Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **9** (2014), No. 1, pp. 1-23

A MOSER-TRUDINGER INEQUALITY FOR THE SINGULAR TODA SYSTEM

LUCA BATTAGLIA[1,a] AND ANDREA MALCHIODI[2,b]

Dedicated to Neil Trudinger with admiration

[1]SISSA, via Bonomea 265, 34136 Trieste, Italy.
[a]E-mail: lbatta@sissa.it
[2]University of Warwick, Mathematics Institute, Zeeman Building, Coventry CV4 7AL and SISSA, via Bonomea 265, 34136 Trieste, Italy.
[b]E-mail: A.Malchiodi@warwick.ac.uk; malchiod@sissa.it

Abstract

In this paper we prove a sharp version of the Moser-Trudinger inequality for the Euler-Lagrange functional of a singular Toda system, motivated by the study of models in Chern-Simons theory. Our result extends those in [14] and [37] for the scalar case, as well as that in [23] for the regular Toda system. We expect this inequality to be a basic tool to attack variationally the existence problem under general assumptions.

1. Introduction

The Moser-Trudinger inequality yields exponential-type embeddings of Sobolev functions in critical dimension. On a compact closed surface Σ the space $H^1(\Sigma)$ embeds compactly into every $L^p(\Sigma)$ for any real $p > 1$: at a more refined level, due to the seminal works [38] and [31] one has the inequality

$$16\pi \log \int_\Sigma e^{u-\overline{u}}\, dV_g \leq \int_\Sigma |\nabla u|^2\, dV_g + C; \qquad u \in H^1(\Sigma), \tag{1}$$

Received July 15, 2013 and in revised form October 4, 2013.
AMS Subject Classification: 35J50, 35J61, 35R01.
Key words and phrases: Toda system, best constants, Moser-Trudinger inequalities, singular Liouville equations.
The authors are supported by the FIRB project *Analysis and Beyond* and by the PRIN *Variational Methods and Nonlinear PDEs*. L.B. acknowledges support from the Mathematics Department at the University of Warwick.

where C is a constant depending only on Σ and its metric g, and where $\overline{u}$ stands for the average of u on the surface.

Inequality (1) has been proven to be fundamental in several contexts such as the Gaussian curvature prescription problem ([2], [11], [10]), mean field equations in fluid dynamics ([18], [19]) and models in theoretical physics ([36], [41]). To give an example, considering a conformal change of metric of the form $\widetilde{g} = e^w g$, the Gaussian curvature of Σ transforms according to the law

$$-\Delta w + 2K_g = 2K_{\widetilde{g}} e^w. \tag{2}$$

If one wishes to prescribe the Gaussian curvature $K_{\widetilde{g}}$ as a given function $K(x)$, then solutions to the problem can be found as critical points of the functional

$$I(u) := \int_\Sigma |\nabla u|^2 dV_g + \int_\Sigma K_g u \, dV_g - \left(\int_\Sigma K_g dV_g \right) \log \left(\int_\Sigma K\, e^u dV_g \right).$$

By means of (1) one can then control the last term in the functional by means of the Dirichlet energy.

More recent versions of (1) include exponential terms with power-type weights, which are motivated by the study of *singular Liouville equations*. For example, given points $p_1, \dots, p_m \in \Sigma$, weights $\alpha_1, \dots, \alpha_m > -1$, and a smooth positive function $h(x)$, a solution of the equation

$$-\Delta w + 2K_g = 2h\, e^w - 4\pi \sum_{j=1}^m \alpha_j \delta_{p_j} \tag{3}$$

yields a conformal metric $\widetilde{g} = e^w g$ with Gaussian curvature h on $\Sigma \setminus \{p_1, \dots, p_m\}$ and with a conical singularity at p_j with opening angle $2\pi(1 + \alpha_j)$.

By the substitution

$$\begin{aligned} w(x) &\mapsto w(x) + 4\pi \sum_{j=1}^m \alpha_j G_{p_j}(x), \\ h(x) &\mapsto \widetilde{h}(x) = h(x) e^{-4\pi \sum_{j=1}^m \alpha_j G_{p_j}(x)}, \end{aligned} \tag{4}$$

(2) transforms into an equation of the form

$$-\Delta w + 2\widetilde{f} = 2\widetilde{h}\, e^w \tag{5}$$

where $\widetilde{f}(x)$ is a smooth function and where

$$\widetilde{h} > 0 \quad \text{on } \Sigma \setminus \{p_1, \dots, p_m\}; \qquad \widetilde{h}(x) \simeq d(x, p_j)^{2\alpha_j} \text{ near } p_j. \tag{6}$$

Although (3) and (5) are perfectly equivalent, the advantage of the latter compared to the former is that the singular structure is absorbed into the factor $\widetilde{h}$, which endows the problem with a variational structure. Similarly to (2), solutions to (5) can be found as critical points of the functional

$$\widetilde{I}(u) := \int_\Sigma |\nabla u|^2 dV_g + \int_\Sigma \widetilde{f}\, u\, dV_g - \overline{K} \log\left(\int_\Sigma \widetilde{h}\, e^u dV_g \right),$$

where $\overline{K} = 2\pi\chi(\Sigma) + 2\pi \sum_{j=1}^m \alpha_j$ is a constant determined by the Gauss-Bonnet formula.

The singular weight $\widetilde{h}$ has indeed an effect on the optimal constant in the corresponding Moser-Trudinger type inequality. In [15], [37] (see also [10] for conical domains) it was shown that

$$16\pi \min\left\{1, 1 + \min_j \alpha_j\right\} \log \int_\Sigma \widetilde{h} e^{u-\overline{u}}\, dV_g \le \int_\Sigma |\nabla u|^2\, dV_g + C; \quad u \in H^1(\Sigma). \tag{7}$$

Notice that, if at least one of the α_j's is negative, say $\alpha_{\overline{j}}$, the constant gets worse, as $\widetilde{h}$ blows-up at $p_{\overline{j}}$. On the other hand when all the weights are positive the constant does not improve: this can be easily seen by the following consideration. The sharpness of the Moser-Trudinger constant $\frac{1}{16\pi}$ can be obtained using the test function

$$\varphi_{\lambda,x}(y) = \log \frac{\lambda^2}{\left(1 + \lambda^2 d(x,y)^2\right)^2}; \qquad x \in \Sigma, \lambda > 0, \tag{8}$$

which makes the two sides of (1) diverge at the same rate. As the conformal volume $e^{\varphi_{\lambda,x}}$ concentrates at x as $\lambda \to +\infty$, there would be no effect from the vanishing of $\widetilde{h}$ if x is a regular point. We also refer to [17], [21] for more general optimal inequalities on singular measure spaces.

Inequality (7) has been useful in finding constant curvature metrics when prescribing conical singularities as it might yield global minima of $\widetilde{I}$, see [37],

[8], as well as in studying general singular mean field equations like

$$-\Delta w + 2f = 2\rho h\, e^{w} - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j}, \tag{9}$$

where f, h are smooth functions, h positive, and ρ is a real parameter, see [4], [3], [29] (see also [12], [13] for a non-variational approach to (9)).

Singular Liouville equations have a role in fluid dynamics, see [39], as well as in the study of Electroweak theory or abelian Chern-Simons vortices, see [36], [41]. For the latter cases, singular points represent zeroes of the scalar wave function involved in the model.

The goal of this paper is to prove a sharp inequality related to a *singular Toda system* arising in Chern-Simons theory, which represents a non-abelian counterpart of (9). Specifically, we consider the following system

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1\right) - \rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1\right) - 4\pi \sum_{j=1}^{m} \alpha_{1,j}(\delta_{p_j} - 1), \\ -\Delta u_2 = 2\rho_2 \left(\frac{h_2 e^{u_2}}{\int_\Sigma h_2 e^{u_2} dV_g} - 1\right) - \rho_1 \left(\frac{h_1 e^{u_1}}{\int_\Sigma h_1 e^{u_1} dV_g} - 1\right) - 4\pi \sum_{j=1}^{m} \alpha_{2,j}(\delta_{p_j} - 1), \end{cases} \tag{10}$$

where h_1, h_2 are smooth positive functions on Σ, and the coefficients $\alpha_{i,j}$ are larger than -1.

