actions [33]. Let α, β be two $\mathbb{C}^*$-actions on a polarized scheme (M, L). Suppose that A_k and B_k are the infinitesimal generators of the actions on $H^0(M, L^k)$, respectively. The inner product (α, β) is given by

$$Tr\left[\left(A_k - \frac{Tr(A_k)I}{d_k}\right)\left(B_k - \frac{Tr(B_k)I}{d_k}\right)\right] = (\alpha,\beta)k^{n+2} + O(k^{n+1}).$$

The *modified Donaldson-Futaki invariant* is given by

$$\mathcal{F}_\beta(\alpha) = \mathcal{F}(\alpha) - \frac{(\alpha,\beta)}{(\beta,\beta)}\mathcal{F}(\beta), \tag{2.4}$$

where $\mathcal{F}(\alpha)$ and $\mathcal{F}(\beta)$ are Donaldson-Futaki invariants of α and β, respectively.

2.3. Notions of K-stabilities

As we said above, the definition of K-stability on Fano manifolds was first given by Tian [37]. Based on the algebraic definition of Futaki invariant above, Donaldson established the K-stability of general polarized manifolds. The definition is related to its degenerations, called test configuration [14]. A *test configuration* for a polarized Kähler manifold (M, L) of exponent r consists of

(1) a scheme $\mathcal{W}$ with a $\mathbb{C}^*$-action;

(2) a $\mathbb{C}^*$-equivariant ample line bundle $\mathfrak{L}$ on $\mathcal{W}$;

(3) a $\mathbb{C}^*$-equivariant flat family of schemes

$$\pi : \mathcal{W} \longrightarrow \mathbb{C},$$

where $\mathbb{C}^*$ acts on $\mathbb{C}$ by multiplication. We require that the fibers $(\mathcal{W}_t, \mathfrak{L}|_{\mathcal{W}_t})$ are isomorphic to (M, L^r) for any $t \neq 0$.

Note that since π is $\mathbb{C}^*$-equivariant, the $\mathbb{C}^*$-action can be restricted to the central fiber. A test configuration is called trivial if $\mathcal{W} = M \times \mathbb{C}$ is a product. The following definition of K-stability was given by [14].

Definition 2.1. A polarized Kähler manifold (M, L) is *K-semistable* if for any test-configuration the Futaki invariant of the induced $\mathbb{C}^*$-action on $(\mathcal{W}_0, \mathfrak{L}|_{\mathcal{W}_0})$ is nonnegative. It is called *K-stable* if in addition the equality holds if and only if the test-configuration is trivial.

It has been proved that this K-stability is a necessary condition for the existence of constant scalar curvature metrics in $2\pi c_1(L)$ on a polarized Kähler manifold (M, L) [32, 25].

In the case that Futaki invariant does not vanish, we need a modified notion of K-stability to study extremal metrics. In [33], Szekelyhidi introduced the notion of relative K-stability based on the modified Donaldson-Futaki invariant as a generalization of the K-stability. Let us recall the definition of relative K-stability.

Let χ be the $\mathbb{C}^*$-action induced by the extremal vector field X. We say that a test configuration is *compatible* with χ, if there is $\mathbb{C}^*$-action $\tilde{\chi}$ on $(\mathcal{W}, \mathfrak{L})$ such that $\pi : \mathcal{W} \longrightarrow \mathbb{C}$ is an equivariant map with trivial $\mathbb{C}^*$-action on $\mathbb{C}$ and the restriction of $\tilde{\chi}$ to $(\mathcal{W}_t, \mathfrak{L}|_{\mathcal{W}_t})$ for nonzero t coincides with that of χ on (M, L^r) under the isomorphism. Note that $\mathbb{C}^*$-action α on $\mathcal{W}$ induces $\mathbb{C}^*$-action on the central fibre $M_0 = \pi^{-1}(0)$ and the restricted line bundle $\mathfrak{L}|_{M_0}$. We denote by $\tilde{\alpha}$ and $\tilde{\chi}$ the induced $\mathbb{C}^*$-action of α and χ on $(M_0, \mathfrak{L}|_{M_0})$, respectively.

The relative K-stability is defined as follows.

Definition 2.2 ([33]). A polarized Kähler manifold (M, L) is *relatively K-semistable* if $\mathcal{F}_{\tilde{\chi}}(\cdot) \leq 0$ for any test-configuration compatible with χ. It is called *relatively K-stable* if in addition that the equality holds if and only if the test-configuration is trivial.

As shown in [3] by examples, the relative K-stability may not be enough to ensure the existence of extremal metrics on general manifolds. A refinement of K-stability in the sense $\mathcal{F}_{\tilde{\chi}}(\alpha) \leq -C\|\alpha\|$ was introduced in [35], where C is a positive constant, $\|\cdot\|$ is a norm for the $\mathbb{C}^*$-action α. We will discuss it later, in the setting of toric manifolds, which are the main objects in this article.

Finally, we would like to point out that since Donaldson-Futaki invariant can be defined for polarized schemes, all the above notions are also well defined for schemes. We only stated the definitions for polarized manifolds which is enough for this article.

3. Reduction on Toric Manifolds

A complex manifold M is called *toric*, if there is a complex torus Hamiltonian action $T^n_{\mathbb{C}}$ on M and the action has a dense free orbit, identified with $T^n_{\mathbb{C}} = (\mathbb{C}^*)^n = (S^1)^n \times \mathbb{R}^n$. It is known that Kähler metrics on toric manifolds can be characterised by functions on the associated moment polytopes [1, 24]. With this property, Donaldson built up a program of studying the existence of constant scalar curvature metrics and stabilities on toric manifolds [14]. He obtained a reduction of the problem on the moment polytopes. Later, the reduction was extended to general extremal metrics [44]. In this section, we will describe this reduction, for more details we refer the readers to [1, 2, 19, 14, 24, 44].

3.1. Delzant polytopes and Abreu's formula

Assume that (M, g) is an n-dimensional toric Kähler manifold with a torus action $T \cong (\mathbb{C}^*)^n$. Then the open dense orbit of T in M induces an global coordinates $(w_1, \ldots, w_n) \in (\mathbb{C}^*)^n$. To do the reduction we use the affine logarithmic coordinates $z_i = \log w_i = \xi_i + \sqrt{-1}\eta_i$. If g is a $(S^1)^n$-invariant Kähler metric, ω_g is determined by a convex function ψ_0 which depends only on $\xi_1, \ldots, \xi_n \in \mathbb{R}^n$ in the coordinates $(z_1, \ldots, z_n)$, namely $\omega_g = 2\sqrt{-1}\partial\bar{\partial}\psi_0$ on $(\mathbb{C}^*)^n$. Since the torus action T is Hamiltonian, there exists a moment map $m : M \to \mathbb{R}^n$, and the image is a convex polytope in $\mathbb{R}^n$. Moreover, the moment map can be given by

$$(m_1, \ldots, m_n) = \left(\frac{\partial\psi_0}{\partial\xi_1}, \ldots, \frac{\partial\psi_0}{\partial\xi_n}\right),$$

that is the gradient of ψ_0. Denote the image by $P = D\psi_0(\mathbb{R}^n)$. Then P is a convex polytope. This polytope is independent of the choice of the metric g in the class $[\omega_g]$. However, P can not be an arbitrary polytope in $\mathbb{R}^n$. It satisfies several special conditions, usually called *Delzant's conditions*, which can be stated as follows [19, 1]:

(1) There are exactly n edges meeting at each vertex p.

