EXCITED RANDOM WALKS:
RESULTS, METHODS, OPEN PROBLEMS

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Dedicated to Professor S.R.S. Varadhan
on the occasion of his 70-th birthday.

Abstract

We consider a class of self-interacting random walks in deterministic or random environments, known as excited random walks or cookie walks, on the d-dimensional integer lattice. The main purpose of this paper is two-fold: to give a survey of known results and some of the methods and to present several new results. The latter include functional limit theorems for transient one-dimensional excited random walks in bounded i.i.d. cookie environments as well as some zero-one laws. Several open problems are stated.

1. Model Description

Random walks (RWs) and their scaling limits are probably the most widely known and frequently used stochastic processes in probability theory, mathematical physics, and applications. Studies of a RW in a random medium are an attempt to understand which macroscopic effects can be seen and modeled by subjecting the RW's dynamics on a microscopic level to various kinds of noise, for example, allowing it to interact with a random

environment or its own history (through path restrictions, reinforcement, excitation etc.).

The Markov, or memory-less, property of simple RWs lies at the heart of the classical approach to these and much more general processes. But it also imposes a restriction on the applicability of such models, as many real life processes certainly have memory. Over the last decades several types of non-markovian RWs appeared in the literature and became active areas of research; for example, self-avoiding RWs, edge or vertex reinforced RWs, and excited RWs (ERWs), also known as "cookie walks".[1] The reader interested in the first two types of models is referred to the surveys [Tóth99] and [Pem07]. At the time when [Pem07] was written, ERWs had just appeared and for this reason were only briefly mentioned [Pem07, p. 51].

The main purpose of this article is two-fold. First, we give a survey of results and some of the methods concerning ERWs. Second, we include several new theorems, see e.g. Theorems 2.5, 3.16, 6.6, and 6.7. We also state several open problems. While we aim at presenting all major results known for ERWs on $\mathbb{Z}^d, d \geq 1$, the choice of methods explained in some detail has been influenced by our personal preferences and contributions to this area.

Primarily, we shall be concerned with discrete time ERWs on $\mathbb{Z}^d, d \geq 1$, even though the construction can be readily extended to other graphs (see [Vol03, GMP08, BS09] for trees, [Zer06, Dol11] for strips) or continuous time-space processes (see [RS11, RS] for so-called excited Brownian motion). Broadly speaking, one considers a certain underlying and presumably well-understood process and modifies its dynamics for the first few visits to each site. These modifications can be thought of as stacks of *cookies* placed on each site of the lattice. Each cookie encodes a probability distribution on $\mathbb{Z}^d$. The walker consumes a cookie at his current location and makes a move according to the distribution prescribed by that cookie. Upon reaching a site where all cookies have already been eaten or were not there to begin with, the walker makes a move in accordance with the original underlying dynamics. Below we consider processes whose underlying dynamics is the simple symmetric RW and for which the cookie stacks are random themselves (i.i.d. or stationary and ergodic) and induce transitions between nearest

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[1]According to Itai Benjamini [personal communication], the name "excited random walk" for the model studied in the seminal paper [BW03] was suggested by Oded Schramm. The notion of "cookies" in this context was introduced later in [Zer05].

neighbors. (For a model with cookies inducing (long-range) non-nearest neighbor transitions see [RS10].)

We start with the description of a relatively general model which allows infinite cookie stacks. Let $\mathcal{E} := \{\pm e_j \mid j \in \{1, 2, \ldots, d\}\}$ be the set of unit coordinate vectors in $\mathbb{Z}^d$ and denote by $\mathcal{M}_{\mathcal{E}}$ the set of probability measures on $\mathcal{E}$, i.e. vectors with $2d$ non-negative entries which sum up to 1. Such vectors are called *cookies*. The set of cookie environments is denoted by

$$\Omega := \mathcal{M}_{\mathcal{E}}^{\mathbb{Z}^d \times \mathbb{N}}.$$

(Here $\mathbb{N} = \{1, 2 \ldots\}$.) The elements of Ω are written as $\omega = (\omega(z, e, i))_{z \in \mathbb{Z}^d, e \in \mathcal{E}, i \in \mathbb{N}}$ with $(\omega(z, e, i))_{e \in \mathcal{E}}$ being the i-th cookie at z. It is consumed by the walker upon the i-th visit to z, if there is such a visit, and provides the walker with the transition probabilities from z to $z + e$ in the next step. More precisely, for fixed $\omega \in \Omega$ and $x \in \mathbb{Z}^d$ an ERW starting at x in the environment ω is a process $X := (X_n)_{n \geq 0}$ on a suitable probability space $(\Omega', \mathcal{F}', P_{x,\omega})$ which satisfies

$$P_{x,\omega}[X_0 = x] = 1 \quad \text{and} \tag{1.1}$$
$$P_{x,\omega}\left[X_{n+1} = X_n + e \mid (X_i)_{0 \leq i \leq n}\right] = \omega\left(X_n, e, \#\{i \in \{0, 1, \ldots, n\} \mid X_i = X_n\}\right)$$

for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $e \in \mathcal{E}$. Here $\#A$ denotes the cardinality of the set A.

The cookie environment ω may be chosen at random itself according to a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the canonical product Borel σ-algebra. Averaging the so-called *quenched* measure $P_{x,\omega}$ over the environment ω we obtain the *averaged* (often also called *annealed*) measure $P_x[\cdot] := \mathbb{E}[P_{x,\omega}[\cdot]]$ on $\Omega \times \Omega'$. The expectation operators corresponding to $P_{x,\omega}, \mathbb{P}$, and P_x are denoted by $E_{x,\omega}, \mathbb{E}$, and E_x, respectively.

A common assumption about $\mathbb{P}$ will be:

(IID)   the family $(\omega(z, \cdot, \cdot))_{z \in \mathbb{Z}^d}$ of cookie stacks is i.i.d. under $\mathbb{P}$.

A weaker condition is:

(SE)   the family $(\omega(z, \cdot, \cdot))_{z \in \mathbb{Z}^d}$ is under $\mathbb{P}$ stationary
and ergodic[2] with respect to the shifts on $\mathbb{Z}^d$.

To escape degenerate situations we shall often assume one of the following *ellipticity conditions*, called weak ellipticity, ellipticity, and uniform ellipticity.

(WEL) For all $z \in \mathbb{Z}^d, e \in \mathcal{E} : \mathbb{P}\left[\forall i \in \mathbb{N} : \omega(z, e, i) > 0\right] > 0.$

  (EL) For all $z \in \mathbb{Z}^d, e \in \mathcal{E}$ and $i \in \mathbb{N} : \mathbb{P}$-a.s. $\omega(z, e, i) > 0.$

(UEL) There is $\kappa > 0$ such that for all $z \in \mathbb{Z}^d, e \in \mathcal{E}$ and $i \in \mathbb{N} : \mathbb{P}$-a.s. $\omega(z, e, i) \geq \kappa.$

Obviously, (UEL)$\Rightarrow$(EL)$\Rightarrow$(WEL).

If $\omega \in \Omega$ is such that $\omega(z, e, i)$ does not depend on z, e, and i then necessarily $\omega(z, e, i) = 1/(2d)$ and hence X is, under $P_{x,\omega}$, the simple symmetric RW on $\mathbb{Z}^d$ starting at x. Since the consumption of a cookie $(\omega(z, e, i))_{e \in \mathcal{E}} = (1/(2d))_{e \in \mathcal{E}}$ does not change the dynamics of the simple symmetric RW, which is our underlying process, such a cookie will be called a *placebo cookie*.

If we allow dependence of $\omega(z, e, i)$ on e but neither on z nor i we obtain all nearest neighbor random walks on $\mathbb{Z}^d$ starting at x whose increments are i.i.d..

If we let $\omega(z, e, i)$ depend on e and z but still not on i then we get all nearest neighbor Markov chains on $\mathbb{Z}^d$. Choosing $\omega \in \Omega$ at random (assuming usually (IID) or (SE)) yields all nearest neighbor *RWs in random environments* (RWRE) on $\mathbb{Z}^d$, see e.g. [Zei04, Szn04]. (However, note that although RWRE can be thought of as a special case of ERW, the present paper does not attempt to survey RWRE results.)

All these processes are markovian. However, as soon as $\omega(z, e, i)$ starts depending on i, X loses the Markov property. The walker's behavior may depend on the number of visits to his current location. Such class of processes seems to be too general to be considered in a coherent way and might not be

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[2]i.e. every $A \in \mathcal{F}$ which is invariant under all shifts on $\mathbb{Z}^d$ satisfies $\mathbb{P}[A] \in \{0, 1\}.$

related anymore to any well-understood underlying process. Thus one has to impose further conditions on $\mathbb{P}$.

One way to re-establish the connection to the underlying process, in our case to the simple symmetric RW, is to limit the number of non-placebo cookies per site. To this purpose, we denote by

$$M(z) := \inf\{j \in \mathbb{N}_0 \mid \forall e \in \mathcal{E} \; \forall i > j : \omega(z, e, i) = 1/(2d)\} \qquad (1.2)$$

the number of cookies at site $z \in \mathbb{Z}^d$, if we do not count placebo cookies which are only followed by more placebo cookies. (Here $\inf \emptyset = \infty$.) Note that under assumption (IID) (resp. (SE)) $M(z)$, $z \in \mathbb{Z}^d$, is an i.i.d. (resp. stationary and ergodic) family of random variables. Again, if $M \equiv 0$ then $(X_n)_{n \geq 0}$ is the simple symmetric RW, whereas for RWRE we have $M(z) \in \{0, \infty\}$.

In the next four short subsections we present the most commonly studied measures $\mathbb{P}$ on Ω.

1.1. The original ERW model and a modification (no excitation after the first visit)

ERWs were introduced by Benjamini and Wilson in [BW03] for $\mathbb{Z}^d, d \geq 1$, with emphasis on $d \geq 2$. There the walker gets a bias in the direction e_1 upon the first visit to a site, whereas upon subsequent visits to the same site he jumps to a uniformly chosen neighbor. More precisely, in the notation introduced above, there is a $p \in (1/2, 1]$ such that $\mathbb{P} = \delta_\omega$, where for all $z \in \mathbb{Z}^d$,

$$\text{(BW)} \quad \begin{aligned} &\omega(z, e_1, 1) = \frac{p}{d}, \quad \omega(z, -e_1, 1) = \frac{1-p}{d}, \text{ and} \\ &\omega(z, e, i) = \frac{1}{2d} \quad \text{if } i \in \mathbb{N} \text{ and } e \in \mathcal{E} \setminus \{e_1, -e_1\} \quad \text{or} \quad \text{if } i \geq 2. \end{aligned}$$

In this special environment ω one can compare the ERW with a simple RW $(Y_n)_{n \geq 0}$ by coupling these processes so that $(X_n - Y_n) \cdot e_1$ is non-decreasing in n and $X_n \cdot e_i = Y_n \cdot e_i$ for all $i = 2, \ldots, d$ and $n \geq 0$.

A natural extension of this model is obtained by fixing an arbitrary direction $\ell \in \mathbb{R}^d \setminus \{0\}$ and assuming that all the cookies induce a drift in that

direction. In [MPRV, Th. 1.2] the setting is extended to measures $\mathbb{P}$ which satisfy (IID), (UEL), and for which there is an $\ell \in \mathbb{R}^d \backslash \{0\}$ such that

$$(\text{MPRV}_\ell) \qquad \begin{aligned} &\exists \lambda > 0 : \sum_{e \in \mathcal{E}} \omega(0, e, 1) \; e \cdot \ell \geq \lambda \quad \mathbb{P}\text{-a.s. and} \\ &\omega(0, e, i) = \omega(0, -e, i) \;\; \mathbb{P}\text{-a.s. for all } e \in \mathcal{E}, \; i \geq 2. \end{aligned}$$

1.2. Any number of ℓ-positive cookies per site

One way to further generalize the models described in Section 1.1 is to lift the restriction on the number of non-placebo cookies, still requiring each cookie to induce a non-negative drift in the same general direction. Such a model was introduced in [Zer05] for $\mathbb{Z}$ and in [Zer06] for $\mathbb{Z}^d, d \geq 2$ (and for strips as a graph in between $\mathbb{Z}$ and $\mathbb{Z}^2$). It is assumed that $\mathbb{P}$ satisfies (SE) if $d = 1$ and (IID) and (UEL) if $d \geq 2$ and that for some $\ell \in \mathbb{R}^d \backslash \{0\}$,

$$(\text{POS}_\ell) \qquad \sum_{e \in \mathcal{E}} \omega(0, e, i) \; e \cdot \ell \geq 0 \quad \mathbb{P}\text{-a.s. for all } i \in \mathbb{N}.$$

A useful fact in this setting is that $(X_n \cdot \ell)_{n \geq 0}$ is a $P_{0,\omega}$-submartingale.

Cookies which satisfy the inequality in (POS_ℓ) are called $(\ell$-$)positive$. (Negativity is defined by the opposite inequality.)

1.3. Boundedly many positive or negative cookies per site $(d = 1)$

This model was introduced in [BS08a, KZ08]. Assume (IID), (WEL), and that the number of non-placebo cookies and their positions within the cookie stacks are bounded, i.e.

(BD) there is a non-random $M \in \mathbb{N}$ such that $\forall \, z \in \mathbb{Z}^d$: $\mathbb{P}$-a.s. $M(z) \leq M$.

This model is probably the most studied. Currently, there is a rather complete picture, which includes criteria for recurrence and transience, laws of large numbers, ballisticity, functional limit theorems, and large deviations.

This model can also be considered in higher dimensions, but so far there has been little progress for $\mathbb{Z}^d, d \geq 2$. Practically all work that is done on this model for $d = 1$ uses a connection with branching processes with migration.

This connection allows to translate main questions for this non-markovian model into questions about branching processes, which are markovian. The branching process approach was also useful in considering ERW on trees (see [BS09]) but it completely breaks down on $\mathbb{Z}^d$, $d \geq 2$.

1.4. RW perturbed at extrema ($d = 1$)

One of the historically first studied non-markovian RWs which fits into the above setting is the so-called *RW perturbed at extrema*, see e.g. [Dav99] and the references therein. This is a nearest-neighbor process $(Y_n)_{n \geq 0}$ on $\mathbb{Z}$ which starts at 0 and satisfies

$$P[Y_{n+1} = Y_n + 1 \mid (Y_i)_{0 \leq i \leq n}] = \begin{cases} p & \text{if } n \geq 1 \text{ and } Y_n = \max_{i \leq n} Y_i, \\ q & \text{if } n \geq 1 \text{ and } Y_n = \min_{i \leq n} Y_i, \\ 1/2 & \text{otherwise} \end{cases}$$

for given $p, q \in (0, 1]$. It has been noted in [BV, Sec. 1] that this walk can be viewed as an ERW under the averaged measure as follows: Let $(M_z)_{z \in \mathbb{Z}}$ be independent random variables w.r.t. some probability measure P such that $P[M_z = k] = p(1-p)^{k-1}$ for $k, z \geq 1$, $P[M_z = k] = q(1-q)^{k-1}$ for $k \geq 1, z \leq -1$ and $M_0 \equiv 0$. Set for all $z \in \mathbb{Z}$ and $i \in \mathbb{N}$,

$$\widetilde{\omega}(z, 1, i) = \begin{cases} 1 & \text{if } z < 0 \text{ and } i < M_z \text{ or if } z > 0 \text{ and } i = M_z, \\ 0 & \text{if } z < 0 \text{ and } i = M_z \text{ or if } z > 0 \text{ and } i < M_z, \\ 1/2 & \text{if } i > M_z \end{cases}$$

and let $\mathbb{P}$ be the distribution of $\widetilde{\omega}$ on Ω. (Note that $(M_z)_{z \in \mathbb{Z}}$ and $(M(z))_{z \in \mathbb{Z}}$ as defined in (1.2) have the same distribution unless $p = 1/2$ or $q = 1/2$.) Then the ERW X has under P_0 the same distribution as $(Y_n)_{n \geq 0}$. However, from the point of view of ERW as introduced above this measure $\mathbb{P}$ seems a bit unnatural since it does not satisfy (SE). Therefore, this model will not play an important role in this paper. For a variant of this model which does satisfy (SE) see [Pin10] and Remark 3.14.

Let us describe how the present paper is organized. At the end of this section we introduce some notation. In Section 2 we deal with the basic question when the range of the ERW is finite and when it is infinite. This will

be useful in Section 3, which considers recurrence and transience. There we discuss zero-one laws for (directional) recurrence and transience and collect the known criteria which distinguish between these two cases. In Section 4 we establish strong laws of large numbers, i.e. the existence of a P_0-a.s. limit v of X_n/n as $n \to \infty$. Section 5 is devoted to the question whether $v = 0$ or $v \neq 0$. We also discuss in this section in more detail the connection to branching processes with migration mentioned above, which is a useful tool when $d = 1$. In Section 6 we consider one-dimensional and functional limit laws for convergence in distribution. In the final section we quote some results which did not fit into any of the previous sections.

Sections 3−6 are divided into two subsections each. The first subsection deals with the one-dimensional situation, the second one considers the multi-dimensional case.

