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THE SYMMETRIC POSITIVE SOLUTIONS OF THREE-POINT BOUNDARY VALUE PROBLEMS FOR NONLINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we are concerned with the existence of symmetric positive solutions for second-order differential equations. Under suitable conditions, the existence of symmetric positive solutions are established by using Krasnoselskii's fixed-point theorems.

1. Introduction

Most of the recent results on the positive solutions are concerned with single equation and simple boundary condition (see [1, 2, 3, 4, 5]). As far as the author knows, there are few results on the symmetric positive solutions. It should be mentioned that Sun [6] discussed the following boundary value problem:

$$\begin{aligned} &u''(t)+a(t)f(t,u(t))=0, 0<t<1,\\ &u(0)=u(1-t), u'(0)-u'(1)=u(\frac{1}{2}), \end{aligned} \tag{1.1}$$

by using Krasnoselskii's fixed-point theorems, the existence of symmetric positive solutions is shown under certain conditions on f. Yang and Sun [7]

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considered the boundary value problem of differential equations

$$\begin{aligned} -u''(x) &= f(x,v), \\ -v''(x) &= g(x,u), \\ u(0) &= u(1) = 0, \\ v(0) &= v(1) = 0. \end{aligned} \tag{1.2}$$

using the degree theory, the existence of a positive solution of (1.2) is established. Motivated by the work of [6] and [7], we concern with the existence of symmetric positive solutions of the boundary value problems.

$$\begin{aligned} -u''(t) &= f(t,v), \\ -v''(t) &= g(t,u), \\ u(t) = u(1-t), \alpha u'(0) - \beta u'(1) &= \gamma u(\frac{1}{2}), \\ v(t) = v(1-t), \alpha v'(0) - \beta v'(1) &= \gamma v(\frac{1}{2}), \end{aligned} \tag{1.3}$$

where $f, g : [0,1] \times R^+ \to R^+$ are continuous, both $f(\cdot, u)$ and $g(\cdot, u)$ are symmetric on $[0,1]$, $f(x,0) \equiv 0$, $g(x,0) \equiv 0$, $\mid \beta - \alpha \mid \leq \mid \frac{\gamma}{2} \mid$, $\beta + \alpha \geq 2\gamma$, $\alpha, \beta \geq 0, \gamma \neq 0$. The arguments for establishing the symmetric positive solutions of (1.3) involve properties of the functions in Lemma 2.5 that play a key role in defining certain cones. A fixed point theorem due to Krasnoselskii is applied to yield the existence of symmetric positive solutions of (1.3).

This paper contains three sections besides the Introduction. In Section 2, we present some necessary definitions and preliminary lemmas that will be used to prove our main results. In Section 3, we discuss the existence of at least one and at least two symmetric positive solutions for BVP (1.3). Finally, we give some examples to illustrate our results in Section 4.

2. Preliminaries

In this section, we present some necessary definitions and preliminary lemmas that will be used in the proof of the results.

Definition 2.1. Let E be a real Banach space. A nonempty closed set $P \subset E$ is called a cone of E if it satisfies the following conditions:

(1) $x \in P, \lambda > 0$ implies $\lambda x \in P$;
(2) $x \in P, -x \in P$ implies $x = 0$.

Definition 2.2. The function u is said to be concave on $[0,1]$ if $u(rt_1+(1-r)t_2) \geq ru(t_1)+(1-r)u(t_2), r, t_1, t_2 \in [0,1]$.

Definition 2.3. The function u is said to be symmetric on $[0,1]$ if $u(t) = u(1-t), t \in [0,1]$.

Definition 2.4. The function (u,v) is called a symmetric positive solution of the BVP (1.3) if u and v are symmetric and positive on $[0,1]$, and satisfy the BVP (1.3).

We shall consider the real Banach space $C[0,1]$, equipped with norm $\| u \|= \max_{0\leq t\leq 1} \mid u(t) \mid$. Denote $C^+[0,1] = \{u \in C[0,1] : u(t) \geq 0, t \in [0,1]\}$.

