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ESTIMATE OF AN INCLUSION IN A BODY WITH DISCONTINUOUS CONDUCTIVITY

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This work is dedicated to Professor Neil Trudinger for his 70th birthday.

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Abstract

We study the problem of estimating the size of an inclusion embedded inside a two dimensional body with discontinuous conductivity by one voltage-current measurement. This problem is practically important because the conductivity of a human body is discontinuous. The proofs rely on quantitative uniqueness estimates for the conductivity equation with discontinuous coefficients.

1. Introduction

An important clinical problem is to estimate the size of a cancerous tumor inside an organ by noninvasive methods. In this paper, we study this problem by the method of electrical impedance tomography (EIT) with one measurement. Previous works on this problem assumed that the conductivity of the studied body is Lipschitz continuous (see, for example, [5, 6]). However, this is not guaranteed in reality, for example, the conductivities of heart, liver, intestines are 0.70 (S/m), 0.10 (S/m), 0.03 (S/m), respectively. In this paper, we show that in the two dimensional case, the assumption on the regularity of the conductivity can be weaken.

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We briefly outline the framework, following [6]. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary. Assume that the background conductivity $\sigma(x)$ is elliptic, i.e. for some $\lambda > 0$,

$$\lambda^{-1} |y|^2 \leq \langle \sigma(x) y, y \rangle \leq \lambda |y|^2, \quad \forall y \in \mathbb{R}^2, \text{ a.e. } x \in \Omega. \tag{1.1}$$

Let D be a subdomain of Ω and $\tilde{\sigma}$ be a matrix-valued function on D with bounded measurable coefficients, representing the conductivity of the inclusion. Let v be the electric potential with boundary value ϕ, i.e.

$$\begin{cases} \operatorname{div}((\sigma(x)\chi_{\Omega\setminus\bar{D}} + \tilde{\sigma}(x)\chi_D)\nabla v) = 0 & \text{in} \quad \Omega, \\ v = \phi \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

The energy required to maintain voltage potential ϕ on $\partial\Omega$ is

$$W := \int_{\partial\Omega} \phi \langle \sigma \nabla v, \nu \rangle \, ds.$$

Let u be the electric potential with the same boundary value when there is no inclusion, i.e.

$$\begin{cases} \operatorname{div}(\sigma(x)\nabla u) = 0 & \text{in} \quad \Omega, \\ u = \phi \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Similarly, we define the energy

$$W_0 := \int_{\partial\Omega} \phi \langle \sigma \nabla u, \nu \rangle \, ds.$$

In [6], it is shown that if σ is Lipschitz continuous and for some $\zeta, \eta > 0$ either

$$(1+\eta)\sigma \leq \tilde{\sigma} \ \leq \ \zeta\sigma \quad \text{a.e. in } \Omega, \tag{1.4}$$

or

$$\zeta\sigma \leq \tilde{\sigma} \ \leq \ (1-\eta)\sigma \quad \text{a.e. in } \Omega, \tag{1.5}$$

then the size of D can be estimated using the *normalized power gap* $\left|\frac{W-W_0}{W_0}\right|$. More precisely, the following estimate holds

$$K_1 \left| \frac{W - W_0}{W_0} \right| \leq |D| \leq K_2 \left| \frac{W - W_0}{W_0} \right|^{\frac{1}{p}}, \tag{1.6}$$

where $p > 1$, K_1 and K_2 are constants depending on a priori data. If moreover D satisfies the fatness condition (4.3), then a better estimate holds

$$K_1 \left| \frac{W - W_0}{W_0} \right| \leq |D| \leq K_2 \left| \frac{W - W_0}{W_0} \right|. \tag{1.7}$$

We will show that in two dimension, the method of [6] works even when σ is only piecewise Hölder continuous. Essentially, this is because in two dimension, the three-ball and doubling inequalities for solutions of (1.3) hold for bounded σ; and a gradient estimate needed in proving the propagation of smallness for ∇u was proved in [15] for piecewise Hölder σ (in any dimension).

We would like to mention that size estimates have also been derived for other systems, for example, [2] for the isotropic elasticity, [16, 17, 18] for the isotropic/anisotropic thin plate, [11, 10] for the shallow shell.

The paper is organized as follows. In next section, we define several notations and list several assumptions used in the paper. In Section 3, we prove some quantitative estimates for solutions of (1.3). In Section 4, we prove (1.6) and (1.7).

