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A BAR OPERATOR FOR INVOLUTIONS IN A COXETER GROUP

G. LUSZTIG

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, U.S.A.
E-mail: gyuri@math.mit.edu

0. Introduction and Statement of Results

0.0. In [6] it was shown that the vector space spanned by the involutions in a Weyl group carries a natural Hecke algebra action and a certain bar operator. These were used in [6] to construct a new basis of that vector space, in the spirit of [2], and to give a refinement of the polynomials $P_{y,w}$ of [2] in the case where y, w were involutions in the Weyl group in the sense that $P_{y,w}$ was split canonically as a sum of two polynomials with cofficients in $\mathbf{N}$. However, the construction of the Hecke algebra action and that of the bar operator, although stated in elementary terms, were established in a non-elementary way. (For example, the construction of the bar operator in [6] was done using ideas from geometry such as Verdier duality for l-adic sheaves.) In the present paper we construct the Hecke algebra action and the bar operator in an entirely elementary way, in the context of arbitrary Coxeter groups.

Let W be a Coxeter group with set of simple reflections denoted by S. Let $l : W \to \mathbf{N}$ be the standard length function. For $x \in W$ we set $\epsilon_x = (-1)^{l(x)}$. Let $\le$ be the Bruhat order on W. Let $w \mapsto w^*$ be an automorphism of W with square 1 which leaves S stable, so that $l(w^*) = l(w)$ for any $w \in W$. Let $\mathbf{I}_* = \{w \in W; w^{*-1} = w\}$. (We write w^{*-1} instead of $(w^*)^{-1}$.) The elements of $\mathbf{I}_*$ are said to be $*$-*twisted involutions* of W.

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Let u be an indeterminate and let $\mathcal{A} = \mathbf{Z}[u, u^{-1}]$. Let $\mathfrak{H}$ be the free $\mathcal{A}$-module with basis $(T_w)_{w \in W}$ with the unique $\mathcal{A}$-algebra structure with unit T_1 such that

(i) $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$ and

(ii) $(T_s + 1)(T_s - u^2) = 0$ for all $s \in S$.

This is an Iwahori-Hecke algebra. (In [6], the notation $\mathfrak{H}'$ is used instead of $\mathfrak{H}$.)

Let M be the free $\mathcal{A}$-module with basis $\{a_w; w \in \mathbf{I}_*\}$. We have the following result which, in the special case where W is a Weyl group or an affine Weyl group, was proved in [6] (the general case was stated there without proof).

Theorem 0.1. *There is a unique $\mathfrak{H}$-module structure on M such that for any $s \in S$ and any $w \in \mathbf{I}_*$ we have*

(i) $T_s a_w = u a_w + (u+1) a_{sw}$ *if* $sw = ws^* > w$;

(ii) $T_s a_w = (u^2 - u - 1) a_w + (u^2 - u) a_{sw}$ *if* $sw = ws^* < w$;

(iii) $T_s a_w = a_{sws^*}$ *if* $sw \neq ws^* > w$;

(iv) $T_s a_w = (u^2 - 1) a_w + u^2 a_{sws^*}$ *if* $sw \neq ws^* < w$.

The proof is given in §2 after some preparation in §1.

Let $^- : \mathfrak{H} \to \mathfrak{H}$ be the unique ring involution such that $\overline{u^n T_x} = u^{-n} T_{x^{-1}}^{-1}$ for any $x \in W, n \in \mathbf{Z}$ (see [2]). We have the following result.

Theorem 0.2. (a) *There exists a unique* $\mathbf{Z}$*-linear map* $^- : M \to M$ *such that* $\overline{hm} = \bar{h}\bar{m}$ *for all* $h \in \mathfrak{H}, m \in M$ *and* $\overline{a_1} = a_1$. *For any* $m \in M$ *we have* $\overline{\overline{m}} = m$.

(b) *For any* $w \in \mathbf{I}_*$ *we have* $\overline{a_w} = \epsilon_w T_{w^{-1}}^{-1} a_{w^{-1}}$.

The proof is given in §3. Note that (a) was conjectured in [6] and proved there in the special case where W is a Weyl group or an affine Weyl group; (b) is new even when W is a Weyl group or affine Weyl group.

