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INVARIANTS OF A COMPLEX COTANGENT LINE FIELD

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Abstract

We study a complex manifold M together with a smooth complex line sub-bundle E of its (1,0)-cotangent bundle. E is assumed to satisfy a certain integrability condition and a non-degeneracy condition. We attach to the structure (M, E) an invariant generalized connection on a principal bundle P over M of adapted coframes. The total space of E minus its zero section has a natural almost complex structure. We determine when it is actually a complex structure.

0. Introduction

Let M be an n-dimensional complex manifold, $n \geq 2$, having a smooth complex line sub-bundle $E \subset T^*_{(1,0)}M$ of its (1,0)-cotangent bundle. We assume that E is ∂-integrable, and Levi non-degenerate, as described below in the first paragraph of section one. Our goal is to study the (local) biholomorphic invariants of such (M, E).

Aside from the non-degeneracy condition, (M, E) may be considered as a complex analogue of the conormal bundle of a real codimension-one foliation of a real manifold. Our motivation, however, stems from the relation of such (M, E) to certain fundamental solutions for the $\overline{\partial}$-operator [5].

Locally E is spanned by a non-zero (1,0)-form θ, which is determined up to a non-zero factor, $\theta \to v\theta$. Coframes (1.2) and their dual frames on M satisfying (1.5) below are said to be adapted to E. They form a principal fiber bundle P over M with structure group G (1.9). The main result is the following theorem, which is proved in Section 2. It solves the biholomorphic

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equivalence problem for such structures (M,E) in the sense of E. Cartan. (Greek indices will run from 1 to $n-1$ in this work.)

Theorem 0.1. *On the bundle P there exist global invariant complex one-forms θ^α, θ, ϕ, ω_β^α, which, together with their complex conjugates, span the complexified cotangent bundle of P. Any biholomorphic map f between two such structures (M,E) and (M',E') lifts to a diffeomorhism $\tilde{f}$ of the bundles P, P' preserving the corresponding forms.*

The theorem may be interpreted as defining a Cartan connection on the bundle P. We investigate its curvature in section 3.

The differential forms of the theorem, if declared to be of type (1,0) define an almost complex structure on P, which is non-integrable, in general. We do not consider its integrability here. Instead, the global forms θ^α, θ pull down via local sections to give (1,0) coframes on M. The map π factors through the principal line bundle $\hat{E}$ associated to E, which is just E with its zero section deleted. Local sections of $P \to \hat{E}$ pull down the forms θ^α, θ, ϕ to forms defining an almost complex structure on $\hat{E}$. Again, this structure is non-integrable, in general. We give conditions for it to be integrable in Corollary 2.1, at the end of Section 2.

The invariant theory of (M,θ),for a fixed θ, was developed in [5]. The relation between the geometries, or G-structures, of (M,θ) and (M,E) recalls that between Riemannian and conformal geometries [1], [3]; or even more closely, that between pseudo-hermitian [4] and pseudo-conformal [2] geometries. However, as as we shall see here, these analogies do not run too deeply, since the G-structure of (M,E) is of first, not second order, and hence considerably simpler.

1. Local structure on M

We let the complex line bundle $E \subset T^*_{(1,0)}M$ be spanned locally by a non-zero (1,0)-form θ on M. Then E is **∂-integrable** if

$$\partial\theta = \theta \wedge \phi' \,, \tag{1.1}$$

for some (1,0)-form ϕ' on M. E is Levi **non-degenerate**, if the (1,1)-form $\overline{\partial}\theta$ is non-degenerate on the complex tangent hyperplane field, $\{\theta = 0\} \subset$

$T_{(1.0)}M$. These two conditions are easily seen to be invariant under $\theta \mapsto v\theta$, so are properties of E.

A well-known example is the line bundle E spanned by the (1,0)-form $\theta = \partial \log |z|^2$ on $\mathbf{C}^n - \{0\}$. From it we get the Bochner-Martinelli (n,n-1)-form, $\theta \wedge (\overline{\partial}\theta)^{n-1} \neq 0$. The possibility of such a construction for more general fundamental solutions for the $\overline{\partial}$-operator is the motivation for the present work, as it was for [5].

