

SIGNATURE PAIRS OF POSITIVE POLYNOMIALS

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Abstract

A well-known theorem of Quillen says that if $r(z, \bar{z})$ is a bihomogeneous polynomial on \mathbb{C}^n positive on the sphere, then there exists d such that $r(z, \bar{z})\|z\|^{2d}$ is a squared norm. We obtain effective bounds relating this d to the signature of r . We obtain the sharp bound for $d = 1$, and for $d > 1$ we obtain a bound that is of the correct order as a function of d for fixed n . The current work adds to an extensive literature on positivity classes for real polynomials. The classes Ψ_d of polynomials for which $r(z, \bar{z})\|z\|^{2d}$ is a squared norm interpolate between polynomials positive on the sphere and those that are Hermitian sums of squares.

1. Introduction

Let $r(z, \bar{z})$ be a real polynomial on \mathbb{C}^n . A basic question one can ask is whether $r(z, \bar{z}) \geq 0$. One way to show that a polynomial is nonnegative is to write it as a sum of Hermitian squares

$$\sum_{j=1}^N |f_j(z)|^2 \tag{1.1}$$

for holomorphic polynomials f_j , i.e., as the squared norm $\|f(z)\|^2$ of a holomorphic mapping f . There exist, however, nonnegative polynomials that

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cannot be written as a squared norm; to construct an easy example, take r nonnegative but with zero set a real hypersurface. For a much more subtle example, consider Example VI.3.6 in [4]:

$$r(z, \bar{z}) = (|z_1 z_2|^2 - |z_3|^4)^2 + |z_1|^8. \quad (1.2)$$

This polynomial is non-negative, its zero set is a complex line, and yet it cannot even be written as a quotient of squared norms.

Thus the condition that a real polynomial is a squared norm is too restrictive, and one is motivated to formulate other less restrictive positivity conditions. See [4, 11, 1, 2, 5] and the references within. A theorem of Quillen [11], proved independently by Catlin and D'Angelo [1], states that if a bihomogeneous polynomial

$$r(z, \bar{z}) = \sum_{|\alpha|=|\beta|=m} c_{\alpha\beta} z^\alpha \bar{z}^\beta \quad (1.3)$$

is positive on the unit sphere, then there exists an integer d such that $r(z, \bar{z})(\|z\|^2)^d = r(z, \bar{z})\|z\|^{2d}$ is a squared norm, and hence r is a quotient of squared norms. Thus one obtains a Hermitian analogue of Hilbert's 17th problem.

With this motivation, we define a set of positivity classes Ψ_d of bihomogeneous polynomials by

$$\Psi_d = \{r : r(z, \bar{z})\|z\|^{2d} \text{ is a Hermitian sum of squares}\}, \quad (1.4)$$

$$\Psi_\infty = \bigcup_{d=0}^{\infty} \Psi_d. \quad (1.5)$$

Ψ_0 consists of the squared norms themselves, and, by the theorem mentioned above, Ψ_∞ contains the polynomials positive on the sphere. It is not difficult (Proposition 6.1) to construct polynomials that show

$$\Psi_0 \subsetneq \Psi_1 \subsetneq \Psi_2 \subsetneq \Psi_3 \subsetneq \dots \quad (1.6)$$

Every real polynomial r has a holomorphic decomposition

$$r(z, \bar{z}) = \sum_{j=1}^{N_+} |f_j(z)|^2 - \sum_{j=1}^{N_-} |g_j(z)|^2 \quad (1.7)$$

for holomorphic polynomials f_j, g_j . When N_+ and N_- are minimal (which occurs when $f_1, \dots, f_{N_+}, g_1, \dots, g_{N_-}$ are linearly independent), we say that r has signature pair (N_+, N_-) and rank $N_+ + N_-$. While f and g are not unique, the signature pair (N_+, N_-) is.

We will be particularly concerned with Ψ_1 . This class is connected to the study of proper holomorphic mappings between balls in complex Euclidean spaces of different dimensions. For example, if $f: \mathbb{C}^n \rightarrow \mathbb{C}^N$ is a polynomial that takes the unit ball to the unit ball properly, then $\|f(z)\|^2 - 1 = p(z, \bar{z})(\|z\|^2 - 1)$. In particular, if f is of degree d and f_d is the degree d part of f , then $\|f_d(z)\|^2 = p_{d-1}(z, \bar{z})\|z\|^2$. Polynomials in Ψ_1 also arise when studying the second fundamental form of more general mappings between balls. See the recent work by Ebenfelt [6] and the references within. For example, by proving that $p(z, \bar{z})\|z\|^2$ must be of rank at least n , Huang [8] proved that all proper mappings between balls that are sufficiently smooth on the boundary are equivalent to the linear embeddings if $N < 2n - 1$.

Our main result for the positivity class Ψ_1 is the following.

Theorem 1.1. *Let $r(z, \bar{z})$ be a real polynomial on \mathbb{C}^n , $n \geq 2$, and suppose that $r(z, \bar{z})\|z\|^2$ is a squared norm. Let (N_+, N_-) be the signature pair of r . Then*

(i)

$$\frac{N_-}{N_+} < n - 1. \quad (1.8)$$

(ii) *The above inequality is sharp, i.e., for every $\varepsilon > 0$ there exists r with $\frac{N_-}{N_+} \geq n - 1 - \varepsilon$.*

Remark 1.1. The case for $n = 1$ is trivial; $\|z\|^2 = |z|^2$ and so $r(z, \bar{z})|z|^2$ has the same signature as r . Therefore, if $r(z, \bar{z})|z|^2$ is a squared norm, then r is a squared norm and $N_- = 0$.

When $d > 1$, the combinatorics becomes more involved. We obtain the following bound.

Theorem 1.2. *Let $r(z, \bar{z})$ be a real polynomial on \mathbb{C}^n , $n \geq 2$, $d \geq 1$, and suppose that $r(z, \bar{z}) \|z\|^{2d}$ is a squared norm. Let (N_+, N_-) be the signature pair of r . Then*

(i)

$$\frac{N_-}{N_+} \leq \binom{n-1+d}{d} - 1. \quad (1.9)$$

(ii) *For each fixed n , there exists a positive constant C_n such that for each d there is a polynomial $r \in \Psi_d$ with $\frac{N_-}{N_+} \geq C_n d^{n-1}$.*

Since $\binom{n-1+d}{d}$ is a polynomial in d of degree $n-1$, the second item says that the bound we obtain is of the correct order, although we do not believe it to be sharp for all n (it is sharp when $n = 2$).

For bihomogeneous polynomials we obtain bounds for the ratios of positive and negative eigenvalues for the classes Ψ_d . A very interesting problem is to find the smallest d so that a positive polynomial is in Ψ_d ; see the work of To and Yeung [12]. An upper bound must involve the magnitude of the coefficients. To see this, consider an example from [5]:

$$(|z|^2 + |w|^2)^4 - \lambda |zw|^4. \quad (1.10)$$

As $\lambda \rightarrow 16$, one requires larger and larger d . On the other hand, our results give an effective lower bound on d given the numbers N_- and N_+ .

