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CR FUNCTIONS AND THEIR FORMAL EXPANSIONS

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Abstract

We give a short overview of some current work on the Borel and the unique continuation property in CR structures and prove that CR functions in minimal integrable CR structures cannot vanish arbitrarily fast.

1. Introduction

The goal of this paper is to provide a short and self-contained survey of some recent results concerning the local properties of smooth CR functions. We will discuss the questions of how much a Taylor series of a CR function determines the function as well as which Taylor series can be attained as Taylor series of CR functions. In the case of holomorphic functions this is easily settled; a holomorphic function determines in a unique way a convergent power series at each point, and a convergent power series locally defines a holomorphic function.

In general CR structures, the question turns out to be much more subtle, and intricately linked to fine geometric properties of the structure. The status of the theory we present is not complete by any means; our aim is therefore rather to provide reasons why the investigation of these geometric properties in terms of what we refer to as the *Borel property* and the *unique*

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continuation property is a promising route to follow and we hope to inspire other complex analysts to see some of the questions in this framework.

Another intriguing aspect is that one can formulate many of the ideas and questions encountered here also in the context of general integrable system of first order PDEs, where they have been studied by Barostichi, Cordaro, and Petronilho; we will refer to these more general results, but focus on the CR case in this paper.

In addition to the survey we shall give a somewhat more intrinsic description of the Borel property than the one used in our paper [1] and show that for any minimal integrable CR structure a quantified unique continuation property holds; this is the main new result of this paper:

Theorem 1. *Let $M \subset \mathbb{C}^N$ be a smooth CR manifold, which is minimal at $p \in M$. Then there exists a neighbourhood U of p in M and a continuous function $\beta\colon U \to \mathbb{R}_+ \cup \{0\}$, which satisfies $\beta(p) = 0$ and $\beta(q) \neq 0$ for $q \neq p$, with the following property:*

If φ is a germ of a CR function at p with $|\varphi(q)| \leq \beta(q)$, then $\varphi \equiv 0$.

Let us remark that it is indeed possible to choose β of class C^∞, however we decided not to include a proof of this stronger statement here.

2. The Setting

2.1. Smooth CR manifolds

A smooth CR manifold M is given by a smooth manifold M of dimension $2n + d$ together with a subbundle

$$\mathcal{V} \subset \mathbb{C}TM,$$

of complex dimension n, satisfying $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$. By a slight abuse of notation, we shall refer to M as a CR manifold and have a fixed subbundle $\mathcal{V}$ in mind.

If $U \subset M$ is open, a CR function f on U is a smooth function $f \in \mathcal{E}(U)$ satisfying $Lf = 0$ for every smooth section $L \in \Gamma(U, \mathcal{V})$ (these are referred to as *CR vector fields* on U). We will write $\mathcal{S}(U)$ for the set of all CR functions on U. The CR structure $\mathcal{V}$ is integrable if for every point $p \in M$ there exists a neighbourhood U of p and smooth CR functions $Z_1, \dots, Z_{n+d}$, satisfying

$Z_j(p) = 0$, $j = 1, \dots, n + d = N$ with linearly independent differentials on U; such a family is referred to as a basic family of solutions at p. We refer to n as the CR-dimension of M. If $d = 1$, M is said to be of *hypersurface type.*

A CR manifold M is *minimal* at the point $p \in M$ if there is no proper submanifold $N \subset M$, passing through p, such that $\mathcal{V}|_N \subset \mathbb{C}TN$.

A hypersurface type CR structure comes with a natural a hermitian form

$$\mathcal{L}\colon \mathcal{V} \times \mathcal{V} \to {}^{\mathbb{C}TM}\!\big/_{\mathcal{V} \oplus \bar{\mathcal{V}}},$$

its *Levi form.* It is *Levi-nondegenerate* if $\mathcal{L}$ is nondegenerate, and *strictly pseudoconvex* if it is positive definite.

An integrable CR structure can be locally embedded into $\mathbb{C}^N$ as a smooth generic submanifold by using a basic family of solutions as coordinates; the CR structure then coincides with the one induced as a real submanifold of $\mathbb{C}^N$, where the subbundle $\mathcal{V}$ is given by the tangent $(0,1)$-vector fields. One sets $T^cM = \operatorname{Re}\mathcal{V}$, which is a complex subbundle of the real tangent bundle TM. The Levi form can then be interpreted as a Hermitian form on T^cM. We say that M is *lineally convex* at p if there exists a neighbourhood U of p in $\mathbb{C}^N$ such that $T^c_pM \cap M \cap U = \{p\}$. A complex direction $v \in T^c_pM$ is of finite type k if the complex line through p in direction v is tangent to order at most k to M at p.