We study the local geometry of E via (1,0)-coframe fields of the form

$$\theta^\alpha, 1 \leq \alpha \leq n-1,\ \theta^n = \theta, \tag{1.2}$$

spanning $T^*_{(1,0)}M$ locally. Following the notation set down by Chern-Moser [2], we let Greek indices run from 1 to $n-1$, and use the summation convention. Bars over indices will usually reflect complex conjugation. Thus,

$$\overline{\partial}\theta = \sum_{i,j=1}^{n} h_{i\overline{j}}\theta^i \wedge \theta^{\overline{j}} = \chi + h_{\alpha\overline{n}}\theta^\alpha \wedge \overline{\theta} + \theta \wedge \phi'', \tag{1.3}$$

$$\chi = h_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}, \tag{1.4}$$

where ϕ'' is a (0,1)-form. The $n-1$ by $n-1$ matrix $(h_{\alpha\overline{\beta}})$, a clear analogue of the Levi form of a real hypersurface, is non-degenerate, but need not have any symmetry properties.

The admissible change of coframe, $\tilde{\theta} = \theta$, $\tilde{\theta}^\alpha = \theta^\alpha + \theta v^\alpha$, results in $\tilde{h}_{\alpha\overline{n}} = h_{\alpha\overline{n}} + h_{\alpha\overline{\beta}}v^{\overline{\beta}}$. Thus, we can choose the v^β uniquely to make $\tilde{h}_{\alpha\overline{n}} = 0$. We call such coframes $\{\theta^\alpha, \theta\}$, and their dual frames $\{X_\alpha, X_n \equiv X\}$, with this additional condition, adapted. This condition, which fixes the direction of X, may be expressed as

$$\iota_{\overline{X}}(\overline{\partial}\theta) = \mu\theta, \tag{1.5}$$

for some factor μ. This is clearly independent of the changes $\theta \mapsto v\theta$.

It is also invariant under biholomorphic maps f, which preserve the cotangent line field E. For, if $f^*\theta = \lambda\theta$, then

$$\begin{aligned}\iota_{\overline{f_*X}}(\overline{\partial}\theta) &= \iota_{\overline{f_*X}}(\overline{\partial}[(f^{-1})^*(\lambda\theta)]) = \iota_{\overline{f_*X}}((f^{-1})^*(\overline{\partial}(\lambda\theta)))\\ &= (f^{-1})^*[\iota_{\overline{X}}(\overline{\partial}(\lambda\theta))] = (f^{-1})^*[(\lambda\mu + \overline{X}\lambda)\theta]\\ &= [(\lambda\mu + \overline{X}\lambda)\circ f^{-1}](f^{-1})^*\theta. \end{aligned}\tag{1.6}$$

But $(f^{-1})^*\theta = \theta/(\lambda \circ f^{-1})$, so we get

$$\iota_{\overline{f_* X}}(\overline{\partial}\theta) = \tilde{\mu}\theta, \ \ \tilde{\mu} = (\mu + \lambda^{-1}\overline{X}\lambda) \circ f^{-1}. \tag{1.7}$$

We shall henceforth restrict to such adapted coframes $\{\theta^\alpha, \theta\}$, so that

$$\partial\theta = \theta \wedge \phi', \ \ \overline{\partial}\theta = \chi + \theta \wedge \phi''. \tag{1.8}$$

The principal bundle, $\pi : P \to M$ of adapted coframes has the structure group, $G = \mathbf{C}^* \times Gl(n-1, \mathbf{C})$, reflecting the now admissible changes

$$\theta = \theta_0 v, \ \ \theta^\alpha = \theta_0^\beta U_\beta{}^\alpha. \tag{1.9}$$

The formulae (1.8), together with the integrability of the (almost) complex structure on M, give the following local structure equations on M,

$$\begin{aligned} d\theta &= \chi + \theta \wedge \phi, \ \ \phi = \phi' + \phi'', & (1.10)\\ d\theta^\alpha &= \theta^\beta \wedge \omega_\beta{}^\alpha + \theta \wedge \tau^\alpha, & (1.11)\\ \tau^\alpha &= \theta^{\overline{\beta}} A_{\overline{\beta}}{}^\alpha + \overline{\theta} A_{\overline{n}}{}^\alpha. & (1.12)\end{aligned}$$

For a fixed adapted coframe $\{\theta^\alpha, \theta\}$, χ given by (1,4), and the torsion one-forms τ^α are uniquely determined. The forms ϕ, $\omega_\beta{}^\alpha$ are determined up to changes

$$\begin{aligned} \tilde{\phi} &= \phi + b\theta, & (1.13)\\ \tilde{\omega}_\beta{}^\alpha &= \omega_\beta{}^\alpha + B_\beta{}^\alpha{}_\gamma \theta^\gamma, \ B_\beta{}^\alpha{}_\gamma = B_\gamma{}^\alpha{}_\beta, & (1.14)\end{aligned}$$

as is easily seen from (1.10), (1.11) and Cartan's lemma.

