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SPREADING PROFILE AND NONLINEAR STEFAN PROBLEMS

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Dedicated to Professor Neil Trudinger on the occasion of his 70th birthday

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Abstract

We report some recent progress on the study of the following nonlinear Stefan problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{for } \ x \in \Omega(t), \ t > 0, \\ u = 0 \text{ and } u_t = \mu |\nabla_x u|^2 & \text{for } \ x \in \Gamma(t), \ t > 0, \\ u(0,x) = u_0(x) & \text{for } \ x \in \Omega_0, \end{cases}$$

where $\Omega(t) \subset \mathbb{R}^N$ $(N \geq 1)$ is bounded by the free boundary $\Gamma(t)$, with $\Omega(0) = \Omega_0$, μ is a given positive constant. The initial function u_0 is positive in Ω_0 and vanishes on $\partial\Omega_0$. The class of nonlinear functions $f(u)$ includes the standard monostable, bistable and combustion type nonlinearities.

When $\mu \to \infty$, it can be shown that this free boundary problem converges to the corresponding Cauchy problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{for } \ x \in \mathbb{R}^N, \ t > 0, \\ u(0,x) = u_0(x) & \text{for } \ x \in \mathbb{R}^N. \end{cases}$$

We will discuss the similarity and differences of the dynamical behavior of these two problems by closely examining their spreading profiles, which suggest that the Stefan condition is a stabilizing factor in the spreading process.

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1. Introduction

In this paper, we report some recent advances on the study of the following nonlinear Stefan problem

$$\begin{cases} u_t - \Delta u = f(u) & \text{for } \ x \in \Omega(t), t > 0, \\ u = 0 \text{ and } u_t = \mu |\nabla_x u|^2 & \text{for } \ x \in \Gamma(t), t > 0, \\ u(0,x) = u_0(x) & \text{for } \ x \in \Omega_0, \end{cases} \tag{1.1}$$

where $\Omega(t) \subset \mathbb{R}^N$ $(N \geq 1)$ is bounded by the free boundary $\Gamma(t)$, with $\Omega(0) = \Omega_0$, μ is a given positive constant. We assume that Ω_0 is a bounded domain that agrees with the interior of its closure $\overline{\Omega}_0$, $\partial\Omega_0$ satisfies the interior ball condition, and $u_0 \in C(\overline{\Omega}_0) \cap H^1(\Omega_0)$ is positive in Ω_0 and vanishes on $\partial\Omega_0$. For the nonlinear function f, we make the following assumptions:

(i) $f(0) = 0$ and $f \in C^{1,\alpha}([0, \delta_0])$ for some $\delta_0 > 0$ and $\alpha \in (0, 1)$,

(ii) $f(u)$ is locally Lipschitz in $[0, \infty)$, $f(u) \leq 0$ in $[M, \infty)$ for some $M > 0$.

We note that these conditions are satisfied by standard monostable, bistable and combustion type nonlinearities. The detailed assumptions on these nonlinearities are to be recalled later, and much of the research work to be discussed here is for these three types of nonlinearities.

The physical meaning of the free boundary condition is that, each point $x \in \Gamma(t)$ moves in the direction of the outer normal to $\Gamma(t)$ at x, with velocity $\mu|\nabla_x u(t, x)|$. In the spherically symmetric setting, where $\Gamma(t) = \{x : |x| = h(t)\}$ and $u = u(t, r)$, $r = |x|$, this can be simplified to $h'(t) = -\mu u_r(t, h(t))$.

Problem (1.1) reduces to the classical one phase Stefan problem when $f(u) \equiv 0$, which describes the melting of ice in contact with water, with $u(x, t)$ representing the temperature of the water. In the setting of (1.1), the water region $\Omega(t)$ is surrounded by ice, and the free boundary $\Gamma(t) = \partial\Omega(t)$ represents the interphase between water and ice. A nonlinear Stefan problem of the form (1.1) may arise if water is replaced by a chemically reactive and heat diffusive liquid surrounded by ice, with $f(u)$ representing the reaction. The classical one phase Stefan problem has been extensively investigated in the past 50 years (see, for example, [6, 17, 18, 19, 23, 24, 27] and the references therein). In contrast, the nonlinear Stefan problem is much less studied.

Our motivation to study (1.1) mainly arises from the wish to better understand the spreading of a new or invasive species, where u is viewed as the density of such a species, and the free boundary represents the spreading front, beyond which the species cannot be observed (i.e., the species has density 0).