We also address the analogous question for real polynomials, i.e., what can we say about a polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ if it is known that $(x_1 + \dots + x_n)^d p(x)$ has non-negative coefficients? Pólya proved in [9] that for each p positive on the positive quadrant, there exists a d such that $(x_1 + \dots + x_n)^d p(x)$ has only positive coefficients. Recent work (for example [10]) focuses on finding an upper bound on d given information about p . Our work can be thought of as finding lower bounds on d given the signature of p in a somewhat more general setting.

When we complexify a real polynomial we obtain a Hermitian symmetric polynomial, i.e., one satisfying $r(z, \bar{w}) = \overline{r(w, \bar{z})}$. Hermitian symmetric

polynomials arise naturally in complex geometry, in particular, degree d Hermitian symmetric polynomials arise as globalizable metrics on the d th power of the universal bundle over the complex projective space; see [5].

Questions about multiples of $\|z\|^{2d}$ also arise in several contexts. As mentioned above, Huang [8] proved that $p(z, \bar{z})\|z\|^2$ must have rank at least n . Generalizing this result, in [3] it was shown that the rank of $p(z, \bar{z})\|z\|^{2d}$ is bigger than or equal to the rank of $\|z\|^{2d}$. A theorem of Pfister says the if $p \geq 0$ for a polynomial p of n real variables, there exists a polynomial q such that q^2p is a sum of at most 2^n squared polynomials. Thus [3] shows that Pfister's theorem fails in the Hermitian context.

Finally, ratios of the sort considered have been studied recently by Grundmeier [7] in the context of group invariant hyperquadric CR mappings. In particular, Grundmeier studied the canonically defined group-invariant mappings from the ball to the hyperquadric. This problem can be seen as studying the proportions of positive and negative eigenvalues of group-invariant polynomials of the form $p(z, \bar{z})(\|z\|^2 - 1)$.

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2. Preliminaries

Let $r(z, \bar{z})$ be a real-valued polynomial on \mathbb{C}^n . We use a linear algebra setting. Suppose $\deg r \leq 2D$. Let $\mathcal{Z} = (1, z_1, \dots, z_n, z_1^2, z_1z_2, \dots, z_n^D)^t$ be the vector of all monomials up to degree D , with \mathcal{Z}^* its conjugate transpose. Then $r(z, \bar{z}) = \mathcal{Z}^*C\mathcal{Z}$ where C is a constant Hermitian matrix. The rank of r is the rank of C , and the signature of r is (N_+, N_-) if and only if C has N_+ positive and N_- negative eigenvalues. Therefore, when we apply linear algebra terminology to r we are referring to properties of the matrix C .

When r is diagonal, the f_j and g_j appearing in the holomorphic decomposition (1.7) are monomials. In this case, questions about $r(z, \bar{z})$ and $r(z, \bar{z})\|z\|^{2d}$ for $z \in \mathbb{C}^n$ can be reformulated as questions about polynomials

on $\{x \in \mathbb{R}^n : x_k \geq 0\}$. Indeed, if in (1.7), each f_j, g_j is a monomial $c_\alpha z^\alpha$ for some multi-index α , then

$$|c_\alpha z^\alpha|^2 = |c_\alpha|^2 \prod_{k=1}^n (|z_k|^2)^{\alpha_k}. \quad (2.1)$$

If $m_j: \mathbb{R}^n \rightarrow \mathbb{R}$ is given by $m_j(x) := |c_\alpha|^2 x^\alpha$, then $|f_j(z)|^2 = m_j(|z_1|^2, \dots, |z_n|^2)$. Thus we can study r by studying an associated real polynomial p on $\{x \in \mathbb{R}^n : x_k \geq 0\}$ with N_+ positive and N_- negative coefficients. Observe that $\|z\|^2$ is itself a diagonal polynomial and is associated with $\ell(x) := \sum_{k=1}^n x_k$.

One can therefore formulate the associated problem for real polynomials. We consider real polynomials $p(x)$ for which $p(x)\ell(x)^d$ has only nonnegative coefficients. Such polynomials are nonnegative on $\{x \in \mathbb{R}^n : x_k \geq 0\}$. Since it is not hard to see how to go from a real polynomial $p(x)$ on \mathbb{R}^n to its Hermitian analogue $r(z, \bar{z})$ on \mathbb{C}^n , if we construct a $p(x)$ with signature (N_+, N_-) , we automatically also construct an $r(z, \bar{z})$ with the same signature (N_+, N_-) .

3. Diagonal Case for $d = 1$

In this section we focus on the diagonal case. The combinatorics in this special case gives insight into the general case, and furthermore, we obtain somewhat stronger results.

The polynomials we construct to establish the sense in which our bounds are sharp are all diagonal. Thus the second part of Theorem 1.1 follows immediately from the last part of the next theorem.

Theorem 3.1. *Suppose p is a polynomial on \mathbb{R}^n , $n \geq 2$, and set $\ell(x) := \sum_{j=1}^n x_j$. Suppose $S(x) := p(x)\ell(x)$ has only nonnegative coefficients. Let N_+ denote the number of monomials in p with positive coefficients, and let N_- denote the number of monomials in p with negative coefficients.*

- (i) *If $N_- > 0$, then $N_+ \geq n$.*
- (ii) *$\frac{N_-}{N_+} < n - 1$.*
- (iii) *For every $\varepsilon > 0$, there exists p with $\frac{N_-}{N_+} > n - 1 - \varepsilon$.*

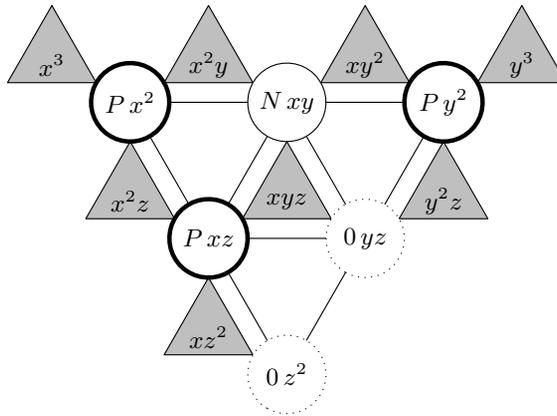


Figure 1: A diagram for a second-degree $p(x, y, z)$ with one negative term.

Before proving the theorem, we describe a useful visualization for our constructions. Consider homogeneous polynomials in $n = 3$ variables. To avoid subscripts x, y, z . Thus we consider polynomials $p(x, y, z)$ such that $S(x, y, z) = p(x, y, z)(x + y + z)$ has only nonnegative coefficients. In Figure 1, we show a diagram for the polynomial $p(x, y, z) = x^2 + y^2 + xz - xy$. We arrange the monomials in a lattice and mark positive coefficients by a P in a thick circle and negative coefficients by an N in a thin circle. In this first diagram, we indicate which monomial each circle represents, though we refrain from doing so for larger diagrams. Zero coefficients are marked with dotted circle and do not really come into play. We also mark by gray triangles the monomials appearing in the product $S(x, y, z) = p(x, y, z)(x + y + z)$. The vertices of each triangle point to monomials of p that contribute to that term of S . We ignore the magnitude of the coefficients; we are only interested in their signs. If a term in the product S receives contributions from both positive and negative terms in p , we can increase the positive coefficients so that the sum of the positive contributions is bigger than the sum of the negative contributions, thus ensuring that S has only positive coefficients. For S to have only positive coefficients, each nonzero term in S must get at least one positive contribution, and hence each triangle must have one vertex pointing to a P in the diagram of p . We do not show triangles that receive no contribution from a term in p .