Lemma 2.5. *Let $y \in C[0,1]$ be symmetric on $[0,1]$, then the three-point BVP*

$$\begin{aligned} u''(t)+y(t) = 0, 0 < t < 1, \\ u(t) = u(1-t), \alpha u'(0)-\beta u'(1) = \gamma u(\frac{1}{2}), \end{aligned} \tag{2.1}$$

has a unique symmetric solution $u(t) = \int_0^1 G(t,s)y(s)ds$, where $G(t,s) = G_1(t,s)+G_2(s)$, here

$$\begin{aligned} G_1(t,s) &= \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \\ G_2(s) &= \begin{cases} (\frac{1}{2}-s)-\frac{1}{2}(1-s)+\dfrac{(\alpha-\beta)(1-s)}{\gamma}+\dfrac{\beta}{\gamma}, & 0 \leq s \leq \frac{1}{2}, \\ -\frac{1}{2}(1-s)+\dfrac{(\alpha-\beta)(1-s)}{\gamma}+\dfrac{\beta}{\gamma}, & \frac{1}{2} \leq s \leq 1. \end{cases} \end{aligned}$$

Proof. From (2.1), we have $u''(t) = -y(t)$. For $t \in [0,1]$, integrating from 0 to t we get

$$u'(t) = -\int_0^t y(s)ds + A_1, \tag{2.2}$$

since $u'(t) = -u'(1-t)$, we can find that

$$-\int_0^t y(s)ds + A_1 = -\int_0^{1-t} y(s)ds - A_1,$$

which leads to

$$\begin{aligned} A_1 &= \frac{1}{2}\int_0^t y(s)ds - \frac{1}{2}\int_0^{1-t} y(s)ds = \frac{1}{2}\int_0^t y(s)ds + \frac{1}{2}\int_0^{1-t} y(1-s)d(1-s) \\ &= \frac{1}{2}\int_0^t y(s)ds + \frac{1}{2}\int_t^1 y(s)ds = \frac{1}{2}\int_0^1 y(s)ds = \int_0^1 (1-s)y(s)ds. \end{aligned}$$

Integrating again we obtain

$$u(t) = -\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds + A_2.$$

From (2.1) and (2.2) we have

$$(\alpha-\beta)A_1 + \beta\int_0^1 y(s)ds = \gamma(-\int_0^{\frac{1}{2}} (\frac{1}{2}-s)ds + \frac{1}{2}\int_0^1 (1-s)y(s)ds + A_2).$$

Thus

$$\begin{aligned} A_2 &= \int_0^{\frac{1}{2}} [(\frac{1}{2}-s) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)]y(s)ds \\ &\quad + \int_{\frac{1}{2}}^1 [\frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)]y(s)ds. \end{aligned}$$

From above we can obtain the BVP (2.1) has a unique symmetric solution

$$\begin{aligned} u(t) &= -\int_0^t (t-s)y(s)ds + t\int_0^1 (1-s)y(s)ds \\ &\quad + \int_0^{\frac{1}{2}} [(\frac{1}{2}-s) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)]y(s)ds \\ &\quad + \int_{\frac{1}{2}}^1 [\frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)]y(s)ds \\ &= \int_0^1 G_1(t,s)y(s)ds + \int_0^1 G_2(s)y(s)ds \\ &= \int_0^1 [G_1(t,s) + G_2(s)]y(s)ds. \end{aligned}$$

This completes the proof. □

Lemma 2.6. *The function $G(t,s)$ satisfies $\frac{3}{4}G(s,s) \le G(t,s) \le G(s,s)$ for $t,s \in [0,1]$ if α,β,γ are defined in* (1.3).

