

OPTIMAL TRANSPORTATION ON THE HEMISPHERE

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Dedicated to Professor Neil Trudinger on the occasion of his 70th birthday

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Abstract

In this paper, we study the optimal transportation on the hemisphere, with the cost function $c(x, y) = \frac{1}{2}d^2(x, y)$, where d is the Riemannian distance of the round sphere. The potential function satisfies a Monge-Ampère type equation with natural boundary condition. In this critical case, the hemisphere does not satisfy the c -convexity assumption. We obtain the *a priori* oblique derivative estimate, and in the special case of two dimensional hemisphere, we obtain the boundary C^2 estimate. Our proof does not require the smoothness of densities.

1. Introduction

Let \mathbb{S}^n be the n -dimensional unit sphere equipped with the standard round metric g and geodesic distance d . Denote the northern hemisphere by $\mathbb{S}_+^n := \mathbb{S}^n \cap \{x_{n+1} \geq 0\}$. Let $c(x, y) = \frac{1}{2}d^2(x, y)$ be the cost function, f, g be two positive densities on \mathbb{S}_+^n , bounded from above and below, and satisfy

$$\int_{\mathbb{S}_+^n} f = \int_{\mathbb{S}_+^n} g. \quad (1.1)$$

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In this paper, we study the optimal transportation from (\mathbb{S}_+^n, f) to (\mathbb{S}_+^n, g) and obtain the *a priori* oblique and boundary estimates without assuming uniform c -convexity of domain and smoothness of densities. Let's briefly recall that in the optimal transportation $(\Omega, f) \rightarrow (\Omega^*, g)$, $f, g > 0$ satisfying $\int_{\Omega} f = \int_{\Omega^*} g$, where Ω, Ω^* are the initial and target domains, the optimal mapping T_u is determined by the potential function u ,

$$Du(x) = -D_x c(x, T_u(x)) \quad (1.2)$$

for a.e. $x \in \Omega$, where the cost function c satisfies conditions (A0)–(A1) in Section 2, and the functions u, v are called potential functions as (u, v) is a maximizer of

$$\sup\{I(\phi, \psi) : (\phi, \psi) \in K\},$$

where

$$\begin{aligned} I(\phi, \psi) &= \int_{\Omega} f(x)\phi(x) + \int_{\Omega^*} g(y)\psi(y), \\ K &= \{(\phi, \psi) \in C(\Omega) \times C(\Omega^*) : \phi(x) + \psi(y) \geq -c(x, y)\}. \end{aligned}$$

When u is smooth, it solves a Monge-Ampère type equation

$$\det [D^2u + D_x^2 c] = |\det D_{xy}^2 c| \frac{f}{g \circ T_u} \quad \text{in } \Omega, \quad (1.3)$$

with a natural boundary condition

$$T_u(\Omega) = \Omega^*. \quad (1.4)$$

In the Euclidean case, when Ω, Ω^* are two bounded domains in \mathbb{R}^n , the global regularity of (1.3)–(1.4) is obtained in [14] by assuming that Ω, Ω^* are uniformly c -convex with respect to each other and the densities f, g are correspondingly smooth. The uniform c -convexity of Ω with respect to Ω^* means that the image $c_y(\cdot, y)(\Omega)$ is uniformly convex in the usual sense for each $y \in \Omega^*$, while analogously Ω^* is uniformly c -convex with respect to Ω , if the image $c_x(x, \cdot)(\Omega^*)$ is uniformly convex for each x in Ω .

In the special case $c(x, y) = -x \cdot y$, the c -convexity is equivalent to the usual convexity, and (1.3) reduces to the standard Monge-Ampère equation with the boundary condition of prescribing the image of gradient mapping,

$$\begin{cases} \det D^2u = h(x, Du) & \text{in } \Omega, \\ Du(\Omega) = \Omega^*. \end{cases} \quad (1.5)$$

The boundary problem (1.5) has been extensively studied by many mathematicians, for example, see [2, 3, 16] and references therein. A crucial assumption in those work is the uniform convexity of domains Ω and Ω^* .

Note that when $c(x, y) = -x \cdot y$, (1.2) implies that $T_u = Du$ for a convex potential u . In our case $c = d^2/2$, where d is the geodesic distance on (\mathbb{S}^n, g) , it is noted in [13] that the optimal mapping can be expressed by the exponential mapping

$$T_u(x) = \exp_x(\nabla_g u(x)),$$

where ∇_g denotes the gradient with respect to the round metric g on \mathbb{S}^n , and u is a c -convex potential. For $\Omega, \Omega^* \subset \mathbb{S}^n$, the condition that Ω is uniformly c -convex with respect to Ω^* is equivalent to the condition that $\exp_y^{-1}(\Omega)$ is uniformly convex in \mathbb{R}^n for each $y \in \Omega^*$, while analogously Ω^* is uniformly c -convex with respect to Ω if $\exp_x^{-1}(\Omega^*)$ is uniformly convex for each x in Ω . However, this is not the case when $\Omega = \Omega^* = \mathbb{S}_+^n$. To see this, one observes that for a point $y_0 \in \mathbb{S}_+^n \cap \{x_{n+1} = 0\}$, one has

$$\exp_{y_0}^{-1}(\Omega) = \{z \in \mathbb{R}^n : z \in B_\pi(0) \cap \{z_n \geq 0\} \text{ or } |z| = \pi\}.$$

The set is not even simply connected hence not a convex set. One can see that for $\Omega = \Omega^* = \mathbb{S}_\varepsilon^n := \mathbb{S}^n \cap \{x_{n+1} \geq \varepsilon\}$, the corresponding sets $\exp_{y_0}^{-1}(\Omega)$ and $\exp_{y_0}^{-1}(\Omega^*)$ are uniformly c -convex to each other for any positive constant $\varepsilon > 0$. Therefore, the hemisphere \mathbb{S}_+^n is a critical case in above sense.