0.3. Let $\underline{\mathcal{A}} = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. We view $\mathcal{A}$ as a subring of $\underline{\mathcal{A}}$ by setting $u = v^2$. Let $\underline{M} = \underline{\mathcal{A}} \otimes_{\mathcal{A}} M$. We can view M as an $\mathcal{A}$-submodule of $\underline{M}$. We extend $^- : M \to M$ to a $\mathbf{Z}$-linear map $^- : \underline{M} \to \underline{M}$

in such a way that $\overline{v^n m} = v^{-n}\overline{m}$ for $m \in M, n \in \mathbf{Z}$. For each $w \in \mathbf{I}_*$ we set $a'_w = v^{-l(w)}a_w \in \underline{M}$. Note that $\{a'_w; w \in \mathbf{I}_*\}$ is an $\underline{\mathcal{A}}$-basis of $\underline{M}$. Let $\underline{\mathcal{A}}_{\leq 0} = \mathbf{Z}[v^{-1}]$, $\underline{\mathcal{A}}_{<0} = v^{-1}\mathbf{Z}[v^{-1}]$, $\underline{M}_{\leq 0} = \sum_{w\in\mathbf{I}_*}\underline{\mathcal{A}}_{\leq 0}a'_w \subset \underline{M}$, $\underline{M}_{<0} = \sum_{w\in\mathbf{I}_*}\underline{\mathcal{A}}_{<0}a'_w \subset \underline{M}$.

Let $\underline{\mathfrak{H}} = \underline{\mathcal{A}} \otimes_{\mathcal{A}} \mathfrak{H}$. This is naturally an $\underline{\mathcal{A}}$-algebra containing $\mathfrak{H}$ as an $\mathcal{A}$-subalgebra. Note that the $\mathfrak{H}$-module structure on M extends by $\underline{\mathcal{A}}$-linearity to an $\underline{\mathfrak{H}}$-module structure on $\underline{M}$. We denote by $^-\colon \underline{\mathcal{A}} \to \underline{\mathcal{A}}$ the ring involution such that $\overline{v^n} = v^{-n}$ for $n \in \mathbf{Z}$. We denote by $^-\colon \underline{\mathfrak{H}} \to \underline{\mathfrak{H}}$ the ring involution such that $\overline{v^n T_x} = v^{-n}T_{x^{-1}}^{-1}$ for $n \in \mathbf{Z}, x \in W$. We have the following result which in the special case where W is a Weyl group or an affine Weyl group is proved in [6, 0.3].

Theorem 0.4. (a) *For any $w \in \mathbf{I}_*$ there is a unique element*

$$A_w = v^{-l(w)} \sum_{y\in\mathbf{I}_*; y\leq w} P^\sigma_{y,w} a_y \in \underline{M}$$

($P^\sigma_{y,w} \in \mathbf{Z}[u]$) *such that $\overline{A_w} = A_w$, $P^\sigma_{w,w} = 1$ and for any $y \in \mathbf{I}_*$, $y < w$, we have* $\deg P^\sigma_{y,w} \leq (l(w) - l(y) - 1)/2$.

(b) *The elements A_w ($w \in \mathbf{I}_*$) form an $\underline{\mathcal{A}}$-basis of $\underline{M}$.*

The proof is given in §4.

0.5. As an application of our study of the bar operator we give (in 4.7) an explicit description of the Möbius function of the partially ordered set $(\mathbf{I}_*, \leq)$; we show that it has values in $\{1, -1\}$. This description of the Möbius function is used to show that the constant term of $P^\sigma_{y,w}$ is 1, see 4.10. In §5 we study the "K-spherical" submodule $\underline{M}^K$ of $\underline{M}$ (where K is a subset of S which generates a finite subgroup W_K of S). In 5.6(f) we show that $\underline{M}^K$ contains any element A_w where $w \in \mathbf{I}_*$ has maximal length in $W_K w W_{K^*}$. This result is used in §6 to describe the action of $u^{-1}(T_s + 1)$ (with $s \in S$) in the basis (A_w) by supplying an elementary substitute for a geometric argument in [6], see Theorem 6.3 which was proved earlier in [6] for the case where W is a Weyl group. In 7.7 we give an inversion formula for the polynomials $P^\sigma_{y,w}$ (for finite W) which involves the Möbius function above and the polynomials analogous to $P^\sigma_{y,w}$ with $*$ replaced by its composition with the opposition automorphism of W. In §8 we formulate a conjecture (see 8.4) relating $P^\sigma_{y,w}$ for certain twisted involutions y, w in an affine Weyl group to the q-analogues of weight multiplicities in [4]. In §9 we show that for $y \le w$ in $\mathbf{I}_*$, $P^{\sigma}_{y,w}$ is equal to the polynomial $P_{y,w}$ of [2] plus an element in $2\mathbf{Z}[u]$. This follows from [6] in the case where W is a Weyl group.