Traditionally the spreading phenomenon is modeled by the Cauchy problem

$$\begin{cases} U_t - \Delta U = f(U) & \text{for } x \in \mathbb{R}^n,\ t > 0, \\ U(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $u_0(x)$ is as in (1.1) but extended to $\mathbb{R}^N$ with value 0 outside Ω_0. In this case, $U(x,t) > 0$ for all $x \in \mathbb{R}^N$ once $t > 0$, but one may specify a certain level set $\Gamma_\delta(t) := \{x : U(t,x) = \delta\}$ as the spreading front, where $\delta > 0$ is small, and $\Omega_\delta(t) := \{x : U(t,x) > \delta\}$ is regarded as the range where the species can be observed. The behavior of (1.2) is much better understood compared with (1.1). Nevertheless, significant progress has been made recently on the understanding of (1.1), as we will see below.

The free boundary problem (1.1) and the Cauchy problem (1.2) are related. It was shown in [8] that if u_μ denotes the unique weak solution of (1.1), with $\Omega_\mu(t) = \{x : u_\mu(t,x) > 0\}$, then as $\mu \to \infty$, $\Omega_\mu(t) \to \mathbb{R}^N$ $(\forall t > 0)$ and

$$u_\mu \to U \text{ in } C_{loc}^{(1+\theta)/2, 1+\theta}((0,\infty) \times \mathbb{R}^N)\ (\forall \theta \in (0,1)),$$

where U is the unique solution of (1.2). Thus the Cauchy problem may be regarded as the limiting problem of (1.1) as $\mu \to \infty$.

The two problems also exhibit fundamentally different behavior. The first difference was observed in Du and Lin [9], where the one space dimension case of (1.1) was considered, with logistic nonlinearity $f(u) = au - bu^2$, a and b are positive constants. Note that in such a case $\Omega(t) = (g(t), h(t))$ is an interval. It was proved in [9] that in this special case problem (1.1) exhibits a spreading-vanishing dichotomy: as $t \to \infty$, either $\Omega(t)$ expands to the entire $\mathbb{R}^1$ and u converges to the positive steady-state a/b (spreading), or $\Omega(t)$ stays bounded and $u \to 0$ (vanishing). Moreover, when spreading happens, there exists $c^* = c^*(\mu) > 0$ such that

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{-g(t)}{t} = c^*. \tag{1.3}$$

The Cauchy problem (1.2) with $f(u) = au - bu^2$, however, behaves very differently. It was shown in the classical work of Aronson and Weinberger [2] that, in any space dimension $N \geq 1$, there exists $c_0 > 0$ independent of N, such that, for any small $\epsilon > 0$,

$$\begin{cases} \lim_{t\to\infty} \max_{|x|\geq(c_0+\epsilon)t} U(t,x) = 0, \\ \lim_{t\to\infty} \max_{|x|\leq(c_0-\epsilon)t} \left|U(t,x) - \frac{a}{b}\right| = 0. \end{cases} \tag{1.4}$$

The number c_0 is usually called the spreading speed, and is determined by certain traveling wave solutions associated to (1.2), which we will discuss in more detail later. Let us point out that, in the special case described above,

$$c_0 = 2\sqrt{a} = \lim_{\mu\to\infty} c^*(\mu).$$

Note that the above result indicates that spreading always happens for (1.2). The behavior of $c^*(\mu)$ was examined numerically in [5], where a deduction of the free boundary condition based on ecological considerations was also given.

A natural question is whether the spreading-vanishing dichotomy for (1.1) with logistic nonlinearity in one space dimension is retained in all space dimensions. In the spherically symmetric setting, this was confirmed in Du and Guo [7]. The extension to the non-symmetric case is highly nontrivial, since in such a case, the smoothness of the free boundary is a difficult question. A first step to address this problem was taken in Du and Guo [8], where it was shown that (1.1) has a unique weak solution $u(t,x)$, with the free boundary understood as $\Gamma(t) = \partial\Omega(t)$, $\Omega(t) = \{x : u(t,x) > 0\}$. As mentioned above, it was also shown in [8] that as $\mu \to \infty$, the weak solution of (1.1) converges to the solution of the corresponding Cauchy problem (1.2). Moreover, for the logistic problem, sufficient conditions for spreading and for vanishing are obtained. However, whether there is a sharp spreading-vanishing dichotomy as in the special cases studied in [9] and [7], was left open, and the regularity of the free boundary and the solution was not considered in [8]. These issues are addressed in Du, Matano and Wang [12], where the following results are obtained, under the assumptions (i) and (ii) above for f.