Figure 2 shows the diagram for a polynomial p with 6 negative coefficients. The key point is that each gray triangle has at least one vertex

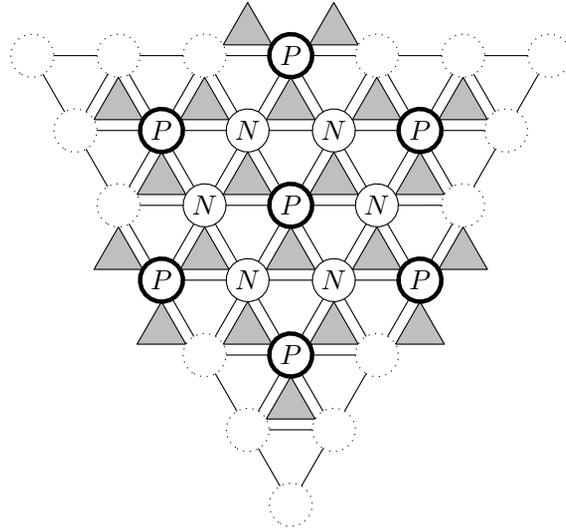


Figure 2: Diagram for an example with 6 negative terms.

pointing to a P in the diagram. It is not hard to argue that, if we have 6 negative terms, we must have at least 7 positive terms. Thus this figure is in some sense optimal. An explicit polynomial having the diagram of Figure 2 is

$$p(x, y, z) = 2xyz^4 + 2x^3z^3 + 2y^3z^3 + 2x^2y^2z^2 + 2x^4yz + 2xy^4z + 2x^3y^3 - x^2yz^3 - xy^2z^3 - x^3yz^2 - xy^3z^2 - x^3y^2z - x^2y^3z. \quad (3.1)$$

With 7 positive and only 6 negative coefficients, N_-/N_+ is far from the predicted bound of 2. Furthermore, this polynomial is already of degree 6. To obtain polynomials with ratio close to the bound, we must take the degree to be very large, and it is impractical to give diagrams for specific examples. However, the pattern in Figure 2 can be extended to obtain our “sharp” examples; the idea is to make the interior of the diagram as in Figure 3. (We omit the triangles.)

In order to make the diagram correspond to a polynomial, we must make this pattern part of a finite diagram. We will show that if we take all terms on the boundary to be positive, no negative terms will appear in S . Because of these boundary terms, we will have slightly more than one positive term for every two negative terms.

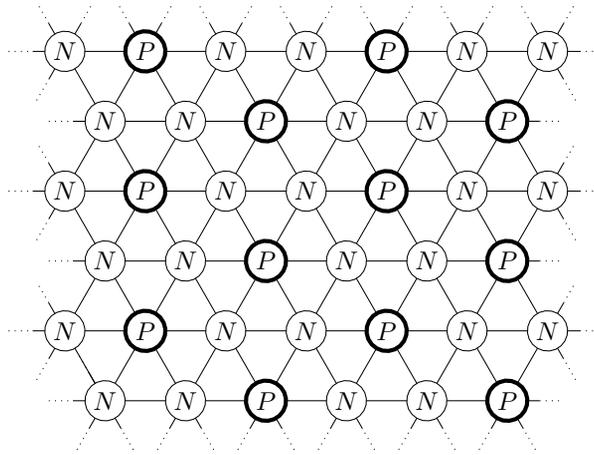


Figure 3: Interior of an optimal diagram for $n = 3$.

Proof of Theorem 3.1. For any polynomial p in n variables, write $p = \sum_{j=0}^D p_j$ where each p_j is homogeneous of degree j . Since we obtain S by multiplying p by a homogeneous polynomial of degree one, if $S = \sum_{j=0}^D S_j$ with S_j homogeneous of degree $j + 1$, S_j is simply $p_j \ell$. One shows easily that, for each statement above, if it holds for each p_j , it holds for p . Thus for the remainder of the proof, we assume all polynomials are homogeneous and that like terms have been collected, so that a polynomial is a sum of distinct monomials.

Proof of (i). Suppose that in p , the coefficient of x^β is negative. This coefficient contributes to the coefficients of n distinct terms in S associated with multi-indices $\beta + \mathbf{e}^k$, $1 \leq k \leq n$, where \mathbf{e}^k is the vector with 1 in the k th position and zero elsewhere. For each k , there must be a multi-index $\alpha(k)$ associated with a positive coefficient in p for which $\alpha(k) + \mathbf{e}^j = \beta + \mathbf{e}^k$ for some j . We claim that if $k_1 \neq k_2$, $\alpha(k_1)$ cannot equal $\alpha(k_2)$. Suppose, on the contrary, that there is a single multi-index α different from β and integers $1 \leq j_1, j_2, k_1, k_2 \leq n$ with $j_1 \neq j_2$ and $k_1 \neq k_2$ such that $\alpha + \mathbf{e}^{j_1} = \beta + \mathbf{e}^{k_1}$ and $\alpha + \mathbf{e}^{j_2} = \beta + \mathbf{e}^{k_2}$. Then $\mathbf{e}^{k_1} - \mathbf{e}^{j_1} - \mathbf{e}^{k_2} + \mathbf{e}^{j_2} = 0$. $\mathbf{e}^{j_1} \neq \mathbf{e}^{k_1}$ since $\alpha \neq \beta$, so it must be that $\mathbf{e}^{k_1} = \mathbf{e}^{k_2}$ and $\mathbf{e}^{j_1} = \mathbf{e}^{j_2}$. This is a contradiction. We conclude that there are indeed at least n distinct multi-indices α for which the coefficient of x^α in p is positive.

Proof of (ii). Let \mathcal{N} be the set of multi-indices α for which the coefficient of x^α in p is negative, and let \mathcal{P} be the set of all multi-indices for which the coefficient of x^α is positive. Since $|\mathcal{N}| = N_-$ and $|\mathcal{P}| = N_+$, the result will follow if whenever \mathcal{N} is nonempty, there exists a function $f: \mathcal{N} \rightarrow \mathcal{P}$ for which $f^{-1}(\{\beta\})$ has at most $n - 1$ elements for each $\beta \in \mathcal{P}$.

Consider $\alpha \in \mathcal{N}$. The negative coefficient c_α in p contributes to n terms in S , among them the one associated with $\alpha' := \alpha + \mathbf{e}^n$. The other multi-indices from p that contribute to this term are $\alpha' - \mathbf{e}^j$ for $1 \leq j \leq n - 1$. In order for the coefficient of $x^{\alpha'}$ in S to be non-negative, there must exist j for which $\alpha' - \mathbf{e}^j \in \mathcal{P}$. We choose j_0 to be minimal with this property and set $f(\alpha) := \alpha' - \mathbf{e}^{j_0} = \alpha + \mathbf{e}^n - \mathbf{e}^{j_0}$.