2. Invariant forms on the bundle P

In passing to the principal bundle $\pi : P \to M$ over M, the coefficients $\{v, U_\beta{}^\alpha\}$ in (1.9) are interpreted as independent fiber coordinates. The forms $\{\theta_0^\alpha, \theta_0\}$ are local forms on P, pulled up from M via the map π, and the forms $\{\theta^\alpha, \theta\}$ are global, intrinsic forms on P. The latter must be completed to a basis by finding further global invariant forms on P, which we shall eventually denote by $\{\phi, \omega_\beta{}^\alpha\}$.

With this notation understood, we have equations (1.10), (1.11), (1.12), for the exterior derivatives of $\{\theta_0^\alpha, \theta_0\}$, with local forms $\{\phi_0, \omega_{0\beta}^{\ \alpha}\}$ pulled up to P. Exterior differentiation of (1.9) shows that, locally on P, we have forms $\{\phi, \omega_\beta{}^\alpha\}$ satisfying (1.10), (1.11), (1.12), and

$$\begin{aligned} \phi &= \phi_0 - dv/v, && (2.1)\\ U_\beta{}^\gamma \omega_\gamma^{\ \alpha} &= \omega_{0\beta}^{\ \gamma} U_\gamma^{\ \alpha} - dU_\beta{}^\alpha, && (2.2)\\ v\tau^\alpha &= \tau_0^\beta U_\beta^{\ \alpha}, \ \ \chi = v\chi_0. && (2.3)\end{aligned}$$

These formulae make it clear that the forms $\{\theta^\alpha, \theta, \phi, \omega_\beta^{\ \alpha}\}$, and their complex conjugates give a local basis for complex one-forms on the manifold P. The forms $\{\phi, \omega_\beta^{\ \alpha}\}$ are not yet uniquely determined, as (1.13), (1.14) show. We proceed to determine them intrinsically. It will then follow that they are global.

We first take the exterior derivative of equation (1.10) to get

$$0 = d\chi + \chi \wedge \phi - \theta \wedge d\phi. \tag{2.4}$$

To compute $d\chi$, we use equation (1.4) and the covariant differential notation,

$$Dh_{\alpha\overline{\beta}} = dh_{\alpha\overline{\beta}} - \omega_\alpha^{\ \gamma} h_{\gamma\overline{\beta}} - h_{\alpha\overline{\gamma}} \omega_{\overline{\beta}}^{\ \overline{\gamma}}. \tag{2.5}$$

Then (2.4) becomes

$$(Dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi) \wedge \theta^\alpha \wedge \theta^{\overline{\beta}} = \theta \wedge (d\phi - h_{\alpha\overline{\beta}}\tau^\alpha \wedge \theta^{\overline{\beta}}) - \overline{\theta} \wedge (h_{\alpha\overline{\beta}}\theta^\alpha \wedge \tau^{\overline{\beta}}). \tag{2.6}$$

From this it follows that each one-form $(Dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi)$ can have no component linearly independent from $\theta^\alpha, \theta, \theta^{\overline{\alpha}}, \overline{\theta}$. For such a term does not appear on the right-hand side of (2.6). Also, there can be no term of the form $\overline{\theta} \wedge \theta^\alpha \wedge \theta^{\overline{\beta}}$ on the left-hand side of (2.6), due to the form (1.12) of τ^α.

Hence, we may write

$$Dh_{\alpha\overline{\beta}} + h_{\alpha\overline{\beta}}\phi = h_{\alpha\overline{\beta},\gamma}\theta^\gamma + h_{\alpha\overline{\beta},\overline{\gamma}}\theta^{\overline{\gamma}} + h_{\alpha\overline{\beta},n}\theta. \tag{2.7}$$

Substitution into (2.6) gives

$$h_{\alpha\overline{\beta},\gamma} = h_{\gamma\overline{\beta},\alpha}, \ \ h_{\alpha\overline{\beta},\overline{\gamma}} = h_{\alpha\overline{\gamma},\overline{\beta}}, \tag{2.8}$$

and

$$h_{\alpha\overline{\beta},n}\theta\wedge\theta^{\alpha}\wedge\theta^{\overline{\beta}}=\theta\wedge(d\phi-h_{\alpha\overline{\beta}}\tau^{\alpha}\wedge\theta^{\overline{\beta}})-\overline{\theta}\wedge(h_{\alpha\overline{\beta}}\theta^{\alpha}\wedge\tau^{\overline{\beta}}). \tag{2.9}$$

