DOOB–MEYER FOR ROUGH PATHS

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Abstract

Recently Hairer–Pillai [Regularity of Laws and Ergodicity of Hypoelliptic SDEs Driven by Rough Paths, to appear in Annals of Probability] proposed the notion of θ-roughness of a path which leads to a deterministic Norris lemma. In the Gubinelli framework (Hölder, level 2) of rough paths, they were then able to prove a Hörmander type result (SDEs driven by fractional Brownian motion, $H > 1/3$). We take a step back and propose a natural "roughness" condition relative to a given p-rough path in the sense of Lyons; the aim being a Doob-Meyer result for rough integrals in the sense of Lyons. The interest in our (weaker) condition is that it is immediately verified for large classes of Gaussian processes, also in infinite dimensions. We conclude with an application to non-Markovian system under Hörmander's condition, in the spirit of Cass–Friz [Densities for rough differential equations under Hörmander's condition, Ann. of Math. (2), 171(3):2115–2141, 2010].

1. Introduction

Consider the indefinite stochastic integral $I := \int_0^{\cdot} \theta\left(t, \omega\right) dB$ where B is a multi-dimensional standard Brownian motion. It will be enough here to consider integrands which are continuous and adapted. Consider also a process $\Lambda = \Lambda_t$ of bounded variation, say continuously differentiable and adapted. It is then a well-known fact from stochastic analysis that $I + \Lambda \equiv 0$ a.s. implies $\theta \equiv 0$ and $\Lambda \equiv 0$ a.s., often referred to as *Doob–Meyer decomposition.* In Malliavin's proof of the Hörmander theorem, i.e. existence of

(smooth) densities of a given stochastic differential equation, under a Lie bracket condition on the driving vector fields, cf. [16], this decomposition is one (among several) key ingredients. It is worth noting that in this application θ and $\dot{\Lambda}$ are functions of some underlying diffusion path, i.e. of the form $\theta(t, \omega) = f(X_t(\omega))$, $\dot{\Lambda} = g(X_t(\omega))$. Actually, showing the existence of a *smooth* density requires a quantitative version of the Doob–Meyer decomposition, known as *Norris lemma* [18]. Both results are usually considered as *probabilistic* in nature and fundamentally related to martingales, quadratic variation etc.

Recently, Hairer–Pillai [11] proposed the notion of θ-roughness of a path which leads to a *deterministic* Norris lemma[1], valid for (level-2, Hölder) rough integrals in the sense of Gubinelli; a variation of the rough integral originally introduced by T. Lyons [13].

It is possible to check that this roughness condition holds for fractional Brownian motion (fBm); indeed in [11] the author show θ-roughness for any $\theta > H$ where H denotes the Hurst parameter. (Recall that Brownian motion corresponds to $H = 1/2$; in comparison, the regime $H < 1/2$ is "rougher" than Bm.) All this turns out to be a key ingredient in their Hörmander type result for stochastic differential equations driven by fBm[2], any $H > 1/3$, solutions of which are in general non-Markovian.

In the present note we take a step back and propose a natural "roughness" condition relative to a given p-rough path (of arbitrary level $[p] = 1, 2, \ldots$) in the sense of Lyons; the aim being a *deterministic* Doob-Meyer result for (general) rough integrals in the sense of Lyons. The interest in our (weaker) condition is that it is immediately verified for large classes of Gaussian processes, also in infinite dimensions. (In essence one only needs a Khintchine law of iterated logarithms for 1-dimensional projections.)

We conclude with an application to non-Markovian systems (existence of densities) under a weak Hörmander's condition, in the spirit of [2]. Smoothness of densities is subject of forthcoming work by Cass, Hairer, Litterer and Tindel.

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[1]See also [12] for some recent work in the context of fractional Brownian motion.

[2]These SDEs are handled with rough path theory and cannot be dealt with by Itô theory.

2. Truly "Rough" Paths and a Deterministic Doob-Meyer Result

Let V, W be Banach-spaces and $L(V, W)$ the space of continuous linear maps from V to W. Let $p \geq 1$. Assume $\mathbf{X} : [0, T] \to V$ to be a p-rough path in the sense of T. Lyons [13, 14] controlled by ω.

Recall that such a rough path consists of a underlying path $X : [0, T] \to V$, together with higher order information which somewhat *prescribes* the iterated integrals $\int_0^\cdot dX_{t_1} \otimes \cdots \otimes dX_{t_k}$ for $1 < k \leq [p]$.