(2) The edges meeting at the vertex p are rational, i.e., each edge is of the form $p + tv_i$, $0 \le t < \infty$, $v_i \in \mathbb{Z}^n$.

(3) The vectors $v_1, \ldots, v_n$ can be chosen to be a basis of $\mathbb{Z}^n$.

As a conclusion, for an n-dimensional compact toric manifold M, together with an associated Kähler class $[\omega_g]$, $(M, [\omega_g])$, there is an associated bounded convex polytope $P \subset \mathbb{R}^n$ satisfying Delzant's conditions. Conversely, from a convex polytope $P \subset \mathbb{R}^n$ satisfying Delzant's conditions, one can recover a toric manifold and the associated Kähler class $(M, [\omega_g])$. See [19] for details.

We will characterize the $(S^1)^n$-invariant Kähler metrics under the polytope coordinates. The polytope P can be represented by a set of inequalities of the form

$$P = \{x \in \mathbb{R}^n : \ \langle x, \ell_i\rangle \le \lambda_i, \ i = 1, 2, \ldots, d\}, \tag{3.1}$$

where ℓ_i is the normal to a face of P, λ_i is a constant, and d is the number of faces of P. Delzant's conditions can be equivalently stated as follows.

(1) There are exactly n faces meeting at each vertex p.

(3) The normals ℓ_i $(i = 1, 2, \ldots, d)$ are vectors in $\mathbb{Z}^n$.

(3) At any given vertex p, let $\ell_{i_1}, \ldots, \ell_{i_n}$ be the normals to the faces at p, then $\det(\ell_{i_1}, \ldots, \ell_{i_n}) = \pm 1$.

Remark 3.1. (i) Note that if $\det(\ell_1, \ldots, \ell_n) = 1$ and if $\ell_i \in \mathbb{Z}^n$, the matrix $(\ell_1, \ldots, \ell_n)$ can be reduced to the unit matrix by Gauss elimination. Therefore $(\ell_1, \ldots, \ell_n)$ is a basis of $\mathbb{Z}^n$.

(ii) The constants $\lambda_1, \ldots, \lambda_d$ are not necessarily integers, and can change continuously. When they are all integers, the associated Kähler class is called integral [24] and from the polytope P we can recover a polarized toric manifold.

(iii) Two different polytopes may correspond to the same toric manifold $(M, [\omega_g])$. Indeed, all Delzant triangles correspond to the complex projective space $\mathbb{CP}^2$.

(iv) We also note that the set of all Kähler classes on a toric manifold M is a finite dimensional convex cone. Moreover, a Kähler class is the first Chern class if and only if $\lambda_i = 1$ for all $i = 1, \ldots, d$ (up to translation of coordinates).

By using the Legendre transformation $\xi = (D\psi_0)^{-1}(x)$, one sees that the function (Legendre dual function) defined by

$$u_0(x) = \langle \xi, D\psi_0(\xi) \rangle - \psi_0(\xi) = \langle \xi(x), x \rangle - \psi_0(\xi(x)), \ \forall\, x \in P$$

is convex. In general, for any G_0-invariant metric $\omega \in [\omega_g]$, $\omega = 2\sqrt{-1}\partial\bar{\partial}\psi$ on $(\mathbb{C}^*)^n$, where ψ is a convex function on $\mathbb{R}^n$. Then one gets a convex function $u_\omega(x)$ on P, also called symplectic potential of ω, by using the above relation while ψ is replaced by ψ. Set

$$\mathcal{C} = \{u = u_0 + f \mid u \text{ is a convex function in } \ P, \ f \in C^\infty(\bar{P})\}.$$

It was shown in [1] that there is a bijection between functions in $\mathcal{C}$ and G_0-invariant Kähler metrics. For any function u in $\mathcal{C}$, it can be explicitly given by [24, 2]:

$$u = \frac{1}{2} \sum_1^d (\lambda_i - \langle \ell_i, x \rangle) log(\lambda_i - \langle \ell_i, x \rangle) + f, \tag{3.2}$$

where f is a function smooth up to boundary of P. We usually say that a function satisfies *Guillemin's boundary condition* if it can be written in the form of (3.2). The scalar curvature of ω can be expressed through its symplectic potential by [1]

$$S(\omega) = -\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j}, \tag{3.3}$$

where (u^{ij}) is the inverse matrix of $(u_{ij}) = (\frac{\partial^2 u}{\partial x_i \partial x_j})$. In x-coordinates, the potential $\theta_X(\omega)$ of the extremal vector field X is independent of the choice of ω in $[\omega_g]$ and is an affine linear function, uniquely determined by the Futaki invariant. We denote it by θ_X. Therefore, on toric manifolds, the existence of extremal metric reduces to finding smooth solutions u to

$$-\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = \bar{S} + \theta_X, \tag{3.4}$$

defined on a Delzant's polytope P such that u satisfies Guillemin's boundary condition, which is given by (3.2). Here $\bar{S}$ is the average of scalar curvature. This equation is called *Abreu's equation.*

3.2. Reduction of Futaki invariant

Let $d\sigma_0$ be the Lebesgue measure on the boundary ∂P and ν be the outer normal vector field on ∂P. Then we define a measure

$$d\sigma = \frac{(\nu, x)}{\lambda_i} d\sigma_0 = \frac{1}{|\ell_i|} d\sigma_0 \tag{3.5}$$

on the face $\langle \ell_i, x\rangle = \lambda_i$ of P. Futaki invariant can be simply expressed on the polytope by [14]

$$\frac{\mathrm{Vol}(P)}{V(M)} \mathcal{F}(\frac{\partial}{\partial z_i}) = -\left(\int_{\partial P} x_i \, d\sigma - \bar{S} \int_P x_i \, dx \right). \tag{3.6}$$

For simplicity, we denote $A := \bar{S} + \theta_X$, and define a linear functional $\mathcal{L}$ by

$$\mathcal{L}(u) = \int_{\partial P} u \, d\sigma - \int_P Au \, dx. \tag{3.7}$$

A is an affine linear function in the polytope coordinates $\{x_1, \dots, x_n\}$, which can be determined as follows. Let $A = a_0 + \sum_1^n a_i x_i$. Then $a_0, a_1, \dots, a_n$

can be determined uniquely by the $n+1$-equation system

$$\mathcal{L}(1)=0,\ \mathcal{L}(x_i)=0,\ i=1,\dots,n. \tag{3.8}$$

3.3. Reduction of K-stability

We consider the relative K-stability of a polarized toric manifold (M,L) which corresponds to an integral polytope P in $\mathbb{R}^n$ (i.e. when λ_i in (3.1) are integers). In [15], Donaldson induced toric degenerations as a class of special test configuration induced by positive rational, piecewise linear functions on P. The reduction of the stability is based on these degenerations.