Proof. For any $t \in [0,1]$ and $s \in [0, \frac{1}{2}]$, we have

$$\begin{aligned} G(t,s) &= G_1(t,s) + G_2(s) \geq G_2(s) = \frac{1}{4}G_2(s) + \frac{3}{4}G_2(s) \\ &= \frac{1}{4}[(\frac{1}{2} - s) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)] \\ &\quad + \frac{3}{4}[(\frac{1}{2} - s) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)] \\ &\geq s(1-s)G_2(s) + \frac{3}{4}G_2(s). \end{aligned}$$

Note that $\mid \beta - \alpha \mid \leq \mid \frac{\gamma}{2} \mid$, $\alpha + \beta \geq 2\gamma$, $\gamma \neq 0$, we obtain $G_2(s) \geq \frac{3}{4}$. Thus $G(t,s) \geq \frac{3}{4}[G_1(s,s) + G_2(s)] = \frac{3}{4}G(s,s)$. As in the same way we can conclude $G(t,s) \geq \frac{3}{4}G(s,s)$ for any $t \in [0,1]$ and $s \in [\frac{1}{2}, 1]$. It is obvious that $G(s,s) \geq G(t,s)$ for $t,s \in [0,1]$. The proof is complete. □

Lemma 2.7. *Let $y \in C^+[0,1]$, then the unique symmetric solution $u(t)$ of the BVP (1.3) is nonnegative on $[0,1]$.*

Proof. Let $y \in C^+[0,1]$. From the fact that $u''(t) = -y(t) \leq 0, t \in [0,1]$, we know that the graph of $u(t)$ is concave on $[0,1]$. From (2.1). We have that

$$\begin{aligned} u(0) = u(1) &= \int_0^{\frac{1}{2}} [(\frac{1}{2} - s) + \frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)]y(s)ds \\ &\quad + \int_{\frac{1}{2}}^1 [\frac{\alpha-\beta}{\gamma}(1-s) + \frac{\beta}{\gamma} - \frac{1}{2}(1-s)]y(s)ds \geq 0. \end{aligned}$$

Note that $u(t)$ is concave, thus $u(t) \geq 0$ for $t \in [0,1]$. This completes the proof. □

Lemma 2.8. *Let $y \in C^+[0,1]$, then the unique symmetric solution $u(t)$ of BVP (1.3) satisfies*

$$\min_{t \in [0,1]} u(t) \geq \frac{3}{4} \parallel u \parallel . \tag{2.3}$$

Proof. For any $t \in [0,1]$, on the one hand, from lemma 2.6 we have that $u(t) = \int_0^1 G(t,s)y(s)ds \leq \int_0^1 G(s,s)y(s)ds$. Therefore,

$$\parallel u \parallel \leq \int_0^1 G(s,s)y(s)ds. \tag{2.4}$$

On the other hand, for any $t \in [0,1]$, from lemma 2.6 we obtain that

$$u(t) = \int_0^1 G(t,s)y(s)ds \geq \frac{3}{4}\int_0^1 G(s,s)y(s)ds \geq \frac{3}{4} \parallel u \parallel . \tag{2.5}$$

From (2.5) and (2.4) we find that (2.3) holds.

Obviously, $(u,v) \in C^2[0,1] \times C^2[0,1]$ is the solution of (1.3) if and only if $(u,v) \in C[0,1] \times C[0,1]$ is the solution of integral equations

$$\begin{cases} u(t) = \int_0^1 G(t,s)f(s,v(s))ds, \\ v(t) = \int_0^1 G(t,s)g(s,u(s))ds. \end{cases} \tag{2.6}$$

Integral equations (2.6) can be transferred to the nonlinear integral equation

$$u(t) = \int_0^1 G(t,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \tag{2.7}$$

Define an integral operator $A : C \to C$ by

$$Au(t) = \int_0^1 G(t,s)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds. \tag{2.8}$$

It is easy to see that the BVP (1.3) has a solution $u = u(t)$ if and only if u is a fixed point of the operator A defined by (2.8).