Theorem 1.1. *For any fixed $t > 0$, $\tilde{\Gamma}(t) := \Gamma(t) \setminus \overline{\mathrm{co}}(\Omega_0)$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^N$, and $\tilde{\Gamma} := \{(t,x) : x \in \tilde{\Gamma}(t),\ t > 0\}$ is a $C^{2,\alpha}$ hypersurface in $\mathbb{R}^{N+1}$. In particular, the free boundary is always $C^{2,\alpha}$ smooth if Ω_0 is convex.*

Here $\overline{\mathrm{co}}(\Omega_0)$ stands for the closed convex hull of Ω_0.

Theorem 1.2. *$\Omega(t)$ is expanding in the sense that $\overline{\Omega}_0 \subset \Omega(t) \subset \Omega(s)$ if $0 < t < s$. Moreover, $\Omega_\infty := \cup_{t>0}\Omega(t)$ is either the entire space $\mathbb{R}^N$, or it is a bounded set. Furthermore, when $\Omega_\infty = \mathbb{R}^N$, for all large t, $\Gamma(t)$ is a smooth closed hypersurface in $\mathbb{R}^N$, and there exists a continuous function $M(t)$ such that*

$$\Gamma(t) \subset \{x : M(t) - \frac{d_0}{2}\pi \leq |x| \leq M(t)\}; \tag{1.5}$$

and when Ω_∞ is bounded, $\lim_{t\to\infty} \|u(t,\cdot)\|_{L^\infty(\Omega(t))} = 0$.

Here d_0 is the diameter of Ω_0.

Theorem 1.3. *If $f(u) = au - bu^2$ with a, b positive constants, then there exists $\mu^* \geq 0$ such that $\Omega_\infty = \mathbb{R}^N$ if $\mu > \mu^*$, and Ω_∞ is bounded if $\mu \in (0, \mu^*]$. Moreover, when $\Omega_\infty = \mathbb{R}^N$, the following holds:*

$$\lim_{t\to\infty} \frac{M(t)}{t} = c^*(\mu),\ \lim_{t\to\infty} \max_{|x|\leq ct} \left|u(t,x) - \frac{a}{b}\right| = 0\ \forall c \in (0, k_0(\mu)),$$

where $c^*(\mu)$ is the same as in (1.3), which is a positive increasing function of μ satisfying $\lim_{\mu\to\infty} c^*(\mu) = 2\sqrt{a}$.

Clearly Theorem 1.3 extends the spreading-vanishing dichotomy of [9] to all space dimensions.

In the rest of this paper, we discuss some further issues associated with the free boundary problem (1.1), and compare the results with that of the Cauchy problem. In section 2, we consider the one space dimension case of (1.1), with $f(u)$ of monostable, bistable and combustion type. We look at the general dynamical behavior of (1.1) and compare it with the corresponding Cauchy problem (1.2). The material here is based on Du and Lou [10]. In section 3, for monostable, bistable and combustion types of nonlinearities, we examine the spreading profile of (1.1) for both the one space dimension case and the case of high space dimensions with spherical symmetry. We will reveal fundamentally different behavior of the free boundary problem from the corresponding Cauchy problem. Our discussions here are based on recent results obtained in Du, Matsuzawa and Zhou [13, 14]. We note that Theorem 1.2 suggests that the spreading behavior of the general case is usually well approximated by the spherically symmetric case.

While leaving the detailed discussions to sections 2 and 3 below, we would like to briefly comment on the key differences between the Cauchy problem (1.2) and the free boundary problem (1.1), and their implications. In the monostable case, $f'(0) > 0$, which indicates that $u \equiv 0$ is an unstable steady-state of (1.2). In the bistable case $f'(0) < 0$ and hence 0 is a stable steady-state, while in the combustion case $f'(0) = 0$ and 0 is weakly stable. These differences in stability of 0 have a profound influence on the spreading behavior of (1.2), especially in the way how the solution approaches the traveling wave profile. The bistable case and the combustion case are known to behave in a similar fashion while the monostable case behaves drastically differently. In sharp contrast, for the free boundary problem (1.1), the spreading profiles for all three types of nonlinearities behave in a rather synchronized manner, and can be treated by a unified approach, suggesting that the Stefan condition in (1.1) can considerably weaken the effect of the steady-state $u \equiv 0$, and thus plays a role of stabilizer in the spreading process.