Recall that a *piecewise linear(PL)* convex function u on P is of the form

$$u=\max\{u^1,\dots,u^r\},$$

where $u^\lambda=\sum a_i^\lambda x_i+c^\lambda, \lambda=1,\dots,r$, for some vectors $(a_1^\lambda,\dots,a_n^\lambda)\in\mathbb{R}^n$ and some numbers $c^\lambda\in\mathbb{R}$. u is called a *rational piecewise linear* convex function if the coefficients a_i^λ and numbers c^λ are all rational.

For a positive rational PL convex function u on P, we choose an integer R so that

$$Q=\{(x,t)\mid x\in P,\ 0<t<R-u(x)\}$$

is a convex polytope in $\mathbb{R}^{n+1}$. Without loss of generality, we may assume that the coefficients a_i^λ are integers and Q is an integral polytope. Otherwise we replace u by lu and Q by lQ for some integer l, respectively. Then the $n+1$-dimensional polytope Q determines an $(n+1)$-dimensional toric variety M_Q with a holomorphic line bundle $\mathfrak{L}\to M_Q$. Note that the face $\bar{Q}\cap\{\mathbb{R}^n\times\{0\}\}$ of Q is a copy of the n-dimensional polytope P, so we have a natural embedding $i: M\to M_Q$ such that $\mathfrak{L}|_M=L$. Decomposing the torus action $T_{\mathbb{C}}^{n+1}$ on M_Q as $T_{\mathbb{C}}^n\times\mathbb{C}^*$ so that $T_{\mathbb{C}}^n\times\{\mathrm{Id}\}$ is isomorphic to the torus action on M, we get $\mathbb{C}^*$-action α by $\{\mathrm{Id}\}\times\mathbb{C}^*$. Hence, we define an equivariant map

$$\pi: M_Q\to\mathbb{CP}^1$$

satisfying $\pi^{-1}(\infty)=i(M)$. One can check that $\mathcal{W}=M_Q\backslash i(M)$ is a test configuration for the pair (M,L), called a *toric degeneration* [14]. This test configuration is compatible to the $\mathbb{C}^*$-action χ induced by the extremal holomorphic vector field X on M. In fact, χ as a group is isomorphic to a one parameter subgroup of $T^n_{\mathbb{C}} \times \{\mathrm{Id}\}$, which acts on $\mathcal{W}$. Since the action is trivial in the direction of α, the test configuration is compatible.

The modified Donaldson-Futaki invariant for a toric degeneration has an explicit formula in polytope coordinates. Indeed, the following proposition relates the K-stability to the positivity of functional (3.7). It was first proved in [14] for the case Futaki invariant vanishes and then extended to general case in [44].

Proposition 3.2. *For a $\mathbb{C}^*$-action α on a toric degeneration on M induced by a positive rational PL-convex function u, we have*

$$\mathcal{F}_{\tilde{\chi}}(\tilde{\alpha}) = -\frac{1}{2Vol(P)}\mathcal{L}(u), \tag{3.9}$$

where χ is the $\mathbb{C}^*$-action induced by the extremal holomorphic vector field X, and $\mathcal{F}_{\tilde{\chi}}(\tilde{\alpha})$ is given by (2.4).

With this proposition, we call (M, L) is *relatively K-stable for toric degenerations* if its associated polytope satisfies $\mathcal{L}(u) \geq 0$ for all rational PL convex functions u on P and if $\mathcal{L}(u) = 0$ for a rational PL convex function u, then u must be a linear function. Then on direction of the Yau-Tian-Donaldson conjecture for toric manifolds can be proved.

Theorem 3.3 ([45]). *Let (M, L) be a polarized toric manifold which admits an extremal metric in $2\pi c_1(L)$. Then (M, L) is relatively K-stable* (*for toric degenerations*).

The theorem is equivalent to that the linear functional $\mathcal{L}$ is positive for all nontrivial rational PL convex functions when equation (3.4) is solvable. In fact, a stronger result was proved in [45], i.e., $\mathcal{L}$ is positive for all nontrivial PL convex functions. Due to this reason, we will use the positivity of $\mathcal{L}$ as the definition of relative K-stability on toric manifolds and we usually omit the words "relative" and "for toric degenerations" for simplicity.

Definition 3.4. We call a polytope P associated to a polarised toric manifold (M, L) is *K-stable* if $\mathcal{L}(u) \geq 0$ for all PL convex functions u on P; and if $\mathcal{L}(u) = 0$ for a PL convex function u, then u must be a linear function.

Remark 3.5. In [14] the above K-stability was defined on polarised toric manifolds, that is the case when the constants λ_i in (3.1) are integers. But obviously this definition can be extended to general polytopes when the constants λ_i in (3.1) are not integers or rational numbers.

The 'if' part of the Yau-Tian-Donaldson conjecture is rather difficult. Most of the recent developments occur in dimension 2. In a series of papers, Donaldson gave a confirmative answer to toric surfaces in the special case that the Kähler class has vanishing Futaki invariant, by a continuity method.

Theorem 3.6 ([15, 16, 17])**.** *A toric surface M admit a constant scalar curvature metric in $2\pi c_1(L)$ if and only if (M, L) is K-stable and Futaki invariant vanishes.*

In the case that Futaki invariant does not vanish, the problem is still open. An important progress has been made recently by [12] when the scalar curvature is positive.

Another interesting problem is how to verify the K-stability condition for a given toric Kähler manifold. The following theorem gives a criterion for the K-stability on toric manifolds.

Theorem 3.7 ([44])**.** *Let (M, L) be an n-dimensional polarized toric manifold and P be the associated polytope. Suppose that for each $i = 1, \ldots, d$, it holds*

$$\bar{S} + \theta_X \leq \frac{n+1}{\lambda_i}, \tag{3.10}$$

where $\lambda_i > 0$ are d numbers defined as in (3.1)*. Then M is relative K-stable.*

Note that $\theta_X \equiv 0$ is equivalent to that the Futaki invariant vanishes. If the Futaki invariant is not zero, (3.10) becomes $\theta_X \leq 1$. It has been verified on toric Fano surfaces [44]. It is interesting to ask whether this condition is satisfied for higher dimensional toric Fano manifolds or not. In case that M is a toric Fano manifold with the vanishing Futaki invariant, condition (3.10) in this theorem is trivial on the canonical Kähler class $2\pi c_1(M)$ since $\bar{S} = n$ and all λ_i are 1. Thus Theorem 3.7 is always true for these toric manifolds. Indeed, the problem of existence of Kähler-Einstein metrics on toric Fano manifolds, has been completely solved by Wang and Zhu [41]. A more general approach to verify the stability of polytopes will be discussed in dimension 2 in the next section.