Notation. For $k \in \mathbb{Z}$ we set $T_k := \inf\{n \geq 0 \mid X_n = k\}$. We need some notation for the environment which is left over after the ERW has eaten some of its cookies. For any $\omega \in \Omega$, $J \in \mathbb{N}_0 \cup \{\infty\}$ and $(x_j)_{j<J} \in (\mathbb{Z}^d)^J$,

$$\psi(\omega, (x_j)_{j<J})(z, e, i) := \omega(z, e, i + \#\{0 \leq j < J \mid x_j = z\})$$

defines the environment $\psi(\omega, (x_j)_{j<J})$ obtained from ω by following the path $(x_j)_{j<J}$ and removing the currently first cookie each time a site is visited. In this context we sometimes need an independent copy of the ERW X, which we denote by X'. By $U \overset{\mathrm{d}}{=} V$ we mean that U and V have the same distribution. Convergence in distribution is denoted by $\Rightarrow$. Saying that $A \subseteq B$ P-a.s. means $P[A \backslash B] = 0$. We write $E[Z, A] := E[Z \cdot \mathbf{1}_A]$ for random variables Z and events A. The integer part of $t \in \mathbb{R}$ is denoted by $[t]$.

2. Finite or Infinite Range

To the best of our knowledge the question whether the range $\{X_n \mid n \geq 0\}$ of the ERW is finite or infinite has not been considered in the literature yet. In this section we give necessary and sufficient criteria for the range to be finite. In particular, under natural conditions the range is either P_0-a.s. finite or P_0-a.s. infinite. This will be useful in Section 3. First, we introduce some notation.

Definition 2.1. For $x \in \mathbb{Z}^d$ and $e \in \mathcal{E}$ write $x \overset{\omega}{\to} x + e$ if and only if $\sum_{i \geq 1} \omega(x, e, i) = \infty$. Define $b_F := \mathbb{P}[\forall e \in F : 0 \overset{\omega}{\not\to} e]$ for $F \subseteq \mathcal{E}$. For $e \in \mathcal{E}$ write b_e instead of $b_{\{e\}}$. The transitive closure in $\mathbb{Z}^d$ of the relation $\overset{\omega}{\to}$ is denoted by $\overset{\omega}{\to}$ as well. Moreover, $\mathcal{C}_x := \{y \in \mathbb{Z}^d \mid x \overset{\omega}{\to} y\}$.

The meaning of $x \overset{\omega}{\to} y$ is illustrated by the following lemma, which follows from the Borel-Cantelli lemma.

Lemma 2.2. $(d \geq 1)$ Let $\omega \in \Omega$ and $x, y \in \mathbb{Z}^d$ with $x \overset{\omega}{\to} y$. Then on the event that the ERW visits x infinitely often, y is $P_{0,\omega}$-a.s. visited infinitely often as well.

Theorem 2.3. $(d = 1, \textbf{range})$ Assume (SE) and (EL).

(a) If $b_1 > 0$ and $b_{-1} > 0$ then the range is P_0-a.s. finite.

(b) If $b_1 = 0$ and $b_{-1} > 0$ then P_0-a.s. $X_n \to +\infty$ as $n \to \infty$.

(c) If $b_1 > 0$ and $b_{-1} = 0$ then P_0-a.s. $X_n \to -\infty$ as $n \to \infty$.

(d) If $b_1 = 0$ and $b_{-1} = 0$ then the range is P_0-a.s. infinite.

As we shall see in the proof in case (a) the walker eventually gets stuck between two essentially reflecting barriers of cookies. A barrier to his left is reflecting to the right and a barrier to his right is reflecting to the left. In case (b) there are only barriers reflecting to the right. They act like valves. Once the walker has passed any of them from left to right he has a positive probability never to penetrate it in the opposite direction. Case (d) is the richest case.

Lemma 2.4. $(d = 1)$ Assume (SE).

Let $b_{-1} > 0$ and assume (EL). Then $P_0[\forall n \geq 0 : X_n \geq 0] > 0$

and $P_0[\inf_{n \geq 0} X_n > -\infty] = 1.$ $\qquad\qquad\qquad\qquad\qquad\qquad$ (2.1)

If $P_0[\forall n \geq 0 : X_n \geq 0] > 0$ then P_0-a.s. $\{\sup_{n \geq 0} X_n = \infty\} \subseteq \{X_n \to \infty\}.$ $\qquad$ (2.2)

***Proof of Lemma* 2.4.** Assume $b_{-1} > 0$. If all the jumps from 0 go to 1 then $X_n \geq 0$ for all $n \geq 0$. Therefore,

$$P_0[\forall n \geq 0 :\ X_n \geq 0] \geq \mathbb{E}[\pi_0], \text{ where } \pi_z := \prod_{i \geq 1} \omega(z, 1, i) = \prod_{i \geq 1} (1 - \omega(z, -1, i)).$$

By $b_{-1} > 0$ and (EL) we have $\mathbb{E}[\pi_0] > 0$, which yields the first statement. Similarly, if $\inf_n X_n = -\infty$ then the walk jumps for each $z \leq 0$ at least once from z to $z - 1$. Therefore, for all $\omega \in \Omega$,

$$P_{0,\omega}\left[\inf_{n \geq 0} X_n = -\infty\right] \leq \prod_{z \leq 0}(1 - \pi_z) \leq \exp\left(-\sum_{z \leq 0}\pi_z\right). \qquad (2.3)$$

By assumption (SE) the sequence $(\pi_z)_{z \leq 0}$ is stationary and ergodic. Since $\mathbb{E}[\pi_0] > 0$ the right-hand side of (2.3) vanishes $\mathbb{P}$-a.s. due to (SE). Hence P_0-a.s. $\inf_n X_n > -\infty$.

For the proof of (2.2) let $k, m \in \mathbb{N}$, $k \leq m$ and let $(\mathcal{F}_n)_{n \geq 0}$ be the filtration generated by X. Then for all $\omega \in \Omega$, $P_{0,\omega}$-a.s. on the event $\{\sup_n X_n = \infty\}$,

$$P_{0,\omega}\left[\liminf_{n \to \infty} X_n \geq k \mid \mathcal{F}_{T_m}\right] \geq P_{0,\omega}\left[\forall n \geq T_m : X_n \geq m \mid \mathcal{F}_{T_m}\right]$$
$$= P_{m,\psi(\omega,(X_n)_{n < T_m})}[\forall n \geq 0 : X_n' \geq m] \qquad (2.4)$$

by the strong Markov property for the Markov chain $((X_i)_{0 \leq i \leq n})_{n \geq 0}$. However, since $\psi(\omega, (X_n)_{n < T_m})(x)$ and $\omega(x)$ differ only at sites $x < m$ the expression in (2.4) is equal to $P_{m,\omega}[\forall n \geq 0 : X_n \geq m]$ which is, due to (SE), $\mathbb{P}$-a.s. for infinitely many $m \in \mathbb{N}$ larger than $\varepsilon := P_0[\forall n \geq 0 : X_n \geq 0]/2$. Hence, on the event $\{\sup_n X_n = \infty\}$, P_0-a.s.,

$$\varepsilon \leq \liminf_{m \to \infty} P_{0,\omega}\left[\liminf_{n \to \infty} X_n \geq k \mid \mathcal{F}_{T_m}\right] = \mathbf{1}_{\{\liminf_{n \to \infty} X_n \geq k\}}$$

by Levy's 0-1 law. Consequently, P_0-a.s. $\{\sup_n X_n = \infty\} \subseteq \{\liminf_{n \to \infty} X_n \geq k\}$. Letting $k \to \infty$ then yields the claim. $\qquad\square$

Proof of Theorem 2.3. Statement (a) follows from (2.1) and its corresponding counterpart for the case $b_1 > 0$. For the proof of (b) observe that by (2.1) P_0-a.s. $\inf_n X_n > -\infty$. Therefore, P_0-a.s.:

$$\{X_n \to \infty\}^c \subseteq \bigcup_{z \in \mathbb{Z}}\{X_n = z \text{ i.o.}\} \overset{\text{L. 2.2}}{\subseteq} \left\{\sup_{n \geq 0} X_n = \infty\right\} \overset{(2.1),(2.2)}{\subseteq} \{X_n \to \infty\},$$

which implies claim (b). Claim (c) follows from (b) by symmetry. In case

(d), P_0-a.s.,

$$\left\{\sup_{n\geq 0}|X_n| < \infty\right\} \subseteq \bigcup_{z\in\mathbb{Z}}\{X_n = z \text{ i.o.}\} \overset{\text{L. 2.2}}{\subseteq} \left\{\sup_{n\geq 0} X_n = \infty\right\}$$

$$\subseteq \left\{\sup_{n\geq 0}|X_n| = \infty\right\},$$

which yields the claim. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

If one strengthens the assumption (SE) to (IID) then a statement similar to Theorem 2.3 can be made also in higher dimensions. The following result builds upon [HSa, Lem. 2.2, 2.3], which deals with RWRE with possibly forbidden directions. It puts the example given in [Zer06, Rem. 1] in a general framework.

Theorem 2.5. ($d \geq 1$, **range**) *Assume* (IID) *and* (EL). *If there is an orthogonal set $F \subset \mathcal{E}$ such that $b_F = 0$ then the range is P_0-a.s. infinite. If there is no such set then the range is P_0-a.s. finite.*

Lemma 2.6. ($d \geq 1$, [**HSa, Lem. 2.3**]) $\mathbb{P}[\#\mathcal{C}_0 = \infty] = 1$ *if and only if there is an orthogonal set $F \subset \mathcal{E}$ such that $b_F = 0$.*

For completeness we include a proof of this lemma.

***Proof of Lemma* 2.6.** If there is an orthogonal set $F \subset \mathcal{E}$ with $b_F = 0$ then there is $\mathbb{P}$-a.s. a nearest-neighbor path $(y_n)_{n\geq 0}$ starting at $y_0 = 0$ with $y_n \overset{\omega}{\to} y_{n+1}$ and $y_{n+1} - y_n \in F$ for all $n \geq 0$. Since F is orthogonal this path is self-avoiding and hence $\{y_n \mid n \geq 0\} \subseteq \mathcal{C}_0$ is infinite. Conversely, assume that there is no such set F. Then

$$\{\#\mathcal{C}_0 < \infty\} \supseteq \left\{\mathcal{C}_0 \subseteq \{0,1\}^d\right\} \supseteq \left\{\forall x \in \{0,1\}^d \; \forall e \in F_x : x \overset{\omega}{\nrightarrow} x + e\right\},$$

where $F_x := \{(-1)^{x_i+1}e_i \mid i = 1, \ldots, d\}$ is the (orthogonal) set of directions pointing from x towards the complement of $\{0,1\}^d$. By assumption $b_{F_x} > 0$ for all $x \in \{0,1\}^d$ and therefore, by independence, $P[\#\mathcal{C}_0 = \infty] < 1$. $\quad\square$

***Proof of Theorem* 2.5.** Following the idea behind [HSa, Lem. 2.2] we shall show that the range is P_0-a.s. infinite if $\mathbb{P}$-a.s. $\#\mathcal{C}_0 = \infty$ and P_0-a.s. finite otherwise. The claim of Theorem 2.5 then follows from Lemma 2.6.

To prove the first implication we assume that $\mathcal{C}_0$ and hence all $\mathcal{C}_x$, $x \in \mathbb{Z}^d$, are $\mathbb{P}$-a.s. infinite. Then P_0-a.s.,

$$\{\#\{X_n \mid n \geq 0\} < \infty\} \subseteq \bigcup_{x \in \mathbb{Z}^d} \{X_n = x \text{ i.o.}\} \stackrel{\text{L. 2.2}}{\subseteq} \bigcup_{x \in \mathbb{Z}^d} \{C_x \subseteq \{X_n \mid n \geq 0\}\}$$
$$\subseteq \{\#\{X_n \mid n \geq 0\} = \infty\}.$$

This proves the first implication. For the opposite implication assume $\mathbb{P}[\#\mathcal{C}_0 < \infty] > 0$. Choose $S \subset \mathbb{Z}^d$ finite such that $\mathbb{P}[\mathcal{C}_0 = S] > 0$. By (EL) and Definition 2.1 there is $\gamma \in \mathbb{N}$, strictly larger than the $\|\cdot\|_\infty$-diameter of S, such that $\mathbb{P}[B_0] > 0$, where for $x \in \mathbb{Z}^d$,

$$B_x := \left\{ \mathcal{C}_x = S + x, \sum_{y \in \mathcal{C}_x, i \in \mathbb{N}, e \in \mathcal{E}: y+e \notin \mathcal{C}_x} \omega(y, y+e, i) \leq \gamma, \right.$$
$$\left. \forall y \in \mathcal{C}_x, i \in \mathbb{N}: \sum_{e \in \mathcal{E}: y+e \notin \mathcal{C}_x} \omega(y, y+e, i) \leq 1 - \gamma^{-1} \right\}.$$

Choose for all $e \in \mathcal{E}$ some $s_e \in S$ which minimizes $S \ni x \mapsto x \cdot e$. Furthermore, define the increasing sequence of stopping times $\sigma_k := \inf\{n \in \mathbb{N} \mid \|X_n\|_\infty = k\gamma\} \leq \infty$, $k \in \mathbb{N}$. Then for all $k \geq 0$, by partitioning,

$$P_0[\sigma_{k+1} < \infty] = \sum_{e \in \mathcal{E}, x: \|x\|_\infty = x \cdot e = k\gamma} P_0\left[\sigma_{k+1} < \infty, X_{\sigma_k} = x, B_{x-s_e}\right] \qquad (2.5)$$
$$+ \sum_{e \in \mathcal{E}, x: \|x\|_\infty = x \cdot e = k\gamma} P_0\left[\sigma_{k+1} < \infty, X_{\sigma_k} = x, B_{x-s_e}^c\right]. \quad (2.6)$$

First observe that for all $e \in \mathcal{E}$ and $x \in \mathbb{Z}^d$ with $\|x\|_\infty = x \cdot e = k\gamma$,

$$k\gamma \leq \|z\|_\infty < (k+1)\gamma \quad \text{for all } z \in S + x - s_e. \qquad (2.7)$$

The summands in (2.6) are easy to handle:

$$P_0\left[\sigma_{k+1} < \infty, X_{\sigma_k} = x, B_{x-s_e}^c\right] \leq \mathbb{E}\left[P_{0,\omega}[\sigma_k < \infty, X_{\sigma_k} = x], B_{x-s_e}^c\right].$$

Observe that $P_{0,\omega}[\sigma_k < \infty, X_{\sigma_k} = x]$ is $\sigma(\omega(z, \cdot, \cdot); \|z\|_\infty < k\gamma)$-measurable, whereas $B_{x-s_e}^c \in \sigma(\omega(z, \cdot, \cdot); \|z\|_\infty \geq k\gamma)$ by the first inequality in (2.7).

Therefore, by (IID), the expression in (2.6) can be estimated from above by

$$\sum_{e \in \mathcal{E}, x: \|x\|_\infty = x \cdot e = k\gamma} \mathbb{E}\left[P_{0,\omega}[\sigma_k < \infty, X_{\sigma_k} = x]\right] \, \mathbb{P}\left[B^c_{x-s_e}\right] = P_0[\sigma_k < \infty] \, \mathbb{P}[B^c_0].$$

(2.8)

We now turn to the right-hand side of (2.5). Due to the second inequality in (2.7) its summands can be estimated from above by

$$P_0\left[\sigma_k < \infty, X_{\sigma_k} = x, B_{x-s_e}, \{X_n : n \geq \sigma_k\} \nsubseteq S + x - s_e\right] \quad (2.9)$$

$$= E_0\left[P_{x,\psi(\omega,(X_n)_{n<\sigma_k})}\left[\{X'_n : n \geq 0\} \nsubseteq S + x - s_e\right], \sigma_k < \infty, X_{\sigma_k} = x, B_{x-s_e}\right],$$

where we used the strong Markov property. By the first inequality in (2.7) we may replace in the last expression $\psi(\omega, (X_n)_{n<\sigma_k})$ by ω and get that the quantity on the right-hand side of (2.9) is equal to

$$\mathbb{E}\left[P_{0,\omega}\left[\sigma_k < \infty, X_{\sigma_k} = x\right] P_{x,\omega}\left[\{X_n : n \geq 0\} \nsubseteq S + x - s_e\right], B_{x-s_e}\right]$$

$$\overset{\text{(IID)}}{=} P_0\left[\sigma_k < \infty, X_{\sigma_k} = x\right] \mathbb{E}\left[P_{0,\omega}\left[\{X_n : n \geq 0\} \nsubseteq S - s_e\right], B_{-s_e}\right]. \quad (2.10)$$

On the event B_{-s_e} any walker who starts inside the cluster $S - s_e = \mathcal{C}_{-s_e}$ at 0 and eventually leaves this cluster has to cross a bond $(y, y + f)$ with $y \in \mathcal{C}_{-s_e}$ and $y + f \notin \mathcal{C}_{-s_e}$. Therefore, on B_{-s_e},

$$P_{0,\omega}\left[\{X_n \mid n \geq 0\} \nsubseteq S - s_e\right]$$

$$\leq 1 - \prod_{y \in \mathcal{C}_{-s_e}, i \in \mathbb{N}} \left(1 - \sum_{f \in \mathcal{E}: y+f \notin \mathcal{C}_{-s_e}} \omega(y, y+f, i)\right)$$

$$\leq 1 - \exp\left(-c \sum_{y \in \mathcal{C}_{-s_e}, i \in \mathbb{N}, f \in \mathcal{E}: y+f \notin \mathcal{C}_{-s_e}} \omega(y, y+f, i)\right) \leq 1 - e^{-c\gamma},$$

where c is a finite constant such that $\ln(1-x) \geq -cx$ for all $0 \leq x \leq 1-\gamma^{-1}$. Summarizing, we get from (2.9) and (2.10) that the sum in (2.5) is at most $P_0\left[\sigma_k < \infty\right](1 - e^{-c\gamma}) \mathbb{P}[B_0]$. Together with (2.8) this implies by induction

$$P_0[\sigma_{k+1} < \infty] \leq (1 - e^{-c\gamma}\mathbb{P}[B_0])P_0[\sigma_k < \infty] \leq (1 - e^{-c\gamma}\mathbb{P}[B_0])^k.$$

Hence there is P_0-a.s. some k with $\sigma_k = \infty$, which implies the finiteness of the range. $\qquad\square$

3. Recurrence and Transience

Since ERW is not a Markov chain the standard definitions of recurrence and transience do not apply right away in this setting.

