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SOME CRITERIA FOR STRONGLY STARLIKE MULTIVALENT FUNCTIONS

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Abstract

By using the method of differential subordination, we obtain some sufficient conditions for strongly p-valent starlikeness.

1. Introduction and Preliminaries

Let $f(z)$ and $g(z)$ be analytic in the unit disk $U = \{z : |z| < 1\}$. The function $f(z)$ is subordinate to $g(z)$ in U, written $f(z) \prec g(z)$, if $g(z)$ is univalent in $U, f(0) = g(0)$ and $f(U) \subset g(U)$.

Let A_p denote the class of functions f of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (p \in N = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in U. A function $f \in A_p$ is said to be p-valent starlike of order α in U if it satisfies

$$\mathrm{Re}\frac{zf'(z)}{f(z)} > p\alpha \quad (z \in U) \tag{1.2}$$

for some $\alpha (0 \le \alpha < 1)$. We denote this class by $S_p^*(\alpha)$. For $-1 \le b < a \le 1$ and $0 < \beta \le 1$, a function $f \in A_p$ is said to be in the class $S_p^*(\beta, a, b)$ if it

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satisfies

$$\frac{zf'(z)}{f(z)} \prec p\left(\frac{1+az}{1+bz}\right)^{\beta}. \tag{1.3}$$

It is easy to know that each function in the class $S_p^*(\beta, a, b)$ is p-valently starlike in U. Also we write

$$S_p^*(\beta, 1, -1) = \widetilde{S}_p^*(\beta) \quad \text{and} \quad S_p^*(1, a, b) = S_p^*(a, b).$$

Note that $S_p^*(1-2\alpha, -1) = S_p^*(\alpha)(0 \leq \alpha < 1)$ and $\widetilde{S}_p^*(\beta)(0 < \beta \leq 1)$ is the class of strongly starlike p-valent functions of order β in U.

A number of results for strongly starlike functions in U have been obtained by several authors (see, e.g., [1, 3-11]). The object of the present paper is to derive some criteria for functions in the class A_p to be strongly starlike p-valent of order β in U.

To prove our results, we need the following lemma due to Miller and Mocanu [2].

Lemma 1.1. *Let $g(z)$ be analytic and univalent in U and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $g(U)$, with $\varphi(w) \neq 0$ when $w \in g(U)$. Set*

$$Q(z) = zg'(z)\varphi(g(z)), \quad h(z) = \theta(g(z)) + Q(z)$$

and suppose that

(i) *$Q(z)$ is starlike univalent in U, and*
(ii) $Re\frac{zh'(z)}{Q(z)} = Re\left\{\frac{\theta'(g(z))}{\varphi(g(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in U)$.

If $p(z)$ is analytic in U, with $p(0) = g(0)$, $p(U) \subset D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(g(z)) + zg'(z)\varphi(g(z)) = h(z), \tag{1.4}$$

then $p(z) \prec g(z)$ and $g(z)$ is the best dominant of (1.4).

Applying Lemma 1.1 we prove

Lemma 1.2. *Let k and m be integers, $m \neq 0, \lambda$ be real, $-1 \leq b < a \leq 1, 0 < \beta \leq 1$ and suppose that one of the following conditions is satisfied:*

(i) $\lambda m > 0, \mu > 0$ *and* $\max\{|k-1|\beta, |k|\beta, |m+k-1|\beta\} \leq 1$;

(ii) $\lambda m > 0, \mu = 0$ *and* $\max\{|k-1|\beta, |m+k-1|\beta\} \le 1$;

(iii) $\lambda = 0, \mu > 0$ *and* $\max\{|k-1|\beta, |k|\beta\} \le 1$;

(iv) $\lambda = \mu = 0$ *and* $|k-1|\beta \le 1$.