On the other hand, to find a toric manifold (M, L) which is K-unstable, it suffices to find a bounded convex Delzant polytope P and a PL convex function u such that $\mathcal{L}(u) < 0$. Examples will be discussed in the final section.

3.4. Uniform K-stability

The K-stability is determined by the positivity of the linear functional $\mathcal{L}$. Let P^* be the union of the interior of P and the interiors of its co-dimension 1 faces. Denote

$$\mathcal{C}_1 = \{u \mid u \text{ is convex on } P^* \text{ and } \int_{\partial P} u < \infty\}. \tag{3.11}$$

The linear functional $\mathcal{L}$ is well defined in $\mathcal{C}_1$. Note that, for $u \in \mathcal{C}_1$, it may not be uniform bounded at the vertices of P, but the value of u at vertices has no effect on the integral $\int_{\partial P} f\, d\sigma$. Without loss of generality, we assume 0 lies in the interior of P. Since $\mathcal{L}$ is invariant when subtracting an affine linear function, and homogeneous with respect to scaling, we consider the set of normalized functions

$$\begin{aligned}\tilde{\mathcal{C}}_1 = \{u \mid\ & u \text{ is a convex function in } P^* \text{ satisfying} \\ & \int_{\partial P} u d\sigma = 1 \text{ and } \inf_P u = u(0) = 0\}.\end{aligned} \tag{3.12}$$

By a simple observation that

$$\int_P u \le C \int_{\partial P} u\, d\sigma,\ u \in \tilde{\mathcal{C}}_1, \tag{3.13}$$

where C is independent of u, $\mathcal{L}$ has a lower bound on $\tilde{\mathcal{C}}_1$.

When proving the existence of constant scalar curvature metrics on toric surfaces [16, 17], Donaldson introduced a refinement of the K-stability. P is called *uniformly K-stable* if

$$\inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u) \ge \lambda_P > 0,$$

where λ_P is a constant depending only on P. In [11], it is proved that the above uniform K-stability is necessary to the existence of extremal metrics on toric manifolds. In particular, in dimension 2, the K-stability and uniform K-stability are equivalent [14, 42].

A different definition of uniform K-stability with $L^{\frac{n}{n-1}}$-norm of convex functions was given in the thesis of Szekelyhidi. In [35], a polytope P is called uniformly K-stable if

$$\mathcal{L}(f) \geq \lambda_P \|f\|_{L^{\frac{n}{n-1}}(P)}$$

for all normalised convex functions. He also showed that there exists a constant such that

$$\|f\|_{L^2(P)} \leq C \int_{\partial P} f \, d\sigma$$

for all non-negative continuous convex functions on P.

The problem of optimal destabilisating test configuration was also studied by Szekelyhidi [34]. He considered the minimum of $\mathcal{L}$ in the space of L^2-convex functions with $\|\cdot\|_{L^2(P)} = 1$. It is proved that if P is K-unstable, then there is a unique convex function u such that the infimum of $\mathcal{L}$ in the considered space is attained at u. Furthermore, it is shown that if u is piecewise linear, then the maximal subpolytopes of P on which u is linear give a standard decomposition of P into semistable pieces. This is an analogous to the Harder-Narasimhan filtration of an unstable vector bundle.

In the next section, one can see that when considering $\mathcal{L}$ in $\tilde{\mathcal{C}}_1$, the destabilisation functions in dimension 2 can be reduced to simple piecewise linear convex functions.

4. The Simple Characterisation on Toric Surfaces

In the next section, we focus on the dimension 2 case, that is, on toric surfaces. A further reduction on the positivity of $\mathcal{L}$ to simple PL convex functions will be established.

Following Donaldson, we say a convex function u is *simple piecewise linear (SPL)* if there is a linear function ℓ such that $u = \max\{0, \ell\}$. If u is simple PL, the set $\mathcal{I}_u = P \cap \{\ell = 0\}$ is called the *crease* of u.

Theorem 4.1. *Let P be a convex polytope in $\mathbb{R}^2$. Then P is (relatively) K-stable if and only if $\mathcal{L}(u) > 0$ for all SPL convex functions with crease $\mathcal{I}_u \neq \emptyset$.*

The theorem contains the following two issues on the simplification of destabilizing functions.

(1) K-semistable case, i.e., if $\mathcal{L}(u) \geq 0$ for all $u \in \mathcal{C}_1$ but there is a nonlinear convex function $u \in \mathcal{C}_1$ such that $\mathcal{L}(u) = 0$. Then there is a SPL convex function $\hat{u}$ such that $\mathcal{L}(\hat{u}) = 0$.

(2) K-unstable case, i.e., if there is $u \in \mathcal{C}_1$ such that $\mathcal{L}(u) < 0$, then there is a SPL convex function $\hat{u}$ such that $\mathcal{L}(\hat{u}) \leq 0$.

(1) was first proved by Donaldson under the assumption $A \geq 0$ (Proposition 5.3.1, [14]). The condition $A \geq 0$ was removed in [42]. (2) was included in [42] as a corollary, but we omitted the details there. By modifying the proof in [42], we give a general proof combing the two issues together here.

As in last section, we assume 0 lies in the interior of P. In particular, we assume 0 lies in the interior of $\{x \in P \mid A(x) = 0\}$ if it is nonempty.

Lemma 4.2. *Let P be a convex polytope $P \subset \mathbb{R}^n$. Suppose $\inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u) \leq 0$. There is a $u \in \tilde{\mathcal{C}}_1$ such that the infimum is attained at u and u is continuous at any co-dimension 1 face of the polytope.*

Proof. Assume $u_k \in \tilde{\mathcal{C}}_1$ be a sequence of functions minimizing $\mathcal{L}$. Then u_k converges to a function u in P and $\int_P u_k \to \int_P u$. We can extend u to any co-dimension 1 face F by letting

$$u(x_0) = \lim_{x \in P, x \to x_0} u(x)$$

for $x_0 \in F$ (the value of u on the codimension 2 edges does not affect the integral $\int_{\partial P} d\sigma$). Note that by $\int_{\partial P} u_k d\sigma = 1$, u_k converges to a function $\tilde{u}$ on the set of all co-dimension 1 faces. If there is a point $x_0 \in F$ at which $u < \tilde{u}$, then we have $\mathcal{L}(u) < \inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u)$. Multiply u by a constant $c > 1$ such that $\int_{\partial P} cu \, d\sigma = 1$. Then $cu \in \tilde{\mathcal{C}}_1$ and

$$\mathcal{L}(u) < c \inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u) \leq \inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u).$$