We rearrange this as

$$\begin{aligned}0= {} & \theta\wedge\{d\phi-h_{\alpha\overline{\beta}}A^{\alpha}_{\overline{\gamma}}\theta^{\overline{\gamma}}\wedge\theta^{\overline{\beta}}-h_{\alpha\overline{\beta},n}\theta^{\alpha}\wedge\theta^{\overline{\beta}}-\overline{\theta}\wedge[h_{\alpha\overline{\beta}}(A^{\overline{\beta}}_{n}\theta^{\alpha}+A^{\alpha}_{\overline{n}}\theta^{\overline{\beta}})]\}\\ & -\overline{\theta}\wedge(h_{\alpha\overline{\beta}}A^{\overline{\beta}}_{\gamma}\theta^{\alpha}\wedge\theta^{\gamma}). \end{aligned}\tag{2.10}$$

Although the two-form $d\phi$ is still unkown, it follows that

$$h_{\alpha\overline{\beta}}A^{\overline{\beta}}_{\gamma}=h_{\gamma\overline{\beta}}A^{\overline{\beta}}_{\alpha}, \tag{2.11}$$

and

$$d\phi=h_{\alpha\overline{\beta}}A^{\alpha}_{\overline{\gamma}}\theta^{\overline{\gamma}}\wedge\theta^{\overline{\beta}}+h_{\alpha\overline{\beta},n}\theta^{\alpha}\wedge\theta^{\overline{\beta}}+\overline{\theta}\wedge[h_{\alpha\overline{\beta}}(A^{\overline{\beta}}_{n}\theta^{\alpha}+A^{\alpha}_{\overline{n}}\theta^{\overline{\beta}})]+\theta\wedge\Phi, \tag{2.12}$$

where Φ is some one-form on P.

Under a substitution (1.13), (1.14) into the (2.5), we get, with obvious notation

$$\tilde{D}h_{\alpha\overline{\beta}}+h_{\alpha\overline{\beta}}\tilde{\phi}=Dh_{\alpha\overline{\beta}}+h_{\alpha\overline{\beta}}\phi-B_{\alpha}{}^{\gamma}{}_{\rho}\theta^{\rho}h_{\gamma\overline{\beta}}-h_{\alpha\overline{\gamma}}B_{\overline{\rho}}{}^{\overline{\gamma}}{}_{\overline{\sigma}}\theta^{\overline{\sigma}}+h_{\alpha\overline{\beta}}b\theta. \tag{2.13}$$

In terms of the coefficients, this is equivalent to

$$\begin{aligned}\tilde{h}_{\alpha\overline{\beta},\rho} &= h_{\alpha\overline{\beta},\rho}-B_{\alpha}{}^{\gamma}{}_{\rho}h_{\gamma\overline{\beta}}, &(2.14)\\ \tilde{h}_{\alpha\overline{\beta},\overline{\sigma}} &= h_{\alpha\overline{\beta},\overline{\sigma}}-h_{\alpha\overline{\gamma}}B_{\overline{\beta}}{}^{\overline{\gamma}}{}_{\overline{\sigma}}, &(2.15)\\ \tilde{h}_{\alpha\overline{\beta},n} &= h_{\alpha\overline{\beta},n}+h_{\alpha\overline{\beta}}b. &(2.16)\end{aligned}$$

We multiply (2.16) on the right by the inverse matrix $h^{\overline{\beta}\alpha}$ of $h_{\alpha\overline{\beta}}$, and sum over α and β, to get

$$\tilde{h}_{\alpha\overline{\beta},n}h^{\overline{\beta}\alpha}=h_{\alpha\overline{\beta},n}h^{\overline{\beta}\alpha}+(n-1)b. \tag{2.17}$$

Because of the symmetry conditions (2.8), we can choose the functions $b, B_{\alpha}{}^{\gamma}{}_{\beta}$ in (1.13), (1.14) uniquely to achieve

$$h_{\alpha\overline{\beta},\overline{\sigma}}=0,\quad h_{\alpha\overline{\beta},n}h^{\overline{\beta}\alpha}=0. \tag{2.18}$$

Notice that (2.18) implies that the one-forms $Dh_{\alpha\overline{\beta}}$ are of type (1,0), when pulled down to the principal bundle $\hat{E}$ of E. This finishes the process of normalization of the forms $\phi, \omega_\beta^{\ \alpha}$. Together with θ^α, θ, they will give a global invariant basis of forms for P.