Fix $\beta \in \mathcal{P}$ and consider $f^{-1}(\{\beta\})$. If α is such a pre-image, $\beta = f(\alpha) = \alpha + \mathbf{e}^n - \mathbf{e}^j$ for some j between 1 and $n - 1$. Thus α must be of the form $\beta - \mathbf{e}^n + \mathbf{e}^j$ for $1 \leq j \leq n - 1$, i.e., $|f^{-1}(\{\beta\})| \leq n - 1$.

Since $p(x)\ell(x)$ has nonnegative coefficients, if we look at the least monomial x^{β_0} in p (according to our monomial order) with nonzero coefficient, we find that it must be positive because it is the only coefficient of p contributing to the coefficient of $x_1 x^{\beta_0}$ in S . Furthermore, $f^{-1}(\{\beta_0\})$ is empty; such a pre-image would be of the form $\beta_0 + \mathbf{e}^j - \mathbf{e}^n$. Since $x^{\beta_0 + \mathbf{e}^j - \mathbf{e}^n} < x^{\beta_0}$ in the monomial order, $x^{\beta_0 + \mathbf{e}^j - \mathbf{e}^n}$ does not appear in p with non-zero coefficient. This proves that the inequality is in fact strict.

Proof of (iii). We construct a family $\{p_D : D \in \mathbb{N}\}$ of polynomials with p_D homogeneous of degree D such that $N_-(p_D)/N_+(p_D) \rightarrow n - 1$ as $D \rightarrow \infty$.

For each multi-index α , set

$$\gamma(\alpha) := \begin{cases} n - 1 & \text{if } \alpha_k = 0 \text{ for some } k \\ n - 1 & \text{if } \alpha_k \geq 1 \text{ for all } k \text{ and } \sum_{k=1}^{n-1} k\alpha_k - D \equiv 0 \pmod{n} \\ -1 & \text{otherwise.} \end{cases} \quad (3.2)$$

We then define

$$p_D(x) := \sum_{|\alpha|=D} \gamma(\alpha)x^\alpha. \quad (3.3)$$

We claim that $S_D = p_D \ell$ has only non-negative coefficients. Consider the term in S_D corresponding to the n -tuple $A = (A_1, A_2, \dots, A_n)$. The coefficient of this term is

$$c(A) := \sum_{k=1}^n \gamma(A - \mathbf{e}^k), \tag{3.4}$$

where we take $\gamma(A - \mathbf{e}^k)$ to be 0 if $A_k = 0$. Since all negative coefficients are equal to -1 and all positive coefficients are equal to $n - 1$, to show that $c(A) \geq 0$, it suffices to show that if there exists k_1 such that $\gamma(A - \mathbf{e}^{k_1}) = -1$, there exists $k_2 \neq k_1$ such that $\gamma(A - \mathbf{e}^{k_2}) = n - 1$.

If $\gamma(A - \mathbf{e}^{k_1}) = -1$, then by our definition of $\gamma(\alpha)$, $A_{k_1} - 1 \geq 1$ and, for all $k \neq k_1$, $A_k \geq 1$. Thus all n of the numbers $\gamma(A - \mathbf{e}^k)$ are non-zero. We consider two cases.

In the first case, suppose there exists $k \neq k_1$ such that $A_k - 1 = 0$. Then $\gamma(A - \mathbf{e}^k) = n - 1$ and $c(A)$ is indeed non-negative.

In the second case, for all k , $A_k - 1 > 0$. For each k , we consider

$$\sum_{j=1}^{n-1} j(A - \mathbf{e}^k)_j - D = \sum_{j=1}^{n-1} jA_j - D - k. \tag{3.5}$$

Since k ranges over $\{1, 2, \dots, n\}$, the n numbers in (3.5) are consecutive and thus range over all congruence classes modulo n . Therefore there exists a k_2 for which $\sum jA_j - D - k_2 \equiv 0 \pmod n$, so that $\gamma(A - \mathbf{e}^{k_2}) = n - 1$. Thus in this case as well, $c(A)$ is non-negative.

Now consider $N_-(p_D)/N_+(p_D)$. By (ii), this is bounded above by $n - 1$. Thus (iii) will follow if we show that this ratio is bounded below by a function of n and D that tends to $n - 1$ as D tends to infinity.

Since we will let $D \rightarrow \infty$, but n is fixed, we may assume without loss of generality that $D > 3n$. As above, write $S_D(x) = \sum_{|A|=D+1} c(A)x^A$. Define a subset of multi-indices A of length $D + 1$ by

$$I(S_D) := \{ A : |A| = D + 1 \text{ and } A_k \geq 2 \text{ for all } k \}. \tag{3.6}$$

These ‘‘interior’’ multi-indices are those for which no zero appears in a multi-index associated with a term in p_D contributing to $c(A)$. Thus exactly n non-zero coefficients from p_D contribute to $c(A)$, with precisely $n - 1$ of

them negative. Since each negative coefficient in p_D contributes to at most n terms in S_D ,

$$nN_-(p_D) \geq (n-1)|I(S_D)|. \quad (3.7)$$

To determine the size of $I(S_D)$, consider the function on $I(S_D)$:

$$g(A) := (A_1 - 2, A_2 - 2, \dots, A_n - 2). \quad (3.8)$$

One checks that g is a bijection between $I(S_D)$ and $\{B = (B_1, \dots, B_n) : B_k \geq 0 \text{ and } |B| = D - 2n + 1\}$. Since $|\{B : B_k \geq 0 \text{ and } |B| = D + 1 - 2n\}|$ equals the number of monomials of degree $D + 1 - 2n$ in n variables,

$$|I(S_D)| = |\{B : B_k \geq 0 \text{ and } |B| = D + 1 - 2n\}| = \binom{D-n}{n-1}. \quad (3.9)$$

Combining with (3.7) gives

$$N_-(p_D) \geq \frac{n-1}{n} \binom{D-n}{n-1}. \quad (3.10)$$

Since p_D has a non-zero coefficient for every monomial of degree D in n variables,

$$\begin{aligned} \frac{N_-(p_D)}{N_+(p_D)} &= \frac{N_-(p_D)}{\binom{D+n-1}{n-1} - N_-(p_D)} \\ &\geq \frac{\frac{n-1}{n} \binom{D-n}{n-1}}{\binom{D+n-1}{n-1} - \frac{n-1}{n} \binom{D-n}{n-1}}. \end{aligned} \quad (3.11)$$

Since n is fixed and we will take a limit as $D \rightarrow \infty$, we need only determine the leading-order term in the numerator and the denominator of the last expression. The numerator is a polynomial in D of degree $n-1$ with leading coefficient $\frac{n-1}{n} \cdot \frac{1}{(n-1)!}$, whereas the denominator is a polynomial in D of degree $n-1$ with leading coefficient $\frac{1}{(n-1)!} - \frac{n-1}{n} \cdot \frac{1}{(n-1)!}$. Thus

$$\lim_{D \rightarrow \infty} \frac{\frac{n-1}{n} \binom{D-n}{n-1}}{\binom{D+n-1}{n-1} - \frac{n-1}{n} \binom{D-n}{n-1}} = \frac{\frac{n-1}{n} \cdot \frac{1}{(n-1)!}}{\frac{1}{(n-1)!} - \frac{n-1}{n} \cdot \frac{1}{(n-1)!}} = n-1. \quad (3.12)$$

This completes the proof of (iii) and of the theorem. \square

4. The general case for $d = 1$

If it were possible to replace an arbitrary r for which $r(z, \bar{z}) \|z\|^2$ is a squared norm with a diagonal \tilde{r} of the same signature for which $\tilde{r}(z, \bar{z}) \|z\|^2$ is a squared norm, the results of the previous section would imply the general results. Although it appears that such a reduction to the diagonal case is not possible, we show that it is possible to replace an r as above with an \tilde{r} with

$$\tilde{r}(z, \bar{z}) = \left\langle \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \mathcal{Z}, \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \mathcal{Z} \right\rangle \tag{4.1}$$

with the same signature as r , but with $\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$ in a partial row-echelon form.