The contradiction follows. □

Lemma 4.3. *Let P be a convex polytope $P \subset \mathbb{R}^n$. Suppose $\inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u) \le 0$ and the minimum is attained at u, then u is a generalized solution to the degenerate Monge-Ampère equation*

$$\det D^2 u = 0. \tag{4.1}$$

Proof. We consider the case that $\{x \in P \mid A(x) = 0\} \cap P$ is not empty. Denote

$$P_+ = \{x \in P \mid A(x) > 0\},\ P_- = \{x \in P \mid A(x) < 0\}.$$

Let

$$u_+(x) = \sup\{\ell(x) \mid \ell \text{ is a linear function with } \ell \le u \text{ in } P_- \cup \partial P\}. \tag{4.2}$$

Then $u_+ = u$ in P_- and on ∂P, and $u_+ \ge u$ in P_+. This implies that $u_+ \in \tilde{\mathcal{C}}_1$. If there is a point $x \in P_+$ such that $u_+(x) > u(x)$, then $\mathcal{L}(u_+) < \mathcal{L}(u)$, in contradiction with the assumption that $\inf_{u \in \tilde{\mathcal{C}}_1} \mathcal{L} = \mathcal{L}(u)$. Hence $u_+ = u$ in P. By (4.2), u satisfies the degenerate Monge-Ampère equation (4.1) in P_+.

Next let

$$u_-(x) = \sup\{\ell(x) :\ \ell \text{ is a supporting function of } u \text{ at some point } x \in P_+\}. \tag{4.3}$$

Then $u_- = u$ in $\overline{P}_+$ and $u_- \le u$ in $\overline{P}_-$. If there is a point $x \in P_-$ such that $u_-(x) < u(x)$, then $\mathcal{L}(u_-) < \mathcal{L}(u)$. Let $\tilde{u}_- = (\int_{\partial P} u_-\, d\sigma)^{-1} u_- \in \tilde{\mathcal{C}}_1$. Since $\int_{\partial P} u_-\, d\sigma \le 1$, we have

$$\mathcal{L}(\tilde{u}_-) < \left(\int_{\partial P} u_-\, d\sigma\right)^{-1} \mathcal{L}(u) \le \inf_{u \in \tilde{\mathcal{C}}_1} \mathcal{L}.$$

The contradiction follows. Hence $u_- = u$ in P.

The above facts imply that u satisfies the degenerate Monge-Ampere equation (4.1) in the whole polytope P. □

Let u be a generalized solution of (4.1) in $\Omega \in \mathbb{R}^n$. For any interior point $z \in \Omega$, let $L_z = \{x_{n+1} = \phi(x), x \in \mathbb{R}^n\}$ be a supporting plane of u at z. By convexity, the set $\mathcal{T} := \{x \in \Omega :\ u(x) = 0\}$ is convex. By (4.1), $\mathcal{T}$ cannot be a single point. The following lemma often used in the study of Monge-Ampere equation.

Lemma 4.4. *An extreme point of $\mathcal{T}$ must be a boundary point of Ω.*

Now we can prove the theorem by showing that $\mathcal{L}$ can always be destabilised by SPL convex functions.

Proposition 4.5. *Let P be a convex polytope $P \subset \mathbb{R}^2$. Suppose $\inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u) \le 0$. Then the infimum can be attained at a SPL convex function.*

Proof. We prove it by contradiction. Suppose that $\mathcal{L}(v) > 0$ for all SPL convex functions in $\tilde{\mathcal{C}}_1$. Then there exists $\sigma_0 > 0$ such that $\mathcal{L}(v) > \sigma_0$ for any SPL convex function $v = \max(0, \ell)$ with $|D\ell| = 1$ and $|\{x \in P \mid v(x) > 0\}| \ge \delta_0$.

By the assumption $\inf_{\tilde{\mathcal{C}}_1} \mathcal{L}(u) \le 0$ and Lemma 4.3, the minimum is attained at u, such that $\det D^2 u = 0$. By Lemma 4.4, the extreme points of $\mathcal{T} = \{x \in P \mid u(x) = 0\}$ are located on ∂P. Assume $n = 2$. Then $\mathcal{T}$ is either a line segment with both endpoint on ∂P, or $\mathcal{T}$ is a polytope (which is a convex subset of P) with vertices on ∂P.

By a rotation of the coordinates, we assume $\mathcal{T}$ is contained in the x_1-axis in the former case, or an edge of $\mathcal{T}$ is contained in the x_1-axis and $\mathcal{T} \subset \{x_2 \le 0\}$ in the latter case. For any point $(x_1, 0) \in P$, let

$$a(x_1) = \lim_{t \searrow 0} \frac{1}{t}(u(x_1, t) - u(x_1, 0)).$$

By convexity the limit exists and is nonnegative. Let $a_0 = \inf a(x_1)$. We must have $a_0 = 0$, otherwise denote

$$\psi(x) = \max(0, x_2), \tag{4.4}$$

and $u_1 = u\chi_-$, $u_2 = (u - a_0\psi)\chi_+$, where $\chi_- = 1$ in $\{x_2 < 0\}$ and $\chi_- = 0$ in $\{x_2 > 0\}$, and $\chi_+ = 1 - \chi_-$. Then ψ is a SPL convex function,

$$u = u_1 + u_2 + a_0\psi,$$

and $u_1 + u_2$ is convex in P. Hence

$$\mathcal{L}(u) = \mathcal{L}(u_1 + u_2) + a_0\mathcal{L}(\psi) > \mathcal{L}(u_1 + u_2),$$

Normalize u_1+u_2 by

$$\widetilde{u_1+u_2} = \left(\int_{\partial P} u_1 + u_2\, d\sigma\right)^{-1} (u_1+u_2)$$

such that $\widetilde{u_1+u_2} \in \tilde{\mathcal{C}}_1$. It is clear that $\int_{\partial P} u_1 + u_2\, d\sigma < 1$. Hence, $\mathcal{L}(\widetilde{u_1+u_2}) < \mathcal{L}(u)$. We reach a contradiction.