From here on, we use $X = (X_1, \dots, X_n, X_{n+1})$ to represent a point on \mathbb{S}_+^n , while $x = (x_1, \dots, x_n)$ represents a point in \mathbb{R}^n . We use the stereographic projection from the south pole to transform \mathbb{S}_+^n into $\Pi(\mathbb{S}_+^n) = B_1(0) \subset \mathbb{R}^n$ by $x = \Pi(X)$ and

$$X = \Pi^{-1}(x) = \left(\frac{2x_1}{1 + |x|^2}, \dots, \frac{2x_n}{1 + |x|^2}, \frac{1 - |x|^2}{1 + |x|^2} \right), \quad (1.6)$$

where $x = (x_1, \dots, x_n) \in B_1(0)$.

Utilizing the ambient Euclidean geometry of \mathbb{R}^{n+1} , it is an elementary calculation that

$$d(X, Y) = \arccos(X \cdot Y), \quad (1.7)$$

for $X, Y \in \mathbb{S}^n$ and d the geodesic distance. Under the stereographic projection Π , one has the optimal transportation from $\Omega = B_1(0)$ to $\Omega^* = B_1(0)$ with the cost function

$$\begin{aligned} \bar{c}(x, y) &= c(\Pi^{-1}(x), \Pi^{-1}(y)) \\ &= \frac{1}{2} \left(\arccos \left(\frac{4(x \cdot y)}{(1 + |x|^2)(1 + |y|^2)} + \frac{(1 - |x|^2)(1 - |y|^2)}{(1 + |x|^2)(1 + |y|^2)} \right) \right)^2, \end{aligned} \quad (1.8)$$

for $x, y \in B_1(0)$. Correspondingly, the potential u and the optimal mapping T will become

$$\bar{u}(x) = u \circ \Pi^{-1}(x) \quad \text{and} \quad \bar{T}(x) = \Pi \circ T \circ \Pi^{-1}(x), \quad \text{for } x \in B_1(0). \quad (1.9)$$

The convexity with respect to \bar{c} is inherited from that of c , namely \bar{u} is \bar{c} -convex if and only if u is c -convex; a domain $E \subset B_1(0)$ is \bar{c} -convex with respect to $E^* \subset B_1(0)$ if and only if $\Pi^{-1}(E)$ is c -convex with respect to $\Pi^{-1}(E^*)$.

Due to the lack of convexity, the standard techniques in dealing with (1.5) are not applicable to (1.3)–(1.4) with the cost function given by (1.8). In this paper, we use an elementary but non-trivial observation of Delanoë and Loeper (see Lemma 2.1) to establish the a priori oblique and boundary estimates. Our main result is the following:

Theorem 1.1. *Assume that the density functions f, g satisfies (1.1) and there exists a constant $\lambda > 0$ such that $\lambda^{-1} < f, g < \lambda$. Then we have the a priori estimate*

$$\sum_{i,k=1}^n -y_i c^{i,k} x_k \geq c_0, \quad (1.10)$$

for all $x \in \Omega$ and $y = T(x)$, where $c_0 > 0$ is a constant depending only on λ .

Moreover, when $n = 2$, we have the a priori boundary estimate

$$\sup_{\partial\Omega} |D^2 u| \leq C, \quad (1.11)$$

for some constant $C > 0$ depending only on λ . If furthermore f, g are smooth, then u is smooth and the optimal mapping T is smooth.

The restriction to the special case $n = 2$ in the statement above is unsatisfactory, but this is the case we can carry out our estimate by using a shortcut for 2×2 inverse matrices. We also remark that, to derive the boundary estimate (1.11), the usual proof relies on the differentiability of the right hand side of the transport equation (1.3), but here we have only used the boundedness assumption of the term.

This paper is organized as follows: In Section 2, we introduce some preliminary notations and results. In Section 3, we prove the oblique estimate (1.10). In Section 4, we prove the boundary C^2 estimate (1.11) in the two-dimensional case.

2. Preliminaries

First, let's recall some basic notions of optimal transportation on a Riemannian manifold \mathcal{M} with the cost function $c(x, y) = \frac{1}{2}d^2(x, y)$.

Definition 2.1. Let \mathcal{M} be a compact Riemannian manifold and $d(\cdot, \cdot)$ be its Riemannian distance function. The c -transform u^c of a function $u : \mathcal{M} \rightarrow \mathbb{R}$ is defined for all $x \in \mathcal{M}$ by

$$u^c(x) = \sup_{y \in \mathcal{M}} \left\{ -\frac{d^2(x, y)}{2} - u(y) \right\}. \quad (2.1)$$

The function u is said to be c -convex if $(u^c)^c = u$.

For a c -convex function u , for any point $x_0 \in \mathcal{M}$, by the above definition there exists $y \in \mathcal{M}$ such that

$$u(x) \geq -\frac{d^2(x, y)}{2} - u^c(y),$$

for all $x \in \mathcal{M}$ with equality holds at $x = x_0$. The function $\varphi(\cdot) = -\frac{d^2(\cdot, y)}{2} - u^c(y)$ is called a c -support of u at $x \in \mathcal{M}$. A function u is c -convex is equivalent to that for any point $x \in \mathcal{M}$ there exists a c -support of u at x . Naturally, the potential function u in (1.2) of optimal transportation is c -convex.