In terms of a defining function $\mathrm{Im\, Z_N = f(Z_1, \dots, Z_{N-1}) + O(Re\, Z_N)}$ for M, we shall say that M is lineally convex along a cone $\mathcal{C}$ of finite order (at most) k if there exists a constant $M > 0$ such that

$$f(cv) \geq M|c|^k$$

for small c and all directions $v \in \mathcal{C}$.

2.2. Smooth and formal CR functions

The set of germs $\mathcal{S}_p$ of CR functions at p is the inductive limit

$$\lim_{\rightarrow} \mathcal{S}(U) = \mathcal{S}_p,$$

where we can take the limit over a decreasing sequence of neighbourhoods U_j with $\bigcap_j U_j = \{p\}$. Each of the spaces is equipped $\mathcal{S}(U)$ with the usual Frechet space structure induced from $\mathcal{E}(U)$, and the inductive limit is in the category of locally convex spaces.

Let $k \in \mathbb{N}$. We declare an equivalence relation on $\mathcal{S}_p$ by setting $f \sim_k g$ if $f - g$ vanishes to order k at p. The set of equivalence classes is denoted by $J_p^k(\mathcal{S})$ and referred to as the set of k-jets of CR functions at p. We note that the subspace $Z_k \subset \mathcal{S}_p$ of germs of functions vanishing to order k at p is closed; hence the quotient is a (finite dimensional) topological vector space. The natural quotient map is denoted by $j_p^k \colon \mathcal{S}_p \to J_p^k(\mathcal{S})$.

The space $J_p^k(\mathcal{S})$ submerges canonically onto $J_p^{k-1}(\mathcal{S})$, since $Z_{k+1} \subset Z_k$. We can therefore consider the inductive limit

$$J_p^\infty(\mathcal{S}) = \lim_{\rightarrow} J_p^k(\mathcal{S}),$$

which we will refer to as the space of *formal CR functions* at p.

There exists a canonical map $j_p^\infty \colon \mathcal{S} \to J_p^\infty(\mathcal{S})$, the kernel of which are the germs of flat CR functions, which we are going to denote by Z_∞.

If M is integrable, or more generally, if there exists a basic family $\{Z_1, \dots, Z_N\}$ of solutions at p, then $J_p^k(\mathcal{S})$, for $k \in \mathbb{N}$, can be identified with the space of polynomials of degree at most k in $Z_1, \dots, Z_N$, and $J_p^\infty(\mathcal{S})$ with the space of formal power series in $Z_1, \dots, Z_N$, i.e.

$$J_p^\infty(\mathcal{S}) = \mathbb{C}[[Z_1, \dots, Z_N]], \quad J_p^k(\mathcal{S}) = \mathbb{C}[[Z_1, \dots, Z_N]] \big/ (Z_1, \dots, Z_N)^k.$$

2.3. The Borel and the unique continuation property

We will say that M possesses the *Borel property* at p if the map $j_p^\infty \colon \mathcal{S}_p \to J_p^\infty(\mathcal{S})$ is onto, i.e. if every formal CR function can be represented by a CR function. In other words, we say that M has the Borel property at p if

$$J_p^\infty(\mathcal{S}) = \mathcal{S}_p \big/ Z_\infty.$$

We say that M has the *unique continuation property* at p if the only flat CR function at p is the trivial one.

2.4. Some examples

A totally real structure has the Borel property at each of its points; indeed, every such structure is integrable (and any system of coordinates will supply a family of basic solutions) and it is well known that every formal power series is the Taylor series of some smooth function, which in turn is a solution.

If an integrable CR structure is *hypoanalytic*, i.e. if every CR function can be written as a holomorphic function of basic solutions, then

$$Z_\infty = \{0\}, \quad \mathcal{S}_p = \mathbb{C}\{Z_1, \ldots, Z_N\};$$

hence the unique continuation property holds at every point, while the Borel property emphatically fails. A classical example of a hypoanalytic CR structure is given by a hypersurface type structure whose Levi form attains both signs at every point.