Since $u > 0$ in $P \cap \{x_2 > 0\}$ and the set $G_\epsilon := \{x \in P :\ u(x) < \epsilon\psi(x)\} \neq \emptyset$, we have

$$G_\epsilon \subset \{0 \leq x_2 < \delta\} \quad \text{with } \delta \to 0 \text{ as } \epsilon \to 0 \tag{4.5}$$

(otherwise by taking limit we would reach a contradiction as $\mathcal{T} \subset \{x_2 \leq 0\}$). Denote

$$\begin{aligned} u_1 &= u\chi_-, \\ u_2 &= (u - \epsilon\psi)\chi_+, \\ \tilde{u}_2 &= \max(u_2, 0). \end{aligned}$$

Then $u = u_1 + u_2 + \epsilon\psi$ and $u_1 + \tilde{u}_2$ is convex in P. Denote $\tilde{u} = u_1 + \tilde{u}_2 + \epsilon\psi$. We have

$$\mathcal{L}(\tilde{u}) = \mathcal{L}(u_1 + \tilde{u}_2) + \epsilon\mathcal{L}(\psi) \geq \mathcal{L}(u_1 + \tilde{u}_2) + \epsilon\sigma_0. \tag{4.6}$$

On the other hand, observing that $0 \leq \tilde{u}_2 - u_2 \leq \epsilon\delta$, we have $u \leq \tilde{u} \leq u + \epsilon\delta$. It follows that

$$\mathcal{L}(\tilde{u}) \leq \mathcal{L}(u) + C\epsilon\delta. \tag{4.7}$$

By (4.6), (4.7), we have

$$\mathcal{L}(u_1 + \tilde{u}_2) \leq \mathcal{L}(u) + \epsilon(\delta - C\sigma_0).$$

Normalize $u_1 + \tilde{u}_2$ by

$$\widetilde{u_1+\tilde{u}_2} = \left(\int_{\partial P} u_1 + \tilde{u}_2\, d\sigma\right)^{-1} (u_1+\tilde{u}_2)$$

such that $\widetilde{u_1+\tilde{u}_2} \in \tilde{\mathcal{C}}_1$. It is clear that $u_1 + \tilde{u}_2 \leq u$, which implies $\int_{\partial P} u_1 + \tilde{u}_2\, d\sigma \leq 1$. But recall that $\delta \to 0$ as $\epsilon \to 0$. Hence when $\epsilon > 0$ is sufficiently small, we obtain $\mathcal{L}(\widetilde{u_1+\tilde{u}_2}) < \mathcal{L}(u)$. The contradiction follows. $\square$

The theorem can be used to verify the (relative) K-stability of polytopes. Namely to verify the (relative) K-stability for a polytope $P \in \mathbb{R}^2$, by Theorem 4.1 it suffices to verify $\mathcal{L}(u) \geq 0$ for all SPL convex functions u. In the last section, we provide examples of stable and unstable polytopes.

5. Examples

In this section, we present examples of unstable Delzant polytopes in dimension 2.

The vertices of a Delzant polytope correspond to the fixed points of $T^2_{\mathbb{C}}$ action on the associated toric surface. A toric surface with 3 or 4 $T^2_{\mathbb{C}}$-fixed points must be $\mathbb{CP}^2$ or a Hirzebruch surface $\mathbb{F}_k(k = 0, 1, 2, \cdots)$, and they all admits extremal metrics in any Kähler class and are K-stable. When the surface has more T^2-fixed points, only partial results are known up to now. For example, on $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ which has 5 fixed points, we only know a family of Kähler classes with symmetry admitting extremal metrics [13]. It is known that every compact toric surface can be obtained from $\mathbb{CP}^2$ or $\mathbb{F}_k$ by a succession of blow-ups at $T^2_{\mathbb{C}}$-fixed points ([21], p.42). More precisely, let M be a toric surface with Kähler class $[\omega_g]$ corresponding to a polytope P. Then a $T^2_{\mathbb{C}}$-fixed point X of M corresponds to a vertex p of the polytope P. A blow-up of M at X is a new toric Kähler surface which corresponds to a convex polytope $\tilde{P}$ obtained by chopping off a corner of the polytope P at p. By applying the result in [4], one sees that on every toric surface, there is a Kähler class which admits an extremal metric [42].

On the other hand, there are examples of unstable polytopes. This means that in the associated Kähler classes, there is no extremal metric. The first example of unstable polytope was found by Donaldson ([14], Section 7.2). It is a symmetric polytope with large number of vertices so that its Futaki invariant vanishes. An interesting question is how the stability is affected by the number of fixed points. In [42], we asked how many vertices at least can destabilise a Delzant polytope, and found an unstable polytope with 9 vertices. Computations in that paper suggests the case of 8 vertices is the borderline case. In this paper we are able to provide a new unstable example with exactly 8 vertices.

5.1. Unstable polytopes

For a given polytope, denote

$$\begin{aligned} b_0 &= \int_{\partial P} d\sigma, \quad b_1 = \int_{\partial P} x_1 d\sigma, \quad b_2 = \int_{\partial P} x_1 d\sigma, \\ v_0 &= \int_P dx, \quad v_1 = \int_P x_1 dx, \quad v_2 = \int_P x_2 dx, \\ v_{11} &= \int_P x_1^2 dx, \quad v_{22} = \int_P x_2^2 dx, \quad v_{12} = \int_P x_1 x_2 dx. \end{aligned}$$

Note that the boundary measure $d\sigma$ is not the standard Lebesgue measure and is given by (3.5).

Let k be a positive integer, β be a positive constant to be determined later and $\alpha = \beta k^2$. Choose $\epsilon_i > 0$ $(i = 1, 2, 3)$ small enough. Let $P_{\epsilon_1,\epsilon_2,\epsilon_3}$ be the polytope with vertices given by

$$\begin{aligned} p_0 &= (-1, -\alpha - 2k), \\ p_1 &= (-1, \alpha + 4), \\ p_2 &= (0, \alpha + 4), \\ p_3 &= (\epsilon_1, \alpha + 4 - \epsilon_1), \\ p_4 &= (\epsilon_1 + \epsilon_2, \alpha + 4 - \epsilon_1 - 2\epsilon_2), \\ p_5 &= (\epsilon_1 + \epsilon_2 + \epsilon_3, \alpha + 4 - \epsilon_1 - 2\epsilon_2 - 3\epsilon_3), \\ p_6 &= (1, \alpha + 3\epsilon_1 + 2\epsilon_2 + \epsilon_3), \\ p_7 &= (1, -\alpha). \end{aligned}$$

One can check that this polytope satisfies Delzant conditions. It can be seen as a four times blow-up surface from the Hirzebruch surface $\mathbb{F}_k$. We prove

Theorem 5.1. *$P_{\epsilon_1,\epsilon_2,\epsilon_3}$ is relative K-unstable when ϵ_1, ϵ_2, ϵ_3 are sufficiently small and β, k are sufficiently large.*

Proof. Note that when $\epsilon_i \to 0$, $i = 1, 2, 3$, $P_{\epsilon_1,\epsilon_2,\epsilon_3}$ converges to a polytope P with vertices given by

$$\begin{aligned} p_0 &= (-1, -\alpha - 2k), \\ p_1 &= (-1, \alpha + 4), \\ p_2 &= (0, \alpha + 4), \\ p_3 &= (1, \alpha), \end{aligned}$$

$$p_4 \;=\; (1, -\alpha).$$

Since we actually concern on the stability of $P_{\epsilon_1,\epsilon_2,\epsilon_3}$, it does not matter whether P satisfies Delzant's condition. We can still have the computations for A and $\mathcal{L}$. It suffices to find a PL function on P such that $\mathcal{L} < 0$.

