Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **8** (2013), No. 4, pp. 481-489

ON A HARDY-LITTLEWOOD THEOREM

E. LIFLYAND[1,a] AND U. STADTMÜLLER[2,b]

The paper is dedicated to the 70th anniversary of N. Trudinger.

[1]Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel.
[a]E-mail: liflyand@math.biu.ac.il
[2] Department of Mathematics, University of Ulm, 89069 Ulm, Germany.
[b]E-mail: ulrich.stadtmueller@uni-ulm.de

Abstract

A known Hardy-Littlewood theorem asserts that if both the function and its conjugate are of bounded variation, then their Fourier series are absolutely convergent. It is proved in the present paper that the same result holds true for functions on the whole axis and their Fourier transforms, with certain adjustments. The proof of the original Hardy-Littlewood theorem is derived from the obtained assertion. It turned out that the former is a partial case of the latter when the function is supposed to be of compact support. A similar result for radial functions is derived from the one-dimensional case.

1. Introduction

The following result is due to Hardy and Littlewood (see [7] or, e.g., [20, Vol.I, Ch.VII, (8.6)]).

Theorem 1.1. *If a (periodic) function f and its conjugate $\widetilde{f}$ are both of bounded variation, then their Fourier series converge absolutely.*

In [20] this result is one of the consequences of the general theory of Hardy spaces, first of all H^1 in the unit disk. Since we are going to generalize the Hardy-Littlewood theorem to functions on the real axis, let us recall certain notions and notations concerning Hardy spaces in this setting. The

Received June 22, 2013 and in revised form August 27, 2013.
AMS Subject Classification: Primary 42A38; Secondary 42A20, 42A50, 42B30, 26B30, 26D15.
Key words and phrases: Fourier integral, Hilbert transform, bounded variation, Lebesgue point.

Fourier transform $\widehat{g}$ of a (complex-valued) function g in $L^1(\mathbb{R})$ is defined by

$$\widehat{g}(t) := \int_{\mathbb{R}} g(x)e^{-itx}dx, \quad t \in \mathbb{R},$$

while its Hilbert transform $\mathcal{H}g$ is defined by

$$\begin{aligned}\mathcal{H}g(x) &:= \frac{1}{\pi}\,(\text{P.V.})\int_{\mathbb{R}} g(x-u)\frac{du}{u} = \frac{1}{\pi}\,(\text{P.V.})\int_{\mathbb{R}} \frac{g(u)}{x-u}\,du \\ &= \frac{1}{\pi}\lim_{\delta\downarrow 0}\int_{\delta}^{\infty}\{g(x-u)-g(x+u)\}\frac{du}{u}, \quad x \in \mathbb{R}.\end{aligned}$$

As is well known, for $g \in L^1(\mathbb{R})$ this limit exists for almost all x in $\mathbb{R}$, and the real Hardy space $H^1(\mathbb{R})$ is defined to be

$$H^1(\mathbb{R}) := \{g \in L^1(\mathbb{R}) : \mathcal{H}g \in L^1(\mathbb{R})\},$$

where $L^1(\mathbb{R})$ is the usual space of integrable functions with norm

$$\|g\|_{L^1} := \int_{\mathbb{R}} |g(x)|\,dx.$$

The Hardy space is endowed with the norm

$$\|g\|_{H^1} := \|g\|_{L^1} + \|\mathcal{H}g\|_{L^1}.$$

If $g \in H^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} g(t)\,dt = 0. \tag{1.1}$$

It was apparently first mentioned in [10]. Since a function f of bounded variation may not be integrable, its Hilbert transform, a usual substitute for the conjugate function, may not exist. One has to use the modified Hilbert transform (see, e.g., [6, Ch.III, §1])

$$\widetilde{\mathcal{H}}f(x) = \;(\text{P.V.})\,\frac{1}{\pi}\int_{\mathbb{R}} f(t)\left\{\frac{1}{x-t}+\frac{t}{1+t^2}\right\}dt.$$

As a singular integral, it behaves like the usual Hilbert transform; the additional term in the integral makes it well-defined near infinity.