While abelian Chern-Simons vortices have been quite studied for some time, see e.g. [7], [9], [32], [34], [35], the treatment of the non-abelian case is more recent, see e.g. [20], [24], [25], [27], [33].

With a change of variable similar to (4) the latter problem transforms into

$$\begin{cases} -\Delta u_1 = 2\rho_1 \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - 1\right) - \rho_2 \left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g} - 1\right), \\ -\Delta u_2 = 2\rho_2 \left(\frac{\widetilde{h}_2 e^{u_2}}{\int_\Sigma \widetilde{h}_2 e^{u_2} dV_g} - 1\right) - \rho_1 \left(\frac{\widetilde{h}_1 e^{u_1}}{\int_\Sigma \widetilde{h}_1 e^{u_1} dV_g} - 1\right), \end{cases} \tag{11}$$

where the functions $\widetilde{h}_i$ satisfy

$$\widetilde{h}_i > 0 \quad \text{on} \quad \Sigma \setminus \{p_1, \dots, p_m\}; \quad \widetilde{h}_i(x) \simeq d(x, p_j)^{2\alpha_{i,j}} \text{ near } p_j, \quad i = 1,\ 2. \tag{12}$$

As for the scalar case one gains the variational structure, with Euler-Lagrange functional

$$J_\rho(u_1,u_2)=\int_\Sigma Q(u_1,u_2)\,dV_g+\sum_{i=1}^{2}\rho_i\left(\int_\Sigma u_i dV_g-\log\int_\Sigma \widetilde{h}_i e^{u_i}dV_g\right), \quad (13)$$

where $Q(u_1,u_2)$ is defined as:

$$Q(u_1,u_2)=\frac{1}{3}\left(|\nabla u_1|^2+|\nabla u_2|^2+\nabla u_1\cdot\nabla u_2\right). \quad (14)$$

Concerning Liouville systems with no singularites, some sharp inequalities were proven in [16], [40] when the matrix of coefficients of the exponential terms is non-negative. For the regular Toda system instead a sharp inequality was found in [23], where it was shown that

$$4\pi\sum_{i=1}^{2}\log\int_\Sigma e^{u_i-\overline{u_i}}\,dV_g\le\int_\Sigma Q(u_1,u_2)\,dV_g+C; \qquad u\in H^1(\Sigma). \quad (15)$$

Notice that one always has the inequality $Q(u_1,u_2)\ge\frac{1}{4}|\nabla u_1|^2$, and hence (15) can be thought of as an extension of (1). Our main result is the following one, which extends both (7) and (15).

Theorem 1.1. *Suppose $p_1,\dots,p_m\in\Sigma$ and $\alpha_{i,j}$, $i=1,2$, $j=1,\dots,m$, satisfy $\alpha_{i,j}>-1$ for all i,j. Then, if $\widetilde{h}_i$ satisfy (12), the following inequality holds*

$$\begin{aligned}4\pi\sum_{i=1}^{2}\min\left\{1,1+\min_j\alpha_{i,j}\right\}\log\int_\Sigma\widetilde{h}_i e^{u_i-\overline{u}_i}\,dV_g\\ \le\int_\Sigma Q(u_1,u_2)\,dV_g+C\quad u_1,u_2\in H^1(\Sigma).\end{aligned} \quad (16)$$

The constants in the above inequality are sharp.

We expect the above result to be a main step for a possible variational approach for the study of (10). In the recent paper [5] the case of non-negative coefficients and positive genus has been treated using simply inequality (15), as the corresponding functions $\widetilde{h}_i$ are uniformly bounded (see also [28] and [30] for the regular case). In more general cases, the full strength of (16) would be needed.

Some steps in the proof of the above theorem follow closely the arguments in [23]: through blow-up analysis one can show with few difficulties that inequality (16) holds for any smaller couple of parameters, and moreover that there exist extremal functions for the corresponding Euler functionals (13). We pass then to the limit for these extremals when the parameters approach the critical ones.

However the presence of singularities might cause in principle a variety of blow-up behaviours (different blow-up rates for the two components, and blow-up at regular or singular points): using a Pohozaev identity from the recent paper [26] we reduce ourselves to two cases only. The former can be brought back to the scalar case, where one can use (7) to get a conclusion; the latter can be solved by using a *local* version of the singular Moser-Trudinger inequality from Adimurthi and Sandeep [1]. The latter argument in particular differs substantially from that in [23], and it also provides a simpler argument for the regular case.

2. Notation and Preliminaries

In this section we provide some useful notation and some known preliminary results which will be used in the proof of the main theorem.

First of all, given two points $x, y \in \Sigma$, we will indicate as $d(x, y)$ the metric distance between x and y on Σ; we will denote as $B_r(p)$ the open metric ball of radius r centered at p.

Given a function $u \in L^1(\Sigma)$, $\overline{u}$ will stand for the average of u on Σ; since we will suppose, from now on, $|\Sigma| = 1$, we can write

$$\overline{u} = \int_\Sigma u dV_g.$$

We denote as x^- the negative part of a real number x, that is

$$x^- := \begin{cases} 0 & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases},$$

and we set, for $i \in \{1, 2\}$,

$$\widetilde{\alpha}_i = -\max_{j \in \{1,\dots,m\}} {\alpha_{i,j}}^-. \tag{17}$$

Notice that, in these terms, the inequality we wish to prove is

$$4\pi\sum_{i=1}^{2}(1+\widetilde{\alpha}_i)\log\int_\Sigma \widetilde{h}_i e^{u_i-\overline{u_i}}dV_g \le \int_\Sigma Q(u_1,u_2)dV_g + C \quad u_1,u_2\in H^1(\Sigma),$$

whereas the singular Chen-Troyanov (7) inequality can be expressed as

$$16\pi(1+\widetilde{\alpha}_i)\log\int_\Sigma \widetilde{h}_i e^{u_i-\overline{u}_i}dV_g \le \int_\Sigma |\nabla u|^2\, dV_g + C; \quad u\in H^1(\Sigma).$$

We then define the i^{th} weight of a point $p\in\Sigma$, for $i\in\{1,2\}$ in the following way

$$p=p_j \quad \Rightarrow \quad \alpha_i(p)=\alpha_{i,j} \qquad p\notin\{p_1,\dots,p_m\} \quad \Rightarrow \quad \alpha_i(p)=0. \tag{18}$$

The definition implies that $\widetilde{h}_i \simeq d(\cdot,p)^{2\alpha_i(p)}$ near p; precisely, it is the only real number such that $\log\widetilde{h}_i - 2\widetilde{\alpha}_i\log d(\cdot,p)$ is bounded in a sufficiently small neighborhood of p.
As anticipated in the introduction, we will prove inequality (16) via blow-up analysis. We define, for a sequence $u_k=(u_{1,k},u_{2,k})$ of solutions of (11), the concentration value of the i^{th} component around a point $p\in\Sigma$ as

$$\sigma_i(p) := \lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(p)}\widetilde{h}_i e^{u_{i,k}}dV_g. \tag{19}$$

Lin, Wei and Zhang in [26] found out, through a Pohožaev identity, that the concentration values satisfy the following condition, which was already pointed out for the regular case in [22].