Definition 2.2. Let u be a c -convex function, the c -normal mapping T_u is defined by

$$T_u(x_0) = \{y \in \mathcal{M} : u(x) \geq \frac{d^2(x_0, y)}{2} - \frac{d^2(x, y)}{2} + u(x_0), \quad \forall x \in \mathcal{M}\}. \quad (2.2)$$

Note that by duality between u and u^c , if $y \in T_u(x_0)$, we have $u^c(y) = -\frac{d^2(x_0, y)}{2} - u(x_0)$, and

$$-D_x c(x_0, y) \in \partial u(x_0),$$

where ∂u is the subgradient of u . If u is C^1 smooth at x_0 , then $T_u(x_0)$ is single valued, and is exactly the mapping given by (1.2). In general, $T_u(x)$ is single valued for almost all $x \in \mathcal{M}$ as u is c -convex and thus twice differentiable almost everywhere by the well-known theorem of Aleksandrov. If $c(x, y) = -x \cdot y$ and \mathcal{M} is Euclidean, then T_u is the normal mapping for convex functions.

We may extend the c -normal mapping to boundary points. Let $x_0 \in \partial \mathcal{M}$ be a boundary point, we denote $T_u(x_0) = \{y \in \mathcal{M} : y = \lim_{k \rightarrow \infty} y_k\}$, where $y_k \in T_u(x_k)$ and $\{x_k\}$ is a sequence of interior points of \mathcal{M} such that $x_k \rightarrow x_0$.

Let \mathcal{U} be a subset of $\mathcal{M} \times \mathcal{M}$, which for simplicity we assume is compact. Denote π_1, π_2 the usual canonical projections. For any $x \in \pi_1(\mathcal{U})$, we denote by \mathcal{U}_x the set $\mathcal{U} \cap \pi_1^{-1}(x)$. Similarly, we can define $\mathcal{U}_y = \mathcal{U} \cap \pi_2^{-1}(y)$, for any $y \in \pi_2(\mathcal{U})$. We introduce the following conditions:

- (A0) The cost function c belongs to $C^4(\mathcal{U})$.
- (A1) For any $(x, y) \in \mathcal{U}$, $(p, q) \in D_x c(\mathcal{U}) \times D_y c(\mathcal{U})$, there exists unique $Y = Y(x, p), X = X(y, q)$, such that $-D_x c(x, Y) = p, -D_y c(X, y) = q$.
- (A2) For any $(x, y) \in \mathcal{U}$, $\det D_{x, y}^2 c \neq 0$.

We recall the definition of c -convexity for domains (see [12]):

Definition 2.3. Let $y \in \pi_2(\mathcal{U})$, a subset Ω of $\pi_1(\mathcal{U}_y)$ is c -convex (resp. uniformly c -convex) with respect to y if the set $\{-D_y c(x, y), x \in \Omega\}$ is a convex (resp. uniformly convex) set of $T_y \mathcal{M}$. Whenever $\Omega \times \Omega^* \subset \mathcal{U}$, Ω is c -convex with respect to Ω^* if it is c -convex with respect to every $y \in \Omega^*$.

Similarly we can define c^* -convexity of domains by exchanging x and y . Without arising any confusion, for simplicity we abuse the notation c -convexity to also mean c^* -convexity by dropping the sup-script. When the cost function $c = d^2/2$, we have $D_x c(x, y) = \exp_x^{-1}(y)$. Therefore, $\Omega^* \subset \mathcal{M}$ is c -convex (resp. uniformly c -convex) with respect to x is equivalent to $\exp_x^{-1}(\Omega^*)$ is convex (resp. uniformly convex).

However, conditions (A0)–(A2) are not satisfied on $\mathbb{S}_+^n \times \mathbb{S}_+^n$ due to the singularities on antipodal points. The next lemma shows that for each point $x \in \partial\mathbb{S}_+^n$, its image under the c -normal mapping of a c -convex function stays uniformly away from its antipodal point \hat{x} . Note that the singularity only occurs on the boundary $\partial\mathbb{S}_+^n = \mathbb{S}_+^n \cap \{x_{n+1} = 0\}$ and the antipodal point $\hat{x} = -x$ for $x \in \partial\mathbb{S}_+^n$.

Lemma 2.1. *Let $T = T_u$ be the c -normal mapping of a c -convex potential u such that $T_{\#}f = g$. Assume that the densities f, g have positive lower and upper bounds. Then there exists a constant $\delta > 0$, such that*

$$d(T(x), \hat{x}) \geq \delta, \quad (2.3)$$

for any $x \in \partial\mathbb{S}_+^n$.

Proof. The proof essentially follows from [4], where the measures and transport maps are defined on the whole sphere \mathbb{S}^n without boundary. We include it here for completeness. Let $x_0 \in \partial\mathbb{S}_+^n$ be a boundary point, and \hat{x}_0 be its antipodal point. We claim that: for almost all $x \in \mathbb{S}_+^n$, $x \neq x_0$,

$$d(T(x), \hat{x}_0) \leq 2\pi \frac{d(T(x_0), \hat{x}_0)}{d(x, x_0)}. \quad (2.4)$$

Then denote $D = \{x \in \mathbb{S}_+^n : d(x, x_0) \geq \pi/2\}$, a subset of \mathbb{S}_+^n . From the preceding inequality, we infer that almost all $x \in D$ are sent by T into $B_\varepsilon(\hat{x}_0)$, where

$$\varepsilon = 2\pi \frac{d(T(x_0), \hat{x}_0)}{\pi/2} = 4d(T(x_0), \hat{x}_0). \quad (2.5)$$

By the measure preserving condition, we then have

$$\int_{B_\varepsilon(\hat{x}_0)} g \geq \int_D f,$$

and thus,

$$d^n(T(x_0), \hat{x}_0) \sup g \geq C \inf f,$$

where C is a constant only depending on n . Since x_0 is arbitrary, we conclude that there is a constant $\delta > 0$ depending on the lower bound of f and upper bound of g such that (2.3) holds.