Correspondingly, the absolute convergence of the Fourier series should be replaced by the integrability of the Fourier transform.

Our work is much in the spirit of the book [3], especially Chapter 8. Roughly speaking, some classes are characterized there for the values $r = 1, 2, \ldots$ of a certain parameter r. Our consideration formally corresponds to the case $r = 0$.

The outline of the paper is as follows. In the next section we formulate and prove the main result. As in the proof of the initial result in [20, Ch.VII, §8] much is based on the Hardy inequality (cf. (8.7) in the cited chapter and (2.4) in the present text). Then we derive the original Hardy-Littlewood theorem from the proven result. In the last section we apply the obtained theorem to derive a similar multidimensional result for radial functions.

2. Main Result

With the above preliminaries in hand, we are in a position to formulate and prove our main result.

Theorem 2.1. *Let f be a function of bounded variation that vanishes at infinity: $\lim\limits_{|t|\to\infty} f(t) = 0$. If its conjugate $\widetilde{\mathcal{H}}f$ is also of bounded variation, then the Fourier transforms of both functions are integrable on $\mathbb{R}$.*

Proof. The only property of a function of bounded variation we really need is that its derivative exists almost everywhere and is integrable; cf. [2, Ch.1]. So, such is $\frac{d}{dx}\widetilde{\mathcal{H}}f$.

More precisely, "almost everywhere" will be specified as at the Lebesgue points. Recall that x is a Lebesgue point of an integrable function g if $g(x)$ is finite and

$$\lim_{t\to 0} \frac{1}{t} \int_x^{x+t} |g(u) - g(x)|\, du = 0,$$

and almost every point is a Lebesgue point of g.

Lemma 2.2. *Under the assumptions of the theorem, we have at any Lebesgue point x of $f'(x)$*

$$\frac{d}{dx}\widetilde{\mathcal{H}}f(x) = \mathcal{H}f'(x). \tag{2.1}$$

***Proof of Lemma* 2.2.** This lemma is a direct analog of the various known results for the Hilbert transform of a function from the spaces different from the space of functions of bounded variation; see, e.g., [14, 3.3.1, Th.1] or [9, 4.8]. Since the assumptions are different, we use different arguments while interchanging limits. Let us start with the right-hand side of (2.1). Integrating by parts, we obtain

$$\begin{aligned}\pi\mathcal{H}f'(x) &= \lim_{\delta\downarrow 0}\left(\int_{-\infty}^{x-\delta}+\int_{x+\delta}^{\infty}\right)\frac{f'(t)}{x-t}\,dt\\ &= \lim_{\delta\downarrow 0}\left[\frac{f(x-\delta)+f(x+\delta)}{\delta}-\left(\int_{-\infty}^{x-\delta}+\int_{x+\delta}^{\infty}\right)\frac{f(t)}{(x-t)^2}\,dt\right]. \quad (2.2)\end{aligned}$$

In order to integrate by parts, we need f to be locally absolutely continuous. It turns out that it is just the case under our assumptions. Consider F to be f on a finite interval, say $(-\pi,\pi]$ for simplicity, and zero otherwise. Its Hilbert transform

$$\mathcal{H}F(x)=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(t)}{x-t}\,dt$$

differs from the classical periodic conjugate function

$$\widetilde{F}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\cot\frac{x-t}{2}\,dt$$

(also understood in the principal value sense) only in kernels. However (see, e.g., [3, (9.0.4)]), the difference of these kernels is quite well behaved:

$$\frac{1}{2}\cot\frac{t}{2}-\frac{1}{t}=\sum_{k\neq 0}\frac{t}{2k\pi(t-2k\pi)}.$$

In particular, the derivative of the right-hand side is integrable. By this, if $\widetilde{F}$ is of bounded variation, then also $\mathcal{H}F$ is. Observe, that for a function of compact support there is no need in modifying the Hilbert kernel, and we can consider the usual Hilbert transform. When a function and its conjugate are both of bounded variation, the function is absolutely continuous, see [20, Ch.VII, (8.2)], exactly as required.