We first establish an elementary proposition.

Proposition 4.1. *If*

$$(\|f(z)\|^2 - \|g(z)\|^2) \|z\|^2 \tag{4.2}$$

is a squared norm, then for every $\lambda \in [0, 1]$

$$(\|f(z)\|^2 - \lambda \|g(z)\|^2) \|z\|^2 \tag{4.3}$$

is also a squared norm.

Proof. For any $\lambda \in [0, 1]$,

$$\begin{aligned} & (\|f(z)\|^2 - \lambda \|g(z)\|^2) \|z\|^2 \\ &= (\|f(z)\|^2 - \|g(z)\|^2) \|z\|^2 + (1 - \lambda) \|g(z)\|^2 \|z\|^2 \\ &= (\|f(z)\|^2 - \|g(z)\|^2) \|z\|^2 + \|\sqrt{1 - \lambda} g \otimes z\|^2. \end{aligned} \tag{4.4}$$

Since a sum of squared norms is itself a squared norm, the claim holds. \square

The next lemma is of critical importance.

Lemma 4.1. *Suppose $r(z, \bar{z}) = \left\langle \begin{bmatrix} I_{N_+} & 0 \\ 0 & -I_{N_-} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z}, \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z} \right\rangle$ has signature pair (N_+, N_-) (so that A and B have rank N_+ and N_- , resp.), and suppose that $r(z, \bar{z}) \|z\|^2$ is a squared norm. Then there exists $\tilde{r}(z, \bar{z}) = \left\langle \begin{bmatrix} I_{N_+} & 0 \\ 0 & -I_{N_-} \end{bmatrix} \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \mathcal{Z}, \begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix} \mathcal{Z} \right\rangle$ with the same signature pair as r such that $\tilde{r}(z, \bar{z}) \|z\|^2$ is also a squared norm and the matrix $\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$ is in row-echelon*

form up to permutation of rows. We will say that such a matrix is in partial row-echelon form.

Proof. For clarity, we suppress the subscripts on our identity matrices and write $r(z, \bar{z}) = \langle [I \ 0; 0 \ -I] \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z}, \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z} \rangle$. Because unitary matrices of the form $\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}$ (with U_1 and U_2 unitary and of dimension $N_+ \times N_+$ and $N_- \times N_-$, resp.) commute with $[I \ 0; 0 \ -I]$, we may write

$$\begin{aligned} r(z, \bar{z}) &= \left\langle \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix}^* \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z}, \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z} \right\rangle \\ &= \left\langle \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z}, \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \mathcal{Z} \right\rangle \\ &= \left\langle \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} U_1 A \\ U_2 B \end{bmatrix} \mathcal{Z}, \begin{bmatrix} U_1 A \\ U_2 B \end{bmatrix} \mathcal{Z} \right\rangle. \end{aligned} \quad (4.5)$$

By choosing the U_i appropriately, we put A and B individually into row-echelon form. We do *not* achieve a *reduced* row-echelon form. We may not be able to eliminate non-zero entries above the pivots, and our pivots need not be 1s. Note that the matrix $C := \begin{bmatrix} A \\ B \end{bmatrix}$ need not be in row-echelon form, even after permuting the rows.

What kinds of transformations can we apply to C to reduce it further? Let T be an $(N_+ + N_-) \times (N_+ + N_-)$ matrix. Then $\langle I' T C \mathcal{Z}, T C \mathcal{Z} \rangle = \langle I' C \mathcal{Z}, C \mathcal{Z} \rangle$ if and only if $T^* I' T = I'$. Consider the leftmost column of C . If it does not have a pivot of either A or B , we set it aside. If it has a pivot of A or a pivot of B , but not both, we again put the column aside. The row containing the pivot may now also be set aside. If we never reach a column with both a pivot of A and a pivot of B , then C is already in the desired form.

Suppose, then, that we reach a column containing both a pivot of A and a pivot of B . Consider the two rows containing the pivots. Both contain only zeros to the left of the pivot. We represent these two rows by the 2×2 matrix

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \quad (4.6)$$

where a_1 and b_1 are non-zero complex numbers and a_2 and b_2 are row vectors containing the rest of the entries of the two rows under consideration. Thus in order to find a transformation T so that $r(z, \bar{z}) = \langle I' T C \mathcal{Z}, T C \mathcal{Z} \rangle$ and TC

has a single pivot in this column, appearing in the position formerly occupied by a_1 , it suffices to find a 2×2 matrix T such that $T^* \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $T \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a'_1 & a'_2 \\ 0 & b'_2 \end{bmatrix}$.

If $T = [t_{ij}]$, the first of these requirements yields

$$|t_{11}|^2 - |t_{21}|^2 = 1 \tag{4.7}$$

$$\bar{t}_{11}t_{12} - \bar{t}_{21}t_{22} = 0 \tag{4.8}$$

$$|t_{22}|^2 - |t_{12}|^2 = 1. \tag{4.9}$$

In order for the second to be satisfied, we require

$$b'_1 = t_{21}a_1 + t_{22}b_1 = 0. \tag{4.10}$$

Thus we need $t_{21} = -t_{22} \frac{b_1}{a_1}$. An elementary calculation shows that T is necessarily of the form $\begin{bmatrix} e^{i\theta}t_{22} & -e^{i\theta}t_{22}\left(\frac{b_1}{a_1}\right) \\ -t_{22}\frac{b_1}{a_1} & t_{22} \end{bmatrix}$ where t_{22} also satisfies

$$|t_{22}|^2 \left(1 - \left| \frac{b_1}{a_1} \right|^2 \right) = 1. \tag{4.11}$$

Thus for this r , if $|a_1| > |b_1|$, it is possible to replace C with a matrix C' of the same rank in which a'_1 is non-zero, but $b'_1 = 0$.