2. Notations and Assumptions

Definition 2.1. Let Ω be an open bounded domain of $\mathbb{R}^2$. Given $0 < \alpha < 1$, we say that $\partial\Omega$ is of class $C^{1,\alpha}$ with parameters r_0, M_0, if for any $P \in \partial\Omega$, there exists a rigid coordinates transform under which $P = 0$ and

$$\Omega \cap B_{r_0}(0) = \{z = (z_1, z_2) \in B_{r_0}(0) : z_2 > \psi(z_1)\},$$

where $\psi(z_1) \in C^{1,\alpha}(-r_0, r_0)$ satisfying $\psi(0) = 0$ and $\nabla\psi(0) = 0$ and

$$\|\psi\|_{C^{1,\alpha}(-r_0,r_0)} \leq M_0.$$

Recall that

$$\begin{aligned} \|\psi\|_{C^{1,\alpha}(-r_0,r_0)} &= \|\psi\|_{L^\infty(-r_0,r_0)} + \|\nabla\psi\|_{L^\infty(-r_0,r_0)} \\ &\quad + \sup_{x,y\in(-r_0,r_0)} \frac{|\nabla\psi(x) - \nabla\psi(y)|}{|x-y|^\alpha}. \end{aligned}$$

We now state the assumptions used in the paper.

Assumptions

- $\Omega \subset \mathbb{R}^2$ is an open bounded $C^{1,\alpha}$ domain with parameters r_0 and M_0.
- There exist disjoint $C^{1,\alpha}$ domains $\Omega_j \subset \Omega, 1 \le j \le m$ such that $\overline{\Omega} = \cup_{j=1}^m \overline{\Omega}_j$ and for some $\mu > 0$, we have $\sigma_j(x) := \sigma(x)\chi_{\Omega_j} \in C^{0,\mu}(\overline{\Omega_j}), 1 \le j \le m$. For $\alpha' = \min\{\mu, \frac{\alpha}{3(\alpha+1)}\}$, let $M_1 = \sup_j \|\sigma_j\|_{C^{0,\alpha'}(\overline{\Omega}_j)}$.
- For any $x \in \overline{\Omega}$, there exist $r > 0$ and an appropriate rotation of coordinates such that the set $(\cup_{j=1}^m \partial\Omega_j) \cap B_r(x)$ consists of the graphs of $\ell(x,r)$ functions of class $C^{1,\alpha}$, whose $C^{1,\alpha}$ norms are bounded by $L(x,r)$. We assume that

$$\mathcal{L} := \sup_{x\in\overline{\Omega}} \inf_{r>0} \left\{ L(x,r) + \ell(x,r) + \frac{1}{r} \right\} < \infty.$$

- $d = \operatorname{dist}(D, \partial\Omega) > 0$.
- For some $\Gamma \subset \partial\Omega$ of positive measure, $\phi|_\Gamma = 0$.

Remark 2.2. The boundaries of subdomains may touch each other. The inclusion D is only required to stay away from the boundary $\partial\Omega$, it may intersect $\partial\Omega_j$'s (see Figure 2.1).

We also define for $h > 0$,

$$\Omega_h = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > h\}.$$

3. Quantitative Uniqueness Estimates

In this section, we prove quantitative uniqueness estimates for solutions of (1.3) that will be used in the next section. We first recall the three ball inequality of [4].

Lemma 3.1. ([4, Theorem 3.11]) *For all $0 < r_1 < r_2 < r_3$, there exist constants $C > 0$ and $0 < \tau < 1$ depending only on λ, $\frac{r_1}{r_3}$, and $\frac{r_2}{r_3}$ such that for any solution of (1.3) in $B_{r_3}(x)$, we have*

$$\|u\|_{L^2(B_{r_2}(x))} \le C\|u\|^{\tau}_{L^2(B_{r_1}(x))}\|u\|^{1-\tau}_{L^2(B_{r_3}(x))}. \tag{3.1}$$