If $p(z)$ *is analytic in* $U, p(0) = 1$, *and* $p(z) \neq 0 (z \in U)$ *when* $m < 0$ (*with* $\lambda < 0$), $k > 0$, *and if*

$$\lambda(p(z))^m + \mu p(z) + \frac{zp'(z)}{(p(z))^k} \prec h(z), \tag{1.5}$$

where

$$h(z) = \lambda \left(\frac{1+az}{1+bz}\right)^{m\beta} + \mu \left(\frac{1+az}{1+bz}\right)^{\beta} + \frac{\beta(a-b)z}{(1+az)^{1+(k-1)\beta}(1+bz)^{1-(k-1)\beta}} \tag{1.6}$$

is (*close-to-convex*) *univalent in* U, *then*

$$p(z) \prec \left(\frac{1+az}{1+bz}\right)^{\beta}$$

and $(\frac{1+az}{1+bz})^{\beta}$ *is the best dominant of* (1.5).

Proof. We choose

$$g(z) = \left(\frac{1+az}{1+bz}\right)^{\beta}, \quad \theta(w) = \lambda w^m + \mu w, \quad \varphi(w) = \frac{1}{w^k}$$

in Lemma 1.1. In view of $-1 \le b < a \le 1$ and $0 < \beta \le 1$, the function $g(z)$ is analytic and convex univalent in U (see [10]). Noting that

$$\operatorname{Re} g(z) > \left(\frac{1-a}{1-b}\right)^{\beta} \ge 0 \quad (z \in U),$$

the functions $\theta(w)$ and $\varphi(w)$ are analytic in $D = \{w : w \neq 0\}$ containing $g(U)$, with $\varphi(w) \neq 0$ when $w \in g(U)$.

Since $|k-1|\beta \le 1$, the function

$$Q(z) = zg'(z)\varphi(g(z)) = \frac{zg'(z)}{(g(z))^k} = \frac{\beta(a-b)z}{(1+az)^{1+(k-1)\beta}(1+bz)^{1-(k-1)\beta}}$$

is starlike univalent in U because

$$\begin{aligned}\mathrm{Re}\frac{zQ'(z)}{Q(z)} &= -1+(1+(k-1)\beta)\mathrm{Re}\frac{1}{1+az}+(1-(k-1)\beta)\mathrm{Re}\frac{1}{1+bz}\\ &> -1+\frac{1+(k-1)\beta}{1+|a|}+\frac{1-(k-1)\beta}{1+|b|}\\ &\geq 0 \quad (z\in U). \qquad (1.7)\end{aligned}$$

Further we have

$$\begin{aligned}&\theta(g(z))+Q(z)\\ &= \lambda\left(\frac{1+az}{1+bz}\right)^{m\beta}+\mu\left(\frac{1+az}{1+bz}\right)+\frac{\beta(a-b)z}{(1+az)^{1+(k-1)\beta}(1+bz)^{1-(k-1)\beta}}\\ &= h(z)\end{aligned}$$

and

$$\frac{zh'(z)}{Q(z)} = \lambda m\left(\frac{1+az}{1+bz}\right)^{(m+k-1)\beta}+\mu\left(\frac{1+az}{1+bz}\right)^{k\beta}+\frac{zQ'(z)}{Q(z)}. \qquad (1.8)$$

If $|m+k-1|\beta\leq 1$, then

$$\left|\arg\left\{\left(\frac{1+az}{1+bz}\right)^{(m+k-1)\beta}\right\}\right| < \frac{|m+k-1|\beta\pi}{2}\leq\frac{\pi}{2} \quad (z\in U). \qquad (1.9)$$

If $|k|\beta\leq 1$, then

$$\left|\arg\left\{\left(\frac{1+az}{1+bz}\right)^{k\beta}\right\}\right| < \frac{|k|\beta\pi}{2}\leq\frac{\pi}{2} \quad (z\in U). \qquad (1.10)$$

Consequently, if one of the conditions (i)-(iv) is satisfied, then it follows from (1.7)-(1.10) that

$$\begin{aligned}\mathrm{Re}\frac{zh'(z)}{Q(z)} &= \lambda m\mathrm{Re}\left\{\left(\frac{1+az}{1+bz}\right)^{(m+k-1)\beta}\right\}+\mu\mathrm{Re}\left\{\left(\frac{1+az}{1+bz}\right)^{k\beta}\right\}+\mathrm{Re}\frac{zQ'(z)}{Q(z)}\\ &> 0 \quad (z\in U).\end{aligned}$$