Theorem 2.1 ([26], Proposition 3.1). *Let* $u_k=(u_{1,k},u_{2,k})\in H^1(\Sigma)^2$ *be solutions of* (11), $\widetilde{\alpha}_i$ *be as in* (18) *and* σ_i *be as in* (19). *Then, it holds*

$$\sigma_1(p)^2-\sigma_1(p)\sigma_2(p)+\sigma_2(p)^2 = 4\pi(1+\widetilde{\alpha}_1(p))\sigma_1(p)+4\pi(1+\widetilde{\alpha}_2(p))\sigma_2(p). \tag{20}$$

In the setting we are considering, a dichotomy between concentration and compactness occurs, similar to the ones in the regular case from Jost-Wang [23], Theorem 3.1. Since the proof of the theorem we are giving is very close to [23], we will only sketch it; we refer to these papers for the details in the regular case.

Theorem 2.2. *Let $\widetilde{h}_i$ as in (12), let $u_k=(u_{1,k},u_{2,k})\in H^1(\Sigma)^2$ be solutions of*

$$\begin{cases} -\Delta u_{i,k} = 2V_{i,k}\widetilde{h}_i e^{u_{i,k}} - V_{3-i,k}\widetilde{h}_{3-i}e^{u_{3-i,k}} + \psi_{i,k} \\ \int_\Sigma \widetilde{h}_i e^{u_{i,k}} dV_g \le C \\ \|\psi_{i,k}\|_{L^p(\Sigma)} \le C \\ V_{i,k} \underset{k\to+\infty}{\to} 1 \ in\ L^\infty(\Sigma) \end{cases} \qquad i\in\{1,2\},$$

for some $p>1$, $C>0$ and define the sets S_i as

$$S_i := \left\{ p\in\Sigma:\ \exists\, x_k \underset{k\to+\infty}{\to} p \text{ such that } u_{i,k}(x_k) \underset{k\to+\infty}{\to} +\infty \right\}.$$

Then, after taking subsequences, one of the following alternatives happens.

1. *For each $i\in\{1,2\}$, either $u_{i,k}$ is bounded in $L^\infty(\Sigma)$ or it tends uniformly to $-\infty$.*
2. *$S_i\neq\emptyset$ for some $i\in\{1,2\}$; in this case, S_i is finite and either $u_{j,k}$ is bounded in $L^\infty_{loc}(\Sigma\backslash(S_1\cup S_2))$ or it converges to $-\infty$ in $L^\infty_{loc}(\Sigma\backslash(S_1\cup S_2))$ for each $j\in\{1,2\}$; moreover, if $S_i\backslash S_{3-i}\neq\emptyset$, then the latter alternative occurs for $u_{i,k}$.*

Proof. (Sketch) Reasoning as in [4] we find that, given $p\in\Sigma$, if for some $i\in\{1,2\}$ one has

$$\limsup_{k\to+\infty}\int_{B_r(p)} V_{i,k}\widetilde{h}_i e^{u_{i,k}} dV_g < 2\pi(1+\alpha_i(p)^-)$$

for sufficiently small r, then $u_{i,k}$ is uniformly bounded from above, and this fact implies the finiteness of the sets S_i. The alternative between being bounded in L^∞ and converging uniformly to $-\infty$ follows by applying a Harnack inequality and the last part of (2) follows by arguing as in [6], Theorem 3. ∎

Finally, as anticipated, we will need a singular Moser-Trudinger inequality on bounded Euclidean domains, from [1]:

Theorem 2.3 ([1], Theorem 2.1). *Let $\Omega \subset \mathbb{R}^2$ a bounded domain containing the origin. Then, for any $\alpha \in (-1, 0]$, it holds*

$$\sup_{u \in H_0^1(\Omega), \int_\Omega |\nabla u(x)|^2 dx \leq 1} \int_\Omega |x|^{2\alpha} e^{4\pi(1+\alpha)u(x)^2} dx \leq C,$$

where C is a constant depending on α and Ω only.

From elementary inequalities we then obtain the following result.

Corollary 2.4. *Let $\Omega \subset \mathbb{R}^2$ a bounded domain containing the origin. Then, for any $\alpha \in (-1, 0]$ and $u \in H_0^1(\Omega)$, it holds*

$$16\pi(1+\alpha) \log \int_\Omega |x|^{2\alpha} e^{u(x)} dx \leq \int_\Omega |\nabla u(x)|^2 dx + C. \tag{21}$$

3. A Moser-Trudinger Inequality

In this section, we are going to prove the following Moser-Trudinger type inequality.

Theorem 3.1. *Let Σ be a closed surface with area $|\Sigma| = 1$, $\widetilde{h}_i$ be as in (12), and $\widetilde{\alpha}_i$ be as in (17). Then, for any $\rho = (\rho_1, \rho_2) \in \mathbb{R}_+^2$ satisfying $\rho_i < 4\pi(1 + \widetilde{\alpha}_i)$ for both $i \in \{1, 2\}$ there exists $C(\rho) > 0$ such that the Euler-Lagrange functional (13) verifies*

$$J_\rho(u) > -C(\rho) \qquad \forall\, u \in H^1(\Sigma)^2$$

Definition 3.2. As in [23], we define the set of admissible parameters Λ as

$$\Lambda := \left\{ \rho \in \mathbb{R}_+^2 : J_\rho \text{ is bounded from below} \right\}.$$

Clearly, Λ preserves the partial order of $\mathbb{R}_+^2$, that is if $\rho \in \Lambda$ then $\widetilde{\rho} \in \Lambda$ until $\widetilde{\rho}_i \leq \rho_i$ for both $i \in \{1, 2\}$; in these terms, Theorem 3.1 is equivalent to saying

$$(0, 4\pi(1 + \widetilde{\alpha}_1)) \times (0, 4\pi(1 + \widetilde{\alpha}_2)) \subset \Lambda.$$

Remark 3.3. One can easily see that Λ is not empty: since it holds

$$\frac{|\nabla u_1|^2 + |\nabla u_2|^2}{6} \leq Q(u_1, u_2)$$

one can apply the scalar Moser-Trudinger inequality (7) to both components to get

$$\left(0,\frac{8}{3}\pi(1+\widetilde{\alpha}_1)\right)\times\left(0,\frac{8}{3}\pi(1+\widetilde{\alpha}_2)\right)\subset\Lambda.$$

To prove Theorem 3.1, some lemmas will be needed. First of all, we notice that when the parameter ρ is in the interior of the set Λ, then the energy functional is not only bounded from below, but even coercive and it has a minimizer; on the other hand, if ρ is on the boundary of Λ, then J_ρ cannot be coercive.

Lemma 3.4. *For any $\rho\in\overset{\circ}{\Lambda}$ there exists a constant C such that*

$$J_\rho(u)\geq\frac{\int_\Sigma\left(|\nabla u_1|^2+|\nabla u_2|^2\right)dV_g}{C}-C$$

Moreover, J_ρ admits a minimizer $u=(u_1,u_2)$ that solves (11).

Proof. Taking $\delta\in\left(0,\frac{d(\rho,\partial\Lambda)}{\sqrt{2}}\right)$, we have $(1+\delta)\rho\in\Lambda$ so $J_{(1+\delta)\rho}(u)\geq-C$; therefore, we can write

$$\begin{aligned}J_\rho(u)&=\frac{\delta}{1+\delta}\int_\Sigma Q(u_1,u_2)dV_g+\frac{J_{(1+\delta)\rho}(u)}{1+\delta}\\&\geq\frac{\delta}{6(1+\delta)}\int_\Sigma\left(|\nabla u_1|^2+|\nabla u_2|^2\right)dV_g-C\end{aligned}$$

and the first claim follows.