Definition 3.1. We call an ERW X satisfying (SE) *recurrent* if it visits P_0-a.s. every site $z \in \mathbb{Z}^d$ infinitely often. It is called *transient* if P_0-a.s. no site is visited infinitely often, i.e. if P_0-a.s. $|X_n| \to \infty$ as $n \to \infty$. For any direction $\ell \in \mathbb{R}^d \backslash \{0\}$ we say that the ERW is *transient in direction* ℓ if P_0-a.s. $X_n \cdot \ell \to \infty$ as $n \to \infty$. We set

$$A_\ell := \left\{ \lim_{n \to \infty} X_n \cdot \ell = \infty \right\}.$$

In $d = 1$ transience in direction 1 (resp. -1) is also called *transience to the right* (*resp. left*).

3.1. Results for $d = 1$

Theorem 3.2. ($d = 1$) *Assume* (SE) *and* (EL). *Then* $P_0[|X_n| \to \infty] = P_0[A_1 \cup A_{-1}] \in \{0, 1\}$.

The proof uses the following lemma. This type of lemma is standard, see e.g. [Zer06, Lem. 9] for references.

Lemma 3.3. ($d = 1$) *Assume* (SE), (WEL), *and* $P_0[A_1] > 0$. *Then also*

$$P_0[\{\forall n \geq 0 : X_n \geq 0\} \cap A_1] > 0. \tag{3.1}$$

Sketch of the proof of Lemma 3.3. First, one shows that for all $z \in \mathbb{Z}$ and $\omega \in \Omega$ such that $\omega(z, 1, i) > 0$ for all $i \geq 1$ and $P_{z,\omega}[A_1] > 0$ one also has

$$P_{z,\omega}[\{\forall n \geq 0 : X_n \geq z\} \cap A_1] > 0. \tag{3.2}$$

The proof of this can be done along the lines of the proof of [Zer05, Lem. 8]. One simply erases the (finitely many) excursions from z to the left of z and visits the sites to the right of z in the same order as before, see Figure 1. For the proof of (3.1) note that by assumptions (SE) and (WEL) there is $\mathbb{P}$-a.s. some $z \geq 0$ such that
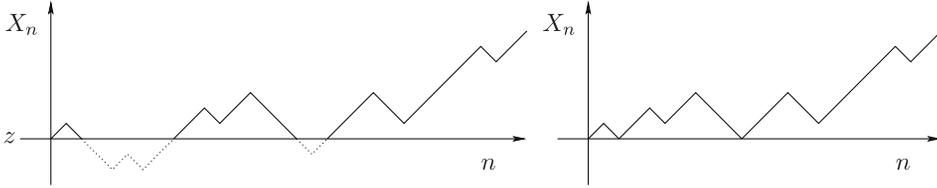
Figure 1: For the proof of (3.2).

$$0 < P_0[A_1, \ \forall i \in \mathbb{N} : \omega(z, 1, i) > 0]$$

$$= E_0 \left[P_{z, \psi(\omega, (X_n)_{n < T_z})}[A_1], \ T_z < \infty, \ \forall i \in \mathbb{N} : \omega(z, 1, i) > 0 \right].$$

Therefore, by (3.2),

$$
\begin{aligned}
0 \ &< \ E_0 \left[P_{z, \psi(\omega, (X_n)_{n < T_z})}[\{\forall n \geq 0 : X'_n \geq z\} \cap A_1], \ T_z < \infty \right] \\
&\leq \ P_z[\{\forall n \geq 0 : X_n \geq z\} \cap A_1]
\end{aligned}
\tag{3.3}
$$

since $\psi(\omega, (X_n)_{n < T_z})(x, \cdot, \cdot) = \omega(x, \cdot, \cdot)$ for $x \geq z$. Due to (SE) the right-hand side of (3.3) is equal to $P_0[\{\forall n \geq 0 : X_n \geq 0\} \cap A_1]$. $\qquad \square$

Proof of Theorem 3.2. Since the ERW has bounded jumps we have P_0-a.s. $\{|X_n| \to \infty\} = A_1 \cup A_{-1}$. Now we consider the four cases of Theorem 2.3. In cases (a), (b) and (c) the 0-1 statement is obvious due to Theorem 2.3. For case (d) we assume $b_1 = b_{-1} = 0$ and $P_0[|X_n| \to \infty] > 0$. Without loss of generality we assume $P_0[A_1] > 0$. By Lemma 3.3 and (EL), $P_0[\forall n \geq 0 : X_n \geq 0] > 0$. Therefore, P_0-a.s.,

$$\{|X_n| \to \infty\}^c \subseteq \bigcup_{z \in \mathbb{Z}} \{X_n = z \text{ i.o.}\} \overset{\text{L. 2.2}}{\subseteq} \left\{ \sup_{n \geq 0} X_n = \infty \right\} \overset{(2.2)}{\subseteq} A_1$$
$$\subseteq \{|X_n| \to \infty\}.$$

This implies $P_0[|X_n| \to \infty] = 1$. $\qquad \square$

Corollary 3.4. $(d = 1)$ *Assume* (SE) *and* (EL). *Then the ERW is either recurrent or transient or has P_0-a.s. finite range.*

Proof. We consider again the four cases of Theorem 2.3. In cases (a)-(c) the claim of the corollary is obviously true. For case (d) assume $b_1 = 0 = b_{-1}$. According to Theorem 3.2, $P_0[|X_n| \to \infty]$ is either 0 or 1. In the latter case

the walk is transient. In the former case there is P_0-a.s. some $z \in \mathbb{Z}$ which is visited infinitely often. By Lemma 2.2 all other sites are P_0-a.s. visited infinitely often as well, i.e. the walk is recurrent.   □

Problem 3.5. $(d = 1)$ The event in Theorem 2.3 that the range is infinite can be rephrased as $\{\sup_n |X_n| = \infty\}$. Dropping there and in Theorem 3.2 the absolute values raises the following problem: find conditions which imply

$$P_0 \left[\sup_{n \geq 0} X_n = \infty \right], \ P_0 \left[\inf_{n \geq 0} X_n = -\infty \right] \ \in \ \{0, 1\} \qquad \text{and/or} \quad (3.4)$$

$$P_0[X_n \to \infty], \ P_0[X_n \to -\infty] \ \in \ \{0, 1\}. \qquad (3.5)$$

Two such sets of conditions are given in Theorem 3.10 below. It is not difficult to see, using Theorems 3.2 and 2.3 and Lemma 2.2, that under the assumptions (SE) and (EL) the two 0-1 statements (3.4) and (3.5) are equivalent. See also Problem 3.20.

Next we present some criteria for recurrence or transience. First we consider the case $M(z) \leq 1$, $z \in \mathbb{Z}$, where the walk is not getting excited about the sites which it has visited before.

Proposition 3.6. $(d = 1)$ *Assume* (SE), (WEL) *and* $\mathbb{P}$-*a.s.* $M(0) \leq 1$. *Then the ERW is recurrent.*

One could prove this statement by combining a monotonicity argument as in Proposition 4.2 below (see e.g. [Zer05, Th. 16]) with Theorem 3.10 (3.9) below. However, we present here a more direct proof in the spirit of [BW03, Section 2].

Lemma 3.7. *Let* $(Z_n)_{n \in \mathbb{N}}$ *be a stationary and ergodic sequence of non-negative random variables with* $E[Z_1] > 0$. *Then* $\sum_n Z_n/n = \infty$ *almost surely.*

Proof of Lemma 3.7. By the ergodic theorem one can recursively choose positive integers $(n_k)_{k \in \mathbb{N}}$ such that for all $k \geq 1$, $n_k \geq s_{k-1} := n_1 + \ldots + n_{k-1}$ and

$$P[B_k^c] \leq k^{-2}, \quad \text{where} \quad B_k := \left\{ \frac{1}{n_k} \sum_{n=s_{k-1}+1}^{s_k} Z_n \geq E[Z_1]/2 \right\}.$$

By the Borel-Cantelli lemma there is a random $K \in \mathbb{N}$ such that almost surely all events B_k, $k \geq K$, occur. Hence, since $Z_n \geq 0$ almost surely,

$$\sum_{n \geq 1} \frac{Z_n}{n} \geq \sum_{k \geq K} \sum_{n=s_{k-1}+1}^{s_k} \frac{Z_n}{n} \geq \sum_{k \geq K} \frac{n_k}{s_k} \frac{1}{n_k} \sum_{n=s_{k-1}+1}^{s_k} Z_n \geq \sum_{k \geq K} \frac{E[Z_1]}{4} = \infty. \quad \square$$

***Proof of Proposition* 3.6.** By Theorem 2.3 (d), P_0-a.s. $\sup_{n \geq 0} |X_n| = \infty$. Therefore,

$$P_0[\{X_n = 0 \text{ i.o.}\}^c] \leq P_0 \left[\{X_n = 0 \text{ i.o.}\}^c \cap \left\{ \sup_{n \geq 0} X_n = \infty \right\} \right] \qquad (3.6)$$

$$+ P_0 \left[\{X_n = 0 \text{ i.o.}\}^c \cap \left\{ \inf_{n \geq 0} X_n = -\infty \right\} \right]. \; (3.7)$$

The term on the right-hand side of (3.6) is equal to

$$P_0 \left[\bigcup_{L \in \mathbb{N}} \{\forall k \geq L : T_k < \infty\} \cap \{\forall n \geq T_L : X_n > 0\} \right]$$

$$= \lim_{L \to \infty} \mathbb{E} \left[P_{0,\omega}[T_L < \infty] \prod_{k \geq L} P_{k,\omega_k'}[T_{k+1} < T_0] \right], \qquad (3.8)$$

where $\omega_k' := \psi(\omega, (0, 1, \ldots, k-1))$. Here we used that $P_{0,\omega}$-a.s. $P_{k,\psi(\omega,(X_n)_{n<T_k})}$ $[T_{k+1} < T_0] = P_{k,\omega_k'}[T_{k+1} < T_0]$. Observe that

$$P_{k,\omega_k'}[T_{k+1} < T_0] = 1 - \omega(k, -1, 1) P_{k-1,\omega_{k+1}'}[T_{k+1} > T_0] = 1 - \frac{2\omega(k, -1, 1)}{k+1}$$

by the gambler's ruin problem for the simple symmetric RW on $\mathbb{Z}$. Therefore, Lemma 3.7 applied to $Z_{n+1} := 2\omega(n, -1, 1)$ yields that the infinite product in (3.8) $\mathbb{P}$-a.s. vanishes. Therefore, the term on the right-hand side of (3.6) is zero. By symmetry the same holds for the expression in (3.7). Consequently, by Lemma 2.2, any $z \in \mathbb{Z}$ is P_0-a.s. visited infinitely often. $\qquad \square$

Remark 3.8. ($d = 1$, first return time to the origin) [AR05, Section 3.3] deals with model (BW) for $d = 1$ and any $p \in (0, 1)$ and employs a physical approach to show that $P_1[T_0 > n] \sim n^{p-1}$ as $n \to \infty$.

If one allows more than just one cookie per site then the walk may become transient. Whether this happens or not depends under certain con-

ditions on the parameter

$$\delta := \mathbb{E}\left[\sum_{e\in\mathcal{E}, i\geq 1} \omega(0, e, i)e\right]$$

if it exists as an element of $(\mathbb{R} \cup \{\pm\infty\})^d$. (We shall see in Sections 5 and 6 that for $d = 1$ the parameter δ characterizes other phase transitions as well.) To interpret δ, observe that after consuming a cookie $\omega(z, \cdot, i)$ the walk is displaced on average by $\sum_e \omega(z, e, i)e$, which we call the *drift* stored in that cookie. Thus the parameter δ is, when it exists, the expected total drift stored in a cookie stack.

Remark 3.9. $(d = 1,$ **w.l.o.g.** $\delta \geq 0)$ Observe that for $d = 1$ replacing ω by $1 - \omega$ switches the sign of δ and that for each ω the distribution of X under $P_{0,1-\omega}$ is the same as of $(-X_n)_{n\geq 0}$ under $P_{0,\omega}$. Hence when investigating the behavior of X one may restrict oneself to the case $\delta \geq 0$.

Theorem 3.10. $(d = 1,$ **recurrence, transience [Zer05, Th. 12] [KZ08, Th. 1]**) *Assume either*

$$(\text{SE}), (\text{POS}_1) \text{ and } \mathbb{P}[\omega(0, 1, \cdot) = (1, 1/2, 1/2, \ldots)] < 1 \qquad or \qquad (3.9)$$

$$(\text{IID}), (\text{BD}), \text{ and } (\text{WEL}). \qquad (3.10)$$

Then the ERW is recurrent if $\delta \in [-1, 1]$, transient to the right if $\delta > 1$ and transient to the left if $\delta < -1$.

Note that in the case $\mathbb{P}[\omega(0, 1) = (1, 1/2, 1/2, \ldots)] = 1$ one has trivially P_0-a.s. $X_n = n$, which makes the walk transient to the right although $\delta = 1$. Observe also that the above criterion for recurrence/transience of ERW is quite different from the one for one-dimensional RWRE due to Solomon, see e.g. [Zei04, Th. 2.1.2].

Idea of the proof of Theorem 3.10 in case (3.9). Let $\delta \geq 0$ and assume that the walk is either recurrent or transient to the right. Consider

$$M_n := X_n - D_n, \text{ where } D_n := \sum_{m=0}^{n-1} (2\omega\left(X_m, 1, \#\{k \leq m \mid X_k = X_m\}\right) - 1)$$

is the total drift stored in the cookies which have been consumed by the walker before time n. By the Doob-Meyer decomposition of the submartingale $(X_n)_{n\geq0}$ the process $(M_n)_{n\geq0}$ is a $P_{0,\omega}$-martingale w.r.t. its canonical filtration. A naive application of the optional stopping theorem yields that the expected drift $E_{0,\omega}[D_{T_k}]$ stored in the cookies which have been consumed by the walk by time T_k, $k \geq 0$, is equal to $E_{0,\omega}[X_{T_k}] = k$. We now compare this quantity to the expected total drift $k\delta$ stored in all the cookies $\omega(z,\cdot,i), i \geq 1, 0 \leq z < k$.

If the walk is transient to the right then it consumes only finitely many cookies to the left of 0. Hence, if $\delta < 1$ then even consuming all the cookies between 0 and k before time T_k would not be enough, for k large, to satisfy the required total demand $E_{0,\omega}[D_{T_k}] = k$. Hence the walk cannot be transient to the right if $\delta < 1$.

Conversely, if $\delta > 1$ then the walk cannot afford to return too often to 0 before time T_k because otherwise it would eventually eat most of the cookies between 0 and $k-1$ and thus exceed its dietary restriction $E_{0,\omega}[D_{T_k}] = k$. This indicates that the walk cannot be recurrent in this case.

This argument can be made precise in case (3.9). An even less formal argument is given in [AR05, p. 2569]. □

The proof of Theorem 3.10 in case (3.10) relies on branching processes similar to the ones used in the proof of Theorem 5.2, which is sketched below.

Problem 3.11. Can one replace in (3.10) the assumption (IID) by (SE)?

Problem 3.12. Compute in the transient case the probability $P_0[\forall n \geq 1 : X_n \neq 0]$ never to return to the starting point. See [Zer05, Th. 18] for an example.

Remark 3.13. (Strips) A result similar to Theorem 3.10 has been shown in [Zer06, Th. 2] for ERWs on strips $\mathbb{Z} \times \{0, 1, \ldots, L-1\}, L \in \mathbb{N}$, under assumptions similar to (3.9).

Remark 3.14. ($d = 1$, RW in "have your cookie and eat it" environments) A self-interacting RW $Y = (Y_n)_{n\geq0}$ on $\mathbb{Z}$ called RW in a "have your cookie and eat it" environment has been introduced in [Pin10]. There, "at each site x, the probability of jumping to the right is $\omega(x) \in [1/2, 1)$, until the first time the process jumps to the left from site x, from which time

onward the probability of jumping to the right" from that site is $1/2$. Here the sequence $(\omega(x))_{x \in \mathbb{Z}}$ (with values in $[1/2, 1)^{\mathbb{Z}}$) is assumed to be stationary and ergodic.

Note that if $\omega(x) = q$ for $x < 0$ the RW Y behaves on the negative integers in the same way as the RW perturbed at extrema, which we described in Section 1.4. There we also showed how a RW perturbed at extrema can be viewed as an ERW. The same applies to Y with the difference that the measure $\mathbb{P}$ which provides the environment for Y does not lack spatial homogeneity but satisfies (SE). Note however that neither (POS$_1$) nor (BD) nor (WEL) are fulfilled by this $\mathbb{P}$. Nevertheless, although $\mathbb{P}$ does not meet the requirements of Theorem 3.10 the conclusion of Theorem 3.10 is still true for this $\mathbb{P}$ as stated in [Pin10, Th. 2]. See also Remark 5.6.