0.6. *Notation.* If Π is a property we set $\delta_{\Pi} = 1$ if Π is true and $\delta_{\Pi} = 0$ if Π is false. We write $\delta_{x,y}$ instead of $\delta_{x=y}$. For $s \in S, w \in \mathbf{I}_*$ we sometimes set $s \bullet w = sw$ if $sw = ws^*$ and $s \bullet w = sws^*$ if $sw \ne ws^*$; note that $s \bullet w \in \mathbf{I}_*$.

For any $s \in S, t \in S, t \ne s$ let $m_{s,t} = m_{t,s} \in [2, \infty]$ be the order of st. For any subset K of S let W_K be the subgroup of W generated by K. If $J \subset K$ are subsets of S we set $W_K^J = \{w \in W_K; l(wy) > l(w)$ for any $y \in W_J - \{1\}\}$, ${}^JW_K = \{w \in W_K; l(yw) > l(w)$ for any $y \in W_J - \{1\}\}$; note that ${}^JW_K = (W_K^J)^{-1}$. For any subset K of S such that W_K is finite we denote by w_K the unique element of maximal length of W_K.

Contents

1. Involutions and Double Cosets

1.1. Let K, K' be two subsets of S such that $W_K, W_{K'}$ are finite and let Ω be a $(W_K, W_{K'})$-double coset in W. Let b be the unique element of minimal length of Ω. Let $J = K \cap (bK'b^{-1})$, $J' = (b^{-1}Kb) \cap K'$ so that $b^{-1}Jb = J'$ hence $b^{-1}W_Jb = W_{J'}$. If $x \in \Omega$ then $x = cbd$ where $c \in W_K^J$, $d \in W_{K'}$ are uniquely determined; moreover, $l(x) = l(c) + l(b) + l(d)$, see Kilmoyer [3, Prop. 29]; see also [1, 2.7.4, 2.7.5]. We can write uniquely $d = zc'$ where

$z \in W_{J'}, c' \in {}^{J'}W_{K'}$; moreover, $l(d) = l(z) + l(c')$. Thus we have $x = cbzc'$ where $c \in W_K^J$, $z \in W_{J'}, c' \in {}^{J'}W_{K'}$ are uniquely determined; moreover, $l(x) = l(c) + l(b) + l(z) + l(c')$. Note that $\tilde{b} := w_K w_J b w_{K'}$ is the unique element of maximal length of Ω; we have $l(\tilde{b}) = l(w_K) + l(b) + l(w_{K'}) - l(w_J)$.

1.2. Now assume in addition that $K' = K^*$ and that Ω is stable under $w \mapsto w^{*-1}$. Then $b^{*-1} \in \Omega$, $\tilde{b}^{*-1} \in \Omega$, $l(b^{*-1}) = l(b)$, $l(\tilde{b}^{*-1}) = l(\tilde{b})$, and by uniqueness we have $b^{*-1} = b$, $\tilde{b}^{*-1} = \tilde{b}$, that is, $b \in \mathbf{I}_*$, $\tilde{b} \in \mathbf{I}_*$. Also we have $J^* = K^* \cap (b^{-1}Kb) = J'$ hence $W_{J'} = (W_J)^*$. If $x \in \Omega \cap \mathbf{I}_*$, then writing $x = cbzc'$ as in 1.1 we have $x = x^{*-1} = c'^{*-1}b(b^{-1}z^{*-1}b)c^{*-1}$ where $c'^{*-1} \in ({}^JW_K)^{-1} = W_K^J$, $c^{*-1} \in (W_{K^*}^{J^*})^{-1} = {}^{J^*}W_{K^*}$, $b^{-1}z^{*-1}b \in b^{-1}W_J b = W_{J^*}$. By the uniqueness of c, z, c', we must have $c'^{*-1} = c$, $c^{*-1} = c'$, $b^{-1}z^{*-1}b = z$. Conversely, if $c \in W_K^J$, $z \in W_{J^*}, c' \in {}^{J^*}W_{K^*}$ are such that $c'^{*-1} = c$ (hence $c^{*-1} = c'$) and $b^{-1}z^{*-1}b = z$ then clearly $cbzc' \in \Omega \cap \mathbf{I}_*$. Note that $y \mapsto b^{-1}y^*b$ is an automorphism $\tau : W_{J^*} \to W_{J^*}$ which leaves J^* stable and satisfies $\tau^2 = 1$. Hence $\mathbf{I}_\tau := \{y \in W_{J^*}; \tau(y)^{-1} = y\}$ is well defined. We see that we have a bijection

(a) $W_K^J \times \mathbf{I}_\tau \to \Omega \cap \mathbf{I}_*$, $(c, z) \mapsto cbzc^{*-1}$.