Definition 1. For fixed $s \in [0, T)$ we call X *"rough at time s"* if (convention $0/0 := 0$)

$$(*) : \forall v^* \in V^* \backslash \{0\} : \limsup_{t \downarrow s} \frac{|\langle v^*, X_{s,t} \rangle|}{\omega(s, t)^{2/p}} = +\infty.$$

If X is rough on some dense set of $[0, T]$, we call it *truly rough*.

Theorem 1. (i) *Let $f \in Lip^\gamma(V, L(V, W))$, $\gamma > p - 1$ (so that the rough integral below is well-defined; cf. [15]) and assume X is rough at time s. Then*

$$\int_s^t f(X)\, d\mathbf{X} = O\left(\omega(s, t)^{2/p}\right) \ as \ t \downarrow s \implies f(X_s) = 0.$$

(i') *As a consequence, if X is truly rough, then*

$$\int_0^\cdot f(X)\, d\mathbf{X} \equiv 0 \ on \ [0, T] \implies f(X_\cdot) \equiv 0 \ on \ [0, T].$$

(i'') *As another consequence, assume $g \in C(V, W)$ and $|t - s| = O(\omega(s, t)^{2/p})$, satisfied e.g. when $\omega(s, t) \asymp t - s$ and $p \geq 2$ (the "rough" regime of usual interest) then*

$$\int_0^\cdot f(X)\, d\mathbf{X} + \int_0^\cdot g(X)\, dt \equiv 0 \ on \ [0, T] \implies f(X_\cdot), g(X_\cdot) \equiv 0 \ on \ [0, T].$$

(ii) *Assume $f \in Lip^\gamma(V \oplus W, L(V, W))$ and that*

$$Z := \begin{pmatrix} X \\ Y \end{pmatrix} \in V \oplus W$$

lifts to a p-rough path $\mathbf{Z} : [0, T] \to V \oplus W$ *and define, somewhat similar to* [4, p.84],

$$\int f(X, Y) d\mathbf{X} := \int h(Z) d\mathbf{Z} \text{ with } h(x, y) = (f(x, y), 0).$$

Then the conclusions from (i), (i') *and* (i"), *with* $f(X)$ *replaced by* $f(X, Y)$, *remain valid.*

Remark 1. Solutions of rough differential equations $dY = V(Y) d\mathbf{X}$ in the sense of Lyons are understood in the integral sense, based on the integral defined in (ii) above. This is our interest in this (immediate) extension of part (i).

Proof. (i) A basic estimate (e.g. [15]) for the W-valued rough integral is

$$\int_s^t f(X) dX = f(X_s) X_{s,t} + O\left(\omega(s, t)^{2/p}\right).$$

By assumption, for fixed $s \in [0, T)$, we have

$$0 = \frac{f(X_s) X_{s,t}}{\omega(s, t)^{2/p}} + O(1) \text{ as } t \downarrow s$$

and thus, for any $w^* \in W^*$,

$$\frac{|\langle v^*, X_{s,t}\rangle|}{\omega(s, t)^{2/p}} := \left|\left\langle w^*, \frac{f(X_s) X_{s,t}}{\omega(s, t)^{2/p}}\right\rangle\right| = O(1) \text{ as } t \downarrow s;$$

where $v^* \in V^*$ is given by $V \ni v \mapsto \langle w^*, f(X_s) v\rangle$ recalling that $f(X_s) \in L(V, W)$. Unless $v^* = 0$, the assumption $(*)$ implies that, along some sequence $t_n \downarrow s$, we have the divergent behavior $|\langle v^*, X_{s,t_n}\rangle| / \omega(s, t_n)^{2/p} \to \infty$, which contradicts that the same expression is $O(1)$ as $t_n \downarrow s$. We thus conclude that $v^* = 0$. In other words,

$$\forall w^* \in W^*, v \in V : \langle w^*, f(X_s) v\rangle = 0.$$

and this clearly implies $f(X_s) = 0$. (Indeed, assume otherwise i.e. $\exists v : w := f(X_s) v \neq 0$. Then define $\langle w^*, \lambda w\rangle := \lambda$ and extend, using Hahn-Banach if necessary, w^* from span$(w) \subset W$ to the entire space, such as to obtain the contradiction $\langle w^*, f(X_s) v\rangle = 1$.)