Every (integrable) strictly pseudoconvex hypersurface type CR structure M possesses the Borel property at each of its points by the results of [1]. Note that such a structure admits an abundance of flat CR functions and therefore does not possess the unique continuation property: One can identify a neighbourhood of $p \in M$ with a hypersurface in $\mathbb{C}^N$ defined by $\mathrm{Im}\, \mathrm{Z_N} = |\mathrm{Z_1}|^2 + \cdots + |\mathrm{Z_{N-1}}|^2 + \mathrm{O}(3)$. Thus the exponentials $\exp(iZ_N^{-k})$ furnish examples of smooth, flat, yet nonzero CR functions.

3. The Borel Property: A necessary and a sufficient condition

The following necessary condition was found by Cordaro, Barostichi, and Petronilho by functional analytic means, and is also valid for general integrable systems:

Theorem 2 ([4], Theorem 2). *If the Borel property holds at a point p, then there exist nontrivial flat CR functions at p.*

The proof of the theorem is in two steps: First, one uses the Baire category theorem to see that if the Borel property holds at p, then there exists a neighbourhood V of p such that $j_p^\infty \colon \mathcal{S}(V) \to J_p^\infty(\mathcal{S})$ is already onto. Both of these spaces are Frechet spaces, and j_p^∞ is continuous. Hence the kernel $K \subset \mathcal{S}(V)$ satisfies that $K^\circ \subset \mathcal{S}(V)^*$ is the image of the transpose map $(j_p^\infty)^*$ (if j_p^∞ is surjective). One can compute the transpose and the dual and see that this leads to a contradiction if $K = \{0\}$.

Thus, if the unique continuation property holds at p, the Borel property necessarily fails at p, and if the Borel property is to hold at p, we therefore need nontrivial flat CR functions.

A convenient way to construct nontrivial flat CR functions is through the use of *peak functions*. Let $p \in M$, and U a neighbourhood of p. We will say that a CR function $\varphi \in \mathcal{S}(U)$ is a *peak function* at p if $\varphi(p) = 0$, $\varphi(q) \neq 0$ for $q \neq p$, and $\arg \varphi \neq -\pi$ on $U \setminus \{p\}$. In particular, the root of such a peak function will satisfy $\mathrm{Re}\sqrt{\varphi(\mathrm{q})} > 0$ for $q \neq p$; but on the other hand, $\sqrt{\varphi}$ is not necessarily smooth at p any longer.

We say that a peak function φ is of *finite type* if there exist constants $\alpha > 0$ and $C > 0$ such that

$$|\varphi(q)| \geq C d(q,p)^\alpha.$$

Note that a peak function gives rise to flat CR functions defined by $\exp(\varphi^{-\beta})$ for appropriate β.

The main necessary result on the Borel property in [1] is

Theorem 3. *If M possesses a peak function of finite type at p, then j_p^∞ is onto, and there exists a neighbourhood V of p and a continuous inverse $F \colon J_p^\infty(\mathcal{S}) \to \mathcal{S}(V)$ to j_p^∞.*

The proof of the theorem reconstructs a smooth CR function from its Taylor series

$$A(Z) = \sum_\alpha A_\alpha Z^\alpha$$

by modifying it with appropriately chosen factors ψ_α to yield a CR function

$$f(q) = \sum_\alpha A_\alpha \psi_\alpha(q) Z(q)^\alpha;$$

the ψ_α satisfy that $\psi_\alpha - 1$ is flat, and their decay is sufficiently rapid.

There is an obvious gap between the statements of Theorem 3 and Theorem 2, and it gives rise to a couple of interesting questions, which we would like to state here:

- Are surjectivity and injectivity of j_p^∞ mutually exclusive?
- If j_p^∞ is not injective, does there exist a peak function at p?
- What is the role of the finite type assumption?

4. Quantifying the Unique Continuation Property

As we have seen, in many interesting circumstances, the unique continuation property necessarily fails. It is thus important to understand how fast a flat CR function can vanish. Let us assume that we are dealing with an embedded CR structure, i.e. a real submanifold $M \subset \mathbb{C}^N$, and that we are looking at the point $p = 0$. We are then asking the following question: Does there exist a continuous function $\beta \neq 0$, defined in a neighbourhood of 0, such that if a CR function φ satisfies $|\varphi(Z)| \leq \beta(|Z|)$ for $z \in M$ close to 0, then $f \equiv 0$?