The only situation left to consider is when $|a_1| \leq |b_1|$. In this case we modify r . Since $r(z, \bar{z}) = \|A\mathcal{Z}\|^2 - \|B\mathcal{Z}\|^2$ and $r(z, \bar{z})\|z\|^2$ is a squared norm, by Proposition 4.1, for any $\lambda \in [0, 1]$, $(\|A\mathcal{Z}\|^2 - \lambda\|B\mathcal{Z}\|^2)\|z\|^2$ is a squared norm, and $\tilde{r}(z, \bar{z}) := \|A\mathcal{Z}\|^2 - \lambda\|B\mathcal{Z}\|^2$ and r have the same signature pair. Observe,

$$\begin{aligned} \tilde{r}(z, \bar{z}) &= \|A\mathcal{Z}\|^2 - \|\sqrt{\lambda}B\mathcal{Z}\|^2 \\ &= \left\langle I' \begin{bmatrix} A \\ \sqrt{\lambda}B \end{bmatrix} \mathcal{Z}, \begin{bmatrix} A \\ \sqrt{\lambda}B \end{bmatrix} \mathcal{Z} \right\rangle. \end{aligned} \tag{4.12}$$

Thus if both a_1 and b_1 are non-zero, but $|a_1| \leq |b_1|$, through an appropriate choice of λ , we can replace r with an \tilde{r} having the same signature pair as r with matrix \tilde{C} having non-zero entries in precisely the same positions as in C , but with the property that $|a_1| > \sqrt{\lambda}|b_1|$. We may thus now apply a transformation T as above to achieve the desired reduction of the matrix

\tilde{C} . Continuing in this manner, we eventually obtain an \tilde{r} with the same signature as the original r , but with the matrix $\begin{bmatrix} \tilde{A} \\ \tilde{B} \end{bmatrix}$ in partial row-echelon form. \square

We can now prove the first part of Theorem 1.1.

Lemma 4.2. *Let $r(z, \bar{z})$ be a real polynomial on \mathbb{C}^n , $n \geq 2$, and suppose that $r(z, \bar{z}) \|z\|^2$ is a squared norm. Let (N_+, N_-) be the signature pair of r . Then*

$$\frac{N_-}{N_+} < n - 1. \quad (4.13)$$

Proof. Let \mathcal{Z}_k denote the vector of all holomorphic monomials in n variables of degree at most k . Order the monomials as above, and note that multiplication by z_j preserves the order. In light of Lemma 4.1, we may assume $r(z, \bar{z}) = \langle I' C \mathcal{Z}_k, C \mathcal{Z}_k \rangle$ where C is in partial row-echelon form and I' is the diagonal matrix $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ with signature (N_+, N_-) .

Let C_j be the matrix defined by

$$(C \mathcal{Z}_k) z_j = C_j \mathcal{Z}_{k+1}. \quad (4.14)$$

Because C is in partial row-echelon form, C_j is as well. Then

$$\begin{aligned} r(z, \bar{z}) \|z\|^2 &= \sum_{j=1}^n |z_j|^2 \langle I' C \mathcal{Z}_k, C \mathcal{Z}_k \rangle \\ &= \sum_{j=1}^n \langle I' C_j \mathcal{Z}_{k+1}, C_j \mathcal{Z}_{k+1} \rangle \\ &= \left\langle \begin{bmatrix} I' & 0 & \cdots & 0 \\ 0 & I' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I' \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \mathcal{Z}_{k+1}, \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{bmatrix} \mathcal{Z}_{k+1} \right\rangle \\ &= \langle \tilde{I}' \tilde{C} \mathcal{Z}_{k+1}, \tilde{C} \mathcal{Z}_{k+1} \rangle. \end{aligned} \quad (4.15)$$

The matrix \tilde{C} is not in partial row-echelon form and is not even of full rank. It is, however, in a special form that we can exploit. Although several rows may have their leading term in the same column, since each C_j is in partial

row-echelon form, each column can contain the leading terms for at most n rows.

At this point, the precise ordering of the rows of \tilde{C} is not important; we are only interested in the numbers of rows associated with positive (resp., negative) entries of I' and the linear relationships between the two sets. We thus re-order the rows. Let P be the matrix containing the nN_+ rows of \tilde{C} associated with positive entries in I' and let Q be the matrix consisting of the nN_- rows associated with negative entries of I' . Since $\tilde{C}^* \tilde{I}' \tilde{C}$ is positive semidefinite, so is

$$\begin{bmatrix} P \\ Q \end{bmatrix}^* \begin{bmatrix} I_{nN_+} & 0 \\ 0 & -I_{nN_-} \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = P^* P - Q^* Q, \quad (4.16)$$

and hence *the rows of Q are in the linear span of the rows of P* . Within P and Q , we may assume that, if the leading term of row i appears in column j , then the leading term of row $i+1$ is either in column j or in some column to the right of column j .

Let m_+ denote the number of columns of P containing the leading term of at least one row of P , and let $e_+ = nN_+ - m_+$. We think of e_+ as the number of “extra” rows. Since we can find m_+ rows of P with leading terms in m_+ distinct columns, $\text{rank}(P^t) \geq m_+$ and $\text{nullity}(P^t) = nN_+ - \text{rank}(P^t) \leq nN_+ - m_+ = e_+$.

More is true; let h denote the number of columns of Q that contain the leading term of a row of Q , but for which the corresponding column of P is not one of the m_+ counted above. We thus have a collection of $m_+ + h$ rows of \tilde{C} that are linearly independent. On the other hand, since all rows of Q are in the linear span of the rows of P , $\text{rank}(P^t) \geq m_+ + h$. Thus

$$\text{nullity}(P^t) = nN_+ - \text{rank}(P^t) \leq nN_+ - m_+ - h = e_+ - h. \quad (4.17)$$

In particular, $h \leq e_+$.

Q has exactly nN_- rows. However, by distinguishing two types of rows of Q , we can estimate the number of rows of Q in terms of the number of rows of P . Our first type of row of Q is one with leading term in one of the h columns counted above. Since no row of P has leading term in such a column, there could be as many as n rows of Q with leading term in a single

such column. Q therefore has at most nh such rows. The second type of row of Q is one with leading term in one of the m_+ columns corresponding to a column of P containing a leading term. Since one of the at most n rows with a leading term in this column must be in P , Q has at most $(n - 1)m_+$ rows of the second type.

This number $(n - 1)m_+$ is still an overestimate for two reasons. First, of the m_+ columns, the left-most has only a single entry, and it appears in P . To see this, consider the initial monomials of the f_j and g_j . Let z^α be the one that comes first in the monomial order. If it were the initial monomial of, say, g_J , then z_1g_J would have an initial monomial z_1z^α coming before the initial monomial of any of the z_kf_j , contradicting the fact that z_1g_J is in the span of the z_kf_j . Thus z^α is the initial monomial of one of the f_j . Since all the f_j have distinct initial monomials and because our monomial order is multiplicative, the left-most column of $\begin{bmatrix} P \\ Q \end{bmatrix}$ containing a non-zero entry is that corresponding to z_1z^α , and it contains precisely one non-zero entry. Second, we must account for the additional e_+ rows in P that also have leading term in one of the $m_+ - 1$ columns. Thus $(n - 1)(m_+ - 1) - e_+$ is still an upper bound for the number of rows in Q of this second type. We find:

$$\begin{aligned} nN_- &\leq nh + (n - 1)(m_+ - 1) - e_+ \\ &\leq ne_+ + (n - 1)m_+ - e_+ - (n - 1) \\ &= (n - 1)(e_+ + m_+) - (n - 1) \\ &< (n - 1)nN_+. \end{aligned} \tag{4.18} \quad \square$$