(i") From the assumptions, $\int_s^t g(X_r)\, dr \leq |g|_\infty |t - s| = O\left(\omega(s,t)^{2/p}\right)$. We may thus use (i) to conclude $f(X_s) = 0$ on $s \in [0, T)$. It follows that $\int_0^{\cdot} g(X_r)\, dr \equiv 0$ and by differentiation, $g(X_\cdot) \equiv 0$ on $[0, T]$.

(ii) Note that $h \in Lip^\gamma(V \oplus W, L(V \oplus W, W))$ fits into the framework of part (i), provided V is replaced by $V \oplus W$. By definition of $\int f(X, Y)\, d\mathbf{X}$

$$
\begin{aligned}
\int_s^t f(X, Y)\, d\mathbf{X} &= \int_s^t h(Z)\, d\mathbf{Z} \\
&= h(Z_s) Z_{s,t} + O\left(\omega(s,t)^{2/p}\right) \\
&= h(X_s, Y_s) \begin{pmatrix} X_{s,t} \\ Y_{s,t} \end{pmatrix} + O\left(\omega(s,t)^{2/p}\right) \\
&= f(X_s, Y_s) X_{s,t} + O\left(\omega(s,t)^{2/p}\right)
\end{aligned}
$$

and the identical proof (for (i'), then (i")) goes through, concluding $f(X_s, Y_s) = 0$. □

Remark 2. The reader may wonder about the restriction to $p \geq 2$ in (i") for Hölder type controls $\omega(s,t) \asymp t - s$. Typically, when $p < 2$, one uses Young theory, thereby avoiding the full body of rough path theory. That said, one can always view a path of finite p-variation, $p < 2$, as rough path of finite 2-variation (iterated integrals are well-defined as Young integrals). Moreover, by a basic consistency result, the respective integrals (Young, rough) coincide. In the context of fBm with Hurst parameter $H \in (1/2, 1)$, for instance, we can take $p = 2$ and note that in this setting fBm is truly rough (cf. below for a general argument based on the law of iterated logarithm). By the afore-mentioned consistency, the Doob–Meyer decomposition of (i") then becomes a statement about Young integrals. Such a decomposition was previously used in [1].

Remark 3. The argument is immediately adapted to the Gubinelli setting of "controlled" paths and would (in that context) yield uniqueness of the derivative process.

Remark 4. In Definition 1, one could replace the denominator $\omega(s,t)^{2/p}$ by $\omega(s,t)^\theta$, say for $1/p < \theta \leq 2/p$. Unlike [11], where $2/p - \theta$ affects the quantitative estimates, there seems to be no benefit of such a stronger condition in the present context.

3. True Roughness of Stochastic Processes

Fix $\rho \in [1, 2)$ and $p \in (2\rho, 4)$. We assume that the V-valued stochastic process X lifts to a random p-rough path. We assume V^* separable which implies separability of the unit sphere in V^* and also (by a standard theorem) separability of V. (Separability of the dual unit sphere in the weak-$*$ topology, guaranteed when V is assumed to be separable, seems not enough for our argument below.)

The following 2 conditions should be thought of as a weak form of a law-of-iterated-logarithm (LIL) lower bound, and a fairly robust form of a LIL upper bound, respectively. As will be explained below, they are easily checked for large classes of Gaussian processes, also in infinite dimensions.

Condition 1. *Set* $\psi(h) = h^{\frac{1}{2\rho}} (\ln\ln 1/h)^{1/2}$. *Assume* (i) *there exists* $c > 0$ *such that for every fixed dual unit vector* $\varphi \in V^*$ *and* $s \in [0, T)$

$$\mathbb{P}\left[\limsup_{t \downarrow s} |\varphi(X_{s,t})| / \psi(t-s) \geq c\right] = 1$$

and (ii) *for every fixed* $s \in [0, T)$,

$$\mathbb{P}\left[\limsup_{t \downarrow s} \frac{|X_{s,t}|_V}{\psi(t-s)} < \infty\right] = 1$$

Theorem 2. *Assume* X *satisfies the above condition. Then* X *is a.s. truly rough.*