In the case of a lineally convex hypersurface, we shall also use a comparison with the peak function ψ which is the defining function of the complex tangent space; i.e. if a defining equation for M is given by $\mathrm{Im\,w} = \mathrm{r(z, \bar{z}, Re\,w)}$, with $r(z,\bar{z},0) > 0$ for $z \neq 0$ close to 0, then $\psi = iw$. Then $|\psi(Z)|$ is comparable to $|Z|$, but anisotropic in the "good" tangent directions $z \in T_0^c M$ and the "bad" direction $\mathrm{Re\,w}$; good tangent directions which are of finite type k have weight k in this distance.

In [2], we provided the following sufficient condition for a CR function to vanish identically:

Theorem 4. *If $M \subset \mathbb{C}^N$ is a lineally convex hypersurface of finite order k along an open cone $\mathcal{C} \subset T^c M$, and φ is a germ of a CR function (of class C^1) on M satisfying*

$$|\varphi(Z)| \leq e^{-\frac{\lambda}{|\psi(Z)|}}$$

for Z close to 0 (in a neighbourhood depending on λ) for all $\lambda > 0$, then $\varphi \equiv 0$.

Before continuing, let us remark on the assumption made about M in Theorem 4. We say that M is of finite commutator type at 0 if the Taylor series of $r(z, \bar{z}, 0)$ is not trivial. The condition that M is lineally convex of finite order along some open cone is equivalent to M being lineally convex and of finite commutator type.

Our Theorem 1 is more general in its scope than Theorem 4, but less explicit in the form of the prescribed decay. At this point, we would also like to point out the unique continuation principle for Gevrey solutions in locally integrable structures obtained in [3]. We will now turn to the proof of Theorem 1. So let from now on $M \subset \mathbb{C}^N$ be a generic smooth submanifold, and $0 \in M$ a minimal point.

We first recall the notion of an analytic disc attached to M and a basic result of Tumanov [5]. An analytic disc attached to M is a holomorphic map $A\colon \mathbb{C} \supset D \to \mathbb{C}^N$ which extends smoothly to bD and satisfies $A(bD) \subset M$. A wedge with edge $M \cap U$ is an open set $\mathcal{W} \subset \mathbb{C}^N$ which contains $U + \Gamma \cap W$, where Γ is an (open) real d-dimensional cone with the property that for $p \in U$ we have $T_pM + \Gamma = \mathbb{C}^N$, and W is an open set. Then Tumanov showed that if 0 is a minimal point of M, then for every small enough neighbourhood U of 0 there exists a wedge $\mathcal{W}$ with edge $M \cap U$ which is filled by discs attached to $M \cap U$.

A smooth CR function f defined near 0 can then be extended to a wedge with edge $M \cap U$ for a small neighbourhood U of 0 by defining its extension on the analytic discs attached to M.

We will also need the following version of Watson's Lemma:

Lemma 4.1. *Let S be an open sector in the complex plane of opening angle α, and $k > \pi/\alpha$. If $H\colon S \cap \{|z| < \varepsilon\}$ is a holomorphic function which satisfies*

$$|H(z)| \leq e^{-\frac{1}{|z|^k}}$$

for z close to 0, then $H = 0$.

Let now U be a small enough neighborhood of 0 in M, and $\mathcal{W}$ an associated wedge with edge $M \cap U$. For any $q \in \mathcal{W}$, we define

$$\mathcal{A}_q = \left\{ h \in \mathcal{O}(D) \cap C^\infty(\overline{D}) \colon h(bD) \subset U, h(0) = q \right\};$$

we have that $\mathcal{A}_q \neq \emptyset$ because of the result of Tumanov. Setting $\|h\|_{bD} = \max\{|h(\zeta)| : \zeta \in bD\}$, we define

$$\ell(q) := \inf\{\|h\|_{bD} : h \in \mathcal{A}_q\},$$

and for any $\rho > 0$

$$\tau(\rho) := \sup\left\{\ell(q) : q \in \overline{B(0,\rho)} \cap \mathcal{W}\right\}.$$

From the definition it is clear that $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function, $\tau(t) > 0$ for $t > 0$, and that $\ell(q) \leq \tau(|q|)$ for all $q \in \mathcal{W}$. Furthermore, we have that $\tau(t) \to 0$ as $t \to 0^+$. Indeed, if for any $\varepsilon > 0$ we denote by $\mathcal{W}_\varepsilon$ the wedge associated to $U_\varepsilon := B(0,\varepsilon) \cap U$ by Tumanov's theorem, we can choose $\delta > 0$ so small that $\overline{B(0,\delta)} \cap \mathcal{W} \subset \mathcal{W}_\varepsilon$; it follows by definition that $\ell(q) < \varepsilon$ for all $q \in \overline{B(0,\delta)} \cap \mathcal{W}$, hence $\tau(\delta) \leq \varepsilon$.