Therefore, it suffices to prove the claim (2.4). Fix $x_0 \in \partial\mathbb{S}_+^n$ and another point $x \in \mathbb{S}_+^n$, define the function

$$F(p) = \frac{1}{2}d^2(p, x) - \frac{1}{2}d^2(p, x_0),$$

for $p \in \mathbb{S}_+^n$. The function F satisfies that [4]

$$\text{grad}_p F(p) = \exp_p^{-1}(x) - \exp_p^{-1}(x_0), \quad (2.6)$$

where the gradient is defined everywhere except \hat{x}_0 . Since our manifold is \mathbb{S}_+^n , by comparison with the Euclidean case,

$$|\text{grad}_p F(p)| \geq d(x_0, x'). \quad (2.7)$$

Let us consider on $\mathbb{S}_+^n \setminus \{\hat{x}_0\}$ the normalized steepest descent equation (with arc-length parameter s):

$$\dot{p}(s) = -\frac{\text{grad}_p F[p(s)]}{|\text{grad}_p F[p(s)]|}.$$

From (2.7), any solution $p(s)$ satisfies

$$\frac{d}{ds} F[p(s)] = -|\text{grad}_p F[p(s)]| \leq -d(x_0, x).$$

It is easy to see that for fixed (x_0, x) , the function $F(p)$ attains its infimum at $p = \hat{x}_0$. Therefore, starting from $p(0) = p_0$, for some $p_0 \neq \hat{x}_0$, the minimum of $p \mapsto F(p)$ is reached by flowing along an integral curve of length $L \geq d(p_0, \hat{x}_0)$. Writing

$$F(p_0) - F(\hat{x}_0) = -\int_0^L \frac{d}{ds} F[p(s)] ds,$$

we then have

$$F(p_0) - F(\hat{x}_0) \geq \int_0^L d(x_0, x) \geq d(x_0, x)d(p_0, \hat{x}_0).$$

It implies that for $x \neq x_0$ and for all $p \in \mathbb{S}_+^n$,

$$d(p, \hat{x}_0) \leq \frac{F(p) - F(\hat{x}_0)}{d(x_0, x)}. \quad (2.8)$$

Next, we show that (2.4) follows from (2.8). We know that the mapping T is a.e. c -monotone, see for example [1, 4], which implies that for almost all $x_0 \in \partial\mathbb{S}_+^n$ and $x \in \mathbb{S}^n$,

$$\frac{1}{2}d^2(x_0, T(x_0)) + \frac{1}{2}d^2(x, T(x)) \leq \frac{1}{2}d^2(x_0, T(x)) + \frac{1}{2}d^2(x, T(x_0)).$$

From the definition of function F , we get

$$F[T(x)] \leq F[T(x_0)].$$

Now, setting $p = T(x)$ in (2.8), we have

$$d(\hat{x}_0, T(x)) \leq \frac{F(T(x)) - F(\hat{x}_0)}{d(x_0, x)} \leq \frac{F(T(x_0)) - F(\hat{x}_0)}{d(x_0, x)},$$

hence, since $p \mapsto F(p)$ is 2π -Lipschitz, we obtain (2.4), namely

$$d(\hat{x}_0, T(x)) \leq 2\pi \frac{d(T(x_0), x_0)}{d(x_0, x)},$$

for almost all $x \in \mathbb{S}_+^n, x \neq x_0$. The proof is finished. \square

We now recall the definition of the cost-sectional curvature [10]; we will also introduce an additional condition on the cost function c , which is crucial for the regularity estimates [12]:

Definition 2.4. Assume that the cost function c satisfies (A0)–(A2) in $\mathcal{U} \subset \mathcal{M} \times \mathcal{M}$. For every $(x_0, y_0) \in \mathcal{U}$, define on $T_{x_0}\mathcal{M} \times T_{x_0}\mathcal{M}$ a real-valued map

$$\mathfrak{S}_c(x_0, y_0)(\xi, \eta) = D_{p_\eta p_\eta x_\xi x_\xi}^4 [(x, p) \rightarrow -c(x, \exp_{x_0}(p))] \Big|_{x_0, p_0 = -\nabla_x c(x_0, y_0)}. \quad (2.9)$$

When ξ, η are unit orthogonal vectors (with respect to the metric g at x_0), $\mathfrak{S}_c(x_0, y_0)(\xi, \eta)$ defines the cost-sectional curvature from x_0 to y_0 in directions (ξ, η) .

In fact, definition (2.9) is equivalent to the following

$$\mathfrak{S}_c(x_0, y_0)(\xi, \eta) = D_{tt}^2 D_{ss}^2 [(t, s) \rightarrow -c(\exp_{x_0}(t\xi), \exp_{x_0}(p_0 + s\eta))] \Big|_{t,s=0}. \quad (2.10)$$

Moreover, the definition of $\mathfrak{S}_c(x_0, y_0)(\xi, \eta)$ is intrinsic, only depends on the points $(x_0, y_0) \in \mathcal{U}$ and vectors (ξ, η) , but not on the choice of local coordinates around x_0 or y_0 , [7, 10]. We are now ready to introduce the condition:

(A3) For any $(x, y) \in \mathcal{U}$, and $\xi, \eta \in \mathbb{R}^n$ with $\xi \perp \eta$,

$$\mathfrak{S}_c(x, y)(\xi, \eta) \geq c_0 |\xi|^2 |\eta|^2, \quad (2.11)$$

where c_0 is a positive constant.