This completes the proof of theorem (0.1).

As mentioned in the introduction, the forms $\theta^\alpha, \theta, \phi$ on P give rise to an almost complex structure on the principal line bundle $\hat{E}$ of E. When is this a complex structure? In view of (1.10) and (1.11), we only have to check that $d\phi$ is in the exterior ideal generated by $\theta^\alpha, \theta, \phi$. Formula (2.12) gives the following.

Corollary 2.1. *This almost complex structure on $\hat{E}$ is integrable, if and only if the coefficients of the torsion forms τ^α satisfy*

$$h_{\alpha\overline{\beta}} A^{\alpha}_{\overline{\gamma}} = h_{\alpha\overline{\gamma}} A^{\alpha}_{\overline{\beta}}, \quad A^{\alpha}_{\overline{n}} = 0. \tag{2.19}$$

3. Curvature

The two-form $\theta \wedge \Phi$ in equation (2.12), which was derived from the exterior derivative of (1.10), is a component of curvature. Since $d\phi = d\phi_0$ locally by (2.1), Φ has the form

$$\Phi = B_\alpha \theta^\alpha + B_{\overline{\alpha}} \theta^{\overline{\alpha}} + B_{\overline{n}} \overline{\theta}. \tag{3.1}$$

To derive the remaining curvature forms, we take the exterior derivative of equation (1.11). This yields the “first Bianchi identity”,

$$\theta^\beta \wedge \Omega_\beta^{\ \alpha} + \theta \wedge D\tau^\alpha = \chi \wedge \tau^\alpha, \tag{3.2}$$

where we have introduced the curvature and covariant differential forms,

$$\Omega_\beta^{\ \alpha} = d\omega_\beta^{\ \alpha} - \omega_\beta^{\ \gamma} \wedge \omega_\gamma^{\ \alpha}, \tag{3.3}$$

$$D\tau^\alpha = d\tau^\alpha - \tau^\beta \wedge (\omega_\beta^{\ \alpha} - \delta_\beta^{\ \alpha} \phi). \tag{3.4}$$

The exterior derivative of (2.2) gives in usual fashion, and obvious notation,

$$U_\beta^{\ \gamma} \Omega_\gamma^{\ \alpha} = \Omega_{0\beta}^{\ \ \gamma} U_\gamma^{\ \alpha}. \tag{3.5}$$

Since the $\Omega_{0\beta}{}^{\gamma}$ are two-forms from the base, it follows that the $\Omega_{\gamma}{}^{\alpha}$ are two-forms in $\theta^{\alpha}, \theta, \theta^{\overline{\alpha}}, \overline{\theta}$. Modulo $\theta, \overline{\theta}$, we have

$$\Omega_{\beta}{}^{\alpha} \equiv P_{\beta}{}^{\alpha}{}_{\rho\sigma}\theta^{\rho} \wedge \theta^{\sigma} + Q_{\beta}{}^{\alpha}{}_{\overline{\rho}\overline{\sigma}}\theta^{\overline{\rho}} \wedge \theta^{\overline{\sigma}} + R_{\beta}{}^{\alpha}{}_{\rho\overline{\sigma}}\theta^{\rho} \wedge \theta^{\overline{\sigma}}, \tag{3.6}$$

where the tensors P and Q are anti-symmetric in ρ, σ. Substitution of (3.6) into (3.2) gives the following Bianchi identies for the coefficients,

$$\begin{aligned} 0 &= P_{\beta}{}^{\alpha}{}_{\rho\sigma} + P_{\rho}{}^{\alpha}{}_{\sigma\beta} + P_{\sigma}{}^{\alpha}{}_{\beta\rho}, & (3.7)\\ 2Q_{\beta}{}^{\alpha}{}_{\overline{\rho}\overline{\sigma}} &= h_{\beta\overline{\rho}}A_{\overline{\sigma}}^{\alpha} - h_{\beta\overline{\sigma}}A_{\overline{\rho}}^{\alpha}, & (3.8)\\ R_{\beta}{}^{\alpha}{}_{\rho\overline{\sigma}} &= R_{\rho}{}^{\alpha}{}_{\beta\overline{\sigma}}. & (3.9) \end{aligned}$$