5. Upper bound on N_-/N_+ for $d > 1$

Lemma 5.1. *Let $r(z, \bar{z})$ be a real polynomial on \mathbb{C}^n , $n \geq 2$, and suppose that $r(z, \bar{z}) \|z\|^{2d}$ is a squared norm. Let (N_+, N_-) be the signature pair of r . Then*

$$\frac{N_-}{N_+} < \binom{n - 1 + d}{d} - 1. \tag{5.1}$$

Proof. We follow the proof of Lemma 4.2. When we multiply $\langle I' C Z_k, C Z_k \rangle$ by $\|z\|^{2d}$ instead of $\|z\|^2$, we obtain $\binom{n-1+d}{d}$ matrices C_j rather than n . More

explicitly, order the degree d multi-indices and let α be the j th multi-index. Let C_j be the matrix defined by

$$(C\mathcal{Z}_k)z^\alpha = C_j\mathcal{Z}_{k+d}. \tag{5.2}$$

As above, since C is in partial row-echelon form and the monomial order is multiplicative, C_j is in partial row-echelon form as well. Then

$$\begin{aligned} r(z, \bar{z}) \|z\|^{2d} &= \left\langle \begin{bmatrix} I' & 0 & \cdots & 0 \\ 0 & I' & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I' \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{\binom{n-1+d}{d}} \end{bmatrix} \mathcal{Z}_{k+d}, \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{\binom{n-1+d}{d}} \end{bmatrix} \mathcal{Z}_{k+d} \right\rangle \\ &= \langle \tilde{I}'\tilde{C}\mathcal{Z}_{k+d}, \tilde{C}\mathcal{Z}_{k+d} \rangle. \end{aligned} \tag{5.3}$$

In the matrix \tilde{C} , each column contains the leading term of at most $\binom{n-1+d}{d}$ rows, though, as above, the left-most non-zero column contains only a single non-zero entry since it comes about by multiplying the least monomial z^α in all the f_j by z_1^d . Thus in a manner identical to the above we obtain:

$$\binom{n-1+d}{d}N_- < \left(\binom{n-1+d}{d} - 1 \right) \binom{n-1+d}{d}N_+. \tag{5.4}$$

□

Remark 5.1. The proof does not use anything about $\|z\|^{2d}$ except that it is a squared norm, its matrix of coefficients is diagonal, and it has rank $\binom{n-1+d}{d}$. Therefore we also obtain the following statement.

Corollary 5.1. *Let $r(z, \bar{z})$ be a real polynomial on \mathbb{C}^n , and consider $s(z, \bar{z}) = \sum_{j=1}^L |z^{\alpha_j}|^2$, where $\alpha_1, \dots, \alpha_L$ are distinct multi-indices. Suppose $r(z, \bar{z})s(z, \bar{z})$ is a squared norm. If (N_+, N_-) is the signature pair of r , then*

$$\frac{N_-}{N_+} < L - 1. \tag{5.5}$$

6. A class of examples for $d > 1$

Lemma 5.1 merely gives an upper bound for N_-/N_+ for $d > 1$; it remains to determine whether the result is sharp.

We first discuss the case $n = 2$. Lemma 5.1 gives $N_-/N_+ < d$, which we claim is sharp for all d . To prove this, we construct a family $\{p_D\}$ of polynomials in two real variables such that $p_D(x, y)(x + y)^d$ has all non-negative coefficients and the ratio $N_-(p_D)/N_+(p_D)$ of negative to positive coefficients tends to d as $D \rightarrow \infty$. The idea of the construction is quite simple; define $p_D(x, y) = \sum c_j x^{D-j} y^j$ where the first and last coefficients are positive and the interior coefficients repeat a pattern of d negatives followed by a positive.

More explicitly, suppose $D = (d + 1)m$ for $m \in \mathbb{N}$ and define $p_D(x, y) = \sum_{j=0}^D \gamma(D - j, j) x^{D-j} y^j$, where

$$\gamma(D - j, j) = \begin{cases} 2^d - 1 & j \equiv 0 \pmod{d + 1} \\ -1 & \text{otherwise} \end{cases}. \quad (6.1)$$

For this family, $N_-(p_D)/N_+(p_D) = dD/(D + d + 1)$, which tends to d as $D \rightarrow \infty$. It only remains to verify as we did in the proof of part (iii) of Theorem 3.1 that the coefficients of p_D have been chosen so that S_D has all nonnegative coefficients. We omit the details.

When $n = 3$, Lemma 5.1 gives

$$\frac{N_-}{N_+} < \binom{n - 1 + d}{d} - 1 = \frac{(d + 1)(d + 2)}{2} - 1. \quad (6.2)$$

When $d = 1$, this gives $N_-/N_+ < 2$, which we know to be sharp. It remains open whether (5.1) is sharp for $d > 1$.

Remark 6.1. For $n = 3$, we were able to construct polynomials p for which $S := p \cdot \ell^d$ has all non-negative coefficients and with

$$\frac{N_-(p)}{N_+(p)} \geq \left\lfloor \frac{(d + 2)^2}{3} \right\rfloor - 1 - \varepsilon. \quad (6.3)$$

We omit the details; we simply mention that the construction can be done by considering the diagram to be an infinite plane and by using a pattern of

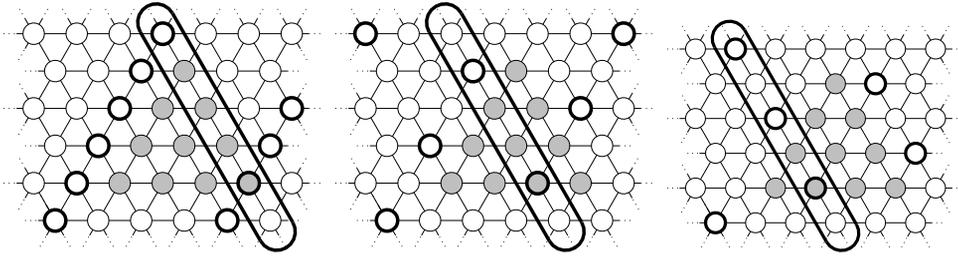


Figure 4: Diagrams illustrating the first step in the induction for the proof of Lemma 6.1. Here, $\nu = 0$, $\nu = 1$, and $\nu = 2$; and $d = 3$.

P s generated by two generalized knight moves. Computer experimentation suggests this bound may, in fact, be optimal. Therefore, we suspect (6.2) is not sharp.

Next we find examples that show that the bound (5.1) is of the right order, i.e., for a fixed n , of order d^{n-1} . This lemma is the last part of the proof of Theorem 1.2.