Let $P = \{u \in C^+[0,1] : u(t)$ is symmetric, concave on $[0,1]$ and $\min\limits_{0\leq t\leq 1} u(t) \geq \frac{3}{4} \parallel u \parallel\}$. It is obvious that P is a positive cone in $C[0,1]$. □

Lemma 2.9. *If the operator A is defined as in* (2.8)*, then $A : P \to P$ is completely continuous.*

Proof. It is obvious that Au is symmetric on $[0,1]$. Note that $(Au)''(t) - f(t,v(t)) \leq 0$, so we have that Au is concave. Thus from lemma 2.6 and non-negativity of f and g,

$$\begin{aligned} Au(t) &= \int_0^1 G(t,s)f\Big(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi\Big)ds \\ &\leq \int_0^1 G(s,s)f\Big(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi\Big)ds, \end{aligned}$$

then

$$\| Au \| \leq \int_0^1 G(s,s) f\Big(s, \int_0^1 G(s,\xi) g(\xi, u(\xi)) d\xi\Big) ds.$$

On the other hand,

$$Au \geq \frac{3}{4} \int_0^1 G(s,s) f\Big(s, \int_0^1 G(s,\xi) g(\xi, u(\xi)) d\xi\Big) ds \geq \frac{3}{4} \| Au \| .$$

Thus, $A(P) \subset P$. Since $G(t,s)$, $f(t,u)$ and $g(t,u)$ are continuous, it is easy to see that $A : P \to P$ is completely continuous. The proof is complete. $\square$

Lemma 2.10 (see [8]). *Let E be a Banach space and $P \subset E$ is a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $A : P \bigcap (\overline{\Omega_2} \backslash \Omega_1) \to P$ be a completely continuous operator. In addition suppose either*

(1) $\| Au \| \leq \| u \|$, $\forall u \in P \bigcap \partial\Omega_1$ *and* $\| Au \| \geq \| u \|$, $\forall u \in P \bigcap \partial\Omega_2$ *or*

(2) $\| Au \| \leq \| u \|$, $\forall u \in P \bigcap \partial\Omega_2$ *and* $\| Au \| \geq \| u \|$, $\forall u \in P \bigcap \partial\Omega_1$

holds. Then A has a fixed point in $P \bigcap (\overline{\Omega_2} \backslash \Omega_1)$.

Lemma 2.11 (see [8]). *Let E be a Banach space and $P \subset E$ is a cone in E. Assume that Ω_1, Ω_2 and Ω_3 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, $\overline{\Omega_2} \subset \Omega_3$ and let $A : P \bigcap (\overline{\Omega_3} \backslash \Omega_1) \to P$ be a completely continuous operator. In addition suppose either*

(1) $\| Au \| \geq \| u \|$, $\forall u \in P \bigcap \partial\Omega_1$;

(2) $\| Au \| \leq \| u \|$, $Au \neq u$, $\forall u \in P \bigcap \partial\Omega_2$;

(3) $\| Au \| \geq \| u \|$, $\forall u \in P \bigcap \partial\Omega_3$

holds. Then A has at least two fixed-points x_1, x_2 in $P \bigcap (\overline{\Omega_3} \backslash \Omega_1)$, and furthermore $x_1 \in P \bigcap (\Omega_2 \backslash \Omega_1)$, $x_2 \in P \bigcap (\overline{\Omega_3} \backslash \overline{\Omega_2})$.

3. Existence of Positive Solutions

In this section, we study the existence of positive solutions for BVP (1.3). First we give the following assumptions:

(H_1) $\lim\limits_{u \to 0^+} \sup\limits_{0 \leq x \leq 1} \dfrac{f(t,u)}{u} = 0$, $\lim\limits_{u \to 0^+} \sup\limits_{0 \leq x \leq 1} \dfrac{g(t,u)}{u} = 0$;

(H_2) $\lim\limits_{u\to\infty} \inf_{0\le x\le 1} \dfrac{f(t,u)}{u}=\infty,\ \lim\limits_{u\to\infty} \inf_{0\le x\le 1} \dfrac{g(t,u)}{u}=\infty;$

(H_3) $\lim\limits_{u\to 0^+} \inf_{0\le x\le 1} \dfrac{f(t,u)}{u}=\infty,\ \lim\limits_{u\to 0^+} \inf_{0\le x\le 1} \dfrac{g(t,u)}{u}=\infty;$