Remark 3.15. ($d = 1$, **RWRE as underlying process**) In [Bau] the simple symmetric RW as underlying process is replaced by a RWRE which is transient, say, to the left. On the first $M_z \geq 0$ visits to $z \in \mathbb{Z}$ the ERW is deterministically pushed to $z + 1$, only on later visits to that site the RWRE environment takes effect. Here the random environment and $(M_z)_{z \in \mathbb{Z}}$ are i.i.d. with $\mathbb{P}[M_0 = 0] > 0$. Sufficient criteria for transience to the left, recurrence or transience to the right of the resulting ERW are given in terms of the tail of the distribution of M_0.

3.2. Results for $d \geq 2$

The following result is the multidimensional analogue to Theorem 3.2. It generalizes Kalikow's zero-one law for directional transience of multidimensional RWRE as stated in [ZM01, Prop. 3], see also [Zei04, Th. 3.1.2] and [HSa, Th. 1.3] for versions with different hypotheses.

Theorem 3.16. ($d \geq 1$, **Kalikow-type zero-one law**)
Assume (IID), (EL) *and let* $\ell \in \mathbb{R}^d \backslash \{0\}$. *Then* $P_0[|X_n \cdot \ell| \to \infty] = P_0[A_\ell \cup A_{-\ell}] \in \{0, 1\}$.

The proof follows the one given in [ZM01, Prop. 3]. It uses the following two lemmas. The first one is the multidimensional analogue of Lemma 3.3.

Lemma 3.17. ($d \geq 1$) *Assume* (SE), (EL), $\ell \in \mathbb{R}^d \backslash \{0\}$, *and* $P_0[A_\ell] > 0$. *Then* $P_0[\{\forall n \geq 0 : X_n \cdot \ell \geq 0\} \cap A_\ell] > 0$.

The proof of this lemma is similar to the proof of Lemma 3.3 outlined above and is identical to that of [Zer06, Lem. 9]. (The general assumption (UEL) in [Zer06] is not needed for the proof of [Zer06, Lem. 9] and can be replaced by (EL).)

Let us denote for any interval $I \subseteq \mathbb{R}$ by S_I the slab $\{x \in \mathbb{Z}^d : x \cdot \ell \in I\}$. The second lemma gives conditions under which the ERW cannot visit any slab $S_{[u,w]}$, $u < w$, infinitely often without ever visiting both neighboring half spaces.

Lemma 3.18. $(d \geq 1)$ *Assume* (IID), $\mathbb{E}[\omega(0, e, 1)] > 0$ *for all* $e \in \mathcal{E}$ *and let* $\ell \in \mathbb{R}^d \backslash \{0\}$ *and* $u, w \in \mathbb{R}$ *with* $u < w$. *Furthermore assume that the range of the ERW is* P_0-*a.s. infinite. Then*

$$P_0[\{X_n \cdot \ell \geq u \text{ i.o.}\} \cap \{\forall n \geq 0 : X_n \cdot \ell \leq w\}] = 0. \qquad (3.11)$$

***Proof of Lemma* 3.18.** The proof is a bit more involved than the one of the corresponding statement [ZM01, Lem. 4] for RWRE. Denote by A the event considered in (3.11) and let F be the event that the walker visits only finitely many distinct elements of $S_{[u,w]}$. It is enough to show that $P_0[A \cap F] = 0 = P_0[A \cap F^c]$.

First, we consider $P_0[A \cap F^c]$. Without loss of generality assume $\ell \cdot e_1 > 0$. Then there is $N \in \mathbb{N}$ such that for all $x \in S_{[u,w]}$ we have $(x + Ne_1) \cdot \ell > w$. On the event $A \cap F^c$ the walker visits infinitely many sets $S_{[u,w]} \cap (\mathbb{Z} \times \{y\})$, $y \in \mathbb{Z}^{d-1}$, since each such set is finite. Each time the walker visits such a set for the first time he has, due to (IID), independently of his past the P_0-probability $\mathbb{E}[\omega(0, e_1, 1)]^N$ to walk in the next N steps in direction e_1 thus reaching the half space $S_{(w,\infty)}$. Having infinitely many independent such chances the walker will P_0-a.s. not miss all of them. Therefore, $P_0[A \cap F^c] = 0$.

Our treatment of $P_0[A \cap F]$ deviates from the corresponding step in the proof of [ZM01, Lem. 4]. The proof is by contradiction. Assume that $P_0[A \cap F] > 0$ and recall Definition 2.1. Then there is some $k \in \mathbb{N}$ such that $P_0[A \cap F_k] > 0$, where F_k is the event that the walker (a) visits at most k elements of $S_{[u,w]}$, (b) has infinite range, (c) does not cross any directed edge (y, z) infinitely often unless $y \xrightarrow{\omega} z$, and (d) visits every element of $\mathcal{C}_y$ infinitely often whenever $y \in \mathbb{Z}^d$ is visited infinitely often. (The events in (b)-(d) have full P_0-measure.)

Denote for $x \in \mathbb{Z}^d$ by $B_x \subseteq \Omega$ the event that there is a self-avoiding nearest-neighbor path $(y_n)_{n \geq 0}$ starting at $y_0 = x$ such that for all $n \in \mathbb{N}$,

$$x \in \mathcal{C}_{y_n}, \tag{3.12}$$

$$\mathcal{C}_{y_n} \subseteq S_{(-\infty, x \cdot \ell]} \quad \text{and} \tag{3.13}$$

$$\#\left(\mathcal{C}_{y_n} \cap S_{\{x \cdot \ell\}}\right) \leq k. \tag{3.14}$$

If B_x occurs denote by $(y_n(x))_{n \geq 0}$ a path with these properties. Choose it according to some deterministic rule if there are several such paths.

On the event $A \cap F_k$ there is at least one random vertex x which is visited infinitely often and maximizes $x \cdot \ell$. The maximal number of such vertices is at most k due to (a). For each such x the event B_x occurs. Indeed, by König's lemma [Kön27, Lemma A] for directed graphs and (b) there is a random self-avoiding nearest-neighbor path $(y_n)_{n \geq 0}$ starting at $y_0 = x$ such that all its directed edges (y_n, y_{n-1}), $n \in \mathbb{N}$, are crossed infinitely many times. Any such path satisfies (3.12)-(3.14) for all $n \in \mathbb{N}$. (3.12) holds since $y_m \overset{\omega}{\to} y_{m-1}$ for all $m \in \mathbb{N}$ due to (c). Moreover, since x maximizes $y \mapsto y \cdot \ell$ among all vertices which are visited infinitely often (d) implies (3.13). And since the walker visits at most k vertices which maximize $y \mapsto y \cdot \ell$ we also have (3.14).

Therefore, $0 < P_0[A \cap F_k] \leq \mathbb{P}[\bigcup_x B_x]$. Hence $\mathbb{P}[B_x] > 0$ for some $x \in \mathbb{Z}^d$. Due to (IID), $(\mathbf{1}_{B_x})_{x \in \mathbb{Z}^d}$ is stationary under $\mathbb{P}$ w.r.t. the shifts on $\mathbb{Z}^d$. Hence $\mathbb{P}[B_0] = \mathbb{P}[B_x] > 0$. Choose $m \in \mathbb{N}$ with $m + 1 \geq 3k/\mathbb{P}[B_0]$. Since $(\mathbf{1}_{B_x})_{x \in \mathbb{Z}^d}$ is also ergodic there is by the ergodic theorem, see e.g. [Ell06, Th. A.11.5], $\mathbb{P}$-a.s. some random $L \in \mathbb{N}$ such that

$$(m+1) \sum_{x \in [-L,L]^d} \mathbf{1}_{B_x} > \frac{m+1}{2} \mathbb{P}[B_0] \#[-L,L]^d \geq k \#[-L-m, L+m]^d. \tag{3.15}$$

For all $n = 0, \ldots, m$ and all $x \in [-L, L]^d$ for which B_x occurs we have $y_n(x) \in [-L - m, L + m]^d$ since $(y_n(x))_{n=0,\ldots,m}$ is a nearest-neighbor path starting at x. By (3.15) there are strictly more than $k \#[-L-m, L+m]^d$ such pairs (n, x). It follows from the pigeonhole principle that there are $y \in [-L-m, L+m]^d$ and pairwise distinct pairs $(n_0, x_0), \ldots, (n_k, x_k) \in \{0, \ldots, m\} \times [-L, L]^d$ such that $y = y_{n_i}(x_i)$ for all $i = 0, \ldots, k$. By (3.12) and (3.13), $x_0, \ldots, x_k \in \mathcal{C}_y \subseteq S_{(-\infty, \min_i x_i \cdot \ell]}$. Therefore, $x_0 \cdot \ell = \ldots = x_k \cdot \ell$ and hence $x_0, \ldots, x_k \in \mathcal{C}_y \cap S_{\{x_0 \cdot \ell\}}$. However, since every path $(y_n(x_i))_{n \geq 0}$,

$i = 0, \ldots, k$, is self-avoiding the sites $x_0, \ldots, x_k$ are pairwise distinct as well. But this contradicts (3.14). $\qquad\square$

***Sketch of the proof of Theorem* 3.16.** By Theorem 2.5 the range of the ERW is either finite or infinite. In the first case the statement is obvious. In the second case the proof goes along the same lines as the one of [ZM01, Prop. 3]. Since the increments of the ERW are uniformly bounded P_0-a.s. exactly one of the three events A_ℓ or $A_{-\ell}$ or $\bigcup_{u<w} A_{u,w}$ occurs, where $A_{u,w} := \{X_n \cdot \ell \in [u, w] \text{ i.o.}\}$. Suppose that $P_0[A_\ell] > 0$ and let $u < w$. By Lemma 3.18 we have P_0-a.s. $\sup_n X_n \cdot \ell = \infty$ on $A_{u,w}$. However, each time the process $(X_n \cdot \ell)_n$ reaches a new maximum $x > w$ it has by Lemma 3.17 and (IID) the same positive chance never to fall back again below x. Consequently, after a geometric number of trials $(X_n \cdot \ell)_n$ has reached a level larger than w below which it will never fall again. Hence, P_0-a.s. $A_{u,w} \subseteq A_{u,w}^c$, i.e. $P_0[A_{u,w}] = 0$. $\qquad\square$

The main result of [BW03] is that under assumption (BW) the ERW is transient in direction e_1 whenever $d \geq 2$. For the proof, one first couples the ERW X in the way described in Section 1.1 to a simple symmetric RW $Y = (Y_n)_{n \geq 0}$. Then one considers so-called *tan points* of Y. These are sites $x \in \mathbb{Z}^d$ which are visited by Y prior to any other point on the "sun ray" $\{x + ke_1 \mid k \in \mathbb{N}\}$. Every time when Y reaches a tan point, X reaches a fresh site and eats a non-placebo cookie which pushes it in direction e_1. Showing that Y has enough tan points then implies transience of X in direction e_1.

However, it seems difficult to adapt the method of tan points to other settings in which the drift is not along a coordinate direction or the excitement occurs at later visits. However, by combining the martingale approach explained after Theorem 3.10 and the method of the environment viewed from the particle this result was extended in [Zer06, Th. 1] to the following more general result.

Theorem 3.19. ($d \geq 2$, directional transience)
Assume (IID) *and* (UEL) *and let* $\ell \in \mathbb{R}^d \backslash \{0\}$ *such that* (POS$_\ell$) *holds with* $\delta \cdot \ell > 0$. *Then the ERW is transient in direction* ℓ, *i.e.* $P_0[A_\ell] = 1$.

Problem 3.20. ($d \geq 2$) Find conditions which imply the zero-one law $P_0[A_\ell] \in \{0, 1\}$ for all $\ell \in \mathbb{R}^d \backslash \{0\}$ (cf. Problem 3.5). Are e.g. (IID), (BD) and (UEL) sufficient (just as for $d = 1$, see (3.10) of Theorem 3.10)?

For RWRE this zero-one law holds under (IID) and (EL) if $d = 2$, see [ZM01, Th. 1]. According to [HSa, Th. 1.5] the assumption (EL) can be dropped. For $d \geq 3$ it is an important open problem, see e.g. [DR10, Open Problem 1.4].

Not much is known in general about recurrence and transience in the sense of Definition 3.1 for $d \geq 2$.

Problem 3.21. ($d \geq 2$) Assume (IID) and (UEL). Is the ERW either recurrent or transient?

Remark and Problem 3.22. ($d \geq 2$, **balanced ERW**) The work [BKS11] introduces so-called $M(d_1, d_2)$-RW. This is an ERW on $\mathbb{Z}^{d_1+d_2}$ which upon the first visit to a vertex performs a d_1-dimensional simple symmetric RW step within the first d_1-coordinates and upon later visits to that same vertex performs a simple symmetric RW step within the last d_2-coordinates. It is proved that the $M(2,2)$-RW is transient and conjectured that the $M(1,2)$- and the $M(2,1)$-walk are transient as well while the $M(1,1)$-walk is believed to be recurrent.

More generally, consider measures $\mathbb{P}$ which are balanced in the sense that $\mathbb{P}$-a.s. $\omega(z, e, i) = \omega(z, -e, i)$ for all $z \in \mathbb{Z}^d, e \in \mathcal{E}, i \in \mathbb{N}$. It follows from recent work [PPS, Th. 1.2, Prop. 1.4] that any balanced ERW satisfying (UEL) in $d \geq 3$ returns P_0-a.s. only finitely often to 0 if there are at most $\max\{2, (d-1)/2\}$ cookies $\omega_1, \ldots, \omega_k \in \mathcal{M}_{\mathcal{E}}$ such that for all $z \in \mathbb{Z}^d$ and $i \in \mathbb{N}$ there is $\mathbb{P}$-a.s. some $j \in \{1, \ldots, k\}$ such that $\omega(z, \cdot, i) = \omega_j$.

One may wonder, whether any balanced ERW satisfying (IID) (or (SE)) and (UEL) is recurrent if $d = 2$ and transient if $d \geq 3$. For RWRE this is true, see [Zei04, Th. 3.3.22].

Remark 3.23. ($d = 3$, **ERW against a wall**) [ABK08] deals with an ERW in a special spatially non-homogeneous environment ω on $\mathbb{Z}^3$. The walk starts at 0, is pushed down whenever it reaches a new site, but is prohibited to enter the lower half space (i.e. $\omega((x, y, z), -e_3, 1) = 1$ and $\omega((x, y, 0), -e_3, i) = 0$ for all $x, y, z \in \mathbb{Z}, z \neq 0, i \in \mathbb{N}$) and otherwise behaves like a simple symmetric RW. It is shown, in particular, that this walk returns a.s. infinitely often to its starting point.

A similar statement regarding recurrence of an ERW which is "excited to the origin" is made in [Koz07, (1)]. In [Koz07] one can find related open problems.

4. Strong Law of Large Numbers

We say that the ERW X satisfies a strong law of large numbers if there is a non-random $v \in \mathbb{R}^d$, called the *velocity* or *speed* of the walk, such that

$$\lim_{n \to \infty} \frac{X_n}{n} = v \quad P_0\text{-a.s..}$$

4.1. Results for $d = 1$

Theorem 4.1. ($d = 1$, **law of large numbers**) *If* (SE) *and* (POS$_1$) *hold then* X *satisfies a strong law of large numbers with speed* $v \in [0, 1]$. *If* (SE) *and the 0-1 law* (3.4) *hold then* X *satisfies a strong law of large numbers with speed* $v \in [-1, 1]$.

Proof. For the first statement see [Zer05, Th. 13]. For the second statement observe that although the proof of [KZ08, Prop. 13] has been stated under stronger conditions it provides a proof of the second claim if $P_0 [\sup_n X_n = \infty] = 1$ or $P_0 [\inf_n X_n = -\infty] = 1$. In the remaining case, when both probabilities are 0, one trivially has P_0-a.s. $X_n/n \to 0$.   □

Next we discuss some properties of the speed as a function of $\mathbb{P}$. We call a cookie $\omega_2(z, \cdot, i)$ *stronger* than another cookie $\omega_1(z, \cdot, i)$ if it pushes its consumer more to the right, i.e. if $\omega_2(z, 1, i) \geq \omega_1(z, 1, i)$. One should expect that making cookies stronger does not decrease the speed of the walk. More precisely, an environment ω_2 is called stronger than another environment ω_1 if each cookie $\omega_2(z, \cdot, i)$ in ω_2 is stronger than the corresponding cookie $\omega_1(z, \cdot, i)$ in ω_1. A probability measure $\mathbb{P}_2$ on Ω is called stronger than another probability measure $\mathbb{P}_1$ if there is a probability measure on $\{(\omega_1, \omega_2) \in \Omega^2 \mid \omega_2 \text{ is stronger than } \omega_1\}$ with first marginal $\mathbb{P}_1$ and second marginal $\mathbb{P}_2$. The following statement is contained in [Zer05, Th. 17] under the additional assumption (POS$_1$).