2. Spreading and Vanishing in One Space Dimension

In one space dimension, (1.1) reduces to the following problem

$$\begin{cases} u_t = u_{xx} + f(u), & g(t) < x < h(t),\ t > 0, \\ u(t, g(t)) = u(t, h(t)) = 0, & t > 0, \\ g'(t) = -\mu\, u_x(t, g(t)), & t > 0, \\ h'(t) = -\mu\, u_x(t, h(t)), & t > 0, \\ -g(0) = h(0) = h_0, \quad u(0, x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{2.1}$$

where $x = g(t)$ and $x = h(t)$ are the moving boundaries to be determined together with $u(t,x)$, μ is a given positive constant. We assume that

$$f : [0,\infty) \to \mathbb{R}^1 \text{ is } C^1,\ f(0) = 0 \text{ and } f(u) \le 0 \text{ in } [M,\infty) \text{ for some } M > 0. \tag{2.2}$$

Moreover, the initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$\mathscr{X}(h_0) := \Big\{\phi \in C^2([-h_0,h_0]) : \ \phi(-h_0) = \phi(h_0) = 0,\ \phi'(-h_0) > 0, \\ \phi'(h_0) < 0,\ \phi(x) > 0 \text{ in } (-h_0,h_0)\Big\}.$$

For any given $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, by a (classical) solution of (2.1) on the time-interval $[0,T]$ we mean a triple $(u(t,x), g(t), h(t))$ belonging to $C^{1,2}(G_T) \times C^1([0,T]) \times C^1([0,T])$, such that all the identities in (2.1) are satisfied pointwisely, where

$$G_T := \big\{(t,x) : t \in (0,T],\ x \in [g(t),h(t)]\big\}.$$

In what follows, the solution may be simply denoted by (u,g,h).

Under the assumption (2.2), our problem (2.1) always has a unique classical solution which is defined for all time $t > 0$. Moreover, $g'(t) < 0$ and $h'(t) > 0$. We will use the notations

$$g_\infty := \lim_{t\to\infty} g(t), \quad h_\infty := \lim_{t\to\infty} h(t).$$

Taking advantage of the assumption that the space dimension is one, and hence the zero number argument can be used, we have the following result (see [10]), which gives more information than Theorem 1.2 when spreading happens.

Theorem 2.1. *Suppose that* (2.2) *holds and* (u,g,h) *is the unique solution of* (2.1). *Then* (g_∞,h_∞) *is either a finite interval or* $(g_\infty,h_\infty) = \mathbb{R}^1$. *Moreover, if* (g_∞,h_∞) *is a finite interval, then* $\lim_{t\to\infty} u(t,x) = 0$, *and if* $(g_\infty,h_\infty) = \mathbb{R}^1$ *then either* $\lim_{t\to\infty} u(t,x)$ *is a nonnegative constant solution of*

$$v_{xx} + f(v) \ = \ 0,\ x \in \mathbb{R}^1, \tag{2.3}$$

or

$$u(t,x) - v(x+\gamma(t)) \ \to \ 0 \ \textit{as}\ t \to \infty,$$

where v is an evenly decreasing positive solution of (2.3)*, and* $\gamma : [0,\infty) \to [-h_0, h_0]$ *is a continuous function.*

By an evenly decreasing function we mean a function $v(x)$ satisfying $v(-x) = v(x)$ which is strictly decreasing in $[0,\infty)$.

Next we focus on three types of nonlinearities:

(f_M) monostable case, (f_B) bistable case, (f_C) combustion case.

In the monostable case (f_M), we assume that f is C^1 and it satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad (1-u)f(u) > 0 \text{ for } u > 0, u \neq 1.$$

Clearly $f(u) = u(1-u)$ belongs to (f_M). This kind of nonlinearity was first investigated by Fisher [15] and Kolmogorov-Petrovskii-Piskunov [25], and is known as Fisher's equation or the KPP equation, which was used to describe the propagation of advantageous genes in a population.

In the bistable case (f_B), we assume that f is C^1 and it satisfies

$$f(0) = f(\theta) = f(1) = 0, \quad f(u) \begin{cases} < 0 & \text{in } (0,\theta), \\ > 0 & \text{in } (\theta, 1), \\ < 0 & \text{in } (1,\infty) \end{cases}$$

for some $\theta \in (0,1)$, $f'(0) < 0$, $f'(1) < 0$ and

$$\int_0^1 f(s)ds > 0.$$

A typical bistable $f(u)$ is $u(u-\theta)(1-u)$ with $\theta \in (0, \frac{1}{2})$. Such a nonlinearity appears in various applications including mathematical ecology, population genetics and physics. See, for example, [15, 28, 1, 2, 16] and the references therein.