Next we consider equation (2.7), which we write more explicitly as

$$dh_{\alpha\overline{\beta}} = \omega_{\alpha}{}^{\gamma}h_{\gamma\overline{\beta}} + h_{\alpha\overline{\gamma}}\omega_{\overline{\alpha}}^{\overline{\gamma}} - h_{\alpha\overline{\beta}}\phi + h_{\alpha\overline{\beta},\gamma}\theta^{\gamma} + h_{\alpha\overline{\beta},n}\theta. \tag{3.10}$$

We take the exterior derivative of this, and use the equation itself to remove all $dh_{\alpha\overline{\beta}}$ terms. This straight-forward, though lengthy, process yields the following, where again we compute modulo $\theta, \overline{\theta}$.

$$\begin{aligned} 0 \equiv\ & \Omega_{\alpha}{}^{\gamma}h_{\gamma\overline{\beta}} + h_{\alpha\overline{\gamma}}\Omega_{\overline{\beta}}^{\overline{\gamma}} & (3.11)\\ & +(Dh_{\alpha\overline{\beta},\sigma} + h_{\alpha\overline{\beta},\sigma}\phi) \wedge \theta^{\sigma} - h_{\alpha\overline{\beta}}h_{\rho\overline{\sigma}}A_{\overline{\gamma}}^{\rho}\theta^{\overline{\gamma}} \wedge \theta^{\overline{\sigma}} & (3.12)\\ & +(h_{\alpha\overline{\beta},n}h_{\rho\overline{\sigma}} - h_{\alpha\overline{\beta}}h_{\rho\overline{\sigma},n})\theta^{\rho} \wedge \theta^{\overline{\sigma}}. & (3.13) \end{aligned}$$

Here we have introduced the covariant differential

$$Dh_{\alpha\overline{\beta},\sigma} = dh_{\alpha\overline{\beta},\sigma} - \omega_{\alpha}{}^{\gamma}h_{\gamma\overline{\beta},\sigma} - \omega_{\overline{\beta}}^{\overline{\gamma}}h_{\alpha\overline{\gamma},\sigma} - \omega_{\sigma}{}^{\gamma}h_{\alpha\overline{\beta},\gamma}. \tag{3.14}$$

As before (2.7) we have

$$Dh_{\alpha\overline{\beta},\sigma} + h_{\alpha\overline{\beta},\sigma}\phi = h_{\alpha\overline{\beta},\sigma\gamma}\theta^{\gamma} + h_{\alpha\overline{\beta},\sigma n}\theta + h_{\alpha\overline{\beta},\sigma\overline{\gamma}}\theta^{\overline{\gamma}} + h_{\alpha\overline{\beta},\sigma}\overline{\theta}. \tag{3.15}$$

Comparison of coefficients, mod $\theta, \overline{\theta}$, gives

$$\begin{aligned} 0 &= P_{\alpha}{}^{\gamma}{}_{\rho\sigma}h_{\gamma\overline{\beta}} + h_{\alpha\overline{\gamma}}Q_{\overline{\beta}}^{\overline{\gamma}}{}_{\rho\sigma} + (1/2)(h_{\alpha\overline{\beta},\sigma\rho} - h_{\alpha\overline{\beta},\rho\sigma}), & (3.16)\\ 0 &= Q_{\alpha}{}^{\gamma}{}_{\overline{\rho}\overline{\sigma}}h_{\gamma\overline{\beta}} + h_{\alpha\overline{\gamma}}P_{\overline{\beta}}^{\overline{\gamma}}{}_{\overline{\rho}\overline{\sigma}} - (1/2)h_{\alpha\overline{\beta}}(h_{\gamma\overline{\sigma}}A_{\overline{\rho}}^{\gamma} - h_{\gamma\overline{\rho}}A_{\overline{\sigma}}^{\gamma}), & (3.17)\\ 0 &= R_{\alpha}{}^{\gamma}{}_{\rho\overline{\sigma}}h_{\gamma\overline{\beta}} - h_{\alpha\overline{\gamma}}R_{\overline{\beta}}^{\overline{\gamma}}{}_{\overline{\sigma}\rho} - h_{\alpha\overline{\beta},\rho\overline{\sigma}} + h_{\alpha\overline{\beta},n}h_{\rho\overline{\sigma}} - h_{\alpha\overline{\beta}}h_{\rho\overline{\sigma},n}. & (3.18) \end{aligned}$$

Notice that by (3.8) and (3.16) or (3.17) the tensors Q and P are already determined by the torsion τ^{α}. Thus, only the tensor R carries additional information.

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