Thus $h(z)$ is (close-to-convex) univalent in U. The other conditions of Lemma 1.1 are seen to be satisfied. Therefore, by using Lemma 1.1, we conclude that $p(z)\prec g(z)$ and $g(z)$ is the best dominant of (1.5). The proof of Lemma 1.2 is completed. □

2. Main Results

Theorem 2.1. *Let* $-1 \le b < a \le 1$ *and* $\lambda \ge 0$. *If* $f \in A_p$ *satisfies* $f(z)f'(z) \ne 0(0 < |z| < 1)$ *and*

$$\left|1 + \frac{zf''(z)}{f'(z)} + \left(\frac{\lambda}{p} - 1\right)\frac{zf'(z)}{f(z)} - \lambda\right| < \frac{a-b}{1+|b|}\left(\lambda + \frac{1}{1+|a|}\right) \quad (z \in U), \tag{2.1}$$

then $f \in S_p^*(a, b)$.

Proof. For $f \in A_p$ satisfying $f(z)f'(z) \ne 0(0 < |z| < 1)$, the function

$$p(z) = \frac{zf'(z)}{pf(z)}$$

is analytic in U with $p(0) = 1$ and $p(z) \ne 0(z \in U)$. By taking

$$k = m = \beta = 1, \quad \lambda \ge 0 \quad \text{and} \quad \mu = 0$$

in Lemma 1.2, (1.5) and (1.6) become

$$\begin{aligned} \lambda p(z) + \frac{zp'(z)}{p(z)} &= 1 + \frac{zf''(z)}{f'(z)} + \left(\frac{\lambda}{p} - 1\right)\frac{zf'(z)}{f(z)} \\ &\prec \frac{(a-b)z}{1+bz}\left(\lambda + \frac{1}{1+az}\right) + \lambda \\ &= h(z). \end{aligned} \tag{2.2}$$

Since $h(z) - \lambda$ is univalent in $U, h(0) = \lambda$, and

$$|h(z) - \lambda| \ge \left|\frac{(a-b)z}{1+bz}\right| \operatorname{Re}\left(\lambda + \frac{1}{1+az}\right) \ge \frac{a-b}{1+|b|}\left(\lambda + \frac{1}{1+|a|}\right)$$

for $|z| = 1(z \ne -\frac{1}{a}, -\frac{1}{b})$, it follows from (2.1) that the subordination (2.2) holds. Hence an application of Lemma 1.2 yields

$$p(z) \prec \frac{1+az}{1+bz},$$

that is, $f \in S_p^*(a, b)$. □

Remark 2.1. If we let $a = 1, b = 0$ and $\lambda = \frac{p}{\alpha}(\alpha > 0)$, then Theorem 2.1 reduces to the result of Yang [12, Theorem 2].

Theorem 2.2. *Let m be an integer, $m \neq 0, \lambda m \geq 0$ and $0 < \beta < \frac{1}{|m|}$. If $f \in A_p$ satisfies $f(z)f'(z) \neq 0 (0 < |z| < 1)$ and*

$$\left|\left(\frac{pf(z)}{zf'(z)}\right)^m \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \lambda\right)\right| < B \quad (z \in U), \tag{2.3}$$

where

$$B = \sqrt{x_0^{m\beta}\left(\lambda^2 + \frac{\beta^2}{4}\left(x_0 + \frac{1}{x_0} + 2\right)\right)} > |\lambda| \quad (\lambda \ real)$$

and x_0 is the positive root of the equation

$$\beta(1 + m\beta)x^2 + 2m(2\lambda^2 + \beta^2)x - \beta(1 - m\beta) = 0, \tag{2.4}$$

then $f \in \widetilde{S}_p^*(\beta)$ and the bound B in (2.3) is the largest number such that