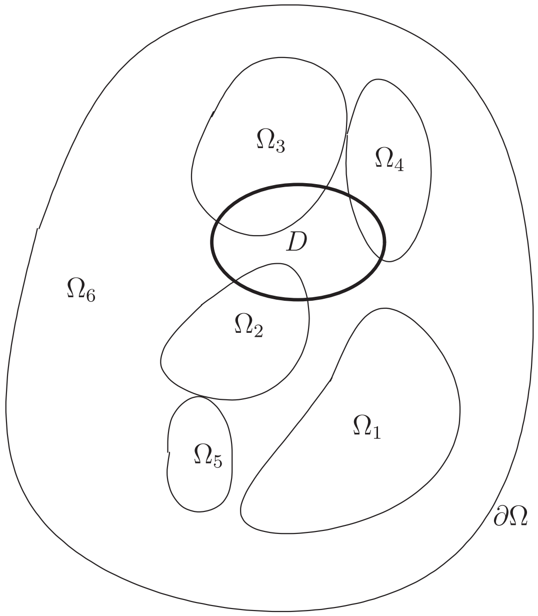

Figure 2.1: Ω_j's may touch each other and D is allowed to intersect the interfaces.

Using this three-ball inequality, we can prove

Lemma 3.2 (propagation of smallness)**.** *Assume that the assumptions in Section 2 holds. Let $u \in H^1(\Omega)$ be the solution of* (1.3)*. For any $\rho > 0$ and every $x \in \Omega_{4\rho}$, we have*

$$\int_{B_\rho(x)} |\nabla u|^2 \geq C \int_\Omega |\nabla u|^2, \tag{3.2}$$

where C depends on Ω, Γ, λ, α, μ, r_0, M_0, M_1, $\mathcal{L}$, ρ, and $\frac{\|\phi\|_{H^2(\partial\Omega)}}{\|\phi\|_{H^{1/2}(\partial\Omega)}}$.

Proof. We follow the arguments presented in [6, Lemma 2.2]. We first observe that it suffices to consider the case ρ is small, so we can assume that Ω_ρ is connected. Using Caccioppoli and Poincaré inequalities, we can deduce from Lemma 3.1 that

$$\|\nabla u\|_{L^2(B_{3r}(x))} \leq C\|\nabla u\|^{\tau}_{L^2(B_r(x))}\|\nabla u\|^{1-\tau}_{L^2(B_{4r}(x))}. \tag{3.3}$$

Given $x, y \in \Omega_{4\rho}$, let γ be a curve in $\Omega_{4\rho}$ joining x and y. We define a sequence x_k's as follows: Let $x_1 = x$. For $k > 1$, let $x_k = \gamma(t_k)$ where

$t_k = \max\{t : |\gamma(t) - x_{k-1}| = 2\rho\}$ if $|x_k - y| > 2\rho$; otherwise let $x_k = y$, $N = k$ and stop the process. Note that since the balls $B_\rho(x_k)$ are disjoint, $N \leq N_0 = \frac{|\Omega|}{\pi\rho^2}$. Using (3.3), noting that $B_\rho(x_{k+1}) \subset B_{3\rho}(x_k)$ because $|x_{k+1} - x_k| \leq 2\rho$, we can deduce that

$$\frac{\|\nabla u\|_{L^2(B_\rho(x_{k+1}))}}{\|\nabla u\|_{L^2(\Omega)}} \leq C\left(\frac{\|\nabla u\|_{L^2(B_\rho(x_k))}}{\|\nabla u\|_{L^2(\Omega)}}\right)^\tau.$$

By induction, we obtain

$$\frac{\|\nabla u\|_{L^2(B_\rho(y))}}{\|\nabla u\|_{L^2(\Omega)}} \leq C^{1/(1-\tau)}\left(\frac{\|\nabla u\|_{L^2(B_\rho(x))}}{\|\nabla u\|_{L^2(\Omega)}}\right)^{\tau^N}. \tag{3.4}$$

Since we can cover $\Omega_{5\rho}$ by no more than $\frac{|\Omega|}{2\rho^2}$ balls of radius ρ, we obtain

$$\frac{\|\nabla u\|_{L^2(\Omega_{5\rho})}}{\|\nabla u\|_{L^2(\Omega)}} \leq C\left(\frac{\|\nabla u\|_{L^2(B_\rho(x))}}{\|\nabla u\|_{L^2(\Omega)}}\right)^{\tau^{N_0}}, \tag{3.5}$$

where C depends on λ, $|\Omega|$, and ρ.