Proposition 4.2. ($d = 1$, **monotonicity of** v) *If $\mathbb{P}_2$ is stronger than $\mathbb{P}_1$ and if the ERW satisfies under both measures $\mathbb{E}_i[P_{0,\omega}[\cdot]]$, $i = 1, 2$, a law of large numbers with speed v_i then $v_2 \geq v_1$.*

Proof. First, we show that for $i = 1, 2$,

$$\lim_{k \to \infty} \frac{k}{T_k} = \max\{v_i, 0\} \qquad \mathbb{E}_i[P_{0,\omega}[\cdot]]\text{-a.s..} \qquad (4.1)$$

If $\sup_n X_n = \infty$ then the sequence $(k/T_k)_{k \geq 0}$ is a subsequence of $(X_n/n)_{n \geq 0}$. Therefore, $\mathbb{E}_i[P_{0,\omega}[\cdot]]$-a.s. $\{\sup_n X_n = \infty\} \subseteq \{\lim_k k/T_k = v_i \geq 0\}$. On the other hand, if $\sup_n X_n < \infty$ then $T_k = \infty$ for some k and thus $\mathbb{E}_i[P_{0,\omega}[\cdot]]$-a.s. $\{\sup_n X_n < \infty\} \subseteq \{\lim_k k/T_k = 0 \geq v_i\}$. In any case (4.1) holds. From this one can conclude like in the proof of [Zer05, Th. 17] that $\max\{v_2, 0\} \geq \max\{v_1, 0\}$. (The general assumption (POS$_1$) present in [Zer05] is not needed for this conclusion.) Analogously, by considering T_{-k} instead of T_k we obtain $\min\{v_2, 0\} \geq \min\{v_1, 0\}$. Hence $v_2 \geq v_1$. □

Remark 4.3. ($d = 1$, **monotonicity of** v **w.r.t. swapping cookies**) In [HSa12, Section 5] yet another monotonicity property for v is discussed. It is shown that "one cannot decrease the (lim sup)-speed of a cookie RW by swapping stronger cookies in a pile with weaker cookies that appear earlier in the same pile (and doing this at each site)". For the precise statement we refer to [HSa12, Th. 5.1].

Remark 4.4. ($d = 1$, **continuity of** v) In [BS08a] deterministic environments ω of the form $\omega(z, 1, \cdot) = (p_1, \ldots, p_M, 1/2, 1/2, \ldots)$ are considered, where $M \in \mathbb{N}$ and $\bar{p} = (p_1, \ldots, p_M) \in [1/2, 1)^M$ are fixed. It is shown in [BS08a, Th. 1.2] that the corresponding speed v depends continuously on $\bar{p}$. It is plausible that the method of proof and result may work under the assumptions (IID), (BD) and (WEL) as well.

4.2. Results for $d \geq 1$

In the (IID) setting a regeneration structure, which goes back to [KKS75] and has been heavily used to study RWRE, see e.g. [Zei04, Section 3.2], can be straightforwardly extended to include ERW, see e.g. [Zer05, p. 114, Rem. 3] and [BR07, Section 3]. See also Figure 2 for $d = 1$.
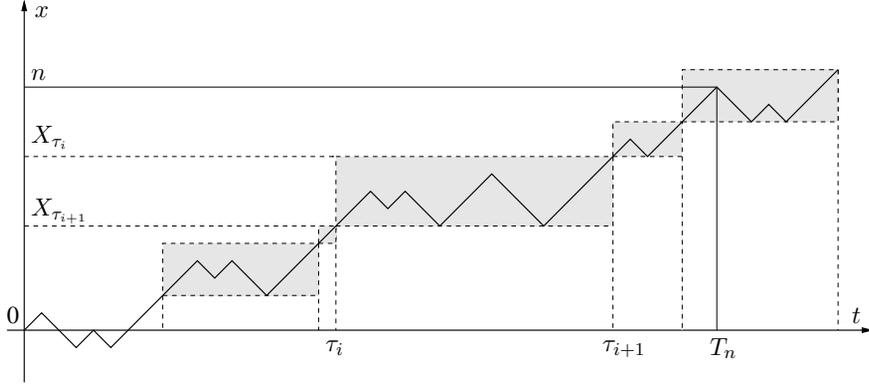
Figure 2: Regeneration structure: sizes and contents of the shaded boxes are i.i.d..

Lemma 4.5. ($d \geq 1$, **regeneration structure**) *Assume* (WEL) *if* $d = 1$ *and* (EL) *if* $d \geq 2$. *Furthermore assume* (IID), *let* $\ell \in \mathbb{R}^d \backslash \{0\}$, *and assume* $P_0[A_\ell] > 0$. *Then there are* $P_0[\,\cdot\,|\,A_\ell]$-*a.s. infinitely many random times* $n \geq 0$, *so-called* regeneration times, *such that* $X_m \cdot \ell < X_n \cdot \ell$ *for all* $m < n$ *and* $X_m \cdot \ell \geq X_n \cdot \ell$ *for all* $m \geq n$. *Call the increasing enumeration of these times* $(\tau_k)_{k \in \mathbb{N}}$. *Then the random* $\bigcup_{n \in \mathbb{N}} (\mathbb{Z}^d)^n$-*valued vectors* $(X_n)_{0 \leq n \leq \tau_1}$, $(X_n - X_{\tau_i})_{\tau_i \leq n \leq \tau_{i+1}}$ $(i \geq 1)$ *are independent w.r.t.* $P_0[\,\cdot\,|\,A_\ell]$. *Moreover, the vectors* $(X_n - X_{\tau_i})_{\tau_i \leq n \leq \tau_{i+1}}$ $(i \geq 1)$ *have the same distribution under* $P_0[\,\cdot\,|\,A_\ell]$ *as* $(X_n)_{0 \leq n \leq \tau_1}$ *under* $P_0[\,\cdot\,|\,\forall n\ X_n \cdot \ell \geq 0]$. *Also* $E_0[(X_{\tau_2} - X_{\tau_1}) \cdot \ell\,|\,A_\ell] < \infty$.

Note that the assumption (UEL) present in [Zei04, Section 3.2] is not needed in Lemma 4.5. Instead the ellipticity conditions needed are inherited from Lemma 3.3 for $d = 1$ and Lemma 3.17 for $d \geq 2$. This regeneration structure can be used in the same way as for RWRE, see e.g. [Zei04, (3.2.8)], to prove the following directional law of large numbers.

Theorem 4.6. ($d \geq 1$, **directional law of large numbers**) *Under the assumptions of Lemma 4.5 the following holds:* P_0-*a.s. on the event* A_ℓ,

$$\lim_{n \to \infty} \frac{X_n \cdot \ell}{n} = v_\ell := \frac{E_0[(X_{\tau_2} - X_{\tau_1}) \cdot \ell\,|\,A_\ell]}{E_0[\tau_2 - \tau_1\,|\,A_\ell]} \in [0, 1]. \qquad (4.2)$$

The challenge is to determine whether v_ℓ is strictly positive or not, i.e. whether $E_0[\tau_2 - \tau_1\,|\,A_\ell]$ is finite or not, see Section 5.

Problem 4.7. ($d \geq 1$) Assume (IID), (EL), $\ell \in \mathbb{R}^d \backslash \{0\}$. Does $X_n \cdot \ell/n$ converge P_0-a.s. on $\{|X_n| \to \infty\}^c$ to 0 as $n \to \infty$? This would generalize [Zer02, Th. 1].

Theorem 4.8. ($d \geq 1$, **law of large numbers**) *Assume* (WEL) *if* $d = 1$ *and* (EL) *if* $d \geq 2$. *Furthermore assume* (IID) *and let* $\ell_1, \ldots, \ell_d$ *be a basis of* $\mathbb{R}^d$ *such that* $P_0[A_{\ell_i} \cup A_{-\ell_i}] = 1$ *for all* $i = 1, \ldots, d$. *Then there are* $v \in \mathbb{R}^d$ *and* $c \geq 0$ *such that* P_0-*a.s.*

$$\lim_{n \to \infty} \frac{X_n}{n} \in \{v, -cv\}.$$

If, in addition, $P_0[A_{\ell_i}] = 1$ *for all* $i = 1, \ldots, d$ *then* X *satisfies a strong law of large numbers with velocity* $v \in \mathbb{R}^d$ *such that* $v \cdot \ell_i \geq 0$ *for all* $i = 1, \ldots, d$.

Proof. The proof of the first statement is analogous to the proof of [DR10, Th. 1.5] and uses Theorem 4.6. The additional assumption $P_0[A_{\ell_i} \cup A_{-\ell_i}] = 1$, which is not needed in [DR10, Th. 1.5], is due to Problem 4.7. The second statement follows directly from Theorem 4.6. $\square$

Remark 4.9. Note that by Theorem 3.19 the assumption $P_0[A_{\ell_i}] = 1$ in Theorem 4.8 is satisfied if (MPRV$_\ell$) holds for some $\ell \in \mathbb{R}^d \backslash \{0\}$ and ℓ_i is chosen from a sufficiently small neighborhood of ℓ.

Remark 4.10. ($d \geq 6$, **LLN via cut points**) Using cut points, a law of large numbers for RWRE has been proved in [BSZ03, Th. 1.4] in the case where at least 5 coordinates of the walk jointly form a standard RW. This method was generalized in [HSu12, Th. 1.1], see also [Hol12, Th. 2.1], to prove a law of large numbers for similar high-dimensional ERWs in the (IID) setting.

Remark 4.11. (d large, **monotonicity, continuity and differentiability of** $v \cdot e_1$) For the original model (BW), it has been shown in [HH10, Th. 1.1] by lace expansion techniques, that the first coordinate $v \cdot e_1$ is monotonically increasing in the drift parameter p if $d \geq 9$. The same approach also allows to prove a weak law of large numbers for $d \geq 6$, see [HH12, Th. 2.3]. [Hol12] assumes (IID) and $\mathbb{P}$-a.s. $\omega(0, e_j, i) = 1/(2d)$ for all $j = 2, \ldots, d$. Using the same expansion technique, it is shown in [Hol12, Th. 2.3] that the velocity is in an appropriate sense continuous in the drift parameters $\mathbb{E}[\omega(0, e_1, i)]$ if $d \geq 6$ and even differentiable if $d \geq 8$. Strict monotonicity for $d \geq 12$ is also

considered. For similar statements regarding monotonicity and continuity in the context discussed in Remark 4.10 see [HSu12, Th. 1.2].

5. Ballisticity

An ERW is called *ballistic* if it satisfies a strong law of large numbers with non-zero velocity v.

5.1. Results for $d = 1$

For recurrence and transience, the number 1, as a value for $|\delta|$ or $M(0)$, played a crucial role, see Theorem 3.10 and Proposition 3.6. Similarly, as we are about to see, the number 2 plays a special role for ballisticity.

Proposition 5.1. ($d = 1$, **no speed with two cookies**) *Assume* (SE) *and* $M(0) \leq 2$ $\mathbb{P}$*-a.s. and the ellipticity condition* $\mathbb{P}[\omega(0,1,1) < 1, \omega(1,1,1) < 1], \mathbb{P}[\omega(0,-1,1) < 1, \omega(1,-1,1) < 1] > 0$. *If the ERW satisfies a strong law of large numbers with speed v then $v = 0$.*

Proof. Under the additional assumption (POS_1) the strong law of large numbers with speed 0 has been proven in [Zer05, Th. 19]. Replacing cookies with negative drift by placebo cookies and using Proposition 4.2 therefore yields $v \leq 0$. By symmetry also $v \geq 0$. □

Theorem 5.2. ($d = 1$, **(non-)ballisticity [MPV06, Th. 1.1, 1.3], [BS08a, Th. 1.1], [KZ08, Th. 2]**) *Assume* (IID), (BD), *and* (WEL). *Then $v < 0$ if $\delta < -2$, $v = 0$ if $\delta \in [-2, 2]$ and $v > 0$ if $\delta > 2$.*

Problem 5.3. Can one replace in Theorem 5.2 the assumption (BD) by (POS_1)?

Idea of the proof of Theorem 5.2. The proof relies on a connection with branching processes, which we now describe. The main ideas go back to [Har52, Sec. 6], [Kni63], [KKS75], and [Tóth96]. They were introduced into the study of ERW in [BS08a]. We follow the exposition given in [KM11].

By Remark 3.9 we may assume without loss of generality that $\delta \geq 0$. It will be convenient to use the following coin-tossing construction of X as characterized in (1.1). Let $(\Omega', \mathcal{F}')$ be some measurable space equipped

with a family of probability measures $P_{0,\omega}$, $\omega \in \Omega$, and ± 1-valued random variables $B_i^{(k)}$, $k \in \mathbb{Z}$, $i \in \mathbb{N}$, which are for all $\omega \in \Omega$ independent under $P_{0,\omega}$ with distribution given by $P_{0,\omega}[B_i^{(k)} = \pm 1] = \omega(k, \pm 1, i)$. The events $\{B_i^{(k)} = 1\}$ (resp. $\{B_i^{(k)} = -1\}$), $i \in \mathbb{N}$, $k \in \mathbb{Z}$, will be called "successes" (resp. "failures"). Then an ERW X starting at 0 in the environment ω can be defined on the probability space $(\Omega', \mathcal{F}', P_{0,\omega})$ by

$$X_0 := 0 \quad \text{and} \quad X_{n+1} := X_n + B_{\#\{0 \le r \le n | X_r = X_n\}}^{(X_n)} \quad \text{for} \quad n \ge 0.$$

By Theorem 3.10, $P_0[T_n < \infty] = 1$ for all $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ let

$$J_{n,k}^{\downarrow} := \sum_{j=0}^{T_n-1} \mathbf{1}_{\{X_j=k,\ X_{j+1}=k-1\}} \quad \text{and} \quad J_{n,k}^{\uparrow} := \sum_{j=0}^{T_n-1} \mathbf{1}_{\{X_j=k,\ X_{j+1}=k+1\}}$$

be the number of jumps from k to $k-1$ resp. from k to $k+1$ before time T_n. Observe that for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$,

$$J_{n,k}^{\uparrow} = J_{n,k+1}^{\downarrow} + \mathbf{1}_{0 \le k < n}. \tag{5.1}$$

Therefore,

$$T_n = \sum_{k \in \mathbb{Z}} J_{n,k}^{\uparrow} + J_{n,k}^{\downarrow} \overset{(5.1)}{=} n + 2 \sum_{k \le n} J_{n,k}^{\downarrow} = n + 2 \sum_{0 \le k \le n} J_{n,k}^{\downarrow} + 2 \sum_{k < 0} J_{n,k}^{\downarrow}.$$

The last sum is bounded above by the total time spent by X below 0. When $\delta > 1$, i.e. X is transient to the right, the time spent below 0 is P_0-a.s. finite, and the growth of T_n is determined by $J_{n,k}^{\downarrow}$ for $0 \le k \le n$. Thus, it is enough to consider the "backward" process $\left(J_{n,n}^{\downarrow}, J_{n,n-1}^{\downarrow}, \ldots, J_{n,0}^{\downarrow}\right)$. Obviously, $J_{n,n}^{\downarrow} = 0$ for every $n \in \mathbb{N}$.

Moreover, denote by $F_m^{(k)}$ the number of "failures" in the sequence $B^{(k)}$ before the m-th "success". Since the last departure from k before time T_n leads to $k+1$ we have

$$J_{n,k}^{\downarrow} = F_{J_{n,k}^{\uparrow}}^{(k)} \overset{(5.1)}{=} F_{J_{n,k+1}^{\downarrow}+1}^{(k)}$$

for all $0 \le k < n$. Since the sequence $((F_m^{(k)})_{m \in \mathbb{N}})_{k \in \mathbb{Z}}$ is i.i.d. under P_0 we conclude that the distribution of $\left(J_{n,n}^{\downarrow}, J_{n,n-1}^{\downarrow}, \ldots, J_{n,0}^{\downarrow}\right)$ under P_0 coincides

with that of $(V_0, V_1, \ldots, V_n)$, where $V = (V_k)_{k \geq 0}$ is the Markov chain defined by

$$V_0 := 0, \quad V_{k+1} := F^{(k)}_{V_k+1} \quad \text{for } k \geq 0.$$

Thus it suffices to study V. Observe that V is a branching process with the following properties: (i) V has exactly 1 immigrant in each generation. The immigration occurs before the reproduction. (ii) One can enumerate the individuals in the k-th generation in such a way that the number $\zeta^{(k)}_m$ of offspring of the m-th individual in generation k is given by the number of failures between the $(m-1)$-th and m-th success in the sequence $B^{(k)}$. (Here the time of the 0-th success is 0.) In particular, if $V_k \geq M$ then the offspring distribution of each individual after the M-th one is Geom(1/2) (i.e. geometric on $\{0\} \cup \mathbb{N}$ with parameter 1/2).

Therefore, we can write for $k \geq 0$,

$$V_{k+1} = \sum_{m=1}^{M \wedge (V_k+1)} \zeta^{(k)}_m + \sum_{m=1}^{V_k - M + 1} \xi^{(k)}_m, \tag{5.2}$$

where $\xi^{(k)}_m$, $k \geq 0$, $m \geq 1$, are i.i.d. Geom(1/2) random variables and the vectors $\left(\zeta^{(k)}_1, \zeta^{(k)}_2, \ldots, \zeta^{(k)}_M \right)$, $k \geq 0$, are i.i.d. under P_0 and independent of $(\xi^{(k)}_m)_{k \geq 0, m \geq 1}$. For each $k \geq 0$ the random variables $\zeta^{(k)}_m, m = 1, \ldots, M$, are in general neither independent nor identically distributed. Define the length σ^V of the life cycle and the total progeny A^V over a single life cycle by

$$\sigma^V := \inf\{j \geq 1 \,|\, V_j = 0\} \quad \text{and} \quad A^V := \sum_{j=0}^{\sigma^V - 1} V_j.$$

These two quantities partially characterize the regeneration structure of transient ERW (see Lemma 4.5 with $d = 1$ and [KZ08, Lem. 12, (30)]), namely,

$$\sigma^V \overset{\mathrm{d}}{=} X_{\tau_2} - X_{\tau_1}, \quad A^V \overset{\mathrm{d}}{=} \frac{\tau_2 - \tau_1 - (X_{\tau_2} - X_{\tau_1})}{2}. \tag{5.3}$$

Detailed information about the tail behavior of σ^V and A^V is the key to the proof of Theorem 5.2 as well as to the results about scaling limits of T_n and X_n (see Section 6). There is a large number of papers which study the extinction probabilities of branching processes with migration. Unfortunately,

they do not seem to include this particular setting. In [KZ08] existing results were adopted by introducing appropriate modifications of the process V. In [BS08a] most and in [KM11] all of the required results about branching processes were obtained directly.