First we estimate A on P. It is a affine linear function that can be uniquely determined by the polytope by (3.8). Denote by $A = a_0 + a_1x_1 + a_2x_2$. By computation,

$$\begin{aligned}
b_0 \;&=\; 4\alpha + 2k + 8, \quad\; b_1 = -2k - 4, \quad\; b_2 = -2\alpha k + 4\alpha - 2k^2 - 2k + 14,\\
v_0 \;&=\; 4\alpha + 2k + 6, \quad\; v_1 = -\frac{2}{3}k - \frac{4}{3}, \quad v_2 = -2\alpha k + 6\alpha - \frac{4}{3}k^2 + \frac{32}{3},\\
v_{11} \;&=\; \frac{4}{3}\alpha + \frac{2}{3}k + \frac{5}{3},\; v_{12} = \frac{2}{3}\alpha k - \frac{4}{3}\alpha + \frac{2}{3}k^2 - \frac{10}{3},\\
v_{22} \;&=\; \frac{4}{3}\alpha^3 + 2\alpha^2 k + 6\alpha^2 + \frac{8}{3}\alpha k^2 + \frac{4}{3}k^3 + \frac{64}{3}\alpha + \frac{80}{3}.
\end{aligned}$$

Substituting them into (3.8), we have a linear equation system

$$\begin{aligned}
&4\alpha + 2k + 8\\
&\quad=\; (4\alpha + 2k + 6)a_0 + (-\frac{2}{3}k - \frac{4}{3})a_1 + (-2\alpha k + 6\alpha - \frac{4}{3}k^2 + \frac{32}{3})a_2,\\
&-2k - 4\\
&\quad=\; (-\frac{2}{3}k - \frac{4}{3})a_0 + (\frac{4}{3}\alpha + \frac{2}{3}k + \frac{5}{3})a_1 + (\frac{2}{3}\alpha k - \frac{4}{3}\alpha + \frac{2}{3}k^2 - \frac{10}{3})a_2,\\
&-2\alpha k + 4\alpha - 2k^2 - 2k + 14\\
&\quad=\; (-2\alpha k + 6\alpha - \frac{4}{3}k^2 + \frac{32}{3})a_0 + (\frac{2}{3}\alpha k - \frac{4}{3}\alpha + \frac{2}{3}k^2 - \frac{10}{3})a_1\\
&\qquad\; + (\frac{4}{3}\alpha^3 + 2\alpha^2 k + 6\alpha^2 + \frac{8}{3}\alpha k^2 + \frac{4}{3}k^3 + \frac{64}{3}\alpha + \frac{80}{3})a_2.
\end{aligned}$$

By the first two equations, we have

$$a_0 = \frac{\alpha^2 + O(\alpha k) + O(\alpha^2 k)a_2}{\alpha^2}, \quad a_1 = \frac{O(\alpha k) + O(\alpha^2 k)a_2}{\alpha^2 + O(\alpha k)}.$$

Substituting them to the third equation, we have

$$a_2 = O(\alpha^{-2}).$$

Note that $\alpha = \beta k^2$. Again substituting a_2 into the first two equations, we have

$$a_0 = 1 + \frac{3\beta - 1}{6\beta^2}k^{-2} + O(k^{-3}), \quad a_1 = -\frac{1}{\beta}k^{-1} + O(k^{-2}).$$

Now we let $u_t = \max\{x_2 - \alpha + tk, 0\}$ and $P_t = \{x \in P \mid u_t > 0\}$. Then we have

$$\begin{aligned} \mathcal{L}(u_t) &= \int_{\partial P_t} u_t d\sigma - \int_{P_t} A u_t dx, \\ \frac{d\mathcal{L}(u_t)}{dt} &= k\left(\int_{\partial P_t} d\sigma - \int_{P_t} A dx\right). \end{aligned}$$

It is clear that

$$\int_{\partial P_0} d\sigma = 6, \int_{P_0} dx = 6, \int_{P_0} x_1 dx = -\frac{4}{3}.$$

For k sufficiently large and $0 << t << k$, we have

$$\begin{aligned} \frac{d\mathcal{L}(u_t)}{dt} &= k\left(\int_{\partial P_t} d\sigma - \int_{P_t} A dx\right) \\ &= k\left[\int_{\partial P_0} d\sigma - \int_{P_0} A dx + (2 - 2a_0)tk - a_2 t^2 k^2\right] \\ &= k\left[6 - 6a_0 - \left(\int_{P_0} x_1 dx\right) a_1 + O(k^{-2}) + (2 - 2a_0)tk - a_2 t^2 k^2\right] \\ &\le -\frac{4}{3\beta} - \frac{3\beta - 1}{3\beta^2} t + O(t^2 k^{-1}). \end{aligned}$$

Choose $\beta \ge \frac{1}{3}$. Then $\frac{d\mathcal{L}(u_t)}{dt} \le -\frac{4}{3\beta}$ for sufficiently large k. Note that $\mathcal{L}(u_0)$ is uniformly bounded with respect to k. By choosing k sufficiently large, we obtain $\mathcal{L}(u_t) < 0$ for $0 << t << k$. □

5.2. K-stable polytopes

An interesting question is whether all Delzant polytope with 7 or less vertices are stable. However, even though the K-stability of a toric surface can be reduced to the positivity of the functional $\mathcal{L}$ for all SPL functions, the verification is technically a difficult problem, even for the polytopes with small number of vertices. In the case that Futaki invariant vanishes, that is, when A is constant, the verification was carried out in [42]. It was computed that among the toric surfaces with 5 or 6 $T^2_{\mathbb{C}}$-fixed points, $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$ is the only one which admits Kähler classes with vanishing Futaki invariants. In addition, all such associated polytopes have been founded. After scaling, the vertices are given by

$$\begin{aligned} p_0 &= (0,0), \\ p_1 &= (0,h), \\ p_2 &= (s,h+s), \\ p_3 &= (1,h+1), \\ p_4 &= (1,1-t), \\ p_5 &= (t,0), \end{aligned}$$

where the nonnegative parameters t, s, h satisfies

$$s+t \;=\; 1 \tag{5.1}$$

or

$$h = 1-t \;=\; 1-s. \tag{5.2}$$

Let H be the hyperplane divisor of $\mathbb{CP}^2$, and D_1, D_2, D_3 be the three exceptional divisors. Then after a dilation, the Kähler class corresponding to (5.1) is $3H-aD_1-bD_2-(3-a-b)D_3$ and the Kähler class corresponding to (5.2) is $3H-c(D_1+D_2+D_3)$, where a, b, c are positive constants, $a+b<3$, and $c<\frac{3}{2}$. It is verified that $\mathcal{L}$ is positive for all SPL convex functions on these polytopes [42]. Hence, by Theorem 3.6, it implies the existence of constant scalar curvature metrics in the associated Kähler class. In the case when $t=s=\frac{1}{2}$, it is half of the first Chern class on $\mathbb{CP}^2\#3\overline{\mathbb{CP}^2}$. In this case, a constant scalar curvature metric is a Kähler-Einstein metric and was also obtained in [31, 40].