It has been verified [11] that the cost function $c = d^2/2$ over \mathbb{S}^n satisfies (A3) for any x, y such that $d(x, y) < \pi$. Then under the assumptions of Lemma 2.1, we have

Corollary 2.1. *The cost function c satisfies conditions (A0)–(A3) on the graph of T_u , $\mathcal{G}_T := \{(x, T_u(x)) : x \in \mathbb{S}_+^n\}$.*

Corollary 2.2. *Let u be a c -convex potential on \mathbb{S}_+^n . The densities f and g are bounded from above and below. Then there exists a constant $\alpha \in (0, 1)$ such that $u \in C^{1,\alpha}(\overline{\mathbb{S}_+^n})$.*

Proof. Using the stereographic projection, it suffices to show that $\bar{u} \in C^{1,\alpha}(\overline{B_1})$ for some constant $\alpha \in (0, 1)$, where \bar{u} is given in (1.9). By Corollary 2.1, the cost function \bar{c} satisfies (A0)–(A3) on the graph $\mathcal{G}_{\bar{T}} = \{(x, \bar{T}(x)) : x \in B_1(0)\}$. The proof then follows from [9] by using a similar argument. The global $C^{1,\alpha}$ regularity was previously obtained by Loeper in [10] for Euclidean domains and in [11] for spheres \mathbb{S}^n . More recently, Figalli, Kim and McCann [5] reduced (A3) to degenerate case, where $c_0 = 0$ in (2.11), under stronger convexity assumptions on domains.

In the following, we sketch the proof of $\bar{u} \in C^{1,\alpha}(\overline{B_1})$. Let $x_0 \in B_1$ be an interior point and $N_r(x_0) := B_r(x_0) \cap \overline{B_1}(0)$ be a small neighborhood of

x_0 . By Lemma 2.1, for each $y_0 \in \bar{T}(x_0)$, $N_r(x_0)$ is \bar{c} -convex with respect to y_0 when $r > 0$ is sufficiently small. Let $\varphi = \bar{c}(\cdot, y_0) + a_0$ be a c -support of \bar{u} at x_0 , where a_0 is a constant. Then we can define the sub-level set

$$S_{h, \bar{u}}^0(x_0) = \{x \in N_r(x_0) : \bar{u}(x) < \varphi(x) + h\}$$

for $h > 0$ small. Since \bar{c} satisfies (A3), $S_{h, \bar{u}}^0(x_0)$ is \bar{c} -convex with respect to y_0 . Thus by the coordinate transform $x \mapsto D_y \bar{c}(x, y_0)$, $S_{h, \bar{u}}^0(x_0)$ becomes a convex set. By applying the normalization argument in [9], we can obtain $\bar{u} \in C^{1, \alpha}(x_0)$.

For the boundary regularity, by extending \bar{u} to a neighborhood of $B_1(0)$ it can be reduced to the interior case since the arguments in [9, 10] allow that the initial density f vanishes. \square

3. Obliqueness

In this section we focus on the optimal transportation after the stereographic transformation, which is from $\Omega = B_1(0)$ to $\Omega^* = B_1(0)$ with the cost function given in (1.8). We drop off the bars over the functions c, u for simplicity, so that the potential function u satisfies equation (1.3) with boundary condition (1.4), where the optimal mapping T_u is determined by (1.2). We now prove (1.10) in the following lemma.

Recall that a boundary condition for a second order partial differential equation defined on a domain Ω of the form

$$G(x, u, Du) = 0 \quad \text{on } \partial\Omega \tag{3.1}$$

is called *oblique*, (or *degenerate oblique*) if

$$G_p \cdot \nu \geq c_0 > 0, \quad (\text{or } \geq 0) \tag{3.2}$$

for all $(x, z, p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^2$, where c_0 is a positive constant, ν denotes the unit outer normal to $\partial\Omega$. Let $\phi(x) = \frac{1}{2}(|x|^2 - 1)$ and $\phi^*(y) = \frac{1}{2}(|y|^2 - 1)$ be smooth defining functions for Ω and Ω^* , respectively. Then (1.4) can be written as

$$\phi^*(T_u) = 0 \quad \text{on } \partial\Omega,$$

$$\phi^*(T_u) < 0 \quad \text{near } \partial\Omega. \quad (3.3)$$

Set $G(x, u, Du) := \phi^* \circ T_u(x, Du)$. The main estimate in this section is the following

Lemma 3.1. *Under the assumptions of Theorem 1.1, the boundary condition (1.4) satisfies a strict obliqueness estimate (3.2).*

Proof. Let $u \in C^2(\overline{\Omega})$ be an elliptic solution of (1.3)–(1.4), and denote $y = T_u(x)$. By differentiation, we have

$$\phi_k^* D_i y_k \tau_i = 0 \quad \text{on } \partial\Omega \quad (3.4)$$

for any unit tangential vector τ on $\partial\Omega$, and

$$\phi_k^* D_\nu y_k \geq 0 \quad \text{on } \partial\Omega \quad (3.5)$$

where ν is the outer normal to $\partial\Omega$, whence

$$\phi_i^* D_j y_i = \chi \nu_j \quad (3.6)$$

for some $\chi \geq 0$. Consequently, from (1.2)

$$-\phi_i^* c^{i,k} w_{jk} = \chi \nu_j, \quad (3.7)$$

where $\{c^{i,j}\} = \{c_{i,j}\}^{-1}$ and

$$w_{ij} := u_{ij} + c_{ij}. \quad (3.8)$$

At this point we observe that $\chi > 0$ on $\partial\Omega$ since $|\nabla\phi^*| \neq 0$ on $\partial\Omega$ and $\det DT \neq 0$. Using the ellipticity of (1.3) and letting $\{w^{ij}\}$ denote the inverse matrix of $\{w_{ij}\}$, we then have

$$-\phi_i^* c^{i,k} = \chi w^{jk} \nu_j. \quad (3.9)$$

Now writing $G(x, p) = \phi^* \circ T_u(x, p) = \phi^*(y)$, by differentiating

$$G_{p_k} = \phi_i^* D_{p_k} y_i = -\phi_i^* c^{i,k} = \chi w^{jk} \nu_j, \quad (3.10)$$

thus

$$G_p \cdot \nu = \chi w^{ij} \nu_i \nu_j > 0, \quad (3.11)$$

on $\partial\Omega$. Next, we obtain a uniform positive lower bound for $G_p \cdot \nu$ as follows. On the boundary $\partial\Omega \times \partial\Omega^*$, the unit outer normal $\nu_i = x_i$ and $\phi_i^* = y_i$, for $i = 1, \dots, n$. From (3.10), we have

$$G_p \cdot \nu = \sum_{i,j=1}^n -y_i c^{i,j}(x, y) x_j. \quad (3.12)$$

We claim that for $(x, y) \in \partial\Omega \times \partial\Omega^*$, for any $1 \leq i \leq n$

$$\sum_{j=1}^n c_{i,j} y_j = -\arccos(x \cdot y) \frac{x_i}{\sqrt{1 - (x \cdot y)^2}}. \quad (3.13)$$