(H_4) $\lim\limits_{u\to\infty} \sup\limits_{0\le x\le 1} \dfrac{f(t,u)}{u}=0,\ \lim\limits_{u\to\infty} \sup\limits_{0\le x\le 1} \dfrac{g(t,u)}{u}=0;$

(H_5) There exists a constant $R_1>0$, such that $f(s,u)\le \frac{R_1}{\int_0^1 G(s,s)ds}$ for every $(s,u)\in[0,1]\times[\frac{3}{4}R_1,R_1]$

Theorem 3.1. *If* (H_1) *and* (H_2) *are satisfied, then* (1.3) *has at least one symmetric positive solution* $(u,v)\in C^2([0,1],R^+)\times C^2([0,1],R^+)$ *satisfying* $u(t)>0, v(t)>0$.

Proof. From (H_1) there is a number $N_1\in(0,1)$ such that for each $(s,u)\in[0,1]\times(0,N_1)$, one has $f(s,u)\le\eta_1 u$, $g(s,u)\le\eta_1 u$, where $\eta_1>0$ satisfies $\eta_1\int_0^1 G(s,s)ds\le 1$, for every $u\in P$ and $\| u \|=\frac{N_1}{2}$, note that $\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi\le\int_0^1 G(\xi,\xi)g(\xi,u(\xi))d\xi\le\int_0^1\eta_1 G(\xi,\xi)u(\xi)d\xi\le\| u \| =\frac{N_1}{2}<N_1$, then

$$\begin{aligned}
Au(x) &\le \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds\\
&\le \eta_1\int_0^1 G(s,s)\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi ds\\
&\le \eta_1^2\int_0^1 G(s,s)\int_0^1 G(\xi,\xi)u(\xi)d\xi ds\le \| u \| .
\end{aligned}$$

Let

$$\Omega_1=\{u\in C^+[0,1],\| u \|<\frac{N_1}{2}\},$$

then

$$\| Au \|\le\| u \|, u\in P\bigcap\partial\Omega_1. \tag{3.1}$$

From (H_2) there is a number $N_2>2N_1$ for each $(s,u)\in[0,1]\times(N_2,+\infty)$, one has $f(s,u)\ge\eta_2 u$, $g(s,u)\ge\eta_2 u$ where $\eta_2>0$ satisfies $\eta_2\frac{\sqrt{27}}{8}\int_0^1 G(s,s)ds\ge 1$, then, for every $u\in P$ and $\| u \|=2N_2$, from lemma 2.6 and lemma 2.8, we have

$$\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi \ \ge\ \frac{9}{16}\int_0^1 \eta_2 G(\xi,\xi)\ \| u \|\ d\xi$$

$$\geq \frac{9}{6\sqrt{3}} \parallel u \parallel = \sqrt{3}N_2 > N_2,$$

then

$$\begin{aligned}\parallel Au \parallel &= \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \\ &\geq \frac{3}{4}\eta_2 \int_0^1 G(s,s)\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi ds \\ &\geq (\frac{3}{4})^2\eta_2^2 \int_0^1 G(s,s)\int_0^1 G(\xi,\xi)u(\xi)d\xi ds \\ &\geq (\frac{3}{4})^3\eta_2^2 \int_0^1 G(s,s)\int_0^1 G(\xi,\xi) \parallel u \parallel d\xi ds \geq \parallel u \parallel .\end{aligned}$$

Let

$$\Omega_2 = \{u \in C^+[0,1], \parallel u \parallel < 2N_2\},$$

then

$$\parallel Au \parallel \geq \parallel u \parallel, u \in P\bigcap \partial\Omega_2. \tag{3.2}$$

Thus from(3.1), (3.2) and Lemma 2.10, we see that the operator A has a fixed point in $P\bigcap(\overline{\Omega_2}\backslash\Omega_1)$. The proof is complete. □

Theorem 3.2. *If* (H_3) *and* (H_4) *are satisfied, then* (1.3) *has at least one symmetric positive solution* $(u,v) \in C^2([0,1],R^+) \times C^2([0,1],R^+)$ *satisfying* $u(t) > 0, v(t) > 0$.