$$\frac{zf'(z)}{f(z)} \prec p\left(\frac{1-z}{1+z}\right)^\beta. \tag{2.5}$$

Proof. Let m be an integer, $m \neq 0$, and define

$$p(z) = \frac{pf(z)}{zf'(z)},$$

where $f \in A_p$ satisfies $f(z)f'(z) \neq 0 (0 < |z| < 1)$. Then $p(z)$ is analytic in U, $p(0) = 1, p(z) \neq 0 (z \in U)$, and

$$\lambda(p(z))^m + \frac{zp'(z)}{(p(z))^{1-m}} = \left(\frac{pf(z)}{zf'(z)}\right)^m \left(\lambda + \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \ (z \in U). \tag{2.6}$$

Putting

$$k = 1 - m, \ \ a = 1, \ \ b = -1, \ \ \lambda m \geq 0, \ \ \mu = 0 \ \text{ and } \ 0 < \beta \leq \frac{1}{|m|}$$

in Lemma 1.2 and using (2.6), we find that if

$$\left(\frac{pf(z)}{zf'(z)}\right)^m \left(\lambda + \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right) \prec h(z), \tag{2.7}$$

where

$$h(z)=\lambda\left(\frac{1+z}{1-z}\right)^{m\beta}+\frac{2\beta z}{(1+z)^{1-m\beta}(1-z)^{1+m\beta}} \tag{2.8}$$

is (close-to-convex) univalent in U, then

$$p(z)\prec\left(\frac{1+z}{1-z}\right)^{\beta},$$

which gives that $f\in\widetilde{S}_p^*(\beta)$.

Letting $0<\theta<\pi$ and $x=\cot^2\frac{\theta}{2}>0$, we deduce from (2.8) that

$$\begin{aligned}|h(e^{i\theta})|^2 &= \left|\frac{1+e^{i\theta}}{1-e^{i\theta}}\right|^{2m\beta}\left|\lambda+\frac{2\beta e^{i\theta}}{1-e^{2i\theta}}\right|^2\\ &= x^{m\beta}\left(\lambda^2+\frac{\beta^2}{4}\left(x+\frac{1}{x}+2\right)\right)=g(x)\quad(\text{say})\end{aligned}$$

and

$$g'(x)=\frac{\beta}{4}x^{m\beta-2}(\beta(1+m\beta)x^2+2m(2\lambda^2+\beta^2)x-\beta(1-m\beta))\quad(x>0). \tag{2.9}$$

Since $0<\beta<\frac{1}{|m|}$, it follows from (2.9) that the function $g(x)$ takes its minimum value at x_0, where x_0 is the positive root of the equation

$$\beta(1+m\beta)x^2+2m(2\lambda^2+\beta^2)x-\beta(1-m\beta)=0.$$

Thus, in view of $h(e^{-i\theta})=\overline{h(e^{i\theta})}(0<\theta<\pi)$, we have

$$\begin{aligned}\inf_{|z|=1(z\neq\pm1)}|h(z)| &= \min_{0<\theta<\pi}|h(e^{i\theta})|\\ &= \sqrt{x_0^{m\beta}\left(\lambda^2+\frac{\beta^2}{4}\left(x_0+\frac{1}{x_0}+2\right)\right)}=B, \end{aligned}\tag{2.10}$$

which implies that $h(U)$ contains the disk $|w|<B$ for $|h(0)|=|\lambda|<B$. Hence, if the condition (2.3) is satisfied, then the subordination (2.7) holds and thus $f\in\widetilde{S}_p^*(\beta)$.