Further, since f is continuous and vanishes at infinity, we find

$$\begin{aligned}&\frac{d}{dx}\left(\int_{-\infty}^{x-\delta}+\int_{x+\delta}^{\infty}\right)f(t)\left(\frac{1}{x-t}+\frac{t}{1+t^2}\right)dt\\ &= -\left(\int_{-\infty}^{x-\delta}+\int_{x+\delta}^{\infty}\right)\frac{f(t)}{(x-t)^2}\,dt+f(x-\delta)\left[\frac{1}{\delta}+\frac{x-\delta}{1+(x-\delta)^2}\right]\end{aligned}$$

$$-f(x+\delta)\Big[-\frac{1}{\delta}+\frac{x+\delta}{1+(x+\delta)^2}\Big].$$

Combining the last display and (2.2), we get

$$\begin{aligned}\pi\mathcal{H}f'(x) \;=\; \lim_{\delta\downarrow 0}\Big\{&\frac{d}{dx}\Big(\int_{-\infty}^{x-\delta}+\int_{x+\delta}^{\infty}\Big)f(t)\Big(\frac{1}{x-t}+\frac{t}{1+t^2}\Big)\,dt\\ &-f(x-\delta)\frac{x-\delta}{1+(x-\delta)^2}+f(x+\delta)\frac{x+\delta}{1+(x+\delta)^2}\Big\},\end{aligned}$$

which gives

$$\pi\mathcal{H}f'(x)=\lim_{\delta\downarrow 0}\frac{d}{dx}\Big(\int_{-\infty}^{x-\delta}+\int_{x+\delta}^{\infty}\Big)f(t)\Big(\frac{1}{x-t}+\frac{t}{1+t^2}\Big)\,dt,$$

since the rest, by continuity, tends to zero as $\delta\downarrow 0$.

What remains is to change the order of the limit and differentiation. The integrals $\displaystyle\int_{-\infty}^{x-1}+\int_{x+1}^{\infty}$ converge uniformly, and both are independent of δ. Hence, it suffices to study the convergence of

$$\Big(\int_{x-1}^{x-\delta}+\int_{x+\delta}^{x+1}\Big)f(t)\Big(\frac{1}{x-t}+\frac{t}{1+t^2}\Big)\,dt.$$

Since

$$\Big(\int_{x-1}^{x-\delta}+\int_{x+\delta}^{x+1}\Big)\frac{dt}{x-t} \;=\; 0,$$

and

$$\Big(\int_{x-1}^{x-\delta}+\int_{x+\delta}^{x+1}\Big)\frac{t}{1+t^2}\,dt \;\le\; 2,$$

it remains to deal with

$$\Big(\int_{x-1}^{x-\delta}+\int_{x+\delta}^{x+1}\Big)\frac{f(t)-f(x)}{x-t}\,dt. \tag{2.3}$$

Let x be a Lebesgue point of f'. We have

$$\begin{aligned}\int_{x+\delta}^{x+1}\frac{f(t)-f(x)}{x-t}\,dt \;&=\; \int_{x+\delta}^{x+1}\frac{1}{x-t}\int_x^t f'(u)\,du\,dt\\ &=\; -\int_\delta^1\frac{1}{t}\int_x^{x+t}f'(u)\,du\,dt.\end{aligned}$$

The finiteness of $f'(x)$ implies

$$\left|\int_{x+\delta}^{x+1} \frac{f(t)-f(x)}{x-t}\,dt\right| \le \int_\delta^1 \frac{1}{t}\int_x^{x+t} |f'(u)-f'(x)|\,du\,dt + |f'(x)|.$$