In the combustion case (f_C), we assume that f is C^1 and it satisfies

$$f(u) = 0 \text{ in } [0,\theta], \quad f(u) > 0 \text{ in } (\theta, 1), \quad f'(1) < 0, \quad f(u) < 0 \text{ in } [1,\infty)$$

for some $\theta \in (0,1)$, and there exists a small $\delta_0 > 0$ such that

$$f(u) \text{ is nondecreasing in } (\theta, \theta + \delta_0).$$

Such a nonlinearity appears, typically, as a model for combustion; see [31, 22, 3, 32] and the references therein. The value θ is called the "ignition temperature".

Clearly (2.2) is satisfied if f is of (f_M), or (f_B), or (f_C) type. The next three theorems are obtained in [10], which give a good description of the long-time behavior of the solution, and they also reveal the related but different sharp transition behaviors between vanishing and spreading for these three types of nonlinearities.

Theorem 2.2 (The monostable case). *Assume that f is of (f_M) type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval with length no bigger than* $\pi/\sqrt{f'(0)}$ *and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0.$$

Moreover, if $u_0 = \sigma\phi$ with $\phi \in \mathscr{X}(h_0)$, then there exists $\sigma^* = \sigma^*(h_0,\phi) \in [0,\infty]$ such that vanishing happens when $0 < \sigma \le \sigma^*$, and spreading happens when $\sigma > \sigma^*$. In addition,

$$\sigma^* \begin{cases} = 0 & \text{if } h_0 \ge \pi/(2\sqrt{f'(0)}), \\ \in (0,\infty] & \text{if } h_0 < \pi/(2\sqrt{f'(0)}), \\ \in (0,\infty) & \text{if } h_0 < \pi/(2\sqrt{f'(0)}) \text{ and if } f \text{ is globally Lipschitz.} \end{cases}$$

Theorem 2.3 (The bistable case). *Assume that f is of (f_B) type, and $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0,$$

or

(iii) Transition: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and there exists a continuous function* $\gamma : [0,\infty) \to [-h_0, h_0]$ *such that*

$$\lim_{t\to\infty} |u(t,x) - v_\infty(x+\gamma(t))| = 0 \text{ locally uniformly in } \mathbb{R}^1,$$

where v_∞ *is the unique positive solution to*

$$v'' + f(v) = 0 \ (x \in \mathbb{R}^1), \ v'(0) = 0, \ v(-\infty) = v(+\infty) = 0.$$

Moreover, if $u_0 = \sigma\phi$ *for some* $\phi \in \mathscr{X}(h_0)$*, then there exists* $\sigma^* = \sigma^*(h_0, \phi) \in (0, \infty]$ *such that vanishing happens when* $0 < \sigma < \sigma^*$*, spreading happens when* $\sigma > \sigma^*$*, and transition happens when* $\sigma = \sigma^*$*. In addition, there exists* $Z_B > 0$ *such that* $\sigma^* < \infty$ *if* $h_0 \ge Z_B$*, or if* $h_0 < Z_B$ *and* f *is globally Lipschitz.*

Theorem 2.4 (The combustion case)**.** *Assume that* f *is of* (f_C) *type, and* $h_0 > 0$, $u_0 \in \mathscr{X}(h_0)$*. Then either*

(i) Spreading: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = 1 \text{ locally uniformly in } \mathbb{R}^1,$$

or

(ii) Vanishing: (g_∞, h_∞) *is a finite interval and*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} u(t,x) = 0,$$

or

(iii) Transition: $(g_\infty, h_\infty) = \mathbb{R}^1$ *and*

$$\lim_{t\to\infty} u(t,x) = \theta \text{ locally uniformly in } \mathbb{R}^1.$$

Moreover, if $u_0 = \sigma\phi$ *for some* $\phi \in \mathscr{X}(h_0)$*, then there exists* $\sigma^* = \sigma^*(h_0, \phi) \in (0, \infty]$ *such that vanishing happens when* $0 < \sigma < \sigma^*$*, spreading happens* *when* $\sigma > \sigma^*$, *and transition happens when* $\sigma = \sigma^*$. *In addition, there exists* $Z_C > 0$ *such that* $\sigma^* < \infty$ *if* $h_0 \geq Z_C$, *or if* $h_0 < Z_C$ *and* f *is globally Lipschitz.*