Theorem 5.4. ([KM11, Th. 2.1, 2.2]) *Assume* (IID), (BD), *and* (WEL) *and let* $\delta > 0$. *Then there are constants* $C_1, C_2 \in (0, \infty)$ *such that*

$$\lim_{n \to \infty} n^\delta P_0(\sigma^V > n) = C_1 \quad and \quad \lim_{n \to \infty} n^{\delta/2} P_0\left(A^V > n\right) = C_2.$$

For future reference in Section 6 we shall record the following corollary of Theorem 5.4 and (5.3) for transient ERW.

Corollary 5.5. *Assume* (IID), (BD), *and* (WEL) *and let* $\delta > 1$. *Then there are constants* $C_1, C_3 \in (0, \infty)$ *such that*

$$\lim_{n \to \infty} n^\delta P_0(X_{\tau_2} - X_{\tau_1} > n) = C_1, \tag{5.4}$$

$$\lim_{n \to \infty} n^{\delta/2} P_0\left(\tau_2 - \tau_1 > n\right) = C_3. \tag{5.5}$$

Combining (5.5) with Theorem 4.6 we obtain Theorem 5.2. $\qquad\square$

Idea of the proof of Theorem 5.4. The main point is that the process V killed upon reaching 0 is reasonably well described by a simple diffusion. The parameters of such a diffusion can be easily computed at the heuristic level. For $V_k \geq M$, (5.2) implies that

$$V_{k+1} - V_k = \sum_{m=1}^{M} \zeta_m^{(k)} - M + 1 + \sum_{m=1}^{V_k - M + 1} (\xi_m^{(k)} - 1).$$

By [BS08a, Lem. 3.3] or [KZ08, Lem. 17],

$$E_0\left[\sum_{m=1}^{M} \zeta_m^{(k)} - M + 1\right] = 1 - \delta.$$

The term $\sum_{m=1}^{M} \zeta_m^{(k)} - M + 1$ is independent of $\sum_{m=1}^{V_k - M + 1} (\xi_m^{(k)} - 1)$. When V_k is large, the latter is, conditioned on V_k, approximately normal with mean 0 and variance essentially equal to $2V_k$. Therefore, the relevant diffusion

should be given by the following stochastic differential equation:

$$dY_t = (1 - \delta)\, dt + \sqrt{2Y_t}\, dB_t, \quad Y_0 = y > 0, \quad t \in [0, \tau^Y],$$

where $\tau^Y = \inf\{t \geq 0 \,|\, Y_t = 0\}$. Observe that $2Y_t$ is a squared Bessel process of dimension $2(1 - \delta)$. Thus, if $\delta > 0$ then $\tau^Y < \infty$ a.s.. Using the scaling properties of Y_t, it is not hard to compute the tail behavior of τ^Y and $\int_0^{\tau^Y} Y_s\, ds$. The bulk of the work is to transfer these results to σ^V and A^V. □

Remark 5.6. ($d = 1$, "have your cookie and eat it" RW) According to [Pin10, Th. 3] the statement of Theorem 5.2 (except for possibly the critical case $|\delta| = 2$) also holds for the measure $\mathbb{P}$ which reflects "have your cookie and eat it" environments as described in Remark 3.14.

Remark and Example 5.7. ($d = 1$, δ does not characterize ballisticity under assumption (SE)) The following example shows that one cannot replace in Theorem 5.2 assumption (IID) by (SE), not even under the additional assumption (POS$_1$). The first such example in this context was constructed in [MPV06, p. 290]. Thus, while δ characterizes recurrence and transience for $d = 1$ also in the general setting of (SE) it is not sufficient to characterize ballisticity under this assumption.

Let $1/2 < p \leq 1$. Then there is a RWRE with speed $v = 0$ in a stationary and ergodic environment which uses exactly p and $1/2$ as transition probabilities to the right. More formally, there is a measure $\mathbb{P}$ on Ω which satisfies (SE) and

$$\mathbb{P}[\forall i \in \mathbb{N} : \omega(0, 1, i) = p] = 1 - \mathbb{P}[\forall i \in \mathbb{N} : \omega(0, 1, i) = 1/2] > 0$$

such that P_0-a.s. $X_n/n \to 0$ as $n \to \infty$. (Note that (POS$_1$) and $\delta = \infty$ hold.)

To construct such $\mathbb{P}$ let $(B_z)_{z \in \mathbb{Z}}$ be a stationary and ergodic $\{0, 1\}$-valued process on some probability space with probability measure P such that $P[B_0 = 1] > 0$ and $E[\widetilde{Z}] = \infty$, where $\widetilde{Z} := \inf\{z \in \mathbb{N}_0 \,|\, B_{-z} = 1\}$. For example, $(B_z)_{z \in \mathbb{Z}}$ could be a two-sided stationary discrete renewal process with i.i.d. inter-point distances which have an infinite second moment. Then set for all $i \in \mathbb{N}$, $\widetilde{\omega}(z, 1, i) = p$ if $B_z = 1$ and $\widetilde{\omega}(z, 1, i) = 1/2$ if $B_z = 0$ and

let $\mathbb{P}$ be the distribution of $\widetilde{\omega}$ on Ω. It then follows from [Zei04, Th. 2.1.9 (c)] that P_0-a.s. $X_n/n \to 0$ as $n \to \infty$.

To construct an ERW example which also satisfies (BD) one can replace all cookies $\omega(z, \cdot, i)$ with i larger than some fixed M by placebo cookies. By Proposition 4.2 this cannot increase the speed. However, since (POS$_1$) still holds the speed cannot become negative by Theorem 4.1 either and is therefore still 0.

Remark 5.8. ($d = 1$, v **is not a function of** δ.) While under the conditions of Theorem 5.2 the value of δ determines whether v is positive, zero, or negative, it does not determine the value of v, not even under strong assumptions. For example, let $p \in (1/2, 1]$ and $M \in \mathbb{N}$ such that $(2p-1)M > 2$ and define $\omega_{p,M} \in \Omega$ by setting for all $z \in \mathbb{Z}$, $\omega_{p,M}(z, 1, i) := p$ if $i \leq M$ and $\omega_{p,M}(z, 1, i) := 1/2$ if $i > M$. Then the parameter δ corresponding to $\mathbb{P} := \delta_{\omega_{p,M}}$ is equal to $(2p - 1)M$ and therefore the corresponding ERW has speed $v > 0$ due to Theorem 5.2. Now pick $M' \in \mathbb{N}$ such that $M' > \delta/v$ and set $p' := (1 + \delta/M')/2$. Then $\mathbb{P}' := \delta_{\omega_{p',M'}}$ has the same parameter $\delta' = \delta$ as $\mathbb{P}$, yet its induced speed v' is, due to Proposition 4.2, not larger than the speed $v'' = 2p' - 1 = \delta/M' < v$ of the asymmetric nearest neighbor RW on $\mathbb{Z}$ which corresponds to $\mathbb{P}'' := \delta_{\omega_{p',\infty}}$.

5.2. Results for $d \geq 2$

It was observed in [BW03, Th. 5] that ERW under assumption (BW) is ballistic if $d \geq 4$ and left as an open question whether the same holds true in dimensions 3 and 2. In two unpublished preprints this question was answered in the affirmative by Kozma first for $d = 3$ and then also for $d = 2$. The paper [BR07] uses a different method to prove this statement. Its authors obtain a lower bound on the number of tan points to show that the expected time $E_0[\tau_2 - \tau_1]$ between two successive regeneration times is finite which, by (4.2), implies ballisticity.

However, as we already mentioned before Theorem 3.19, methods involving tan points might not be easily adaptable to more general settings. Using a related but more general approach the following result about ballisticity was obtained in [MPRV]. Here we only mention that [MPRV] uses martingale techniques and completely avoids tan points. A key ingredient of the proof is a lower bound on the growth of the range of the ERW [MPRV,

Prop. 4.1]. The positivity condition plays a crucial role in the proofs. (Recall from Remark 4.9 that the strong law of large numbers holds in this setting.)

Theorem 5.9. ($d \geq 2$, **ballisticity, [BR07, Th. 1 (i)], [MPRV, Th. 1.2 (i)]**)
Let $\ell \in \mathbb{R}^d \backslash \{0\}$ and assume (MPRV$_\ell$), (IID), *and* (UEL). *Then the ERW is ballistic and its velocity v satisfies $v \cdot \ell > 0$.*

According to [MPRV, Th. 1.1] this result can also be extended to certain processes which do not fit into the framework considered here. For more recent developments in this direction see [MP].

Remark 5.10. ($d \geq 2$, **positive and negative cookies**) Little is known about ballisticity outside of the setting of Theorem 5.9. In [KZ08, Section 9] an example is given for an ERW in $d \geq 4$ which is ballistic in spite of $\delta = 0$. For $d \geq 9$, [Hol12, Th. 2.2] shows, using lace expansion, that if the drift induced by the first cookie is strong enough the first cookie can overrule the subsequent cookies and determine the direction of the velocity. On the other hand, for $d \geq 2$, if the first cookie is weak enough and all the subsequent cookies induce a drift in the opposite direction then their influence might win, see [Hol12, Lem. 2.4].

6. Limit Laws and Functional Limit Theorems

In this section we discuss limit laws and functional limit theorems for ERWs under the averaged measure P_0. Quenched limit laws are mentioned in Problem and Example 6.17 at the end of this section.

6.1. Results for $d = 1$

There is a rich variety of limit laws and limit processes for ERW in $d = 1$. The limit processes obtained so far in the ERW literature are Brownian motion [KZ08], a certain class of Brownian motions perturbed at extrema [Dol11, DK12], the running maximum of Brownian motion [DK12], as well as, under a different kind of scaling, so-called excited Brownian motions [RS]. As a new result we prove here functional limit theorems for the model of Section 1.3 with $\delta \in (1, 2) \cup (2, 4]$ with limit processes which are strictly

stable with indices $\alpha \in [1/2, 2]$ and skewness parameter 1 (totally skewed to the right). The proof is based on [Lin78].

Let us outline the plan of this subsection. First, we state functional limit theorems in the recurrent case. In the transient regime, we start with convergence of one-dimensional distributions. Then we recall and compare the notions of convergence in J_1 and M_1 on the Skorokhod space. Further, we state and prove, using [Lin78], functional limit theorems for transient ERWs. Finally, we consider convergence to excited Brownian motion.

Let $D[0, \infty)$ be the Skorokhod space, i.e. the space of right-continuous real-valued functions with left limits on $[0, \infty)$ endowed with the standard Skorokhod topology (i.e. J_1, see definition (6.7) below). Denote by $\overset{J_1}{\Rightarrow}$ the weak convergence on $D[0, \infty)$ in J_1. Unless stated otherwise, we shall assume that all processes start at the origin at time 0.

We begin with the recurrent case and define the candidate for the limiting process, the so-called *Brownian motion perturbed at extrema* ([Dav96, PW97, CD99, Dav99]). Let $\alpha, \beta \in (-\infty, 1)$ and let $B = (B(t))_{t \geq 0}$ denote a standard Brownian motion. Then

$$X_{\alpha,\beta}(t) = B(t) + \alpha \sup_{s \leq t} X_{\alpha,\beta}(s) + \beta \inf_{s \leq t} X_{\alpha,\beta}(s) \qquad (6.1)$$

has a pathwise unique solution $X_{\alpha,\beta} = (X_{\alpha,\beta}(t))_{t \geq 0}$ that is adapted to the filtration of B (see e.g. [CD99, Th. 2]). It is called (α, β)-perturbed Brownian motion. In the 90's the main purpose of the study of RWs $(Y_n)_{n \geq 0}$ perturbed at extrema, as introduced in Section 1.4, was to prove functional limit theorems for these walks. And indeed, for certain values of the parameters p and q one has

$$\frac{Y_{[n\cdot]}}{\sqrt{n}} \overset{J_1}{\Rightarrow} X_{\alpha,\beta}(\cdot) \text{ as } n \to \infty,$$

where $\alpha = 2 - 1/p$ and $\beta = 2 - 1/(1-q)$, see e.g. [Dav96, Th. 1.2] for an accessible statement and also [Dav99].

For the ERWs from Sections 1.2 and 1.3 there are the following results:

Theorem 6.1. ($d = 1$, **convergence to Brownian motion perturbed at extrema [Dol11, Th. 1, Rem. 2], [DK12, Th. 1]**) *Let $\delta \in [0, 1)$.*

2013] EXCITED RANDOM WALKS *Assume either*

> (SE), (POS$_1$), (UEL), *and*
>
> $\mathbb{E}\left[\sup\{i \in \mathbb{N}: \ |\omega(0,1,i) - 1/2| \geq \varepsilon\}\right] < \infty$ *for every* $\varepsilon > 0$ $\qquad$ (6.2)

or (IID), (BD), *and* (WEL). *Then under* P_0,

$$\frac{X_{[n\cdot]}}{\sqrt{n}} \overset{J_1}{\Rightarrow} X_{\delta,-\delta}(\cdot) \ \text{ as } n \to \infty.$$

Remark 6.2. (Strips) A similar result holds for recurrent ERWs on strips $\mathbb{Z} \times \{0, 1, \dots, L-1\}$, $L \in \mathbb{N}$, under assumptions similar to (6.2), see [Dol11, Th. 1].

Theorem 6.3. ($d = 1$, convergence to the running maximum of Brownian motion [DK12, Th. 2]) *Assume* (IID), (BD), *and* (WEL) *and let* $\delta = 1$ *and* $S(t) := \max_{s \leq t} B(s)$, $t \geq 0$. *Then there exists a constant* $b > 0$ *such that under* P_0,

$$\frac{X_{[n\cdot]}}{b\sqrt{n}\log n} \overset{J_1}{\Rightarrow} S(\cdot) \ \text{ as } n \to \infty.$$ $\qquad$ (6.3)

Problem 6.4. Replace the assumptions (IID), (BD), and (WEL) of Theorem 6.3 by (6.2). Does the result of Theorem 6.3 remain true? If yes, can one also remove (POS$_1$)?

Observe that in Theorem 6.3 the limiting process is transient while the original process is recurrent. This might seem surprising. For the proof of Theorem 6.3 one first shows that (6.3) holds with $S_{[n\cdot]} := \max_{0 \leq i \leq [n\cdot]} X_i$ instead of $X_{[n\cdot]}$. Then it is argued ([DK12, Lem. 5.2]) that with probability approaching 1 the maximum amount of "backtracking" of X_j from S_j for $j \leq [Tn]$ is negligible on the scale $\sqrt{n}\log n$ for every $T < \infty$.

We turn now to the transient case $\delta > 1$. To describe limit laws of transient ERWs we need the following notation. For $\alpha \in (0, 2]$ and $b > 0$ denote by $Z_{\alpha,b}$ a random variable such that for all $u \in \mathbb{R}$,

$$\log E\left[e^{iuZ_{\alpha,b}}\right] = \begin{cases} -b|u|^\alpha(1 - i\tan(\pi\alpha/2)\operatorname{sign} u), & \text{if } \alpha \neq 1; \\ -b|u|\left(1 + \frac{2i}{\pi}\log|u|\operatorname{sign} u\right), & \text{if } \alpha = 1, \end{cases}$$ $\qquad$ (6.4)

see e.g. [Dur10, (3.7.11)]. Here we agree to set $\operatorname{sign} 0$ to 0. Observe also that $Z_{2,b}$ is just a centered normal random variable with variance $2b$.

Theorem 6.5. ($d = 1$, convergence of one-dimensional distributions)
Assume (IID), (BD), *and* (WEL) *and let v be the speed of the ERW as in Theorem 4.1. The following results hold under P_0.*

(i) *If $\delta \in (1, 2)$ then there is a constant $b > 0$ such that as $n \to \infty$,*

$$\frac{T_n}{n^{2/\delta}} \Rightarrow Z_{\delta/2,b} \quad and \quad \frac{X_n}{n^{\delta/2}} \Rightarrow (Z_{\delta/2,b})^{-\delta/2}.$$

(ii) *If $\delta = 2$ then there is a positive constant c such that*

$$\frac{T_n}{n \log n} \to \frac{1}{c} \quad and \quad \frac{X_n}{n/\log n} \to c \quad in\ probability\ as\ n \to \infty. \quad (6.5)$$

Moreover, there are a positive constant b and functions $D(n) \sim \log n$ and $\Gamma(n) \sim 1/\log n$ such that as $n \to \infty$,

$$\frac{T_n - c^{-1}nD(n)}{n} \Rightarrow Z_{1,b} \quad and \quad \frac{X_n - cn\Gamma(n)}{c^2 n \log^{-2} n} \Rightarrow -Z_{1,b}. \quad (6.6)$$

(iii) *If $\delta \in (2, 4)$ then as $n \to \infty$,*

$$\frac{T_n - v^{-1}n}{n^{2/\delta}} \Rightarrow Z_{\delta/2,b} \quad and \quad \frac{X_n - vn}{v^{1+2/\delta}n^{2/\delta}} \Rightarrow -Z_{\delta/2,b}.$$

(iv) *If $\delta = 4$ then there is a constant $b > 0$ such that as $n \to \infty$,*

$$\frac{T_n - v^{-1}n}{\sqrt{n \log n}} \Rightarrow Z_{2,b} \quad and \quad \frac{X_n - vn}{v^{3/2}\sqrt{n \log n}} \Rightarrow -Z_{2,b}.$$

(v) *If $\delta > 4$ then there is a constant $b > 0$ such that as $n \to \infty$,*

$$\frac{T_n - v^{-1}n}{\sqrt{n}} \Rightarrow Z_{2,b} \quad and \quad \frac{X_n - vn}{v^{3/2}\sqrt{n}} \Rightarrow -Z_{2,b}.$$

Moreover, everywhere above X_n can be replaced by $\inf_{i \geq n} X_i$ or $\sup_{i \leq n} X_i$.