To prove the rest we notice that, if we restrict ourselves to the subset of $H^1(\Sigma)^2$ consisting of all functions satisfying $\int_\Sigma\widetilde{h}_ie^{u_i}dV_g=1$, the energy is coercive because, from Poincaré's inequality and (7)

$$\begin{aligned}\int_\Sigma u_i^2dV_g&=\int_\Sigma(u_i-\overline{u_i})^2\,dV_g+(\overline{u_i})^2\\&\leq C\int_\Sigma|\nabla u_i|^2dV_g+\left(C+\frac{1}{16\pi(1+\widetilde{\alpha}_i)}\int_\Sigma|\nabla u_i|^2dV_g\right)^2\\&\leq C\left(1+\int_\Sigma|\nabla u_i|^2dV_g\right)^2.\end{aligned}$$

Being J_ρ weakly lower-semicontinuous as well, the existence of minimizers follows from the direct methods of calculus of variations. □

Lemma 3.5. *For any $\rho \in \partial\Lambda$ there exists a sequence $\{\widetilde{u}_k\}_{k\in\mathbb{N}} \subset H^1(\Sigma)^2$ verifying*

$$\int_\Sigma \left(|\nabla \widetilde{u}_{1,k}|^2 + |\nabla \widetilde{u}_{2,k}|^2\right) dV_g \underset{k\to+\infty}{\to} +\infty$$
$$\lim_{k\to+\infty} \frac{J_\rho(\widetilde{u}_k)}{\int_\Sigma \left(|\nabla \widetilde{u}_{1,k}|^2 + |\nabla \widetilde{u}_{2,k}|^2\right) dV_g} \le 0.$$

Proof. Suppose by contradiction that

$$\int_\Sigma \left(|\nabla u_{1,k}|^2 + |\nabla u_{2,k}|^2\right) dV_g \underset{k\to+\infty}{\to} +\infty$$
$$\Rightarrow \frac{J_\rho(u_k)}{\int_\Sigma \left(|\nabla u_{1,k}|^2 + |\nabla u_{2,k}|^2\right) dV_g} \ge \theta > 0$$

for any choice of $\{u_k\}$. This would mean that

$$J_\rho(u) \ge \frac{\theta}{2} \int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g - C,$$

hence for any small δ we would get

$$\begin{aligned} J_{(1+\delta)\rho}(u) &= (1+\delta) J_\rho(u) - \delta \int_\Sigma Q(u_1,u_2) dV_g \\ &\ge \left((1+\delta)\frac{\theta}{2} - \frac{\delta}{2}\right) \int_\Sigma \left(|\nabla u_1|^2 + |\nabla u_2|^2\right) dV_g - C \\ &\ge -C \end{aligned}$$

hence $(1+\delta)\rho \in \Lambda$, whereas one clearly has $(1-\delta)\rho \in \Lambda$; this is in contradiction to $\rho \in \partial\Lambda$. □

We then need a basic calculus lemma. Its proof will be omitted, as it can be found in [23] (following an idea of W. Ding).

Lemma 3.6 ([23], Lemma 4.4)**.** *Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{b_k\}_{k\in\mathbb{N}}$ be two sequences of real numbers satisfying*

$$a_k \underset{k\to+\infty}{\to} +\infty \qquad \text{and} \qquad \lim_{k\to+\infty} \frac{b_k}{a_k} \le 0.$$

Then there exists a smooth function $F : [0,+\infty) \to \mathbb{R}$ satisfying, up to sub-

sequences,

$$0 < F'(t) < 1 \quad \textit{for any } t \geq 0 \qquad F'(t) \underset{t\to+\infty}{\to} 0 \qquad F(a_k) - b_k \underset{k\to+\infty}{\to} +\infty.$$

The latter lemma will be applied to the sequences

$$a_k = \int_\Sigma Q(\widetilde{u}_{1,k}, \widetilde{u}_{2,k}) dV_g \qquad\qquad b_k = J_\rho(\widetilde{u}_k)$$

where $\widetilde{u}_k$ is as in Lemma 3.5, and we will consider the auxiliary functional

$$\widetilde{J}_\rho(u) := J_\rho(u) - F\left(\int_\Sigma Q(u_1, u_2) dV_g\right),$$

whose behavior is described by the following lemma.

Lemma 3.7. *For any $\rho \in \overset{\circ}{\Lambda}$ the functional $\widetilde{J}_\rho$ is bounded from below on $H^1(\Sigma)^2$ and its infimum is achieved by a function satisfying*

$$\begin{cases} -\left(1 - \frac{2}{3} g(u)\right) \Delta u_i + \frac{g(u)}{3} \Delta u_{3-i} = 2\rho_i \left(\widetilde{h}_i e^{u_i} - 1\right) - \rho_{3-i} \left(\widetilde{h}_{3-i} e^{u_{3-i}} - 1\right); \\ \displaystyle\int_\Sigma \widetilde{h}_i e^{u_i} dV_g = 1, \end{cases}$$

where $g(u) = F'\left(\int_\Sigma Q(u_1, u_2) dV_g\right)$. On the other hand, if $\rho \in \partial\Lambda$ then $\inf_{H^1(\Sigma)^2} \widetilde{J}_\rho = -\infty$

Proof. For $\rho \in \overset{\circ}{\Lambda}$ one can argue as in Lemma 3.4, yielding lower semi-continuity from the regularity of F and coercivity from the behavior of F' at infinity.

For $\rho \in \partial\Lambda$, taking $\widetilde{u}_k$ as in Lemma 3.5 and applying Lemma 3.6 one gets

$$\widetilde{J}_\rho(\widetilde{u}_k) = b_k - F(a_k) \underset{k\to+\infty}{\to} -\infty.$$

This concludes the proof. ∎

We are now in position to prove the main theorem of this section.

Proof of Theorem 3.1. Suppose by contradiction that

$$(0, 4\pi(1 + \widetilde{\alpha}_1)) \times (0, 4\pi(1 + \widetilde{\alpha}_2)) \not\subset \Lambda;$$

then there is some $\overline{\rho} \in \partial\Lambda$ with $\overline{\rho}_i < 4\pi(1+\widetilde{\alpha}_i)$ for both $i \in \{1,2\}$.

Consider a sequence $\{\rho_k\}_{k\in\mathbb{N}} \in \overset{\circ}{\Lambda}$ with $\rho_k \underset{k\to+\infty}{\to} \overline{\rho}$ and a minimizer u_k for $\widetilde{J}_{\rho_k}$, as in Lemma 3.7; then, $v_k := u_k + \log \rho_k$ solves

$$\begin{cases} -\Delta v_{i,k} = 2\frac{6-5g(v_k)}{6-8g(v_k)+2g(v_k)^2}\left(\widetilde{h}_i e^{v_{i,k}} - \rho_{i,k}\right) \\ \qquad\qquad -\frac{3-4g(v_k)}{3-4g(v_k)+g(v_k)^2}\left(\widetilde{h}_{3-i} e^{v_{3-i,k}} - \rho_{3-i,k}\right); \\ \int_\Sigma \widetilde{h}_i e^{v_{i,k}} dV_g = \rho_{i,k}, \end{cases}$$

with $\frac{6-5g(v_k)}{6-8g(v_k)+2g(v_k)^2}$ and $\frac{3-4g(v_k)}{3-4g(v_k)+g(v_k)^2}$ both uniformly converging to 1, so Theorem 2.2 can be applied to this sequence. The normalization on the integral implies that $u_{i,k}$ cannot tend to $-\infty$ for any $i \in \{1,2\}$; moreover, we can also exclude boundedness in $L^\infty(\Sigma)$ because this would imply convergence to a minimizer $\overline{u}$ of $\widetilde{J}_{\overline{\rho}}$, contradicting Lemma 3.7.

The only case left is the blow-up around at least one point p: Pohožaev's identity (20) implies that if there is a singularity of mass $\alpha_{i,j}$ on p then $\sigma_i \geq 4\pi(1+\alpha_{i,j})$ for some $i \in \{1,2\}$, whereas if p is a regular point then there is a component with a mass of at least 4π around it; in both cases, for such an i we obtain:

$$4\pi(1+\widetilde{\alpha}_i) \leq \lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(p)} \widetilde{h}_i e^{v_{i,k}} dV_g \leq \lim_{k\to+\infty}\int_\Sigma \widetilde{h}_i e^{v_{i,k}} dV_g = \overline{\rho}_i < 4\pi(1+\widetilde{\alpha}_i),$$

that is a contradiction. $\square$

We conclude the section by showing a partial converse of Theorem 3.1, namely that for higher values of the parameter ρ the functional J_ρ is unbounded from below.