By Corollary 1.3 in [15], $\|\nabla u\|^2_{L^\infty(\Omega)} \leq C\|\phi\|^2_{C^{1,1/2}(\Omega)}$, hence by the embedding $H^2(\partial\Omega) \hookrightarrow C^{1,1/2}(\partial\Omega)$, we get

$$\int_{\Omega\setminus\Omega_{5\rho}} |\nabla u|^2 \leq C|\Omega \setminus \Omega_{5\rho}|\|\phi\|^2_{C^{1,\alpha'}(\partial\Omega)} \leq C\rho\|\phi\|^2_{H^2(\partial\Omega)}. \tag{3.6}$$

Here we have used $|\Omega\setminus\Omega_{5\rho}| \lesssim \rho$ since $\partial\Omega$ is Lipschitz.

Using the Poincaré inequality of [9, Theorem 6.1-8 (b)], recalling that $\varphi|_\Gamma = 0$, we have

$$\|\phi\|^2_{H^{1/2}(\partial\Omega)} \leq C\|u\|^2_{H^1(\Omega)} \leq C\|\nabla u\|^2_{L^2(\Omega)}. \tag{3.7}$$

Combining this and (3.6), we see that if ρ is small enough depending on Ω, Γ, λ, r_0, M_0, M_1, α, μ, $\mathcal{L}$, and $\|\phi\|_{H^2(\partial\Omega)}/\|\phi\|_{H^{1/2}(\partial\Omega)}$,

$$\frac{\|\nabla u\|^2_{L^2(\Omega_{5\rho})}}{\|\nabla u\|^2_{L^2(\Omega)}} \geq \frac{1}{2}.$$

The lemma follows from this and (3.5). ☐

Next, we derive a local doubling inequality for solutions of (1.3).

Lemma 3.3. *For any $\rho > 0$, there exist $\delta = \delta(\rho, \lambda) \in (0, \rho)$ and a constant $C = C(\rho, \lambda)$ such that for all $x \in \Omega_\rho$ and $r \in (0, \delta)$ and any non-trivial solution u of* (1.3), *we have*

$$\frac{\|u\|_{L^2(B_{4r}(x))}}{\|u\|_{L^2(B_r(x))}} \leq C \frac{\|u\|_{L^\infty(B_\rho(x))}}{\|u\|_{L^\infty(B_\delta(x))}}. \tag{3.8}$$

Proof. The proof, using the theory of quasiconformal maps, follows the ideas of the proof of Proposition 2 in [1]. We first note that it suffices to consider the case u is real-valued. Let $v \in H^1_{loc}(\Omega)$ be a σ-harmonic conjugate of u, i.e.

$$\nabla v = J \sigma \nabla u$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $f = u + iv$ satisfies

$$\partial_{\bar{z}} f = \nu_1 \partial_z f + \nu_2 \overline{\partial_z f},$$

where

$$\nu_1 = \frac{bc - ad + 1 + i(b - c)}{(a+1)(d+1) - bc}, \quad \nu_2 = \frac{d - a + i(b + c)}{(a+1)(d+1) - bc}.$$

It is easy to check that $|\nu_1| + |\nu_2| \leq \kappa < 1$. Here κ is a constant depending only on λ.

By Bers-Nirenberg representation theorem (see [8], p. 259), there exists a quasiconformal map $\chi : \Omega \to \chi(\Omega)$ and an analytic function $h : \chi(\Omega) \to \mathbb{C}$ such that $f = h \circ \chi$. Furthermore, there exist $K, \alpha > 1$ depending on κ such that

$$K^{-1} |x - y|^\alpha \leq |\chi(x) - \chi(y)| \leq K |x - y|^{\frac{1}{\alpha}}, \quad \forall x, y \in \Omega.$$

Let $\delta = (10K^2)^{-\alpha} \rho^{\alpha^2}$ and $R = (10K)^{-1} \rho^\alpha$, then we have

$$\chi(B_\delta(x)) \subset B_R(\chi(x)) \text{ and } B_{10R}(\chi(x)) \subset \Omega.$$

By Theorem 3.6.2 in [7], there exists an increasing function γ depending only on κ with $\gamma(0)=0$ such that if $x_1,x_2,x_3\in B(x,\delta)$ then

$$\frac{|\chi(x_1)-\chi(x_2)|}{|\chi(x_1)-\chi(x_3)|}\leq \gamma\left(\frac{|x_1-x_2|}{|x_1-x_3|}\right).$$