Since the inner integral on the right-hand side is uniformly bounded on $(0,1)$, we get the uniform convergence of the integrals in (2.3) (the other integral is treated in exactly the same manner). This allows us to apply $\lim\limits_{\delta\downarrow 0}$ and $\frac{d}{dx}$ in any order, which leads to the required relation (2.1) at any Lebesgue point x of $f'(x)$. Since such is almost every point, the proof is complete. □

With this result in hand, the proof of the theorem goes along the following lines. Since the function f is of bounded variation, its derivative f' exists almost everywhere and is integrable. It follows from the boundedness of variation of its conjugate and from Lemma 2.2 that $\mathcal{H}f'(x)$ exists at almost every x and is also integrable. Therefore $f' \in H^1(\mathbb{R})$. We shall now make use of the well-known extension of Hardy's inequality (see, e.g., [5, (7.24)])

$$\int_{\mathbb{R}} \frac{|\widehat{f'}(x)|}{|x|}\,dx \le \|f'\|_{H^1(\mathbb{R})}. \tag{2.4}$$

Observe that the assumptions of the theorem imply the cancelation property (1.1) for f'. Integrating by parts, which is possible since f is locally absolutely continuous, we obtain

$$\widehat{f'}(x) = \int_{\mathbb{R}} f'(t)e^{-itx}dt = ix\int_{\mathbb{R}} f(t)e^{-itx}dt.$$

Hence, the left-hand side of (2.4) is exactly the L^1 norm of the Fourier transform of f. Further, we have $i\operatorname{sign} x\widehat{f'}(x) = \widehat{\mathcal{H}f'}(x)$, which, by Lemma 2.2, is the Fourier transform of $\dfrac{d}{dx}\widetilde{\mathcal{H}}f$. Integrating by parts as above, we conclude that in our situation the left-hand side of (2.4) is also the L^1 norm of the Fourier transform of $\widetilde{\mathcal{H}}f$. The proof is complete. □

3. The Original Hardy-Littlewood Theorem

In this section we derive the proof of the original Hardy-Littlewood theorem from Theorem 2.1. It turned out that the former is a partial case of the latter when the function is supposed to be of compact support.

Beginning the proof of Theorem 1.1, we may consider f to be the 2π-periodic extension of the function F which coincides with f on $(-\pi,\pi]$ and is zero otherwise. Of course, this function is of bounded variation as well. Using the argument after (2.2) in the opposite direction, we are now under the assumptions of Theorem 2.1, and hence the Fourier transform of F is integrable. It follows from this (see [19, Ch.2, §11]) that the Fourier series of its periodic extension, that is, the Fourier series of f, is absolutely convergent. Since the Fourier coefficients of $\widetilde{f}$ are the same modulo as those of f, the Fourier series of $\widetilde{f}$ also converges absolutely.

This fact seems to be a good supplement to Wiener's study of the relations between the Fourier transform of a compactly supported function and the Fourier series of its periodic extension in [19, Ch.2].

4. Radial Case

Let us now apply the obtained results to problems of integrability of the multidimensional Fourier transform of a radial function $f(x) = f_0(|x|)$. Let $\widehat{f}$ denote its usual Fourier transform on $\mathbb{R}^n$. The known Leray's formula (see Lemma 25.1' in [15]) says that when

$$\int_0^\infty \frac{t^{n-1}}{(1+t)^{\frac{n-1}{2}}}\,|f_0(t)|\,dt < \infty, \tag{4.1}$$

the following relation holds

$$\widehat{f}(x) = 2\pi^{\frac{n-1}{2}}\int_0^\infty I(t)\cos|x|t\,dt, \tag{4.2}$$

where the fractional integral I is given by

$$I(t) = \frac{2}{\Gamma\left(\frac{n-1}{2}\right)}\int_t^\infty sf_0(s)(s^2-t^2)^{\frac{n-3}{2}}\,ds.$$

This result has proved to be very convenient for deriving statements for the Fourier transform of a radial function from known one-dimensional results; see, e.g., [11], [12].