References

1. M. Abreu, Kähler geometry of toric varieties and extremal metrics, *Inter. J. Math.*, **9**(1998), 641-651.

2. M. Abreu, Kähler geometry of toric manifolds in symplectic coordinates, in "*Symplectic and Contact Topology: Interactions and Perspectives*", pp. 1-24, Fields Inst. Commun., 35, Amer. Math. Soc., 2003.

3. V. Apostolov, D. Calderbank, P. Gauduchon and C. Tonnesen-Friedman, Hamiltonian 2-forms in Kähler geometry III, Extremal metrics and stability, *Invent. Math.*, **173**(2008), 547-601.

4. C. Arezzo, F. Pacard and M. Singer, Extremal metrics on blow-ups, *Duke Math. J.*, **157**(2011), 1-51.

5. R. J. Berman, K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics, arXiv:1205.6214.

6. E. Calabi, Extremal Kähler metrics, Seminar on differential geometry, *Ann. of Math Studies*, Princeton Univ. Press, **102**(1982), 259-290.

7. E. Calabi, Extremal Kähler metrics, in "*Differential Geometry and Complex Analysis*", eds. I. Chavel and H. M. Farkas, Springer-Verlag, 95-114, 1985.

8. X. X. Chen, S. K. Donaldson and S. Sun, Kähler-Einstein metrics on Fano manifolds I, approximation of metrics with cone singularities, arXiv:1211.4566.

9. X. X. Chen, S. K. Donaldson and S. Sun, Kähler-Einstein metrics on Fano manifolds II, limits with cone angle less than 2π, arXiv:1212.4714.

10. X. X. Chen, S. K. Donaldson and S. Sun, Kähler-Einstein metrics on Fano manifolds III, limits as cone angle approaches 2 and completion of the main proof, arXiv:1302.0282.

11. B. H. Chen, A. M. Li and L. Sheng, Uniform K-stability for extremal metrics on toric varieties, arXiv: 1109.5228.

12. B. H. Chen, A. M. Li and L. Sheng, Extremal metrics on toric surfaces, ar-Viv:1008.2607.

13. X. Chen, C. Lebrun and B. Weber, On conformally Kähler-Einstein manifolds, *J. Amer. Math. Soc.*, **21**(2008), 1137-1168.

14. S. K. Donaldson, Scalar curvature and stability of toric varieties, *J. Diff. Geom.*, **62**(2002), 289-349.

15. S. K. Donaldson, Interior estimates for solutions of Abreu's equation, *Collect. Math.*, **56**(2005), 103-142.

16. S. K. Donaldson, Extremal metrics on toric surfaces: a continuity method, *J. Diff. Geom.*, **79**(2008), 389-432.

17. S. K. Donaldson, Constant scalar curvature metrics on toric surfaces, *Geom. Funct. Anal.*, **19**(2009), 83-136.

18. S. K. Donaldson, Kähler geometry on toric manifolds, and some other manifolds with large symmetry, Handbook of Geometric Analysis, No. 1, 2008.

19. T. Delzant, Hamiltoniens periodique et image convex del'application moment, *Bull. Soc. Math. France*, **116**(1988), 315-339.

20. W. Y. Ding and G. Tian, Kähler-Einstein metrics and the generalized Futaki invariant, *Invent. Math.*, **110**(1992), 315-335.

21. W. Fulton, *Introduction to Toric Varieties*, Princeton Univ. Press, 1993.

22. A. Futaki, An obstruction to the existence of Einstein Kähler metrics, *Invent. Math.*, **73**(1983), 437-443.

23. A. Futaki and T. Mabuchi, Bilinear forms and extremal Kähler vector fields associated with Kähler class, *Math. Ann.*, **301**(1995), 199-210.

24. V. Guillemin, Moment maps and combinatorial invariants of Hamiltonian T^n-spaces, *Progress in Math.*, **122**(1994), Birkhäuser, Boston.

25. T. Mabuchi, K-stability of constant scalar curvature polarization, arXiv:0812.4093.

26. T. Mabuchi, A stronger concept of K-stability, arXiv:0910.4617.

27. T. Mabuchi, The Yau-Tian-Donaldson Conjecture for general polarizations, arXiv:1307.3623.

28. T. Mabuchi, Existence problem of extremal Kähler metrics, arXiv:1307.5203.

29. D. H. Phong and J. Sturm, Lectures on stability and constant scalar curvature, arXiv:0801.4179.

30. J. Ross, Constant scalar curvature metrics on toric manifolds, Lecture notes for a minicourse part of MACK5 workshop, Rome, 2012.

31. Y. T. Siu, The existence of Kähler-Einstein metrics on manifolds with positive anti-canonical line bundle and a suitable finite symmetry group, *Ann. Math.*, **127**(1988), 585-627.

32. J. Stoppa, K-stability of constant scalar curvature Kähler manifolds, *Adv. Math.*, **221**(2009), 1397-1408.

33. G. Székelyhidi, Extremal metrics and K-stability, *Bull. London Math. Soc.*, **39**(2007), 76-84.

34. G. Székelyhidi, Optimal test-configurations for toric varieties, *J. Diff. Geom.*, **80**(2008), 501-523.

35. G. Székelyhidi, Extremal metrics and K-stability, Ph.D. thesis.

36. J. Stoppa and G. Székelyhidi, Relative K-stability of extremal metrics , *J. Eur. Math. Soc.*, **13**(2011), 899-909.

37. G. Tian, Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.*, **130**(1997), 1-39.

38. G. Tian, *Canonical Metrics in Kähler Geometry*, Birkhauser, 2000.

39. G. Tian, K-stability and Kähler-Einstein metrics, arXiv:1211.4669.

40. G. Tian and S. T. Yau, Kähler-Einstein metrics on complex surfaces with $C_1 > 0$, *Comm. Math. Phys.*, **112**(1987), 175-203.

41. X.-J. Wang and X. Zhu, Kähler-Ricci solitons on toric manifolds with positive first Chern class, *Adv. Math.*, **188**(2004), 87-103.

42. X. J. Wang and B. Zhou, Existence and nonexistence of extremal metrics on toric Kähler manifolds. *Adv. Math.*, **226**(2011), 4429-4455.

43. S. T. Yau, Open problems in geometry, *Proc. Symp. Pure. Math.*, **54**(1993), 1-28.

44. B. Zhou and X. H. Zhu, Relative K-stability and modified K-energy on toric manifolds, Adv. Math. 219(2008), 1327-1362.

45. B. Zhou and X. H. Zhu, K-stability on toric manifolds, *Proc. Amer. Math. Soc.*, **136**(2008), 3301-3307.