Proof. The regeneration structure described in Lemma 4.5 and the estimates (5.4) and (5.5) play an important role in the proof. Statements (i)

and (6.5) were proven under additional assumptions, in particular (POS$_1$), in [BS08b, Th. 1.1]. These assumptions were removed in [KM11, Rem. 9.2]. Parts (iii) and (iv) were obtained in [KM11, Th. 1.3]. The second claim of part (v) is contained in [KZ08, Th. 3]. Although part (v) was not stated or proven in [KM11], its proof is almost the same as the one of part (iii) (see [KM11, Proof of Th. 1.3, pp. 593–594]). The only difference is that one can use standard limit theorems for square integrable random variables instead of those quoted in the original proof.

The fluctuation results (6.6) do not seem to appear anywhere in the ERW literature. The proof of (6.6) can be written along the lines of [KKS75, pp. 166–168] taking into account [KM11, Th. 2.1, 2.2, Lem. 9.1]. The fact that $X_{\tau_n} - X_{\tau_{n-1}}$ has infinite second moment (see (5.4)) requires an application of a limit theorem from renewal theory, which is different from the one used in [KKS75, p. 166, line 8 from the bottom]. Namely, the deviations of the number of renewals up to time n have fluctuations of order $\sqrt{n \log n}$ instead of $\sqrt{n}$ (combine [Whi02, Th. 6.3.1] with [GS69, Ch. 9, Sec. 6, Th. 2]). Apart from this, the proof of [KKS75, pp. 166–168] goes through essentially word for word. □

It is natural to expect that the same embedded regeneration structure, which was useful in the proof of Theorem 6.5, will allow us to obtain the corresponding functional limit theorems. Theorem 6.5 suggests that the limit processes should be stable processes, which need not be continuous, and this brings us to the question of choosing an appropriate topology on the Skorokhod space $D[0, \infty)$.

In 1956, A. V. Skorokhod introduced 4 metrics on $D[0, 1]$ ([Sko56]). These metrics generate the corresponding topologies on $D[0, 1]$, which are called J_1, M_1, J_2, and M_2. One of them, namely J_1, is often called "the (standard) Skorokhod topology" and is probably the most widely used in the study of weak convergence of stochastic processes on the Skorokhod space. Nevertheless, when the candidate for the limiting process is not continuous, one should not necessarily expect the convergence in J_1 but only in one of the "less demanding" topologies, in our case M_1. This fact is well-known, for example, in the study of heavy traffic limits in queuing theory (see [Whi02] and references therein).

Let Λ denote the set of continuous strictly increasing mappings of $[0,1]$ onto itself. Recall ([Bil99, Sec. 12], [Whi02, Section 3.3 (3.2)]) that for $x_1, x_2 \in D[0,1]$ the Skorokhod J_1-distance between x_1 and x_2 is defined by

$$\rho_{J_1}(x_1, x_2) := \inf_{\lambda \in \Lambda} \left(\sup_{t \in [0,1]} |x_1(t) - x_2(\lambda(t))| \vee \sup_{t \in [0,1]} |\lambda(t) - t| \right). \qquad (6.7)$$

The Skorokhod M_1-distance $\rho_{M_1}(x_1, x_2)$ is informally defined as "the distance between the completed graphs of x_1 and x_2, Γ_{x_1} and Γ_{x_2}". More precisely ([Whi02, Sec. 3.3]), for $x \in D[0,1]$ let

$$\Gamma_x := \{(z,t) \in \mathbb{R} \times [0,1] : \ z = \alpha x(t-) + (1-\alpha)x(t) \text{ for some } \alpha \in [0,1]\} \subset \mathbb{R}^2,$$

where $x(0-) := x(0)$, and define the order on Γ_x as follows: $(z_1, t_1) \leq (z_2, t_2)$ if either $t_1 < t_2$ or $t_1 = t_2$ and $|z_1 - x(t_1-)| \leq |z_2 - x(t_2-)|$. Denote by Π_x the set of all parametric representations of Γ_x, i.e. continuous non-decreasing (with respect to the above defined order on Γ_x) functions (u,r) taking $[0,1]$ onto Γ_x. For $x_1, x_2 \in D[0,1]$ let

$$\rho_{M_1}(x_1, x_2) := \inf_{(u_i, r_i) \in \Pi_{x_i}, \, i=1,2} \left(\sup_{t \in [0,1]} |u_1(t) - u_2(t)| \vee \sup_{t \in [0,1]} |r_1(t) - r_2(t)| \right).$$

Clearly, J_1-convergence implies M_1-convergence. The converse is not true. For both examples on Figure 3[3] we have $\rho_{M_1}(x_n, x) \to 0$ but $\rho_{J_1}(x_n, x) \nrightarrow 0$ as $n \to \infty$. In the left picture, a single jump of the limiting function is achieved by a double jump in the approximating functions. The right picture illustrates the creation of an unmatched jump in the limiting function by continuous approximating functions. Neither situation is an obstacle for the M_1-convergence but both prevent J_1-convergence.

Note however ([Whi02, Sec. 12.4]) that

$$\text{if } \rho_{M_1}(x_n, x) \to 0 \text{ and } x \in C[0,1] \text{ then } \rho_{J_1}(x_n, x) \to 0. \qquad (6.8)$$

---

[3]Figure 3: $x = \frac{1}{4}\mathbf{1}_{[0,1/2)} + \frac{3}{4}\mathbf{1}_{[1/2,1]}$. Left: $x_n = \frac{1}{4}\mathbf{1}_{[0,1/2-1/n)} + \frac{1}{2}\mathbf{1}_{[1/2-2/n,1/2-1/n)} + \frac{3}{4}\mathbf{1}_{[1/2-1/n,1]}$. Right: $x_n(t) = \begin{cases} 1/4 - 1/n + 2t/(n-2), & \text{if } 0 \leq t < 1/2 - 1/n; \\ 3/4 + n(t - 1/2)/2, & \text{if } 1/2 - 1/n \leq t < 1/2; \\ 3/4 + 2(t - 1/2)/n, & \text{if } 1/2 \leq t \leq 1. \end{cases}$
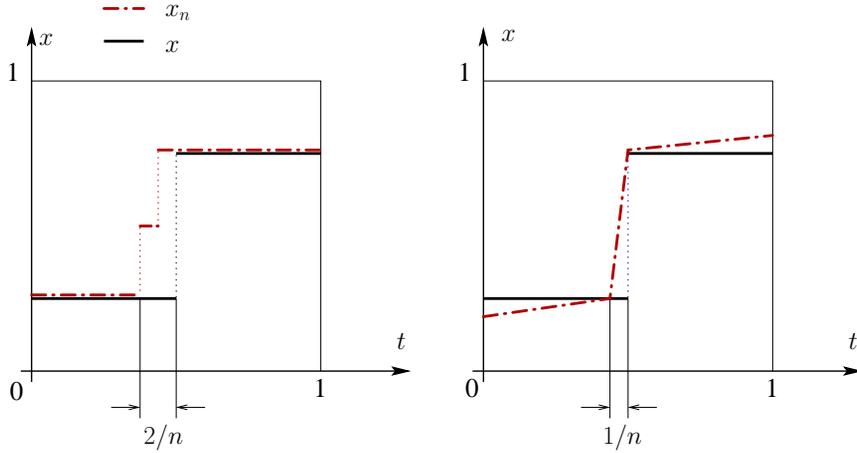
Figure 3: Unmatched jump of the limiting process.

Under natural assumptions (see [Whi02, Sec. 13.6]) the inverse map is continuous with respect to M_1. Consider the (right continuous) inverses of the functions in Figure 3 (see Figure 4) and notice that in the left picture of Figure 4 we have $\rho_{J_1}(x_n^{-1}, x^{-1}) \to 0$ (and also uniform convergence) and in the right picture only $\rho_{M_1}(x_n^{-1}, x^{-1}) \to 0$ as $n \to \infty$ (but neither uniform nor J_1-convergence).
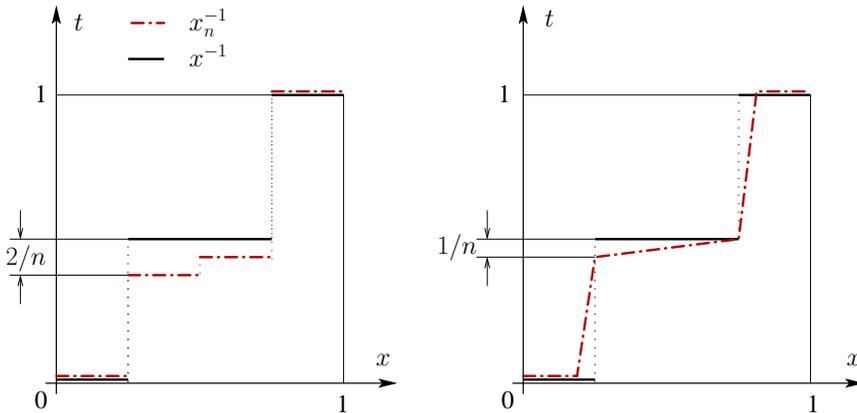

Figure 4: The convergence of inverse mappings.

This example illustrates that "strengthening" of convergence might or might not occur when taking the inverse (compare the statements of Theorems 6.6 and 6.7 for $\delta \in (2, 4)$ below).

We defined the M_1-distance on $D[0,1]$. This definition extends to $D[0,T]$, $T > 0$, in an obvious way. For $x_1, x_2 \in D[0,\infty)$ we let $\rho_{M_1}^T(x_1, x_2)$ denote the M_1-distance between the restrictions of x_1 and x_2 to the space $D[0,T]$. Then (see [Whi02, Sec. 3.3, (3.6)])

$$\rho_{M_1}^\infty(x_1, x_2) := \int_0^\infty e^{-T}(\rho_{M_1}^T(x_1, x_2) \wedge 1)\, dT.$$

The distance $\rho_{J_1}^\infty(x_1, x_2)$ can be defined by replacing M_1 with J_1 everywhere in the preceding paragraph. For further information about the space $D[0,\infty)$ with metric $\rho_{M_1}^\infty$ the reader is referred to [Whi02, Sec. 12.9].

We shall return to ERWs. Using the notation introduced in Theorem 6.5 define processes $\eta_n = (\eta_n(t))_{t\geq 0}$ and $\xi_n = (\xi_n(t))_{t\geq 0}$ as follows.

(i) If $\delta \in (1,2)$ then $\quad \eta_n(t) := \dfrac{T_{[nt]}}{n^{2/\delta}}, \qquad\qquad\qquad \xi_n(t) := \dfrac{X_{[nt]}}{n^{\delta/2}}.$

(ii) If $\delta = 2$ then $\qquad \eta_n(t) := \dfrac{T_{[nt]} - c^{-1}[nt]D(n)}{n}, \; \xi_n(t) := \dfrac{X_{[nt]} - c[nt]\Gamma(n)}{c^2 n \log^{-2} n}.$

(iii) If $\delta \in (2,4)$ then $\eta_n(t) := \dfrac{T_{[nt]} - [nt]/v}{n^{2/\delta}}, \qquad\quad \xi_n(t) := \dfrac{X_{[nt]} - [nt]v}{v^{1+2/\delta} n^{2/\delta}}.$

(iv) If $\delta = 4$ then $\qquad \eta_n(t) := \dfrac{T_{[nt]} - [nt]/v}{\sqrt{n \log n}}, \qquad\quad \xi_n(t) := \dfrac{X_{[nt]} - [nt]v}{v^{3/2}\sqrt{n \log n}}.$

(v) If $\delta > 4$ then $\qquad \eta_n(t) := \dfrac{T_{[nt]} - [nt]/v}{\sqrt{n}}, \qquad\qquad \xi_n(t) := \dfrac{X_{[nt]} - [nt]v}{v^{3/2}\sqrt{n}}.$

Note that $\eta_n(1)$ and $\xi_n(1)$ are the random variables considered in Theorem 6.5. Recall the random variables $Z_{\delta/2,b}$, $b > 0$, which were defined in (6.4).

Theorem 6.6. ($d = 1$, functional limit theorem) *Assume* (IID), (BD), *and* (WEL) *and let* $\delta \in (1,2)$. *Then* $\xi_n \overset{J_1}{\Rightarrow} \xi$ *as* $n \to \infty$ *under* P_0. *Here* ξ *is the inverse of a stable subordinator* η *for which* $\eta(1) \overset{d}{=} Z_{\delta/2,b}$, *and* b *is the same as in part (i) of Theorem 6.5.*

We denote by $\overset{M_1}{\Rightarrow}$ the weak convergence on $D[0,\infty)$ in M_1.

Theorem 6.7. ($d = 1$, functional limit theorem) *Assume* (IID), (BD), *and* (WEL). *If* $\delta \in (2,4)$ *then* $\xi_n \overset{M_1}{\Rightarrow} \xi$ *as* $n \to \infty$ *under* P_0. *Here* ξ *is a stable process with index* $\delta/2$ *such that* $\xi(1) \overset{d}{=} -Z_{\delta/2,b}$. *If* $\delta \geq 4$ *then*

$\xi_n \overset{J_1}{\Rightarrow} \sqrt{2b}\, B$ *as $n \to \infty$. The constant b is the same as in the corresponding part of Theorem 6.5.*

Remark 6.8. (case $\delta = 2$) The case $\delta = 2$ should have been included in Theorem 6.7. However, the results of [Lin78] do not cover this case, and a separate proof is needed.

Remark 6.9. (no J_1-convergence) In the first part of Theorem 6.7 ($\delta \in (2,4)$) the convergence in M_1 can not be replaced with convergence in J_1. The intuitive explanation is that the jumps of the limiting process ξ are unmatched by those of the approximating processes ξ_n since the jumps of ξ_n are bounded by $2/n^{2/\delta}$. This phenomenon was illustrated by examples in Figure 3 except that the jumps in the limiting process ξ are due to centering and, therefore, are negative. See also [Whi02, Section 3.3 and Chapter 6].

Remark 6.10. (convergence of η_n) The weak convergence in M_1 under P_0 of the processes η_n to the stable process η which is characterized by the corresponding one-dimensional distribution of Theorem 6.5 can also be established (see [Lin78, Th. 2(i) with $g(t) \equiv 1$, $q = 1, f(t) = h(t) = t$] for $\delta > 2$ and Remark 6.12 below for $\delta \in (1,2)$), but, since our main interest is in ξ_n, we omit the corresponding results.

For the proofs of Theorems 6.6 and 6.7 we shall need the following result from [Lin78].

Theorem 6.11. ([Lin78, Th. 1, Rem. 2]) *Let $(\zeta_i)_{i \in \mathbb{N}}$ be i.i.d. random variables such that $P[\zeta_1 \leq 0] = 0$, and set $s_0 := 0$, $s_n := \sum_{i=1}^n \zeta_i$ for $n \in \mathbb{N}$. Suppose that $\mu := E[\zeta_1] < \infty$ and the $D[0,\infty)$-valued process $w = (w(t))_{t \geq 0}$ satisfies: $w(0) = 0$ and*

$$(\zeta_n, w(s_{n-1} + \cdot\zeta_n) - w(s_{n-1})), \ n \geq 1, \ \text{are i.i.d.}$$
$$\text{random variables with values in } [0,\infty) \times D[0,1], \tag{6.9}$$

and for some norming constants $a(n)$, $n \in \mathbb{N}$,

$$\frac{w(s_{[n\cdot]})}{a(n)} \overset{M_1}{\Rightarrow} Y(\cdot) \qquad \text{as } n \to \infty,$$

where Y is a stable process with index α. Moreover, define

$$
M_n^{(2)}(w) := \left(\sup_{0=t_0<t_1<t_2<t_3=1} \sum_{i=1}^{3} |w(s_{n-1}+t_i\zeta_n) - w(s_{n-1}+t_{i-1}\zeta_n)| \right) \\
- |w(s_n) - w(s_{n-1})|.
$$

Then

$$
\frac{w(n\cdot)}{\mu^{-1/\alpha}a(n)} \overset{M_1}{\Rightarrow} Y(\cdot) \quad \text{if and only if} \quad \frac{M_{[n\cdot]}^{(2)}(w)}{a(n)} \overset{M_1}{\Rightarrow} 0.
$$

Proof of Theorem 6.6. By (3.10) of Theorem 3.10 the ERW is transient to the right. Therefore, the regeneration structure described in Lemma 4.5 exists and we may set $\zeta_i = X_{\tau_{i+1}} - X_{\tau_i}$ for $i \in \mathbb{N}$ and $w(t) = T_{[t]+X_{\tau_1}} - \tau_1$ for $t \in [0,\infty)$ and have that $(\zeta_i)_{i\geq 1}$ is i.i.d. and $\mu = E_0[\zeta_i] < \infty$. Moreover, (6.9) is satisfied. Therefore, by (5.4) and [Whi02, Th. 4.5.3],

$$
\frac{w(s_{[n\cdot]})}{n^{2/\delta}} = \frac{\tau_{[n\cdot]+1} - \tau_1}{n^{2/\delta}} \overset{J_1}{\Rightarrow} Y(\cdot),
$$

where Y is a stable subordinator. Moreover, $M_{[nt]}^{(2)}(w) \equiv 0$, since $w(t)$ is monotone. By Theorem 6.11 we conclude that $(n/\mu)^{-2/\delta}(T_{[n\cdot]+X_{\tau_1}} - \tau_1) \overset{M_1}{\Rightarrow} Y(\cdot)$ as $n \to \infty$. Since τ_1 is P_0-a.s. finite, the convergence-together theorem [Whi02, Th. 11.4.7] implies that

$$
\frac{T_{[n\cdot]+X_{\tau_1}}}{(n/\mu)^{2/\delta}} \overset{M_1}{\Rightarrow} Y(\cdot). \tag{6.10}
$$

Taking the inverses (see [Whi02, Cor. 13.6.3]), we get $n^{-\delta/2}(S_{[n\cdot]} - X_{\tau_1}) \overset{M_1}{\Rightarrow} \mu Y^{-1}(\cdot)$, where $S_n = \max_{0\leq j\leq n} X_j$. Since τ_1 is P_0-a.s. finite, we conclude that $n^{-\delta/2}S_{[n\cdot]} \overset{M_1}{\Rightarrow} \mu Y^{-1}(\cdot)$. Next we note that Y^{-1} has continuous trajectories ([Ber96, Lem. III.17]). This allows us to strengthen the convergence to J_1 (see (6.8)), i.e. $n^{-\delta/2}S_{[n\cdot]} \overset{J_1}{\Rightarrow} \mu Y^{-1}$. Part (i) of Theorem 6.5 identifies the one-dimensional distributions of the limiting processes.