Proposition 3.8. *If* $\rho_i > 4\pi(1+\widetilde{\alpha}_i)$ *for some* $i \in \{1,2\}$, *then* $\inf_{H^1(\Sigma)^2} J_\rho = -\infty$ *that is*

$$\Lambda \subset (0, 4\pi(1+\widetilde{\alpha}_1)] \times (0, 4\pi(1+\widetilde{\alpha}_2)].$$

Proof. We will show the proof only for $i = 1$, since the same argument works for $i = 2$ as well.

Choosing a point p_1 such that $\widetilde{h}_1 \simeq d(\cdot, p_i)^{2\widetilde{\alpha}_1}$ in its neighborhood, we define for large λ

$$
\begin{aligned}
\varphi_{1,\lambda}(x) &= \log\left(\frac{\lambda^{1+\widetilde{\alpha}_1}}{1+(\lambda d(x,p_1))^{2(1+\widetilde{\alpha}_1)}}\right)^2;\\
\varphi_{2,\lambda}(x) &= -\frac{1}{2}\log\left(\frac{\lambda^{1+\widetilde{\alpha}_1}}{1+(\lambda d(x,p_1))^{2(1+\widetilde{\alpha}_1)}}\right)^2.
\end{aligned}
$$

Using the fact that $\left|\nabla\left(d(x,p_1)^{2(1+\widetilde{\alpha}_1)}\right)\right| \le 2(1+\widetilde{\alpha}_1)d(x,p_1)^{1+2\widetilde{\alpha}_1}$, we obtain

$$
\begin{aligned}
|\nabla\varphi_{1,\lambda}(x)| &= \left|\frac{-2\lambda^{2(1+\widetilde{\alpha}_1)}\left|\nabla\left(d(x,p_1)^{2(1+\widetilde{\alpha}_1)}\right)\right|}{1+(\lambda d(x,p_1))^{2(1+\widetilde{\alpha}_1)}}\right|\\
&\le \frac{4(1+\widetilde{\alpha}_1)\lambda^{2(1+\widetilde{\alpha}_1)}d(x,p_1)^{1+2\widetilde{\alpha}_1}}{1+(\lambda d(x,p_1))^{2(1+\widetilde{\alpha}_1)}}\\
&\le \min\left\{C\lambda^2 d(x,p_1)^{1+2\widetilde{\alpha}_2}, \frac{4(1+\widetilde{\alpha}_1)}{d(x,p_1)}\right\},
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\int_\Sigma Q(\varphi_{1,\lambda},\varphi_{2,\lambda})dV_g = \frac{1}{4}\int_\Sigma |\nabla\varphi_{1,\lambda}|^2 dV_g\\
&\le C\lambda^4\int_{B_{\frac{1}{\lambda}}(p_1)} d(\cdot,p_1)^{2(1+\widetilde{\alpha}_1)}dV_g + 4(1+\widetilde{\alpha}_1)^2\int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)}\frac{dV_g}{d(\cdot,p_1)^2}\\
&\le C + 8\pi(1+\widetilde{\alpha}_1)^2\log\lambda. \qquad (22)
\end{aligned}
$$

Moreover, being

$$
\begin{aligned}
\max\{1,(\lambda d(x,p_1))^{2(1+\widetilde{\alpha}_1)}\} &\le 1+(\lambda d(x,p_1))^{2(1+\widetilde{\alpha}_1)}\\
&\le C\max\{1,(\lambda d(x,p_1))^{2(1+\widetilde{\alpha}_1)}\}, \qquad (23)
\end{aligned}
$$

one gets

$$
\overline{\varphi_{1,\lambda}} = \int_\Sigma(\max\{2(1+\widetilde{\alpha}_1)\log\lambda, -2(1+\widetilde{\alpha}_1)(\log\lambda + 2\log d(\cdot,p_1))\} + O(1))dV_g.
$$

Dividing Σ into the two regions where the above maximum is attained and using the integrability of $\log d(\cdot, p_1)$ in two dimensions one gets

$$\begin{aligned}\overline{\varphi_{1,\lambda}} &= 2(1+\widetilde{\alpha}_1)\log\lambda \int_{B_{\frac{1}{\lambda}}(p_1)} dV_g - 2(1+\widetilde{\alpha}_1)\log\lambda \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)} dV_g \\ &\quad -4(1+\widetilde{\alpha}_1)\int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)} \log d(\cdot,p_1) dV_g + O(1) \\ &= -2(1+\widetilde{\alpha}_1)\log\lambda + O(1), \end{aligned} \tag{24}$$

and clearly $\overline{\varphi_{2,\lambda}} = (1+\widetilde{\alpha}_1)\log\lambda + O(1)$.

For a small but fixed $\delta > 0$ we have, again by (23),

$$\begin{aligned}\int_\Sigma \widetilde{h}_1 e^{\varphi_{1,\lambda}} dV_g &\geq C\int_{B_\delta(p_1)\setminus B_{\frac{1}{\lambda}}(p_1)} d(\cdot,p_1)^{2\widetilde{\alpha}_1} e^{\varphi_{1,\lambda}} dV_g \\ &\geq \frac{C}{\lambda^{2(1+\widetilde{\alpha}_1)}} \int_{B_\delta(p_1)\setminus B_{\frac{1}{\lambda}}(p_1)} \frac{dV_g}{d(\cdot,p_1)^{4+2\widetilde{\alpha}_1}} \\ &\geq C; \end{aligned} \tag{25}$$

on the other hand, we can write that

$$\begin{aligned}\int_\Sigma \widetilde{h}_2 e^{\varphi_{2,\lambda}} dV_g &\geq C\lambda^{1+\widetilde{\alpha}_1}\int_{\Sigma\setminus B_{\frac{1}{\lambda}}(p_1)} \widetilde{h}_2 d(\cdot,p_1)^{2(1+\widetilde{\alpha}_1)} dV_g \\ &\geq C\lambda^{1+\widetilde{\alpha}_1}. \end{aligned} \tag{26}$$

Therefore, from (22), (24), (25),(26) we conclude that

$$J_\rho(\varphi_{1,\lambda},\varphi_{2,\lambda}) \leq 2(1+\widetilde{\alpha}_1)(4\pi(1+\widetilde{\alpha}_1)-\rho_1)\log\lambda + O(1) \underset{\lambda\to\infty}{\to} -\infty,$$

as desired. $\square$

4. The Optimal Inequality

In the last section we are going to discuss the boundedness from below of J_ρ in the only case left, that is when $\rho_i = 4\pi(1+\widetilde{\alpha}_i)$ for some $i \in \{1,2\}$; we will show that $\inf_{H_1(\Sigma)^2} J_\rho > -\infty$ in this case as well.

Theorem 4.1. *Let Σ be a closed surface with area $|\Sigma| = 1$, $\widetilde{h}_i$ be as in* (12)*, $\widetilde{\alpha}_i$ be as in* (17) *and J_ρ be as in* (13)*.*

Then, there exists a constant $C > 0$ such that for any $u \in H^1(\Sigma)^2$

$$J_{4\pi(1+\widetilde{\alpha}_1),4\pi(1+\widetilde{\alpha}_2)}(u) > -C$$

namely

$$\Lambda = (0, 4\pi(1 + \widetilde{\alpha}_1)] \times (0, 4\pi(1 + \widetilde{\alpha}_2)].$$

Theorem 4.1 is equivalent to saying that, given a sequence

$$\rho_k \underset{k\to+\infty}{\nearrow} (4\pi(1 + \widetilde{\alpha}_1), 4\pi(1 + \widetilde{\alpha}_2)),$$

there exists $C > 0$ such that $\inf_{H^1(\Sigma)^2} J_{\rho_k} \geq -C$.