Theorem 4.1. *Let f_0 satisfy* (4.1), *while I and its $n-2$ derivatives be locally absolutely continuous and vanish at zero and infinity. If $I^{(n-1)}$ satisfies the assumptions of Theorem* 2.1, *then the* (*multidimensional*) *Fourier transform of f is Lebesgue integrable over $\mathbb{R}^n$.*

Proof. First, we integrate by parts $n-1$ times in (4.2). Integrated terms vanish. We thus arrive to the formula

$$\widehat{f}(x) = \frac{2\pi^{\frac{n-1}{2}}(-1)^{n-1}}{|x|^{n-1}} \int_0^\infty I^{(n-1)}(t)\cos(\frac{\pi(n-1)}{2} - |x|t)\,dt.$$

Applying now Theorem 2.1 to the integral on the right-hand side and integrating then in the polar coordinates, we complete the proof. □

Remark 4.2. Observe that for $n = 1$ understanding the function I formally as f_0 reduces Theorem 4.1 to the one-dimensional Theorem 2.1.

Acknowledgment

The authors thank a referee for valuable remarks.

References

1. G. Bateman and A. Erdélyi, *Higher Transcendental Functions, Vol. II*, McGraw Hill Book Company, New York, 1953.
2. S. Bochner, *Lectures on Fourier Integrals*, Princeton Univ. Press, Princeton, N.J., 1959.
3. P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Volume 1: One-dimensional theory, Pure and Applied Mathematics, Vol. 40, Academic Press, New York, 1971.
4. J. Cossar, A theorem on Cesàro summability, *J. London Math. Soc.*, **16** (1941), 56-68.
5. J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland, 1985.
6. J. B. Garnett, *Bounded Analytic Functions*, Springer, N.Y., 2007.
7. G. H. Hardy and J. E. Littlewood, Some new properties of Fourier constants, *Math. Ann.*, **97** (1927), 159-209.
8. E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, Berlin, 1969.

9. F. W. King, *Hilbert Transforms*, Vol.1, Enc. Math/ Appl., Cambridge Univ. Press, Cambridge, 2009.

10. H. Kober, *A note on Hilberts operator*, *Bull. Amer. Math. Soc.*, **48:1** (1942), 421-426.

11. E. Liflyand and S. Tikhonov, Extended solution of Boas' conjecture on Fourier transforms, *C. R. Acad. Sci. Paris, Ser. I*, **346** (2008), 1137-1142.

12. E. Liflyand and S. Tikhonov, Two-sided weighted Fourier inequalities, *Ann. Sc. Norm. Super. Pisa CI. Sci.*(5). **XI** (2012), 341-362.

13. E. Liflyand, W. Trebels, On Asymptotics for a Class of Radial Fourier Transforms, *Zeitschrift für Analysis und ihre Anwendungen* (*J. Anal. Appl.*) **17** (1998), 103-114.

14. J. N. Pandey, *The Hilbert Transform of Schwartz Distributions and Applications*, John Wiley & Sons, New York, 1996.

15. S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon & Breach Sci. Publ., New York, 1992.

16. E. M. Stein and R. Shakarghi, *Functional Analysis: Introduction to Further Topics in Analysis*, Princeton Univ. Press, Princeton and Oxford, 2011.

17. E. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

18. R. M. Trigub, E. S. Belinsky, *Fourier Analysis and Appoximation of Functions*, Kluwer, 2004.

19. N. Wiener, *The Fourier Integral and Certain of Its Applications*, Dover Publ., Inc., New York, 1932.

20. A. Zygmund, *Trigonometric Series*, Vol. I, II, Cambridge Univ. Press, Cambridge, U.K., 1966.