For any $r > 0$, we now define $\eta(r) = \sup\{\rho > 0 : \tau(\rho) < r\}$; as before, $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, and it is easy to check that $\eta(t) > 0$ for $t > 0$, $\eta(t) \to 0$ as $t \in 0^+$ and $\eta(\tau(\rho)) \leq \rho$ for all $\rho > 0$.

We can assume that for every v close enough to a fixed complement of T_0M the intersection of $\mathcal{W}$ with the complex line L_v in direction v contains a sector of opening angle α.

Fix any $k > \pi/\alpha$, and set $\beta_1(r) = e^{-\frac{1}{\eta(r)^k}}$; choose any increasing continuous function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\beta \leq \beta_1$. We claim that β satisfies the conditions of Theorem 1.

So let f be a CR function defined on U such that $|f(p)| \leq \beta(|p|)$ for all $p \in U$, and let F be its holomorphic extension to $\mathcal{W}$. By the maximum principle, we have $|F(q)| \leq \max\{|f(h(\zeta))| : \zeta \in bD\}$ for all $q \in \mathcal{W}$ and $h \in \mathcal{A}_q$. Since by assumption $|f(h(\zeta))| \leq \beta(|h(\zeta)|)$ for all $\zeta \in bD$, it follows $|F(q)| \leq \beta(\|h\|_{bD})$ for all $h \in \mathcal{A}_q$, and taking the infimum over $\mathcal{A}_q$ we obtain $|F(q)| \leq \beta(\ell(q))$ for all $q \in \mathcal{U}$.

In view of the previous definitions we now get

$$|F(q)| \leq \beta_1(\ell(q)) \leq \beta_1(\tau(|q|)) = e^{-\frac{1}{\eta(\tau(|q|))^k}} \leq e^{-\frac{1}{|q|^k}}$$

for all $q \in \mathcal{U}$.

Now choose any vector v, $|v| = 1$, which is close enough to the complement to T_0M as above, i.e. such that the complex line $L_v = \{zv : z \in \mathbb{C}\}$ intersects $\mathcal{W}$ in a domain $\mathcal{W}_v$ of L_v ($\mathcal{W}_v = \{zv : z \in \Omega_v\}$ where $\Omega_v \subset \mathbb{C}$, $0 \in b\Omega_v$, contains a sector of opening angle α). Setting $F_v(z) = F(zv)$, it follows from the previous discussion that $|F_v(z)| \leq e^{-\frac{1}{|z|^k}}$ for all $z \in \Omega_v$; by 4.1 we deduce that $F_v \equiv 0$. Hence the restriction to F to each $L_v \cap \mathcal{W}$ vanishes identically, which implies that $F \equiv 0$ (and thus $f \equiv 0$) since $\cup_v(L_v \cap \mathcal{W})$ contains an open subset of $\mathcal{W}$.

5. Closing Remarks

There exist an abundance of natural questions which arise in the context of the Borel property and the unique continuation property. We have already mentioned some of these questions above, in the context of the Borel property.

For the unique continuation property, we would like to point out that for the existence of rapidly vanishing CR functions, a sufficient criterion was obtained in [2]; in particular, it provides the following counterpoint to the unique continuation principle of Theorem 4:

Theorem 5. *If $M \subset \mathbb{C}^N$ is a lineally convex hypersurface of finite order ℓ along an open cone $\mathcal{C} \subset T^cM$, and $k < 1$, then there exists a germ of a smooth CR function φ, $\varphi \neq 0$, on M satisfying*

$$|\varphi(Z)| \leq e^{-\frac{1}{|\psi(Z)|^k}}$$

for Z close to 0.

Actually, we provided this result for a far more general class of "admissible" rates, which is a bit technical to define, so we refrain from doing it here. But a natural question is whether there is a *critical rate*, or rather, a sharp condition to identify vanishing rates which allow for nontrivial CR functions. Such a condition is still missing from the theory.

Another question in that direction is whether it is possible to give a more intrinsic description of the rate defined in the proof of Theorem 1—the definition links the rate to the geometry of M, but it is not clear how one can read off a rate from geometric data.

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