Let $c=\gamma(8)>1$ then for any $x\in\Omega_\rho$ and $r\in(0,\delta)$, there exists $s\in(0,R/c)$ such that if $y=\chi(x)$ then

$$B_s(y)\subset\chi(B_{r/2}(x)) \quad \text{and} \quad \chi(B_{4r}(x))\subset B_{cs}(y). \tag{3.9}$$

Since $\tilde u=\operatorname{Re}h$ is harmonic on $\chi(\Omega)$, by Hadamard's three-circle theorem, there exists an absolute constant C such that

$$\frac{\|\tilde u\|_{L^\infty(B_{cs}(y))}}{\|\tilde u\|_{L^\infty(B_s(y))}}\leq C\frac{\|\tilde u\|_{L^\infty(B_{4R}(y))}}{\|\tilde u\|_{L^\infty(B_{3R}(y))}}.$$

By Theorem 3.1.2 in [7], $|E|=0$ iff $|\chi(E)|=0$, hence (3.9) implies

$$\frac{\|u\|_{L^\infty(B_{4r}(x))}}{\|u\|_{L^\infty(B_{r/2}(x))}}\leq C\frac{\|u\|_{L^\infty(B_{\rho_2}(x))}}{\|u\|_{L^\infty(B_{\rho_1}(x))}}.$$

Here

$$\rho_1=(3R/K)^\alpha=3^\alpha\delta>\delta,\ \ \rho_2=(4KR)^{\frac{1}{\alpha}}=(2/5)^{1/\alpha}\rho<\rho.$$

Using well-known estimates for elliptic equations with measurable coefficients, we have

$$\frac{\|u\|_{L^2(B_{4r}(x))}}{\|u\|_{L^2(B_r(x))}}\leq C\frac{\|u\|_{L^\infty(B_{4r}(x))}}{\|u\|_{L^\infty(B_{r/2}(x))}}\leq C\frac{\|u\|_{L^\infty(B_\rho(x))}}{\|u\|_{L^\infty(B_\delta(x))}}. \qquad \square$$

4. Size Estimates

To begin, we recall the following energy inequalities proved in [6].

Lemma 4.1. [6, Lemma 2.1] *Assume that σ satisfies the ellipticity condition* (1.1). *If either* (1.4) *or* (1.5) *holds, then*

$$C_1\int_D|\nabla u|^2dx\leq|W_0-W|\leq C_2\int_D|\nabla u|^2dx, \tag{4.1}$$

where C_1, C_2 are constants depending only on λ, η, and ζ.

We now state and prove the main theorem.

Theorem 4.2. (i) *Suppose that the assumptions in Section 2 hold. Then there exist constants $K_1, K_2 > 0$ and $p > 1$ depending only on Ω, Γ, λ, α, μ, r_0, M_0, M_1, $\mathcal{L}$, d, η, ζ, ρ, and $\|\phi\|_{H^2(\partial\Omega)}/\|\phi\|_{H^{1/2}(\partial\Omega)}$ such that*

$$K_1 \left|\frac{W_0 - W}{W_0}\right| \le |D| \le K_2 \left|\frac{W_0 - W}{W_0}\right|^{\frac{1}{p}}. \tag{4.2}$$

(ii) *If moreover, there exists $h > 0$ such that*

$$|D_h| \ge \frac{1}{2}|D| \quad \textit{(fatness condition).} \tag{4.3}$$

then

$$K_1 \left|\frac{W_0 - W}{W_0}\right| \le |D| \le K_2 \left|\frac{W_0 - W}{W_0}\right|, \tag{4.4}$$

where K_1 and K_2 depend on the various constants as in (i) *and also on h.*

Proof. The proof closely follows the arguments of [6].

We first establish the lower bound. Let $c = \frac{1}{|\Omega_{d/4}|}\int_{\Omega_{d/4}} u$. By the gradient estimate of [15, Theorem 1.1], the interior estimate of [14, Theorem 8.17] and the Poincaré inequality for the domain $\Omega_{d/4}$, we have

$$\|\nabla u\|_{L^\infty(\Omega_{d/2})} \le C\|u - c\|_{L^\infty(\Omega_{d/3})} \le C\|u - c\|_{L^2(\Omega_{d/4})} \le C\|\nabla u\|_{L^2(\Omega)}.$$

From this, the trivial estimate $\|\nabla u\|^2_{L^2(D)} \le C|D|\|\nabla u\|^2_{L^\infty(\Omega_{d/2})}$ and Lemma 4.1, the lower bound follows.