Proof. Take a dense, countable set of dual unit vectors, say $K \subset V^*$. Since K is countable, the set on which condition (i) holds simultaneously for all $\varphi \in K$ has full measure,

$$\mathbb{P}\left[\forall \varphi \in K : \limsup_{t \downarrow s} |\varphi(X_{s,t})| / \psi(t-s) \geq c\right] = 1$$

On the other hand, every unit dual vector $\varphi \in V^*$ is the limit of some $(\varphi_n) \subset K$. Then

$$\frac{|\langle \varphi_n, X_{s,t} \rangle|}{\psi(t-s)} \leq \frac{|\langle \varphi, X_{s,t} \rangle|}{\psi(t-s)} + |\varphi_n - \varphi|_{V^*} \frac{|X_{s,t}|_V}{\psi(t-s)}$$

so that, using $\overline{\lim}\,(|a| + |b|) \leq \overline{\lim}\,(|a|) + \overline{\lim}\,(|b|)$, and restricting to the above set of full measure,

$$c \leq \overline{\lim_{t \downarrow s}}\frac{|\langle \varphi_n, X_{s,t}\rangle|}{\psi\,(t-s)} \leq \overline{\lim_{t \downarrow s}}\frac{|\langle \varphi, X_{s,t}\rangle|}{\psi\,(t-s)} + |\varphi_n - \varphi|_{V^*}\,\overline{\lim_{t \downarrow s}}\frac{|X_{s,t}|_V}{\psi\,(t-s)}.$$

Sending $n \to \infty$ gives, with probability one,

$$c \leq \overline{\lim_{t \downarrow s}}\frac{|\langle \varphi, X_{s,t}\rangle|}{\psi\,(t-s)}.$$

Hence, for a.e. sample $X = X\,(\omega)$ we can pick a sequence (t_n) converging to s such that $|\langle \varphi, X_{s,t_n}\rangle|\,/\psi\,(t_n - s) \geq c - 1/n$. On the other hand, for any $\theta \geq 1/\,(2\rho)$

$$\begin{aligned}\frac{|\langle \varphi, X_{s,t_n}\,(\omega)\rangle|}{|t_n - s|^\theta} &= \frac{|\langle \varphi, X_{s,t_n}\,(\omega)\rangle|}{\psi\,(t_n - s)}\frac{\psi\,(t_n - s)}{|t_n - s|^\theta}\\ &\geq (c - 1/n)\,|t_n - s|^{\frac{1}{2\rho}-\theta}\,L\,(t_n - s)\\ &\to \infty\end{aligned}$$

since $c > 0$ and $\theta \geq 1/\,(2\rho)$ and slowly varying $L\,(\tau) := (\ln\ln 1/\tau)^{1/2}$ (in the extreme case $\theta = 1/\,(2\rho)$ the divergence is due to the (very slow) divergence $L\,(\tau) \to \infty$ as $\tau = t_n - s \to 0$.) □

3.1. Gaussian processes

The conditions put forward here are typical for Gaussian process (so that the pairing $\langle \varphi, X\rangle$ is automatically a scalar Gaussian process). Sufficient conditions for (i), in fact, a law of iterated logarithm, with equality and $c = 1$ are e.g. found in [17, Thm 7.2.15]. These conditions cover immediately - and from general principles - many Gaussian (rough paths) examples, including fractional Brownian motion ($\rho = 1/\,(2H)$, lifted to a rough path [5, 9]) and the stationary solution to the stochastic heat equation on the torus, viewed as as Gaussian processes parametrized by $x \in [0, 2\pi]$; here $\rho = 1$, the fruitful lift to a "spatial" Gaussian rough path is due to Hairer [10].

As for condition (ii), it holds under a very general condition [9, Thm

A.22]

$$\exists \eta > 0 : \sup_{0 \le s,t \le T} E \exp \left(\eta \frac{|X_{s,t}|_V^2}{|t-s|^{1/\rho}} \right) < \infty.$$

In presence of some scaling, this condition is immediately verified by Fernique's theorem.