Remark 2.5. In [9], to determine whether spreading or vanishing happens for the special monostable nonlinearity, a threshold value of μ was established, which was shown in [9] to be always finite. Here we use σ in $u_0 = \sigma\phi$ as a varying parameter, which appears more natural especially for the bistable and combustion cases, since in these cases the dynamical behavior of (2.1) is more responsive to the change of the initial function than to the change of μ; for example, when $\|u_0\|_\infty \leq \theta$, then vanishing always happens regardless of the value of μ.

Theorems 2.3 and 2.4 above are parallel to Theorems 1.3 and 1.4 in [11], where the Cauchy problem (1.2) in one space dimension was considered. In contrast, Theorem 2.2 is very different from the Cauchy problem version, where a "hair-trigger" phenomenon appears, namely, when f is of (f_M) type, any nonnegative solution of (1.2) is either identically 0, or it converges to 1 as $t \to \infty$ (see [2]). Indeed, the results below will add further support to the observation that, though the monostable case for the Cauchy problem behaves rather differently from the bistable and combustion cases, for the free boundary problem (1.1), the behaviors of these three cases are more synchronized and can often be handled by a unified approach.

We now investigate the spreading speed of (2.1) when spreading happens. It turns out that, like the Cauchy problem, there exists an asymptotic spreading speed, which is determined by the following problem

$$\begin{cases} q_{zz} - cq_z + f(q) = 0 \quad \text{for } z \in (0,\infty), \\ q(0) = 0,\ q_z(0) = c/\mu,\ q(\infty) = 1,\ q(z) > 0 \text{ for } z > 0. \end{cases} \tag{2.4}$$

Proposition 2.6. ([10]) *Assume that* f *is of* (f_M), *or* (f_B), *or* (f_C) *type. Then for each* $\mu > 0$, (2.4) *has a unique solution* $(c, q) = (c^*, q^*)$.

We call q^* a "semi-wave" with speed c^*, since the function $v(t,x) = q^*(c^*t - x)$ satisfies

$$v_t = v_{xx} + f(v)\ (t \in \mathbb{R}^1,\ x < c^*t),\ v(t, c^*t) = 0,\ v(t, -\infty) = 1,$$

and it resembles a wave moving to the right at constant speed c^*, with front at $x = c^*t$. In comparison with the normal traveling wave generated by the solution of

$$Q_{zz} - cQ_z + f(Q) = 0 \text{ for } z \in \mathbb{R}^1,\ Q(-\infty) = 0,\ Q(+\infty) = 1,\ Q(0) = 1/2, \tag{2.5}$$

the generator $q^*(z)$ of $v(t,x)$ here is only defined on the half line $\{z \geq 0\}$. Hence we call it a semi-wave. We notice that at the front $x = c^*t$, we have $c^* = -\mu v_x(t,x)$, namely the Stefan condition in (2.1) is satisfied by $v(t,x)$ at $x = c^*t$.

Making use of the above semi-wave, Du and Lou [10] proved the following result.

Theorem 2.7. *Assume that f is of* (f_M)*, or* (f_B)*, or* (f_C) *type, and spreading happens. Let c^* be given by Proposition 2.6. Then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{-g(t)}{t} = c^*,$$

and for any small $\varepsilon > 0$, there exist positive constants δ, M and T_0 such that

$$\max_{|x|\leq(c^*-\varepsilon)t} |u(t,x) - 1| \leq Me^{-\delta t} \text{ for all } t \geq T_0. \tag{2.6}$$

The asymptotic spreading speed c^* depends on the parameter μ appearing in the free boundary conditions and in (2.4). Therefore we may denote c^* by $c^*(\mu)$ to stress this dependence.

We now compare $c^* = c^*(\mu)$ with the spreading speed c_0 determined by the corresponding Cauchy problem, in the sense of (1.4) with a/b replaced by 1. (Here we understand that c_0 depends on f, and $c_0 = 2\sqrt{a}$ when $f(u) = au - bu^2$.) It is well-known (see, e.g., [2]) that when f is of type (f_M), the spreading speed c_0 is the minimal $c > 0$ such that (2.5) has a solution Q, and when f is of type (f_B) or (f_C), c_0 is the unique $c > 0$ such that (2.5) has a solution Q. Moreover, it was shown in [10] that $c^*(\mu)$ is increasing in μ and

$$\lim_{\mu\to\infty} c^*(\mu) = c_0.$$

This is not surprising though, since the Cauchy problem (1.2) is the limiting problem of (1.1) as $\mu \to \infty$.