Proof. From (H_3), there is a number $\overline{N_3} \in (0,1)$ such that for each $(x,u) \in [0,1] \times (0,\overline{N_3})$, one has $f(s,u) \geq \eta_3 u$, $g(s,u) \geq \eta_3 u$ where $\eta_3 > 0$ satisfies $\frac{\sqrt{27}}{8}\eta_3 \int_0^1 G(s,s)ds \geq 1$. From $g(x,0) \equiv 0$ and the continuity of $g(s,u)$, we know that there exists number $N_3 \in (0,\overline{N_3})$ such that $g(s,u) \leq \frac{\overline{N_3}}{\int_0^1 G(s,s)ds}$ for each $(s,u) \in [0,1] \times (0,N_3]$. Then for every $u \in P$ and $\parallel u \parallel = N_3$, note that $\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi \leq \int_0^1 G(\xi,\xi)\frac{\overline{N_3}}{\int_0^1 G(s,s)ds}d\xi = \overline{N_3}$. Thus

$$\begin{aligned}Au(x) &= \int_0^1 G(t,s)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \\ &\geq \frac{3}{4}\eta_3 \int_0^1 G(s,s)\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi ds\end{aligned}$$

$$\geq (\frac{3}{4})^2\eta_3^2 \int_0^1 G(s,s)\int_0^1 G(\xi,\xi)u(\xi)d\xi ds$$
$$\geq (\frac{3}{4})^3\eta_3^2 \int_0^1 G(s,s)\int_0^1 G(\xi,\xi) \parallel u \parallel d\xi ds \geq \parallel u \parallel .$$

Let

$$\Omega_3 = \{u \in C^+[0,1], \parallel u \parallel < N_3\},$$

then

$$\parallel Au \parallel \geq \parallel u \parallel, u \in P\bigcap \partial\Omega_3. \tag{3.3}$$

From (H_4), there exist $C_1 > 0$ and $C_2 > 0$ such that $f(s,u) \leq \eta_4 u + C_1$, $g(s,u) \leq \eta_4 u + C_2$ for $\forall (s,u) \in [0,1]\times(0,\infty)$, where $\eta_4 > 0$, and $\eta_4\int_0^1 G(\xi,\xi)d\xi \leq 1$. Then, for $u \in C^+[0,1]$ we have

$$\begin{aligned}
Au &= \int_0^1 G(s,t)f(s,\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds\\
&\leq \int_0^1 G(s,s)(\eta_4\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi + C_1)ds\\
&\leq \eta_4\int_0^1 G(s,s)\int_0^1 G(\xi,\xi)g(\xi,u(\xi))d\xi ds + C_3\\
&\leq \eta_4\int_0^1 G(s,s)\int_0^1 G(\xi,\xi)(\eta_4 u + C_2)d\xi ds + C_3\\
&\leq (\eta_4)^2\int_0^1 G(s,s)\int_0^1 G(\xi,\xi) \parallel u \parallel d\xi ds + C_4 \leq \parallel u \parallel + C_4
\end{aligned}$$

Thus $\parallel Au \parallel \leq \parallel u \parallel$ as $\parallel u \parallel \to \infty$.

Let $\Omega_4 = \{u \in E, \parallel u \parallel < L_4\}$. For each $u \in P$ and $\parallel u \parallel = L_4 > L_3$ large enough, we have

$$\parallel Au \parallel \leq \parallel u \parallel, u \in P\bigcap \partial\Omega_4. \tag{3.4}$$

Thus from (3.3), (3.4) and Lemma 2.10, we know that the operator A has a fixed point in $P\bigcap(\overline{\Omega_4}\backslash\Omega_3)$. The proof is complete. □

Theorem 3.3. *If* (H_2), (H_3)*and* (H_5) *are satisfied, then* (1.3) *has at least two symmetric positive solutions* $(u_1,v_1),(u_2,v_2) \in C^2([0,1],R^+)\times C^2([0,1],R^+)$ *satisfying* $u_1(t) > 0, v_1(t) > 0, u_2(t) > 0, v_2(t) > 0$.