Moreover, in view of Lemma 3.4, it suffices to show that the minimizers u_k of J_{ρ_k} verify $J_{\rho_k}(u_k) > -C$; these functions solve

$$\begin{cases} -\Delta u_{i,k} = 2\rho_{i,k}\left(\widetilde{h}_i e^{u_{i,k}} - 1\right) - \rho_{3-i,k}\left(\widetilde{h}_{3-i} e^{u_{3-i,k}} - 1\right) \\ \displaystyle\int_\Sigma \widetilde{h}_i e^{u_{i,k}} dV_g = 1 \end{cases} \qquad i \in \{1, 2\},$$

therefore, as in the proof of Theorem 3.1, we can apply Theorem 2.2 to $v_k := u_k + \log \rho_k$.

As in the proof of Theorem 3.1, the condition on the integral excludes convergence to $-\infty$, whereas if u_k is bounded in $\|\cdot\|_{L^\infty(\Sigma)}$ it converges to a minimizer of $J_{4\pi(1+\widetilde{\alpha}_1),4\pi(1+\widetilde{\alpha}_2)}$ hence the conclusion is trivial, so we may suppose that at least one component blows up.

The following lemma describes the two possible blow-up scenarios.

Lemma 4.2. *Let $\{u_k\}_{k\in\mathbb{N}}$ be a blowing up sequence of minimizers of J_{ρ_k} for some sequence ρ_k such that $\rho_k \underset{k\to+\infty}{\to} (4\pi(1 + \widetilde{\alpha}_1), 4\pi(1 + \widetilde{\alpha}_2))$ and let $\alpha_i(p)$ be as in* (18)*. Then, one of the following happens:*

1. *Only the i^{th} component of u_k blows up, for some $i \in \{1, 2\}$ and it does at a single point p_i with $\alpha_i(p_i) = \widetilde{\alpha}_i$ around it.*
2. *Each component of u_k blows up at a single point p_i satisfying $\alpha_i(p_i) = \widetilde{\alpha}_i$ around it, and $p_1 \neq p_2$.*

Proof. Suppose that only one component blows up, say $u_{1,k}$, and suppose it blows up around a point p_1 satisfying $\alpha_1(p_1) > \widetilde{\alpha}_1$. Then, by (20) we obtain

$$\begin{aligned} 4\pi(1+\alpha_1(p_1)) &= \lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(p_1)}\widetilde{h}_1 e^{v_{1,k}}dV_g \le \lim_{k\to+\infty}\int_\Sigma \widetilde{h}_1 e^{v_{1,k}}dV_g \\ &= 4\pi(1+\widetilde{\alpha}_1), \end{aligned}$$

that is a contradiction; moreover, if the blow-up occurs at two points p_1, p_2, then one similarly gets another contradiction:

$$\begin{aligned} 8\pi(1+\widetilde{\alpha}_1) &= \lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(p_1)\cup B_r(p_2)}\widetilde{h}_1 e^{v_{1,k}}dV_g \le \lim_{k\to+\infty}\int_\Sigma \widetilde{h}_1 e^{v_{1,k}}dV_g \\ &= 4\pi(1+\widetilde{\alpha}_1). \end{aligned}$$

Suppose now that both components blow up at the same point; then, again by (20), $v_{i,k}$ must have a local mass strictly greater than $4\pi(1+\widetilde{\alpha}_i)$ around that point, for some $i\in\{1,2\}$, but this is impossible since the total mass of $v_{i,k}$ is converging to $4\pi(1+\widetilde{\alpha}_i)$; therefore, at any given point only one component may blow up, hence we can argue as in the previous case to get the conclusion. ☐

We will consider first the single-component blow-up in alternative (1).

Lemma 4.3. *Suppose $u_{1,k}$ blows up at p_1 and $u_{2,k}$ does not blow up. Then,*

1. $u_{1,k}-\overline{u_{1,k}} \underset{k\to+\infty}{\to} G_1$ *in* $W^{2,p}_{loc}(\Sigma\backslash\{p_1\})$ *for any* $p\in[1,\frac{1}{-\widetilde{\alpha}_1})$ *and weakly* in* $W^{1,q}(\Sigma)$ *for any* $q\in[1,2)$, *and* G_1 *satisfies*

$$\begin{cases} -\Delta G_1 = 8\pi(1+\widetilde{\alpha}_1)\,(\delta_{p_1}-1) - 4\pi(1+\widetilde{\alpha}_2)(f-1) \\ \int_\Sigma G_1 dV_g = 0 \end{cases} . \tag{27}$$

2. $u_{2,k}-\overline{u_{2,k}} \underset{k\to+\infty}{\to} G_2$ *in* $W^{2,p}_{loc}(\Sigma\backslash\{p_1\})$ *for any* $p\in[1,\frac{1}{-\widetilde{\alpha}_2})$ *and weakly* in* $W^{1,q}(\Sigma)$ *for any* $q\in[1,2)$, *and* G_2 *satisfies*

$$\begin{cases} -\Delta G_2 = 8\pi(1+\widetilde{\alpha}_2)(f-1) - 4\pi(1+\widetilde{\alpha}_1)\,(\delta_{p_1}-1) \\ \int_\Sigma G_2 dV_g = 0 \end{cases} . \tag{28}$$

for some non-negative $f\in L^1(\Sigma)$ *satisfying* $\int_\Sigma f dV_g = 1$.

Proof. First of all, we prove that $u_{i,k}-\overline{u_{i,k}}$ is bounded in $W^{1,q}(\Sigma)$ for $q\in[1,2)$: taking $q'\in(2,+\infty]$ such that $\frac{1}{q'}+\frac{1}{q}=1$,

$$\begin{aligned}\|u_{i,k}-\overline{u_{i,k}}\|_{W^{1,q}(\Sigma)} &\leq C\|\nabla u_{i,k}\|_{L^q(\Sigma)}\\ &= C\sup_{\phi\in W^{1,q'}(\Sigma),\ \|\nabla\phi\|_{L^{q'}}\leq 1}\left|\int_\Sigma\nabla u_{i,k}\cdot\nabla\phi dV_g\right|\\ &\leq C\sup_{\phi\in W^{1,q'}(\Sigma),\ \|\nabla\phi\|_{L^{q'}}\leq 1}\|\Delta u_{i,k}\|_{L^1(\Sigma)}\|\phi\|_{L^\infty(\Sigma)}\\ &\leq C\sup_{\phi\in W^{1,q'}(\Sigma),\ \|\nabla\phi\|_{L^{q'}}\leq 1}\|\Delta u_{i,k}\|_{L^1(\Sigma)}\|\nabla\phi\|_{L^{q'}(\Sigma)}\\ &\leq C.\end{aligned}$$

Moreover, from Theorem 2.2 we know that, in the sense of measure,

$$\widetilde{h}_1e^{u_{1,k}}\underset{k\to+\infty}{\rightharpoonup}\delta_{p_1}\qquad\widetilde{h}_2e^{u_{2,k}}\underset{k\to+\infty}{\rightharpoonup}f\in L^1(\Sigma);$$

therefore, taking G_i satisfying respectively (27), (28), for any fixed $\phi\in W^{1,q'}(\Sigma)$

$$\begin{aligned}&\left|\int_\Sigma\nabla\left(u_{1,k}-\overline{u_{1,k}}-G_1\right)\cdot\nabla\phi dV_g\right|\\ &=\int_\Sigma\left(-\Delta u_{1,k}+\Delta G_1\right)\phi dV_g\\ &\leq C\left|\int_\Sigma\left(2\rho_{1,k}\widetilde{h}_1e^{u_{1,k}}-8\pi(1+\widetilde{\alpha}_1)\delta_{p_1}\right)\phi dV_g\right|\\ &\quad+C\left|\int_\Sigma\left(4\pi(1+\widetilde{\alpha}_2)f-\rho_{2,k}\widetilde{h}_2e^{u_{2,k}}\right)\phi dV_g\right|=o(1).\end{aligned}$$

in a similar way, we get $u_{2,k}-\overline{u_{2,k}}\underset{k\to+\infty}{\overset{*}{\rightharpoonup}}G_2$ in $W^{1,q}(\Sigma)$ and convergence in $W^{2,p}_{loc}(\Sigma\backslash\{p_1\})$ follows from standard elliptic estimates. $\square$

Remark 4.4. From the previous lemma, we deduce that $|\overline{u_{2,k}}|\leq C$, since both $u_{2,k}$ and $u_{2,k}-\overline{u_{2,k}}$ are uniformly bounded in $L^\infty_{loc}(\Sigma\backslash\{p_1\})$; therefore, up to subsequences, the previous convergence result extends to $u_{2,k}$.