Next, we establish the upper bounds.

(i) We will first establish that $|\nabla u|^2$ is an A_p-weight, following the proof of Theorem 1.1 in [13]. Let $\rho = d/5$ and δ be the constant appears in Lemma 3.3. By Caccioppoli inequality and (3.2), for any $x \in \Omega_{5\rho}$ we have

$$\|u - c\|_{L^\infty(B_\delta(x))} \ge C\,\|u - c\|_{L^2(B_\delta(x))} \ge C\,\|\nabla u\|_{L^2(B_{\delta/2}(x))} \ge C\,\|\nabla u\|_{L^2(\Omega)}\,.$$

(Note that C depends also on δ). By interior estimate, we have

$$\|u-c\|_{L^\infty(B_\rho(x))} \leq 2\|u\|_{L^\infty(B_\rho(x))} \leq C\|\varphi\|_{H^{1/2}(\partial\Omega)}.$$

For $r \in (0,\delta)$, applying the doubling inequality of 3.3 to $u-c$ where $c = \frac{1}{|B_r|}\int_{B_r(x)} u$, we get

$$\frac{\|u-c\|_{L^2(B_{2r}(x))}}{\|u-c\|_{L^2(B_r(x))}} \leq C\frac{\|u-c\|_{L^\infty(B_\rho(x))}}{\|u-c\|_{L^\infty(B_\delta(x))}} \leq \frac{C\|\varphi\|_{H^{1/2}(\partial\Omega)}}{\|\nabla u\|_{L^2(\Omega)}} \leq C.$$

At the last inequality we have used (3.8). We note that the constant C depends on various constants, including $\|\varphi\|_{H^2(\partial\Omega)} / \|\varphi\|_{H^{1/2}(\partial\Omega)}$ but is independent of r.

This and the Caccioppoli inequality give

$$r^{-1}\|\nabla u\|_{L^2(B_r(x))} \leq C\|u-c\|_{L^2(B_{2r}(x))} \leq C\|u-c\|_{L^2(B_r(x))}.$$

Combining this with the Poincaré inequality

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|u-c|^2\right)^{\frac{1}{2}} \leq Cr^{-1}\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|\nabla u|^{\frac{3}{2}}\right)^{\frac{2}{3}},$$

we get

$$\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|\nabla u|^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{|B_r(x)|}\int_{B_r(x)}|\nabla u|^{\frac{3}{2}}\right)^{\frac{2}{3}}.$$

This reverse Hölder inequality shows that $|\nabla u|^2$ is an A_p-weight for some $p>1$ (see [12, Chapter 7]).

We cover D with internally nonoverlapping closed squares Q_k, $1 \leq k \leq I$, with side length 2ρ. Since $|\nabla u|^2$ is and A_p-weight, by [12, (7.2)], we have

$$\frac{|D\cap Q_k|}{|Q_k|} \leq C\left(\frac{\int_{D\cap Q_k}|\nabla u|^2}{\int_{Q_k}|\nabla u|^2}\right)^{1/p}.$$

Summing over k and using (3.2), we get

$$|D| \leq C \left(\frac{\int_D |\nabla u|^2}{\min_k \int_{Q_k} |\nabla u|^2} \right)^{1/p} \leq C \left(\frac{\int_D |\nabla u|^2}{\int_\Omega |\nabla u|^2} \right)^{1/p}.$$

The upper bound of $|D|$ now follows from (4.1).

(ii). Let $\rho = \frac{1}{4}\min\{d, h\}$ and cover D_h with internally nonoverlapping closed squares $\{Q_k\}_{k=1}^J$ of side length 2ρ. It is clear that $Q_k \subset D$, hence

$$\begin{aligned}\int_D |\nabla u|^2 dx &\geq \int_{\cup_{k=1}^J Q_k} |\nabla u|^2 dx \geq \frac{|D_h|}{\rho^2} \min_k \int_{Q_k} |\nabla u|^2 dx. \\ &\geq \frac{C|D|}{\rho^2} \int_\Omega |\nabla u|^2 dx.\end{aligned}$$

Here we have used Lemma 3.2 and the fatness condition at the last inequality. The upper bound of $|D|$ follows from this and Lemma 4.1. □

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