For the function

$$f(z) = z^p \exp\left\{p \int_0^z \frac{1}{t}\left(\left(\frac{1-t}{1+t}\right)^\beta - 1\right) dt\right\} \in A_p, \tag{2.11}$$

we find after some computations that $f \in \widetilde{S}_p^*(\beta)$ and that

$$\left(\frac{pf(z)}{zf'(z)}\right)^m \left(\lambda + \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right) = h(z). \tag{2.12}$$

Furthermore we conclude from (2.10) and (2.12) that the bound B in (2.3) is the largest number such that (2.5) holds true. The proof of the theorem is completed. □

Corollary 2.1. *Let m be an integer, $m \neq 0$, $0 < \beta < \frac{1}{|m|}$. If $f \in A_p$ satisfies $f(z)f'(z) \neq 0 (0 < |z| < 1)$ and*

$$\left|\left(\frac{pf(z)}{zf'(z)}\right)^m \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < B_m \quad (z \in U), \tag{2.13}$$

where

$$B_m = \frac{\beta}{\sqrt{(1+m\beta)^{1+m\beta}(1-m\beta)^{1-m\beta}}},$$

then $f \in \widetilde{S}_p^*(\beta)$ and the bound B_m in (2.13) is the largest number such that (2.5) holds true.

Proof. Putting $\lambda = 0$ in Theorem 2.2, we get

$$x_0 = \frac{1-m\beta}{1+m\beta}$$

and

$$B = \frac{\beta}{\sqrt{(1+m\beta)^{1+m\beta}(1-m\beta)^{1-m\beta}}} > 0.$$

Therefore Corollary 2.1 follows immediately from Theorem 2.2. □

Remark 2.2. Nunokawa et al. [4, Main theorem] proved that if $f \in A_1$ is univalent in U and satisfies

$$\left|\frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1\right| < B_1 \quad (z \in U), \tag{2.14}$$

where

$$B_1 = \frac{\beta}{\sqrt{(1+\beta)^{1+\beta}(1-\beta)^{1-\beta}}} \quad (0 < \beta < 1),$$

then $f \in \widetilde{S}_1^*(\beta)$.

Obviously Corollary 2.1 with $p = m = 1$ yields the above result obtained by Nunokawa et al. [4] using another method. Furthermore we have shown that the bound B_1 in (2.14) is the largest number such that

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1-z}{1+z}\right)^{\beta}.$$

Theorem 2.3. *Let $m \in N, \lambda > 0$ and $0 < \beta \leq \frac{1}{m}$. If $f \in A_p$ satisfies $f(z)f'(z) \neq 0 (0 < |z| < 1)$ and*

$$\left|\arg\left\{\left(\frac{pf(z)}{zf'(z)}\right)^m \left(\lambda + \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right\}\right| < \frac{m\beta\pi}{2} + \arctan\frac{\beta}{\lambda} \quad (z \in U), \tag{2.15}$$

then $f \in \widetilde{S}_p^*(\beta)$ and the bound $\frac{m\beta\pi}{2} + \arctan\frac{\beta}{\lambda}$ in (2.15) is the largest number such that (2.5) holds true.

Proof. Let

$$h(z) = \left(\frac{1+z}{1-z}\right)^{m\beta}\left(\lambda + \frac{2\beta z}{1-z^2}\right) \quad (z \in U)$$

for $m \in N, \lambda > 0$ and $0 < \beta \leq \frac{1}{m}$. Then $h(0) = \lambda > 0$ and

$$h(e^{i\theta}) = \left(\cot\frac{\theta}{2}\right)^{m\beta} e^{\frac{m\beta\pi}{2}i}\left(\lambda + \frac{\beta i}{2}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right) \quad (0 < \theta < \pi).$$

From this we have

$$\begin{aligned}\arg h(e^{i\theta}) &= \frac{m\beta\pi}{2} + \arctan\left(\frac{\beta}{2\lambda}\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right)\right) \\ &\geq \frac{m\beta\pi}{2} + \arctan\frac{\beta}{\lambda} \quad (0 < \theta < \pi),\end{aligned}$$

which, in view of $h(e^{-i\theta}) = \overline{h(e^{i\theta})}$, implies that

$$\inf_{|z|=1(z\neq\pm1)} |\arg h(z)| = \frac{m\beta\pi}{2} + \arctan\frac{\beta}{\lambda}.$$