We will now consider the alternative (2) in Lemma 4.2.

When both components blow up, the last lemma has a counterpart; its proof follow closely the proof of Lemma 4.3, and therefore will be omitted.

Lemma 4.5. *Suppose each $u_{i,k}$ blows up at p_i. Then, for both $i \in \{1,2\}$ we have that $u_{i,k} - \overline{u_{i,k}} \underset{k\to+\infty}{\to} G_i$ in $W^{2,p}_{loc}(\Sigma\backslash\{p_i\})$ for any $p \in [1, \frac{1}{-\widetilde{\alpha}_i})$ ($p \in [1,\infty)$ if $\widetilde{\alpha}_i = 0$) and weakly* in $W^{1,q}(\Sigma)$ for any $q \in [1,2)$, and G_i satisfies*

$$\begin{cases} -\Delta G_i = 8\pi(1+\widetilde{\alpha}_i)\left(\delta_{p_i} - 1\right) - 4\pi(1+\widetilde{\alpha}_{3-i})(\delta_{p_{3-i}} - 1); \\ \displaystyle\int_\Sigma G_i dV_g = 0. \end{cases}$$

In the case of both components blowing up, a sort of *localized* Moser-Trudinger inequality is required.

Lemma 4.6. *Suppose each $u_{i,k}$ blows up at p_i. Then, for any small $\delta > 0$ there exists $C = C(\delta) > 0$ such that for both $i \in \{1,2\}$*

$$\frac{1}{4}\int_{B_\delta(p_i)} |\nabla u_{i,k}|^2 dV_g + \rho_{i,k}\overline{u_{i,k}} \geq -C.$$

Proof. We will take δ such that $B_\delta(p_i)$ does not contain any other singular point and we will suppose that $B_\delta(p_i)$ is a flat disk, see [23] (Remark 3.3).

This condition can be achieved through a conformal change of metric which results in a modified Liouville equation. The same estimates on minimizers hold true for the modified equation and one gets lower bounds on the functionals as before.

Consider the solution $\widetilde{w}_{i,k}$ of

$$\begin{cases} -\Delta \widetilde{w}_{i,k} = 0 & \text{on } B_\delta(p_i), \\ \widetilde{w}_{i,k} - u_{i,k} + \overline{u_{i,k}} = 0 & \text{on } \partial B_\delta(p_i); \end{cases}$$

standard elliptic estimates and Lemma 4.5 give

$$\|\widetilde{w}_{i,k}\|_{C^1(B_\delta(p_i))} \leq C\,\|\widetilde{w}_{i,k}\|_{L^\infty(B_\delta(p_i))} \leq C\|u_{i,k} - \overline{u_{i,k}}\|_{L^\infty(\partial B_\delta(p_i))} \leq C.$$

Moreover, we can apply the scalar Moser-Trudinger inequality (21) to $w_{i,k} := u_{i,k} - \overline{u_{i,k}} - \widetilde{w}_{i,k}$, which belongs to $H^1_0(B_\delta(p_i))$:

$$\int_{B_\delta(p_i)} |\nabla w_{i,k}|^2 dV_g - 16\pi(1+\widetilde{\alpha}_i)\log\int_{B_\delta(p_i)} d(\cdot,p_i)^{2\widetilde{\alpha}_i} e^{w_{i,k}} dV_g \geq -C.$$

The construction of $\widetilde{w}_{i,k}$ gives

$$\begin{aligned}
&\int_{B_\delta(p_i)} |\nabla w_{i,k}|^2 dV_g - \int_{B_\delta(p_i)} |\nabla u_{i,k}|^2 dV_g \\
&= \int_{B_\delta(p_i)} \left(2\nabla u_{i,k} \cdot \nabla \widetilde{w}_{i,k} + |\nabla \widetilde{w}_{i,k}|^2\right) dV_g \\
&\le 2|\nabla \widetilde{w}_{i,k}|_{L^\infty(B_\delta(p_i))} \int_{B_\delta(p_i)} |\nabla u_{i,k}| dV_g + \int_{B_\delta(p_i)} |\nabla \widetilde{w}_{i,k}|^2 dV_g \\
&\le C;
\end{aligned}$$

on the other hand, for large k we may suppose that $\int_{B_\delta(p_i)} \widetilde{h}_i e^{u_{i,k}} dV_g \ge \frac{1}{2}$, so

$$\begin{aligned}
\int_{B_\delta(p_i)} d(\cdot, p_i)^{2\widetilde{\alpha}_i} e^{w_{i,k}} dV_g &= e^{-\overline{u_{i,k}}} \int_{B_\delta(p_i)} d(\cdot, p_i)^{2\widetilde{\alpha}_i} e^{u_{i,k} - \widetilde{w}_{i,k}} dV_g \\
&\ge Ce^{-\overline{u_{i,k}}} \int_{B_\delta(p_i)} \widetilde{h}_i e^{u_{i,k} - \widetilde{w}_{i,k}} dV_g \\
&\ge Ce^{-\overline{u_{i,k}}} \int_{B_\delta(p_i)} \widetilde{h}_i e^{u_{i,k}} dV_g \\
&\ge \frac{C}{2} e^{-\overline{u_{i,k}}}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\frac{1}{4} \int_{B_\delta(p_i)} |\nabla u_{i,k}|^2 dV_g + \rho_{i,k} \overline{u_{i,k}} \\
&\ge \frac{1}{4} \int_{B_\delta(p_i)} |\nabla w_{i,k}|^2 dV_g - \rho_{i,k} \log \int_{B_\delta(p_i)} d(\cdot, p_i)^{2\widetilde{\alpha}_i} e^{w_{i,k}} dV_g - C \\
&\ge -C.
\end{aligned}$$

which is the conclusion. □

We have now all the necessary tools to conclude the proof of Theorem 4.1.

***Proof of Theorem* 4.1.** Take a minimizing blowing up sequence u_k and suppose that the first alternative in Lemma 4.2 holds; it is not restrictive to suppose that $u_{1,k}$ blows up.

From Lemma 4.3 and the following remark we know that $\overline{u_{2,k}}$ is uniformly bounded; therefore, using the scalar Moser-Trudinger inequality (7) we obtain

$$\begin{aligned} J_{\rho_k}(u_k) &= \int_\Sigma Q(u_{1,k},u_{2,k})dV_g + \rho_{1,k}\overline{u_{1,k}} + \rho_{2,k}\overline{u_{2,k}} \\ &\geq \int_\Sigma Q(u_{1,k},u_{2,k})dV_g + \rho_{1,k}\overline{u_{1,k}} - C \\ &\geq \frac{1}{4}\int_\Sigma |\nabla u_{1,k}|^2 dV_g + \rho_{1,k}\overline{u_{1,k}} - C \\ &\geq -C. \end{aligned}$$

that concludes the analysis of the first case.