Proof. Let

$$\Omega_5 = \{u \in C^+[0,1], \| u \| < R_1\},$$

then $\forall u \in P \bigcap \partial\Omega_5$, we have $u(s) \in [\frac{3}{4}R_1, R_1]$. From lemma 2.6,lemma 2.8 and (2.4) we can obtain

$$\begin{aligned}
\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi &\geq \frac{3}{4}\int_0^1 G(\xi,\xi)g(\xi,u(\xi))d\xi \geq \frac{3}{4} \| u \| \\
\int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi &\leq \int_0^1 G(\xi,\xi)g(\xi,u(\xi))d\xi \\
&\leq \int_0^1 G(\xi,\xi)d\xi \frac{R_1}{G(s,s)ds} = R_1. \qquad (3.5)
\end{aligned}$$

Thus $Au = \int_0^1 G(s,t)f(s, \int_0^1 G(s,\xi)g(\xi,u(\xi))d\xi)ds \leq \int_0^1 G(s,s)\frac{R_1}{\int_0^1 G(\xi,\xi)d\xi}ds$ $= R_1 = \| u \|$. Then

$$\| Au \| \leq \| u \|, u \in P \bigcap \partial\Omega_5. \qquad (3.6)$$

For another hand, from (H_2) and (H_3), we can choose two right numbers $\widetilde{N_2} \in (R_1, \infty)$, $\widetilde{N_3} \in (0, R_1)$ satisfy

$$\begin{aligned}
\| Au \| &\geq \| u \|, u \in P \bigcap \partial\widetilde{\Omega_2}, \qquad (3.7) \\
\| Au \| &\geq \| u \|, u \in P \bigcap \partial\widetilde{\Omega_3}, \qquad (3.8)
\end{aligned}$$

where $\widetilde{\Omega_2} = \{u \in C^+[0,1], \| u \| < \widetilde{N_2}\}$, $\widetilde{\Omega_3} = \{u \in C^+[0,1], \| u \| < \widetilde{N_3}\}$. Then from lemma 2.11, (3.6), (3.7) and (3.8), A has at least two fixed points in $P \bigcap (\overline{\widetilde{\Omega_2}} \backslash \overline{\Omega_5})$ and $P \bigcap (\Omega_5 \backslash \widetilde{\Omega_3})$, respectively. The proof is complete. □

4. Examples

In this section, we give three examples to illustrate our results.

Example 4.1. Let $f(t,v) = v^2 + \frac{[1+t(1-t)]v^2}{1+v^2}$, $g(t,u) = 2u^2 + \frac{2[1+t(1-t)]u^2}{1+u^2}$, $\alpha = \beta = \gamma = 1$, then the conditions of Theorem 3.1 are satisfied. From Theorem 3.1, BVP(1.3) has at least one symmetric positive solution.

Example 4.2. Let $f(t,v) = v^{\frac{1}{2}} + \frac{[1+t(1-t)]v^2}{1+v^2}$, $g(t,u) = 2u^{\frac{1}{2}} + \frac{2[1+t(1-t)]u^2}{1+u^2}$, $\alpha = \beta = \gamma = 1$, then conditions of Theorem 3.2 are satisfied. From Theorem 3.2, BVP(1.3) has at least one symmetric positive solution.

Example 4.3. Let $f(t,v) = \frac{t(1-t)+1}{4}(v^{\frac{1}{2}} + v^2)$, $g(t,u) = \frac{t(1-t)+1}{5}(u^{\frac{1}{2}} + u^2)$, $\alpha = \beta = \gamma = 1$, then conditions of Theorem 3.3 are satisfied. From Theorem 3.3, BVP(1.3) has at least two symmetric positive solutions.

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