Hence, $\sum_{i=1}^n c^{i,j} x_j = -\frac{y_i \sqrt{1 - (x \cdot y)^2}}{\arccos(x \cdot y)}$ and then

$$G_p \cdot \nu = \frac{|y|^2 \sqrt{1 - (x \cdot y)^2}}{\arccos(x \cdot y)} = \frac{\sqrt{1 - (x \cdot y)^2}}{\arccos(x \cdot y)}. \quad (3.14)$$

By Lemma 2.1, $1 + x \cdot y \geq \varepsilon_0$ for some positive constant ε_0 . Therefore, there exists a constant $c_0 > 0$ such that (3.2) holds. The proof of Lemma 3.1 is completed.

Let us now prove the claim (3.13) at $(x, y) \in \partial B_1 \times \partial B_1$ with the cost function c given in (1.8). Denote

$$\theta = \frac{4(x \cdot y)}{(1 + |x|^2)(1 + |y|^2)} + \frac{(1 - |x|^2)(1 - |y|^2)}{(1 + |x|^2)(1 + |y|^2)}, \quad (3.15)$$

the cost function $c(x, y) = \frac{1}{2} \arccos^2 \theta$. By differentiating, the first order derivatives are

$$c_i = \frac{\partial c}{\partial x_i} = -\frac{\arccos \theta}{\sqrt{1 - \theta^2}} \frac{1}{1 + |y|^2} \left(\frac{4y_i}{1 + |x|^2} - \frac{8(x \cdot y)x_i}{(1 + |x|^2)^2} - \frac{4x_i(1 - |y|^2)}{(1 + |x|^2)^2} \right) \quad (3.16)$$

for all $i = 1, \dots, n$.

At $(x, y) \in \partial B_1 \times \partial B_1$, $|x| = |y| = 1$ and the function $\theta = x \cdot y$, thus

$$c_i = \frac{-\arccos(x \cdot y)}{\sqrt{1 - (x \cdot y)^2}} (y_i - (x \cdot y)x_i). \quad (3.17)$$

Therefore, we obtain the relation $x \cdot D_x c = \sum_{i=1}^n x_i c_i = 0$. We point this out because it will be used on the boundary estimates in the next section.

By differentiating θ in (3.15) with respect to y variable, we have

$$\begin{aligned} \frac{\partial \theta}{\partial y_i} &= \frac{1}{1+|x|^2} \left(\frac{4x_i}{1+|y|^2} - \frac{8(x \cdot y)y_i}{(1+|y|^2)^2} - \frac{4y_i(1-|x|^2)}{(1+|y|^2)^2} \right) \\ &= x_i - (x \cdot y)y_i, \end{aligned} \quad (3.18)$$

for all $(x, y) \in \partial B_1 \times \partial B_1$ and $i = 1, \dots, n$. By a further differentiation of (3.16) with respect to y variable, the mixed second order derivatives are

$$\begin{aligned} c_{i,j} &= \left(\frac{1}{1-\theta^2} - \frac{\theta \arccos \theta}{(1-\theta^2)^{3/2}} \right) \frac{\partial \theta}{\partial y_j} \frac{1}{1+|y|^2} \left(\frac{4y_i}{1+|x|^2} - \frac{8(x \cdot y)x_i}{(1+|x|^2)^2} - \frac{4x_i(1-|y|^2)}{(1+|x|^2)^2} \right) \\ &\quad + \arccos \theta \frac{1}{\sqrt{1-\theta^2}} \frac{2y_j}{(1+|y|^2)^2} \left(\frac{4y_i}{1+|x|^2} - \frac{8(x \cdot y)x_i}{(1+|x|^2)^2} - \frac{4x_i(1-|y|^2)}{(1+|x|^2)^2} \right) \\ &\quad - \arccos \theta \frac{1}{\sqrt{1-\theta^2}} \frac{1}{1+|y|^2} \frac{1}{1+|x|^2} \left(4\delta_{ij} - \frac{8x_i x_j}{1+|x|^2} + \frac{8x_i y_j}{1+|x|^2} \right). \end{aligned} \quad (3.19)$$

Combining (3.18) with (3.19) and noting that $\theta = x \cdot y$, $|x| = |y| = 1$, we have

$$\begin{aligned} c_{i,j} &= \left(\frac{1}{1-\theta^2} - \frac{\theta \arccos \theta}{(1-\theta^2)^{3/2}} \right) (x_j - (x \cdot y)y_j) (y_i - (x \cdot y)x_i) \\ &\quad + \frac{\arccos \theta}{\sqrt{1-\theta^2}} (y_j (y_i - (x \cdot y)x_i) - (\delta_{ij} - x_i x_j + x_i y_j)). \end{aligned} \quad (3.20)$$

Therefore, the sum in (3.13) becomes

$$\begin{aligned} \sum_{j=1}^n c_{i,j} y_j &= \left(\frac{1}{1-\theta^2} - \frac{\theta \arccos \theta}{(1-\theta^2)^{3/2}} \right) (x \cdot y - x \cdot y) (y_i - (x \cdot y)x_i) \\ &\quad + \frac{\arccos \theta}{\sqrt{1-\theta^2}} ((y_i - (x \cdot y)x_i) - (y_j - (x \cdot y)x_i + x_i)) \\ &= -\arccos(x \cdot y) \frac{x_i}{\sqrt{1-(x \cdot y)^2}}. \end{aligned} \quad (3.21)$$

Thus we have proved the claim (3.13), hence (3.17); which in term established Lemma 3.1. \square

4. Boundary C^2 Estimate

In this section we prove the boundary C^2 estimate (1.11) in the two dimensional case. Recall that $\Omega, \Omega^* = B_1(0)$ and the boundary condition is written as

$$\phi^*(T_u) = 0 \quad \text{on } \partial B_1(0), \quad (3.1)$$

where $\phi^*(y) = \frac{1}{2}(|y|^2 - 1)$, and $T_u = T(\cdot, Du)$ is the optimal mapping.