Suppose now that both components blow up; then, we may conclude by applying Lemma 4.6:

$$\begin{aligned} J_{\rho_k}(u_k) &= \int_\Sigma Q(u_{1,k},u_{2,k})dV_g + \rho_{1,k}\overline{u_{1,k}} + \rho_{2,k}\overline{u_{2,k}} \\ &\geq \sum_{i=1}^{2}\left(\int_{B_\delta(p_i)} Q(u_{1,k},u_{2,k})dV_g + \rho_{i,k}\overline{u_{i,k}}\right) \\ &\geq \sum_{i=1}^{2}\left(\frac{1}{4}\int_{B_\delta(p_i)} |\nabla u_{i,k}|^2 dV_g + \rho_{i,k}\overline{u_{i,k}}\right) \\ &\geq -C. \end{aligned}$$

This concludes the proof. □

References

1. Adimurthi and K. Sandeep, A singular Moser-Trudinger embedding and its applications, *NoDEA Nonlinear Differential Equations Appl.*, **13**(2007), No.5-6, 585-603.

2. T. Aubin, Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire, *J. Funct. Anal.*, **32**(1979), No. 2, 148–174.

3. D. Bartolucci, F. De Marchis and A. Malchiodi, Supercritical conformal metrics on surfaces with conical singularities, *Int. Math. Res. Not. IMRN*, **24**(2011), 5625-5643.

4. D. Bartolucci and A. Malchiodi, An improved geometric inequality via vanishing moments, with applications to singular liouville equations, *Comm. Math. Phys.*, to appear, d.o.i. 10.1007/s00220-013-1731-0.

5. L. Battaglia, A. Jevnikar, A. Malchiodi and D. Ruiz, A general existence result for the toda system on compact surfaces, http://arxiv.org/abs/1306.5404.

6. H. Brezis and F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, *Comm. Partial Differential Equations*, **16**(1991), No. 8-9, 1223-1253.

7. L. A. Caffarelli and Y. S. Yang, Vortex condensation in the Chern-Simons Higgs model: an existence theorem, *Comm. Math. Phys.*, **168**(1995), No. 2, 321-336.

8. A. Carlotto and A. Malchiodi, Weighted barycentric sets and singular Liouville equations on compact surfaces, *J. Funct. Anal.*, **262**(2012), No. 2, 409-450.

9. D. Chae and O. Yu. Imanuvilov, The existence of non-topological multivortex solutions in the relativistic self-dual Chern-Simons theory, *Comm. Math. Phys.*, **215**(2000), No. 1, 119-142.

10. S.-Y. A. Chang and P. C. Yang, Conformal deformation of metrics on S^2, *J. Differential Geom.*, **27**(1988), No.2, 259-296.

11. S.-Y. A. Chang and P. C. Yang, Prescribing Gaussian curvature on S^2, *Acta Math.*, **159**(1987), No. 3-4, 215-259.

12. C.-C. Chen and C.-S. Lin, Topological degree for a mean field equation on Riemann surfaces, *Comm. Pure Appl. Math.*, **56**(2003), No.12, 1667-1727.

13. C.-C. Chen and C.-S. Lin, Mean field equations of Liouville type with singular data: sharper estimates, *Discrete Contin. Dyn. Syst.*, **28**(2010), No. 3, 1237-1272.

14. W. X. Chen, A Trüdinger inequality on surfaces with conical singularities, *Proc. Amer. Math. Soc.*, **108**(1990), No. 3, 821-832.

15. X. Chen, Remarks on the existence of branch bubbles on the blowup analysis of equation $-\Delta u = e^{2u}$ in dimension two, *Comm. Anal. Geom.*, **7**(1999), No. 2, 295-302.

16. M. Chipot, I. Shafrir and G. Wolansky, On the solutions of Liouville systems, *J. Differential Equations*, **140**(1997), No. 1, 59-105.

17. A. Cianchi, Moser-Trudinger trace inequalities, *Adv. Math.*, **217**(2008), No. 5, 2005-2044.

18. W. Ding, J. Jost, J. Li and G. Wang, Existence results for mean field equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **16**(1999), No. 5, 653-666.

19. Z. Djadli, Existence result for the mean field problem on Riemann surfaces of all genuses, *Commun. Contemp. Math.*, **10**(2008), No. 2, 205-220.

20. G. Dunne, *Self-dual Chern-Simons Theories*, Lecture notes in physics. New series m: Monographs. Springer, 1995.

21. L. Fontana and C. Morpurgo, Adams inequalities on measure spaces, *Adv. Math.*, **226**(2011), No. 6, 5066-5119.

22. J. Jost, C. Lin and G. Wang, Analytic aspects of the Toda system. II, Bubbling behavior and existence of solutions, *Comm. Pure Appl. Math.*, **59**(2006), No. 4, 526-558.

23. J. Jost and G. Wang, Analytic aspects of the Toda system. I, A Moser-Trudinger inequality, *Comm. Pure Appl. Math.*, **54**(2001), No. 11, 1289-1319.

24. H.-C. Kao and K. Lee, Self-dual SU(3) Chern-Simons Higgs systems, *Phys. Rev. D* (3), **50**(1994), No. 10, 6626-6632.

25. K. Lee, Self-dual nonabelian Chern-Simons solitons, *Phys. Rev. Lett.*, **66**(1991), No. 5, 553-555.

26. C. Lin, J. Wei and L. Zhang, Classification of blowup limits for SU(3) singular Toda systems, *preprint*, 2013.

27. M. Lucia and M. Nolasco, SU(3) Chern-Simons vortex theory and Toda systems, *J. Differential Equations*, **184**(2002), No. 2, 443-474.

28. A. Malchiodi and C. B. Ndiaye, Some existence results for the Toda system on closed surfaces, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei* (9) *Mat. Appl.*, **18**(2007), No. 4, 391-412.

29. A. Malchiodi and D. Ruiz, New improved Moser-Trudinger inequalities and singular Liouville equations on compact surfaces, *Geom. Funct. Anal.*, **21**(2011), No. 5, 1196-1217.

30. A. Malchiodi and D. Ruiz, A variational analysis of the Toda system on compact surfaces, *Comm. Pure Appl. Math.*, **66**(2013), No. 3, 332-371.

31. J. Moser, A sharp form of an inequality by N, Trudinger, *Indiana Univ. Math. J.*, **20**(1970/71), 1077-1092.

32. M. Nolasco and G. Tarantello, Double vortex condensates in the Chern-Simons-Higgs theory, *Calc. Var. Partial Differential Equations*, **9**(1999), No. 1, 31-94.

33. M. Nolasco and G. Tarantello, Vortex condensates for the SU(3) Chern-Simons theory, *Comm. Math. Phys.*, **213**(2000), No. 3, 599-639.

34. J. Spruck and Y. S. Yang, Topological solutions in the self-dual Chern-Simons theory: existence and approximation, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **12**(1995), No. 1, 75-97.

35. G. Tarantello, Multiple condensate solutions for the Chern-Simons-Higgs theory. *J. Math. Phys.*, 37(8, 3769-3796, 1996.

36. G. Tarantello, *Selfdual gauge field vortices*, Progress in Nonlinear Differential Equations and their Applications, 72. Birkhäuser Boston Inc., Boston, MA, 2008. An analytical approach.

37. M. Troyanov, Prescribing curvature on compact surfaces with conical singularities, *Trans. Amer. Math. Soc.*, **324**(1991), No. 2, 793-821.

38. N. S. Trudinger, On imbeddings into Orlicz spaces and some applications, *J. Math. Mech.*, **17**(1967), 473-483.

39. A. Tur and V. Yanovsky, Point vortices with a rational necklace: new exact stationary solutions of the two-dimensional Euler equation, *Phys. Fluids*, **16**(2004), No. 8, 2877-2885.

40. G. Wang, Moser-Trudinger inequalities and Liouville systems, *C. R. Acad. Sci. Paris Sér. I Math.*, **328**(1999), No. 10, 895-900.

41. Y. Yang, *Solitons in field theory and nonlinear analysis*, Springer Monographs in Mathematics. Springer-Verlag, New York, 2001.