Lemma 6.1. *Fix $d > 1$ and $n > 2$. There exists a polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ for which $S_d(p) := p \cdot \ell^d$ has all non-negative coefficients and with*

$$\frac{N_-(p)}{N_+(p)} \geq \left(\frac{1}{2^{\frac{n(n-1)}{2}}} \right) d^{n-1} = C(d, n). \tag{6.4}$$

Proof. The proof is by induction on the number of variables n . When $n = 2$, we proved above that we can find polynomials for which $p \cdot \ell^d$ has non-negative coefficients with the ratio N_-/N_+ arbitrarily close to d . Thus there exists a polynomial for which the ratio exceeds $d/2$. Thus the result holds for $n = 2$.

We proceed to the inductive step. To simplify notation, we dehomogenize by setting $x_n = 1$. We therefore seek nonhomogeneous polynomials $p(x_1, \dots, x_{n-1})$ such that the product $p(x_1, \dots, x_{n-1})(x_1 + \dots + x_{n-1} + 1)^d$ has nonnegative coefficients. Suppose that for $n - 1$ there exists p' such that

$$\frac{N_-(p')}{N_+(p')} \geq C(d, n - 1) - \varepsilon/2. \tag{6.5}$$

That is, p' is a nonhomogeneous polynomial in $(n-1)-1 = n-2$ variables and multiplying by $(x_1 + \dots + x_{n-2} + 1)^d$ yields a polynomial with nonnegative

coefficients.

Let $x = (x', x_{n-1})$ where $x' \in \mathbb{R}^{n-2}$ so that $p'(x') = \sum_{\alpha} \gamma'(\alpha) x'^{\alpha}$. We define

$$p(x) = \sum_{j=0}^k \sum_{\alpha} \gamma(\alpha, j) x'^{\alpha} x_{n-1}^j \quad (6.6)$$

for appropriately chosen coefficients $\gamma(\alpha, j)$ and for k large. For each j between 1 and $k-1$, take $\gamma(\alpha, j) = \gamma'(\alpha)$. In other words, for each of these j we simply repeat the pattern of positives and negatives from p' . For $j=0$ and $j=k$, we take sufficiently large positive coefficients to guarantee that $p\ell^d$ has only non-negative coefficients.

When $n=3$, the situation is illustrated in the first diagram of Figure 4. In the diagram, thick circles are positive coefficients and thin circles are negative coefficients, as before. A “row” in the diagram corresponding to a fixed power of x_2 (a fixed j) is marked with a thick line. Finally, the shaded circles are the coefficients that contribute to a single coefficient in $p \cdot \ell^3$. Therefore any such triangle (or simplex in higher dimensions) must contain a positive coefficient, as it does in our diagram. By translating this triangle, we can see the different collections of terms in p that contribute to different monomials in S . Notice that we cannot place this triangle any differently so that it includes only negative terms. Further notice that on the marked “row” we have a diagram for $n=2$. This is how we are using the inductive hypothesis. The diagram illustrates only what happens in the “interior” and not on the boundary, where $j=0$ or $j=k$.

By taking a large enough degree to make the contribution to N_+ from the “rows” $j=0$ and $j=k$ arbitrarily small in the ratio, we find

$$\frac{N_-(p)}{N_+(p)} \geq C(d, n-1) - \varepsilon. \quad (6.7)$$

We can now improve upon this technique; suppose that instead of using p' that satisfied (6.5) for d we take a p' satisfying the equation for $d-1$. We can then take $\gamma(\alpha, j) = \gamma'(\alpha)$ only for *even* j between 1 and k and can take all $\gamma(\alpha, j)$ for odd j to be negative. After possibly making the positive coefficients larger, we conclude that $p \cdot \ell^d$ has positive coefficients. This process is illustrated in the second diagram of Figure 4. Notice that only

every second “row” contains positives, and that we took the positives to be closer together by exactly one on the rows that do contain positives.

Again by making the degree large enough we obtain a p such that

$$\frac{N_-(p)}{N_+(p)} \geq 1 + 2C(d - 1, n - 1) - \varepsilon. \tag{6.8}$$

By repeating this procedure (as illustrated by skipping two “rows” in the last diagram of Figure 4) we can lower d by ν to obtain a p such that

$$\frac{N_-(p)}{N_+(p)} \geq \nu + (\nu + 1)C(d - \nu, n - 1) - \varepsilon. \tag{6.9}$$

Picking $\nu = \lfloor \frac{d}{2} \rfloor$ we obtain a polynomial with

$$\frac{N_-(p)}{N_+(p)} \geq \left\lfloor \frac{d}{2} \right\rfloor + \left(\left\lfloor \frac{d}{2} \right\rfloor + 1 \right) C \left(\left\lfloor \frac{d}{2} \right\rfloor, n - 1 \right) - \varepsilon. \tag{6.10}$$

Let us prove $C(d, n) \geq C_n d^{n-1}$ by induction. For $n = 2$, we have seen that we can take $C_2 = \frac{1}{2}$. Assume the bound $C(d, n - 1) \geq C_{n-1} d^{n-2}$ holds for $n - 1$. We compute for $n > 2$,

$$\left\lfloor \frac{d}{2} \right\rfloor + \left(\left\lfloor \frac{d}{2} \right\rfloor + 1 \right) C \left(\left\lfloor \frac{d}{2} \right\rfloor, n - 1 \right) - \varepsilon \geq \left(\frac{d}{2} \right) C_{n-1} \left(\frac{d}{2} \right)^{n-2} = \frac{C_{n-1}}{2^{n-1}} d^{n-1}. \tag{6.11}$$

We are allowed to drop the ε because we are dropping $\lfloor \frac{d}{2} \rfloor$ from the right-hand side. Therefore we can take $C_n = \frac{C_{n-1}}{2^{n-1}}$ and $C_2 = \frac{1}{2}$ to obtain $C_n = \frac{1}{2^{n(n-1)/2}}$, and therefore (6.4) holds. \square

We have finished the proof of Theorem 1.2. As our final proposition, we show that the classes Ψ_j are distinct for all j .

Proposition 6.1. *For all $j = 0, 1, 2, \dots$,*

$$\Psi_j \subsetneq \Psi_{j+1}. \tag{6.12}$$

Proof. As above, we need only construct real polynomials. Define

$$q_k(x) = x_1^k + x_2^k + x_2^{k-1}x_3 + x_2^{k-1}x_4 + \dots + x_2^{k-1}x_n - \varepsilon x_1 x_2^{k-1}. \tag{6.13}$$

The only monomial of

$$-\varepsilon x_1 x_2^{k-1} (x_1 + x_2 + \cdots + x_n)^d \quad (6.14)$$

that does not appear in

$$(x_2^k + x_2^{k-1} x_3 + x_2^{k-1} x_4 + \cdots + x_2^{k-1} x_n) (x_1 + x_2 + \cdots + x_n)^d \quad (6.15)$$

for all $d = 1, 2, \dots$, is the term $-\varepsilon x_1^{d+1} x_2^{k-1}$. It appears in $x_1^k (x_1 + x_2 + \cdots + x_n)^d$ when $k = d + 1$, but not for any smaller d . By taking $\varepsilon > 0$ small enough we obtain that $q_k \cdot \ell^{d+1}$ has all positive coefficients in this case.

Therefore q_{d+1} is in Ψ_{d+1} , but not in Ψ_d . Notice that $q_{d+1} \notin \Psi_d$ even if we make the negative coefficient arbitrarily small. \square

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