We may thus conclude that the asymptotic spreading speed $c^*(\mu)$ of the free boundary problem is always smaller than the asymptotic spreading speed c_0 of the corresponding Cauchy problem, and as $\mu \to \infty$, $c^*(\mu) \to c_0$. We observe that $c^*(\mu)$ is always the unique value of $c > 0$ such that (2.4) has a solution q, while c_0 is the unique value of $c > 0$ such that (2.5) has a solution Q when f is of type (f_B) or (f_C), but when f is of type (f_M), for every $c \geq c_0$, (2.5) has a solution Q. This phenomenon suggests that the instability of the steady-state $u \equiv 0$ in the monostable case has a significant influence on the behavior of (2.5) which determines the spreading speed of the Cauchy problem, but its influence is considerably weakened in (2.4) which determines the spreading speed of the free boundary problem. This suggests that the Stefan condition works against the influence of the instability of $u \equiv 0$ in the monostable case.

3. Spreading Profile

The profound differences between the free boundary problem (1.1) and the Cauchy problem (1.2) are further revealed when we examine their spreading profiles. We first recall some known results for the Cauchy problem (1.2). In one space dimension, a classical result of Fife and McLeod [16] states that for f of type ($\mathrm{f_B}$), and for appropriate initial function u_0 that guarantees $U(t,x) \to 1$ as $t \to \infty$, where U is the unique solution to (1.2), the spreading profile of U is described by

$$\begin{aligned} |U(t,x) - Q_{c_0}(c_0 t + x + C_-)| &< Ke^{-\omega t} \text{ for } x < 0, \\ |U(t,x) - Q_{c_0}(c_0 t - x + C_+)| &< Ke^{-\omega t} \text{ for } x > 0. \end{aligned}$$

Here (c_0, Q_{c_0}) is the unique solution of (2.5), $C_\pm \in \mathbb{R}^1$, and K, ω are suitable positive constants.

The monostable case of (1.2) behaves very differently. As mentioned before, in such a case, there exists $c_0 > 0$ such that (2.5) has a unique solution Q_c for every $c \geq c_0$, and it has no solution for $c < c_0$ (see [2]). Moreover, there is an essential difference on how the solution of (1.2) approaches the traveling waves: When ($\mathrm{f_M}$) holds and furthermore $f(u) \leq f'(0)u$ for $u \in (0,1)$ (so

f falls to the so called "pulled" case), as $t \to \infty$, there exist constants $C_\pm$ such that

$$\lim_{t\to\infty} \max_{x\geq 0} \left| U(t,x) - Q_{c_0}(c_0 t - \frac{3}{c_0}\log t - x + C_+) \right| = 0,$$

and

$$\lim_{t\to\infty} \max_{x\leq 0} \left| U(t,x) - Q_{c_0}(c_0 t - \frac{3}{c_0}\log t + x + C_-) \right| = 0.$$

The term $\frac{3}{c_0}\log t$ is known as the logarithmic Bramson correction term; see [4, 21, 26, 29] for more details.

For space dimension $N \geq 2$, if $u_0(x)$ is spherically symmetric and hence the unique solution U of (1.2) is spherically symmetric ($U = U(t,|x|)$), results in [20, 30] indicate that the Bramson correction term for the monostable case (with some extra conditions on f) becomes

$$\frac{N+2}{c_0}\log t \text{ (for the pulled case of } f),$$

or

$$\frac{N-1}{c_0}\log t \text{ (for the pushed case of } f),$$

that is, there exists some constant C such that for the pulled case of f,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| U(t,|x|) - Q_{c_0}\Big(c_0 t - \frac{N+2}{c_0}\log t + C - |x|\Big) \right| = 0,$$

and for the pushed case of f,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| U(t,|x|) - Q_{c_0}\Big(c_0 t - \frac{N-1}{c_0}\log t + C - |x|\Big) \right| = 0.$$

In the bistable case (as well as the combustion case), the Fife-McLeod result should be changed to (see [30])

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^N} \left| U(t,|x|) - Q_{c_0}\Big(c_0 t - \frac{N-1}{c_0}\log t + L - |x|\Big) \right| = 0,$$

where L is some constant.