It is convenient to denote the vector field $\beta = (\beta_1, \beta_2)$ where

$$\beta_k := \frac{\partial \phi^*}{\partial p_k} = \phi_i^* D_{p_k} y_i = -\phi_i^* c^{i,k}. \quad (3.2)$$

Differentiating along any tangential vector field τ on ∂B_1 , we have

$$0 = \phi_k^* D_i y_k \tau_i = -\phi_k^* c^{k,j} w_{ji} \tau_i = w_{\tau\beta} \quad \text{on } \partial B_1. \quad (3.3)$$

Let ν be the unit outer normal of ∂B_1 , by differentiating

$$0 \leq \phi_k^* D_i y_k \nu_i = -\phi_k^* c^{k,j} w_{ji} \nu_i = w_{\nu\beta} \quad \text{on } \partial B_1. \quad (3.4)$$

Here and below we use the notation $w_{\xi\eta}$ to denote $w_{ij}\xi_i\eta_j$ even if ξ and η are not unit vector fields.

Suppose $w_{\xi\xi}$ takes its maximum over ∂B_1 and unit vector ξ at $x_0 \in \partial B_1$. Note that we may write ξ in terms of a tangential component $\tau(\xi)$ and a component in the direction of β , namely

$$\xi = \tau(\xi) + \frac{\nu \cdot \xi}{\beta \cdot \nu} \beta$$

where

$$\tau(\xi) = \xi - (\nu \cdot \xi)\nu - \frac{\nu \cdot \xi}{\beta \cdot \nu} \beta^T$$

and

$$\beta^T = \beta - (\beta \cdot \nu)\nu.$$

Thanks to the oblique estimate in Lemma 3.1, we have

$$|\tau(\xi)|^2 = 1 - \left(1 - \frac{|\beta^T|^2}{(\beta \cdot \nu)^2}\right) (\nu \cdot \xi)^2 - 2(\nu \cdot \xi) \frac{\beta^T \cdot \xi}{\beta \cdot \nu} \leq C.$$

Thus,

$$\begin{aligned} w_{\xi\xi} &= w_{\tau(\xi)\tau(\xi)} + \frac{2\nu \cdot \xi}{\beta \cdot \nu} w_{\tau\beta} + \frac{(\nu \cdot \xi)^2}{(\beta \cdot \nu)^2} w_{\beta\beta} \\ &\leq |\tau(\xi)|^2 w_{\tau_0\tau_0} + \frac{(\nu \cdot \xi)^2}{(\beta \cdot \nu)^2} w_{\beta\beta}, \end{aligned} \quad (3.5)$$

Namely, it suffices to control $w_{\tau_0\tau_0}$ and $w_{\beta\beta}$.

From (3.17) and (1.2), $x \cdot Du \equiv 0$ on ∂B_1 . Without loss of generality, we may assume $x_0 = (0, 1)$ and locally ∂B_1 can be represented by $x_2 = \rho(x_1) = \sqrt{1 - |x_1|^2}$. By tangential differentiation at x_0 ,

$$\begin{aligned} 0 &= u_1 + x_k u_{k1} + (u_2 + x_k u_{k2}) \rho', \\ 0 &= 2u_{11} + x_k u_{k11} + (u_2 + x_k u_{k2}) \rho'' \\ &= 2u_{11} + u_{211} + (u_2 + u_{22}). \end{aligned}$$

As e_1, e_2 are the unit tangential and outer normal vectors at x_0 , respectively, we have

$$u_{\nu 11} \leq O(1 + w_{ii}). \quad (3.6)$$

We now tangentially differentiate the boundary condition $\phi^*(T_u)$ twice in the e_1 direction at x_0 , to obtain

$$\phi_{ij}^* \frac{\partial y_i}{\partial x_1} \frac{\partial y_j}{\partial x_1} + \phi_i^* \frac{\partial^2 y_i}{\partial x_1^2} + \phi_i^* \frac{\partial y_i}{\partial x_2} = 0,$$

and thus

$$\sum_i \left(c^{i,k} w_{k1} \right)^2 - y_i \frac{\partial}{\partial x_1} \left(c^{i,k} w_{k1} \right) - y_i c^{i,k} w_{k2} = 0,$$

which implies

$$\begin{aligned} w_{11}^2 &\leq y_i \left(c_{1,}^{i,k} w_{k1} + c^{i,k} D_1 w_{k1} \right) + w_{\beta 2} \\ &= y_i c_{1,}^{i,k} w_{k1} + y_i c^{i,k} (u_{k11} + c_{k11,} + c_{k1,p} c^{p,q} w_{q1}) + w_{\beta 2} \\ &= u_{\beta 11} + O(1 + w_{ii}). \end{aligned} \quad (3.7)$$