Example 1. d-dimensional fBm is a.s. truly rough (in fact, H-rough)

In order to apply this in the context of (random) rough integration, we need to intersect the class of truly rough Gaussian processes with the classes of Gaussian processes which admit a rough path lift. To this end, we recall the following standard setup [9]. Consider a continuous d-dimensional Gaussian process, say X, realized as coordinate process on the (not-too abstract) Wiener space $(E, \mathcal{H}, \mu)$ where $E = C\left([0,T], \mathbb{R}^d\right)$ equipped with μ is a Gaussian measure s.t. X has zero-mean, independent components and that $V_{\rho\text{-var}}\left(R, [0,T]^2\right)$, the ρ-variation in 2D sense of the covariance R of X, is finite for $\rho \in [1,2)$. (In the fBm case, this condition translates to $H > 1/4$). From [9, Theorem 15.33] it follows that we can lift the sample paths of X to p-rough paths for any $p > 2\rho$ and we denote this process by **X**, called the *enhanced Gaussian process*. In this context, modulo a deterministic time-change, condition (ii) will always be satisfied (with the same ρ). The non-degeneracy condition (i), of course, cannot be expect to hold true in this generality; but, as already noted, conditions are readily available [17].

Example 2. Q-Wiener processes are a.s. truly rough. More precisely, consider a separable Hilbert space H with ONB (e_k), $(\lambda_k) \in l^1$, $\lambda_k > 0$ for all k, and a countable sequence (β^k) of independent standard Brownians. Then the limit

$$X_t := \sum_{k=1}^{\infty} \lambda_k^{1/2} \beta_t^k e_k$$

exists a.s. and in L^2, uniformly on compacts and defines a Q-Wiener process, where $Q = \sum \lambda_k \langle e_k, \cdot \rangle$ is symmetric, non-negative and trace-class. (Conversely, any such operator Q on H can be written in this form and thus gives rise to a Q-Wiener process.) By Brownian scaling and Fernique, condition

(ii) is obvious. As for condition (i), let φ be an arbitrary unit dual vector and note that $\varphi\left(X.\right)/\sigma_\varphi$ is standard Brownian provided we set

$$\sigma_\varphi^2 := \sum \lambda_k \left\langle \varphi, e_k \right\rangle^2 > 0.$$

By Khintchine's law of iterated logarithms for standard Brownian motion, for fixed φ and s, with probability one,

$$\limsup_{t\downarrow s} \left|\varphi\left(X_{s,t}\right)\right|/\psi\left(t-s\right) \geq \sqrt{2}\sigma_\varphi.$$

Since $\varphi \mapsto \sigma_\varphi^2$ is weakly continuous (this follows from $(\lambda) \in l^1$ and dominated convergence) and compactness of the unit sphere in the weak topology, $c := \inf \sigma_\varphi > 0$, and so condition (ii) is verified.

Let us quickly note that Q-Wiener processes can be naturally enhanced to rough paths. Indeed, it suffices to define the $H \otimes H$-valued "second level" increments as

$$(s,t) \mapsto \mathbb{X}_{s,t} := \sum_{i,j} \lambda_i^{1/2} \lambda_j^{1/2} \left(\int_s^t \beta_{s,\cdot}^i \circ d\beta^j\right) e_i \otimes e_j.$$

which essentially reduces the construction of the "area-process" to the Lévy area of a 2-dimensional standard Brownian motion. (Alternatively, one could use stochastic integration against Q-Wiener processes.) Rough path regularity, $\left|\mathbb{X}_{s,t}\right|_{H\otimes H} = O\left(\left|t-s\right|^{2\alpha}\right)$ for some $\alpha \in (1/3, 1/2]$ (in fact: any $\alpha < 1/2$), is immediate from a suitable Kolmogorov-type (or Garsia-Rodemich-Rumsey) criterion (e.g. [8, 9]).

Variations of the scheme are possible of course, it is rather immediate to define Q-Gaussian processes in which $\left(\beta^k\right)$ are replaced by $\left(X^k\right)$, a sequence of independent Gaussian processes, continuous each with covariance uniformly of finite ρ-variation, $\rho < 2$.

Let us insist that the (random) rough integration against Brownian, or Q-Wiener processes) is well-known to be consistent with Stratonovich stochastic integration (e.g. [14, 9, 8]). In fact, one can also construct a rough path lift via Itô-integration, in this case (random) rough integration (now against a "non-geometric" rough path) coincides with Itô-integration.