Thus $h(U)$ contains the sector

$$|\arg w| < \frac{m\beta\pi}{2} + \arctan\frac{\beta}{\lambda}.$$

The remaining part of the proof of the theorem is similar to that in the proof of Theorem 2.2 and so we omit it. Also the function $f(z)$ defined by (2.11) shows that the bound in (2.15) is the largest number such that (2.5) holds true. □

Setting $m = \lambda = 1$, Theorem 2.3 reduces to the following:

Corollary 2.2. *Let* $0 < \beta \leq 1$. *If* $f \in A_p$ *satisfies* $f'(z) \neq 0 (0 < |z| < 1)$ *and*

$$\left|\arg\left(1 - \frac{f(z)f''(z)}{(f'(z))^2}\right)\right| < \frac{\beta\pi}{2} + \arctan\beta \quad (z \in U), \tag{2.16}$$

then $f \in \widetilde{S}_p^*(\beta)$ *and the bound* $\frac{\beta\pi}{2} + \arctan\beta$ *in* (2.16) *is the largest number such that* (2.5) *holds true.*

If we let

$$0 < \beta < \frac{1}{m} \quad \text{and} \quad \frac{m\beta\pi}{2} + \arctan\frac{\beta}{\lambda} = \frac{\pi}{2},$$

then Theorem 2.3 yields

Corollary 2.3. *Let* $m \in N$ *and* $0 < \beta < \frac{1}{m}$. *If* $f \in A_p$ *satisfies* $f(z)f'(z) \neq 0 (0 < |z| < 1)$ *and*

$$Re\left\{\left(\frac{f(z)}{zf'(z)}\right)^m \left(\beta\tan\frac{m\beta\pi}{2} + \frac{zf'(z)}{f(z)} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right)\right\} > 0 \quad (z \in U), \tag{2.17}$$

then $f \in \widetilde{S}_p^*(\beta)$ *and the result is sharp.*

Theorem 2.4. *Let* $m \in N$ *and* $0 < \beta < \frac{1}{m}$. *If* $f \in A_p$ *satisfies* $f(z)f'(z) \neq 0$ $(0 < |z| < 1)$ *and*

$$\left|\lambda\left(\frac{zf'(z)}{pf(z)}\right)^m + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \rho\right| < \rho \quad (z \in U), \tag{2.18}$$

where

$$\rho = \beta \tan\left(\frac{(1+m\beta)\pi}{4}\right) \quad \text{and} \quad 0 < \lambda < 2\rho, \tag{2.19}$$

then $f \in \widetilde{S}_p^*(\beta)$.

Proof. By taking

$$k=1,\ \ m\in N,\ \ a=1,\ \ b=-1,\ \ \lambda>0,\ \ \mu=0,\ \ 0<\beta<\frac{1}{m}$$

and

$$p(z) = \frac{zf'(z)}{pf(z)}$$

in Lemma 1.2, we see that if

$$\lambda(p(z))^m + \frac{zp'(z)}{p(z)} = \lambda\left(\frac{zf'(z)}{pf(z)}\right)^m + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \prec h(z), \tag{2.20}$$

where

$$h(z) = \lambda\left(\frac{1+z}{1-z}\right)^{m\beta} + \frac{2\beta z}{1-z^2} \tag{2.21}$$

is (close-to-convex) univalent in U, then $f \in \widetilde{S}_p^*(\beta)$.

Letting $0<\theta<\pi$ and $x=\cot\frac{\theta}{2}>0$, it follows from (2.21) that

$$h(e^{i\theta}) = \lambda\left(\frac{1+e^{i\theta}}{1-e^{i\theta}}\right)^{m\beta} + \frac{2\beta e^{i\theta}}{1-e^{2i\theta}}$$

$$= \lambda x^{m\beta}\cos\frac{m\beta\pi}{2} + i\left(\lambda x^{m\beta}\sin\frac{m\beta\pi}{2} + \frac{\beta}{2}\left(x+\frac{1}{x}\right)\right) \quad (x>0).$$