Let us assume that the maximal double-tangential term $w_{\tau\tau}$ occurs at

x_0 in a tangential direction e_1 , i.e. $w_{11}(x_0)$. Hence, $D_\tau w_{11}(x_0) = 0$, which gives

$$u_{\tau 11} \leq O(1 + w_{ii}). \quad (3.8)$$

Therefore, from (3.6) and (3.8)

$$u_{\beta 11} \leq O(1 + w_{ii}),$$

and by (3.7)

$$w_{11}^2 \leq O(1 + w_{ii}).$$

Using the fact that $\lambda^{-1} < \det w_{ij} < \lambda$ for some constant $\lambda > 0$, we conclude that

$$w_{11}(x_0) \leq C. \quad (3.9)$$

It remains to bound $w_{\beta\beta}(x_0)$. By contradiction, we may assume $w_{\beta\beta}(x_0)$ is arbitrarily large. Note that we can decompose $\nu (= e_2)$ in terms of

$$\nu = -\frac{1}{\beta \cdot \nu} (\beta - (\beta \cdot \nu)\nu) + \frac{1}{\beta \cdot \nu} \beta.$$

There exists a matrix $A = (a_{ij})$ such that at x_0 ,

$$\begin{bmatrix} 1 & 0 \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \tau \\ \beta \end{bmatrix} = \begin{bmatrix} \tau \\ \nu \end{bmatrix},$$

where $0 < a_{22} = \frac{1}{\beta \cdot \nu} \leq C$ by the obliqueness, and thus $\det A \leq C$. From the decomposition,

$$\begin{aligned} w_{11} &= w_{\tau\tau}, & w_{12} &= a_{21}w_{\tau\tau}, \\ w_{22} &= a_{22}^2 w_{\beta\beta} + a_{21}^2 w_{\tau\tau}. \end{aligned}$$

Since $|a_{21}|, w_{\tau\tau}$ are bounded, w_{22} will be arbitrarily large if $w_{\beta\beta}$ is (by assumption).

Next we invoke the dual problem: Let u^* denote the c -transform of u , defined for $y = T_u(x) \in \Omega^*$ by

$$u^*(y) = -c(x, y) - u(x).$$

It follows that

$$Du^*(y) = -c_y(x, y) = -c_y(T_{u^*}^*(y), y),$$

where

$$T_{u^*}^*(y) = (T_u)^{-1}(y),$$

and the dual equation is

$$\begin{aligned} |\det D_y(T_{u^*}^*)| &= g(y)/f(T_{u^*}^*) \quad \text{in } \Omega^*, \\ T_{u^*}^*(\Omega^*) &= \Omega. \end{aligned}$$

Furthermore, by differentiation at $y = T_u(x)$,

$$w^{ij}(x) = w_{kl}^*(y)c^{k,i}c^{l,j}(x, y), \quad (3.10)$$

where $w_{kl}^*(y) = u_{y_k y_l}^*(y) + c_{,kl}(x, y)$ and (w^{ij}) is the inverse of (w_{ij}) . By a similar analysis as for (3.9), we have

$$w_{\tau^* \tau^*}^*(y_0) \leq C, \quad (3.11)$$

where $y_0 = T_u(x_0)$ and τ^* is the tangential direction at y_0 .

Define

$$\tilde{\tau}_k := \tau_i^* c_{i,k}(x_0, y_0). \quad (3.12)$$

Then by (3.10) and (3.11) we have

$$\begin{aligned} C &\geq w_{ij}^* \tau_i^* \tau_j^* = \left(w_{ij}^* c^{i,k} c^{j,l} \right) \tilde{\tau}_k \tilde{\tau}_l = w^{kl} \tilde{\tau}_k \tilde{\tau}_l \\ &= w^{11} \tilde{\tau}_1^2 + 2w^{12} \tilde{\tau}_1 \tilde{\tau}_2 + w^{22} \tilde{\tau}_2^2. \end{aligned}$$

It is easy to see that the last two terms are bounded because of (3.3) and (3.9). If we can show $\tilde{\tau}_1^2 \geq \delta_0$ for some constant $\delta_0 > 0$, then we have a contradiction as $w^{11} = w_{22}/\det w_{ij}$ will become arbitrary large (by assumption).

At (x_0, y_0) , by the obliqueness estimate (1.10)

$$-c^{2,1}y_1 - c^{2,2}y_2 \geq c_0,$$

where $c_0 > 0$ is constant. This is equivalent to

$$\frac{1}{\det c_{i,j}} (c_{2,1}y_1 - c_{1,1}y_2) \geq c_0,$$

and

$$(c_{2,1}y_1 - c_{1,1}y_2)^2 \geq c_0^2 (\det c_{i,j})^2 =: \delta_0 > 0.$$

At y_0 , the tangential $\tau^* = (y_2, -y_1)$. From (3.12)

$$\tilde{\tau}_1 = c_{1,1}\tau_1^* + c_{2,1}\tau_2^* = c_{1,1}y_2 - c_{2,1}y_1,$$

and thus we obtain

$$\tilde{\tau}_1^2 \geq \delta_0 > 0.$$

The above contradiction implies that $w_{\beta\beta}(x) \leq C$ for all $x \in \partial B_1$. Therefore, by (3.5) we conclude the estimate (1.11).

By Corollary 2.1, the cost function c satisfies the condition (A3). We now observe by the work pioneered by Trudinger-Wang [14] in the subject, one can obtain the global C^2 and higher order estimates under the further assumption that the densities f and g are C^2 and smooth.

To see this, from [14], we have the estimate

$$\sup_{\Omega} |D^2u| \leq C(1 + \sup_{\partial\Omega} |D^2u|), \quad (3.13)$$

combining this with (1.11), we obtain the global C^2 estimate.

Once the second derivatives are bounded, equations (1.3)–(1.4) are uniformly elliptic. This combined with the obliqueness estimate (1.10) yields global $C^{2,\alpha}$ estimates [8]. Moreover, the higher order estimates follow from the theory of linear elliptic equations with oblique boundary conditions [6] and thus Theorem 1.1 is proved.

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