Finally, we would like to replace the maximal process S with the original walk X. It is enough to consider restrictions of both processes to intervals $[0,T]$, $T > 0$, ([Bil99, Th. 16.7]). To simplify notation, we consider without loss of generality the time interval $[0,1]$ instead of $[0,T]$. The tail estimate

(5.5) for the variables of the i.i.d. sequence $(\tau_{i+1} - \tau_i)$, $i \geq 1$, implies that for every $\nu > 0$ there is $K > 0$ such that $P_0[\tau_{[Kn^{\delta/2}]} \leq n] < \nu$ for all $n \in \mathbb{N}$. For every $\varepsilon > 0$ we have

$$
P_0 \left[\max_{0 \leq t \leq 1} (S_{[nt]} - X_{[nt]}) > \varepsilon n^{\delta/2} \right]
$$

$$
\leq \quad \nu + P_0 \left[\max_{0 \leq m \leq n} (S_m - X_m) > \varepsilon n^{\delta/2}, \; \tau_{[Kn^{\delta/2}]} > n \right]
$$

$$
\leq \quad \nu + P_0 \left[\max_{1 \leq i \leq [Kn^{\delta/2}]} (X_{\tau_i} - X_{\tau_{i-1}}) > \varepsilon n^{\delta/2} \right]
$$

$$
\leq \quad \nu + P_0[X_{\tau_1} > \varepsilon n^{\delta/2}] + P_0 \left[\max_{2 \leq i \leq [Kn^{\delta/2}]} (X_{\tau_i} - X_{\tau_{i-1}}) > \varepsilon n^{\delta/2} \right]
$$

$$
\overset{(5.4)}{\leq} \quad 2\nu + 1 - \left(1 - \frac{2C_1}{(\varepsilon n^{\delta/2})^\delta} \right)^{Kn^{\delta/2}} < 3\nu
$$

for all sufficiently large n. Now apply the convergence-together theorem. $\square$

Remark 6.12. (convergence of η_n for $\delta \in (1,2)$) From (6.10) it is only one step from the conclusion that $\eta_n \overset{M_1}{\Rightarrow} \eta$, where η is a stable subordinator with $\eta(1) \overset{d}{=} Z_{\delta/2,b}$. Indeed, a straightforward computation shows that for every $T > 0$ the J_1- (and, therefore, M_1-) distance between $n^{-2/\delta} T_{[n\cdot]+X_{\tau_1}}$ and $n^{-2/\delta} T_{[n\cdot]}$ on $D[0,T]$ converges to 0 in P_0-probability as $n \to \infty$. The claim follows by the convergence-together theorem and part (i) of Theorem 6.5.

Proof of Theorem 6.7. The statement of Theorem 6.7 for $\delta > 4$ is contained in [KZ08, Th. 3]. Thus, we assume that $\delta \in (2,4]$ (even though the proof works for all $\delta > 2$). Set $\zeta_i = \tau_{i+1} - \tau_i$, $i \geq 1$, and $w(t) = X_{[t]+\tau_1} - v[t] - X_{\tau_1}$, $t \geq 0$. By Lemma 4.5 $(\zeta_i)_{i \geq 1}$ is i.i.d. and (6.9) holds. Moreover, $\mu = E_0[\zeta_i] < \infty$ by (5.5).

By Theorem 4.6, $E_0[X_{\tau_2} - X_{\tau_1}] = \mu v$. Let $a(n) := n^{2/\delta}$ if $\delta \in (2,4)$ and $a(n) := \sqrt{n \log n}$ if $\delta = 4$. We shall show the weak convergence in J_1 of

$$
\frac{w(s_{[n\cdot]})}{a(n)} = \frac{X_{\tau_{[n\cdot]+1}} - v(\tau_{[n\cdot]+1} - \tau_1) - X_{\tau_1}}{a(n)}. \tag{6.11}
$$

Indeed, by (5.4), (5.5), [Whi02, Th. 4.5.3], and [GS69, Ch. 9, Sec. 6, Th. 2]

$$
\frac{X_{\tau_{[n\cdot]+1}} - X_{\tau_1} - \mu v[n\cdot]}{a(n)} \overset{J_1}{\Rightarrow} 0 \quad \text{and} \quad \frac{\tau_{[n\cdot]+1} - \tau_1 - \mu[n\cdot]}{a(n)} \overset{J_1}{\Rightarrow} \widetilde{Y}(\cdot),
$$

where $\widetilde{Y}$ is a stable process with index $\delta/2$. The convergence-together theorem implies that the process in (6.11) converges in J_1 to $Y := -v\widetilde{Y}$. Finally, we need to check the condition $(a(n))^{-1}M^{(2)}_{[n\cdot]}(w) \overset{M_1}{\Rightarrow} 0$. The following two simple lemmas accomplish the task. (We state these lemmas for all $\delta > 2$ setting $a(n) = \sqrt{n}$ when $\delta > 4$.)

Lemma 6.13. *Let $\delta > 2$. Then, under the conditions stated above,*

$$M^{(2)}_n(w) \le 4(X_{\tau_{n+1}} - X_{\tau_n}) \quad \text{for all } n \ge 1.$$

Proof. Let $w(t) = w_1(t) - w_2(t)$, where $w_1(t) := X_{[t]+\tau_1}$ and $w_2(t) := v[t] + X_{\tau_1}$. Notice that the latter process is increasing. Then for $n \in \mathbb{N}$ and $0 = t_0 < t_1 < t_2 < t_3 = 1$,

$$\Big(\sum_{i=1}^{3}|w(s_{n-1}+t_i\zeta_n)-w(s_{n-1}+t_{i-1}\zeta_n)|\Big)-|w(s_n)-w(s_{n-1})|$$

$$\le\Big(\sum_{i=1}^{3}|w_1(s_{n-1}+t_i\zeta_n)-w_1(s_{n-1}+t_{i-1}\zeta_n)|+w_2(s_{n-1}+t_i\zeta_n)-w_2(s_{n-1}+t_{i-1}\zeta_n)\Big)$$

$$-\Big(w_2(s_n)-w_2(s_{n-1})-|w_1(s_n)-w_1(s_{n-1})|\Big) \le 4|w_1(s_n)-w_1(s_{n-1})|.$$

This implies the desired estimate. $\qquad\square$

Lemma 6.14. *Let $\delta > 2$. Then, under the conditions stated above,*

$$\max_{1\le m\le n}\frac{X_{\tau_{m+1}}-X_{\tau_m}}{a(n)} \to 0 \quad \text{as } n\to\infty \text{ in } P_0\text{-probability.}$$

Proof. For every $\varepsilon > 0$,

$$P_0\left[\max_{1\le m\le n}\frac{X_{\tau_{m+1}}-X_{\tau_m}}{a(n)} > \varepsilon\right] \le 1-(1-P_0[X_{\tau_2}-X_{\tau_1} > \varepsilon a(n)])^n$$

$$\overset{(5.4)}{\le} 1-\Big(1-\frac{2C_1}{(\varepsilon a(n))^\delta}\Big)^n \to 0 \text{ as } n\to\infty. \quad\square$$

By Theorem 6.11 we obtain $(a(n))^{-1}(X_{[n\cdot]+\tau_1}-v[n\cdot]-X_{\tau_1}) \overset{M_1}{\Rightarrow} \mu^{-2/\delta}Y(\cdot)$ as $n\to\infty$. Since τ_1 is a.s. finite and X moves in unit steps, we conclude

that

$$\frac{X_{[n\cdot]} - v[n\cdot]}{a(n)} \overset{M_1}{\Rightarrow} \mu^{-2/\delta} Y(\cdot).$$

Parts (iii) and (iv) of Theorem 6.5 identify the one-dimensional distribution of the limiting stable process. If $\delta = 4$ the process Y is a constant multiple of the standard Brownian motion and, therefore, is continuous. By (6.6), for this case we can replace M_1- with J_1-convergence. This completes the proof of Theorem 6.7. $\qquad\square$

Finally we quote a functional limit theorem in a different spirit, which can be found in [RS]. Here $\mathbb{P}$ is not fixed but scaled as well. Roughly speaking, the cookies are being diluted, becoming more and more like placebo cookies while the total drift stored in a cookie stack remains the same.

Theorem 6.15. ($d = 1$, **excited Brownian motion as limit of ERW [RS, Th. 1.4]**) *Assume that $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous and bounded by some finite constant C. Define for all $k \geq C, k \in \mathbb{N}$, a spatially homogeneous environment $\omega_k \in \Omega$ by setting for all $z \in \mathbb{Z}, i \in \mathbb{N}$,*

$$\omega_k(z, 1, i) = 1 - \omega_k(z, -1, i) = \frac{1}{2}\left(1 + \frac{1}{2k}\varphi\left(\frac{i}{2k}\right)\right)$$

and let $(X_{k,n})_{n\geq 0}$ be an ERW in the deterministic environment ω_k. Furthermore, let $(Y(t))_{t\geq 0}$ be the solution, called excited Brownian motion*, to the stochastic differential equation*

$$dY(t) = dB(t) + \varphi(L(t, Y(t)))\, dt, \quad Y_0 = 0,$$

where L is the local time process of Y. Then as $k \to \infty$,

$$\frac{X_{k,[4k^2\cdot]}}{2k} \overset{J_1}{\Rightarrow} Y(\cdot).$$

6.2. Results for $d \geq 1$

The law of large numbers considered in Theorem 5.9 can be complemented by an averaged central limit theorem.

Theorem 6.16. ($d \geq 2$, **convergence to Brownian motion [BR07, Th. 1 (ii)], [MPRV, Th. 1.2 (ii)]**) *Let $\ell \in \mathbb{R}^d\backslash\{0\}$ and assume (MPRV$_\ell$),*

(IID) *and* (UEL). *Then there exists a non-degenerate $d \times d$ matrix G such that with respect to P_0,*

$$\frac{X_{[n\cdot]} - [n\cdot]v}{\sqrt{n}} \overset{J_1}{\Rightarrow} B_G(\cdot) \quad as\ n \to \infty,$$

where B_G is the d-dimensional Brownian motion with covariance matrix G.

A central limit theorem in the original model (BW) was also obtained in [HH12, Th. 2.2] by the expansion technique mentioned in Remarks 4.11 and 5.10.

Problem and Example 6.17. ($d \geq 1$, **quenched CLT**) Find measures $\mathbb{P}$ (not including those which model RWRE) such that for $\mathbb{P}$-almost all ω the distribution under $P_{0,\omega}$ of T_n or X_n, properly centered and normalized as needed, converges weakly to some non-degenerate distribution as $n \to \infty$.

(a) Here is a somewhat degenerate example for $d = 1$: Assume (IID), $\sigma^2 := \mathbb{E}[\omega(0,1,1)\ \omega(0,-1,1)] > 0$ and let $\mathbb{P}$-a.s. $\omega(0,1,i) = 1$ for $i \in \{2,3\}$. It is not difficult to see that for $\mathbb{P}$-almost all ω the random variables $\Delta_k := T_{k+1} - T_k$, $k \geq 0$, are independent under $P_{0,\omega}$ with $P_{0,\omega}[\Delta_k = 1] = \omega(k,1,1)$ and $P_{0,\omega}[\Delta_k = 3] = \omega(k,-1,1)$ for $k \geq 1$ and $P_{0,\omega}[T_1 = 2j+1] = \omega(-j,1,1) \prod_{k=0}^{j-1} \omega(-k,-1,1)$ for $j \in \mathbb{N}_0$. By the Lindeberg-Feller central limit theorem (see e.g. [Dur10, Th. 3.4.5] with $X_{n,k} := (\Delta_k - E_{0,\omega}[\Delta_k])/\sqrt{n}$) we have $\mathbb{P}$-a.s. the following convergence in $P_{0,\omega}$-distribution:

$$\frac{T_n - E_{0,\omega}[T_n]}{2\sigma\sqrt{n}} \Rightarrow Z \quad as\ n \to \infty,\ \text{where}\ Z \sim \mathcal{N}(0,1).$$

Note that here the centering is random, i.e. depends on ω.

(b) Consider measures $\mathbb{P}$ under which not only the stacks of cookies are i.i.d. like in (IID) but also the cookies within the stacks, i.e. under which the $2d$-valued vectors $\omega(z,\cdot,i)$, $z \in \mathbb{Z}^d, i \in \mathbb{N}$, are i.i.d.. Obviously, under the averaged measure P_0 this ERW has the same distribution as a standard RW with i.i.d. increments distributed according to $(\mathbb{E}[(\omega(x,e,i)])_{e \in \mathcal{E}}$ and thus satisfies a central limit theorem, unless in degenerate cases. Nevertheless it is not obvious that the distribution of X under $P_{x,\omega}$ satisfies for $\mathbb{P}$-almost all ω a central limit theorem as well. A similar problem has been intensively studied in the setting of RWs in space-time random environments, see e.g. [DL09] and

the references therein. The difference between this well-established model and the new model proposed here is that in the new model the transition probabilities would not depend on the absolute time but on the local time.

7. Further Limit Theorems

In [Ras04] a large deviation principle for a large class of self-interacting RWs on $\mathbb{Z}^d, d \geq 1$, including some ERWs, is proved. A stronger result for ERWs in the setting discussed in Section 1.3, whose proof uses branching processes, is given in [Pet12].

Theorem 7.1. ($d = 1$, large deviation principle under the averaged measure [Ras04], [Pet12]) *Assume* (IID), (BD) *and* (WEL). *Then there is a continuous, convex function* $I : [-1, 1] \to [0, \infty)$ *such that for any Borel set* $B \subseteq [-1, 1]$,

$$- \inf_{x \in B^\circ} I(x) \leq \liminf_{n \to \infty} P_0 \left[\frac{X_n}{n} \in B \right] \leq \limsup_{n \to \infty} P_0 \left[\frac{X_n}{n} \in B \right] \leq - \inf_{x \in \bar{B}} I(x),$$

where B° *denotes the interior and* $\bar{B}$ *the closure of* B. *Moreover, if* $\delta > 2$ *and* v *denotes the speed of the walk then* $I(x) = 0$ *for all* $x \in [0, v]$. *More precisely, for all* $x \in (0, v)$,

$$\lim_{n \to \infty} \frac{\log P_0[X_n/n < x]}{\log n} = 1 - \frac{\delta}{2}.$$

For further properties of the rate function I see [Pet12, Lem. 5.1].

The maximum local time of transient ERW under the conditions of Section 1.3 is considered in [RR]. In particular, the following is shown.

Theorem 7.2. ($d = 1$, maximum local time [RR, Th. 1.2 (ii), Cor. 1.4]) *Assume* (IID), (BD) *and* (WEL) *and let* $\delta > 1$. *Denote by* $\xi_n^* := \max_{x \in \mathbb{Z}} \#\{i \in \{0, \ldots, n\} \mid X_i = n\}$ *the largest number of visits to a single site by time* n. *Then* P_0-*a.s. for all* $\alpha > 1/\delta$,

$$\lim_{n \to \infty} \frac{\xi_n^*}{n^{1/(2 \vee \delta)} (\log n)^\alpha} = 0 \quad and \quad \lim_{n \to \infty} \frac{(\log n)^\alpha \, \xi_n^*}{n^{1/(2 \vee \delta)}} = \infty.$$

Thus $(\xi_n^*)_{n \geq 0}$ behaves in the non-ballistic transient regime $\delta \in (1,2]$ similarly to its counterpart for the simple symmetric RW whereas in the ballistic regime $\delta > 2$ the exact value of δ plays an important role. The proof uses branching processes.

Acknowledgment

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