Further we deduce that for $x>0$,

$$0 \ < \ \mathrm{Re}\frac{1}{h(e^{i\theta})} = \frac{\lambda x^{m\beta}\cos\frac{m\beta\pi}{2}}{\left(\lambda x^{m\beta}\cos\frac{m\beta\pi}{2}\right)^2 + \left(\lambda x^{m\beta}\sin\frac{m\beta\pi}{2} + \frac{\beta}{2}\left(x+\frac{1}{x}\right)\right)^2}$$

$$\leq \ \frac{\lambda x^{m\beta}\cos\frac{m\beta\pi}{2}}{\left(\lambda x^{m\beta}\cos\frac{m\beta\pi}{2}\right)^2 + \left(\lambda x^{m\beta}\sin\frac{m\beta\pi}{2} + \beta\right)^2}$$

$$= \frac{\lambda \cos \frac{m\beta\pi}{2}}{(\lambda^2 x^{m\beta} + \beta^2 x^{-m\beta}) + 2\lambda\beta \sin \frac{m\beta\pi}{2}}$$

$$\leq \frac{\cos \frac{m\beta\pi}{2}}{2\beta\left(1 + \sin \frac{m\beta\pi}{2}\right)} = \frac{1}{2\rho} \quad (0 < \theta < \pi), \tag{2.22}$$

where ρ is given by (2.19). Noting that $h(e^{-i\theta}) = \overline{h(e^{i\theta})}(0 < \theta < \pi)$, (2.22) leads to

$$|h(e^{i\theta}) - \rho|^2 - \rho^2 = |h(e^{i\theta})|^2 \left(1 - 2\rho \mathrm{Re} \frac{1}{h(e^{i\theta})}\right) \geq 0 \quad (0 < |\theta| < \pi). \tag{2.23}$$

In view of $0 < h(0) = \lambda < 2\rho$, (2.23) implies that

$$\{w : |w - \rho| < \rho\} \subset h(U).$$

Consequently, if the condition (2.18) is satisfied, then the subordination (2.20) holds, and so $f \in \widetilde{S}_p^*(\beta)$. □

Corollary 2.4. *Let* $m \in N$ *and* $0 < \beta < \frac{1}{m}$. *If* $f \in A_p$ *satisfies* $f(z)f'(z) \neq 0 (0 < |z| < 1)$ *and*

$$\left|\beta \left(\frac{zf'(z)}{pf(z)}\right)^m + 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} - \rho\right| < \rho \quad (z \in U), \tag{2.24}$$

where ρ *is given as in Theorem* 2.4, *then* $f \in \widetilde{S}_p^*(\beta)$ *and the bound in* (2.24) *is the largest number such that*

$$\frac{zf'(z)}{f(z)} \prec p \left(\frac{1+z}{1-z}\right)^\beta.$$

Proof. Note that $0 < \beta < \rho$. Putting $\lambda = \beta$ in Theorem 2.4 and using (2.24), it follows that $f \in \widetilde{S}_p^*(\beta)$.

For the function

$$f(z) = z^p \exp\left\{p \int_0^z \frac{1}{t}\left(\left(\frac{1+t}{1-t}\right)^\beta - 1\right) dt\right\} \in A_p,$$

it is easy to verify that $f \in \widetilde{S}_p^*(\beta)$,

$$\beta\left(\frac{zf'(z)}{pf(z)}\right)^m+1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}=\beta\left(\left(\frac{1+z}{1-z}\right)^{m\beta}+\frac{2z}{1-z^2}\right)=h_1(z)\ \text{(say)}$$

and

$$\begin{aligned}\lim_{z\to i}|h_1(z)-\rho| &= \beta\left|e^{\frac{m\beta\pi}{2}i}+i-\tan\left(\frac{(1+m\beta)\pi}{4}\right)\right| \\ &= \beta\tan\left(\frac{(1+m\beta)\pi}{4}\right) \\ &= \rho. \qquad (2.25)\end{aligned}$$

The proof of Corollary 2.4 is completed. □

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