4. An Application

Let X be a continuous d-dimensional Gaussian process which admits a rough path lift in the sense described at the end of the previous section. Assume in addition that the Cameron-Martin space $\mathcal{H}$ has *complementary Young regularity* in the sense that $\mathcal{H}$ embeds continuously in the space of continuous paths, from $[0, T]$ to $\mathbb{R}^d$, with finite q-variation for $\frac{1}{p} + \frac{1}{q} > 1$. Note $q \leq p$ for μ is supported on the paths of finite p-variation. This is true in great generality with $q = \rho$ whenever $\rho < 3/2$ and also for fBm (and variations thereof) for all $H > 1/4$. Complementary Young regularity of the Cameron-Martin space is a natural condition, in particular in the context of Malliavin calculus and has been the basis of non-Markovian Hörmander theory, the best results up to date were obtained in [2] (existence of density only, no drift, general non-degenerate Gaussian driving noise) and then [11] (existence of a smooth density, with drift, fBm $H > 1/3$). We give a quick proof of existence of density, *with drift,* with general non-degenerate Gaussian driving noise (including fBm $H > 1/4$). To this end, consider the rough differential equation

$$dY = V_0\left(Y\right) dt + V\left(Y\right) d\mathbf{X}$$

subject to a *weak* Hörmander condition at the starting point. (Vector fields, on $\mathbb{R}^e$, say are assumed to be bounded, with bounded derivatives of all orders.) In the drift free case, $V_0 = 0$, conditions on the Gaussian driving signal X where given in [2] which guarantee existence of a density. With no need of going into full detail here, the proof (by contradiction) follows a classical pattern which involves a deterministic, non-zero vector z such that[3]

$$z^T J_{0\leftarrow \cdot}^{\mathbf{X}(\omega)}\left(V_k\left(Y_\cdot\left(\omega\right)\right)\right) \equiv 0 \text{ on } [0, \Theta\left(\omega\right))$$

,every $k \in \{1, \dots, d\}$ for some a.s. positive random time Θ. Here J

(This follows from a global non-degeneracy condition, which, for instance, rules out Brownian bridge type behavior, and a 0-1 law, see conditions 3,4 in [2]). From this

$$\int_0^\cdot z^T J_{0\leftarrow t}^{\mathbf{X}}\left([V, V_k]\left(Y_t\right)\right) d\mathbf{X} + \int_0^\cdot z^T J_{0\leftarrow t}^{\mathbf{X}}\left([V_0, V_k]\left(Y_t\right)\right) dt \equiv 0$$

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[3] $J_{0\leftarrow \cdot}^{\mathbf{X}(\omega)}$ denotes the Jacobian of the inverse flow; cf. [2, 9].

on $[0, \Theta(\omega))$; here $V = (V_1, \ldots V_d)$ and V_0 denote smooth vector fields on $\mathbb{R}^e$ along which the RDEs under consideration do not explode. Now we assume the driving (rough) path to be truly rough, at least on a positive neighborhood of 0. Since $Z := (X, Y, J)$ can be constructed simultaneously as rough path, say $\mathbf{Z}$, we conclude with Theorem 1, (iii):

$$z^T J_{0\leftarrow\cdot}^{\mathbf{X}} \left([V_l, V_k] (Y_\cdot) \right) \equiv 0 \equiv z^T J_{0\leftarrow\cdot}^{\mathbf{X}} \left([V_0, V_k] (Y_t) \right).$$

Usual iteration of this argument shows that z is orthogonal to $V_1, \ldots, V_d$ and then all Lie-brackets (also allowing V_0), always at y_0. Since the weak-Hörmander condition asserts precisely that all these vector fields span the tangent space (at starting point y_0) we then find $z = 0$ which is the desired contradiction. We note that the true roughness condition on the driving (rough) path replaces the support type condition put forward in [2]. Let us also note that this argument allows a pain-free handling of a drift vector field (not including in [2]); examples include immediately fBm with $H > 1/4$ but we have explained above that far more general driving signals can be treated. In fact, it transpires true roughness of Q-Wiener processes (and then, suitable generalizations to Q-Gaussian processes) on a separable Hilbert space $\mathbb{H}$ allows to obtain a Hörmander type result where the Q-process "drives" countably many vector fields given by $V : \mathbb{R}^e \to L(\mathbb{H}, \mathbb{R}^e)$.

The Norris type lemma put forward in [11] suggests that the argument can be made quantitative, at least in finite dimensions, thus allowing for a Hörmander type theory (existence of smooth densities) for RDE driven by general non-degenerate Gaussian signals. In [11] the authors obtain this result for fBm, $H > 1/3$.

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