In sharp contrast, the spreading profile of the free boundary problem is much more synchronized for all three types of nonlinearities. In [13], for space dimension one, the following result is obtained.

Theorem 3.1. *Suppose that f is of type* ($\mathrm{f_M}$), *or* ($\mathrm{f_B}$), *or* ($\mathrm{f_C}$) *and* (u,g,h) *is the unique solution to* (2.1) *for which spreading happens. Let* (c^*, q_{c^*}) *be given by Proposition 2.6. Then there exist* $\hat{H}, \hat{G} \in \mathbb{R}$ *such that*

$$\begin{aligned}\lim_{t\to\infty}(h(t)-c^*t-\hat{H})=0,\ \lim_{t\to\infty}h'(t) &= c^*,\\ \lim_{t\to\infty}(g(t)+c^*t-\hat{G})=0,\ \lim_{t\to\infty}g'(t) &= -c^*,\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\sup_{x\in[0,\,h(t)]}|u(t,x)-q_{c^*}(h(t)-x)| &= 0,\\ \lim_{t\to\infty}\sup_{x\in[g(t),\,0]}|u(t,x)-q_{c^*}(x-g(t))| &= 0.\end{aligned}$$

For higher space dimensions, [14] considered the radially symmetric case of (1.1), which has the form

$$\begin{cases} u_t=\Delta u+f(u), & 0<r<h(t),\ t>0,\\ u_r(t,0)=0,\ u(t,h(t))=0, & t>0,\\ h'(t)=-\mu u_r(t,h(t)), & t>0,\\ h(0)=h_0,\ \ u(0,r)=u_0(r), & 0\le r\le h_0,\end{cases} \tag{3.1}$$

where $\Delta u = u_{rr}+\frac{N-1}{r}u_r$, $r=h(t)$ is the moving boundary to be determined, h_0 is a positive constant. The initial function u_0 is chosen from

$$\mathscr{K}(h_0) := \Big\{\psi\in C^2([0,h_0]) : \psi'(0)=\psi(h_0)=0,\ \psi(r)>0 \text{ in } [0,h_0)\ \Big\}.$$

For any given $h_0>0$ and $u_0\in\mathscr{K}(h_0)$, by a classical solution of (3.1) on the time-interval $[0,T]$ we mean a pair $(u(t,r),h(t))$ belonging to $C^{1,2}(D_T)\times C^1([0,T])$, such that all the identities in (3.1) are satisfied pointwisely, where

$$D_T := \big\{(t,r) : t\in(0,T],\ r\in[0,h(t)]\big\}.$$

If f is of monostable, or bistable, or combustion type, it is known that (3.1) has a classical solution defined for all $t>0$. Simple sufficient conditions can be easily obtained to guarantee that spreading happens for (3.1), namely $h(t)\to\infty$ as $t\to\infty$, and $\lim_{t\to\infty}u(t,r)=1$ locally uniformly for $r\in[0,\infty)$. The following result of [14] describes the spreading profile of (3.1).

Theorem 3.2. *Suppose that f is of type* $(\mathrm{f_M})$, *or* $(\mathrm{f_B})$, *or* $(\mathrm{f_C})$ *and* (u,h) *is the unique solution to* (3.1) *for which spreading happens. Let* (c^*, q_{c^*}) *be given by Proposition 2.6. Then there exists* $\hat{h} \in \mathbb{R}^1$ *such that*

$$\lim_{t\to\infty}\left[h(t) - c^*t - c_N\log t - \hat{h}\right] = 0,\ \lim_{t\to\infty} h'(t) = c^*,$$

and

$$\lim_{t\to\infty}\sup_{r\in[0,h(t)]}\left|u(t,r) - q_{c^*}(h(t)-r)\right| = 0,$$

where

$$c_N = \frac{N-1}{\zeta\, c^*},\quad \zeta = 1 + \frac{c^*}{\mu^2\int_0^\infty q_{c^*}'(z)^2 e^{-c^*z}dz}.$$

Let us note that, in sharp contrast to the Cauchy problem, for the free boundary problem, the spreading profile for all three types of nonlinearities can be described in a unified fashion. The distinct spreading behavior for the monostable case of the Cauchy problem, caused by the instability of $u \equiv 0$, has disappeared in the free boundary problem. This adds further support to the view that the Stefan condition plays the role of a stabilizer in the modeling of spreading processes.

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