1.3. In the setup of 1.2 we assume that $s \in S$, $K = \{s\}$, so that $K' = \{s^*\}$. In this case we have either

$$sb = bs^*,\ J = \{s\},\ \Omega \cap \mathbf{I}_* = \{b, bs^* = \tilde{b}\},\ l(bs^*) = l(b) + 1, \text{ or}$$

$$sb \neq bs^*,\ J = \emptyset,\ \Omega \cap \mathbf{I}_* = \{b, sbs^* = \tilde{b}\},\ l(sbs^*) = l(b) + 2.$$

1.4. In the setup of 1.2 we assume that $s \in S, t \in S, t \neq s$, $m := m_{s,t} < \infty$, $K = \{s, t\}$, so that $K^* = \{s^*, t^*\}$. We set $\beta = l(b)$. For $i \in [1, m]$ we set $\mathbf{s}_i = sts\cdots$ (i factors), $\mathbf{t}_i = tst\cdots$ (i factors).

We are in one of the following cases (note that we have $sb = bt^*$ if and only if $tb = bs^*$, since $b^{*-1} = b$).

(i) $\{sb, tb\} \cap \{bs^*, bt^*\} = \emptyset$, $J = \emptyset$, $\Omega \cap \mathbf{I}_* = \{\xi_{2i}, \xi'_{2i} (i \in [0, m]), \xi_0 = \xi'_0 = b, \xi_{2m} = \xi'_{2m} = \tilde{b}\}$ where $\xi_{2i} = \mathbf{s}_i^{-1} b \mathbf{s}_i^*, \xi'_{2i} = \mathbf{t}_i^{-1} b \mathbf{t}_i$, $l(\xi_{2i}) = l(\xi'_{2i}) = \beta + 2i$.

(ii) $sb = bs^*$, $tb \neq bt^*$, $J = \{s\}$, $\Omega \cap \mathbf{I}_* = \{\xi_{2i}, \xi_{2i+1} (i \in [0, m-1])\}$ where $\xi_{2i} = \mathbf{t}_i^{-1} b \mathbf{t}_i^*$, $l(\xi_{2i}) = \beta + 2i$, $\xi_{2i+1} = \mathbf{t}_i^{-1} b \mathbf{s}_{i+1}^* = \mathbf{s}_{i+1}^{-1} b \mathbf{t}_i^*$, $l(\xi_{2i+1}) = \beta + 2i + 1$, $\xi_0 = b, \xi_{2m-1} = \tilde{b}$.

(iii) $sb \ne bs^*$, $tb = bt^*$, $J = \{t\}$, $\Omega \cap \mathbf{I}_* = \{\xi_{2i}, \xi_{2i+1}(i \in [0, m-1])\}$ where $\xi_{2i} = \mathbf{s}_i^{-1} b \mathbf{s}_i^*$, $l(\xi_{2i}) = \beta + 2i$, $\xi_{2i+1} = \mathbf{s}_i^{-1} b \mathbf{t}_{i+1}^* = \mathbf{t}_{i+1}^{-1} b \mathbf{s}_i^*$, $l(\xi_{2i+1}) = \beta + 2i + 1$, $\xi_0 = b, \xi_{2m-1} = \tilde{b}$.

(iv) $sb = bs^*$, $tb = bt^*$, $J = K$, m odd, $\Omega \cap \mathbf{I}_* = \{\xi_0 = \xi_0' = b, \xi_{2i+1}, \xi_{2i+1}'(i \in [0, (m-1)/2]), \xi_m = \xi_m' = \tilde{b}\}$ where $\xi_1 = sb$, $\xi_3 = tstb$, $\xi_5 = ststsb, \ldots$; $x_1' = tb$, $x_3' = stsb$, $x_5' = tststb, \ldots$; $l(\xi_{2i+1}) = l(\xi_{2i+1}') = \beta + 2i + 1$.

(v) $sb = bs^*$, $tb = bt^*$, $J = K$, m even, $\Omega \cap \mathbf{I}_* = \{\xi_0 = \xi_0' = b, \xi_{2i+1}, \xi_{2i+1}'(i \in [0, (m-2)/2]), \xi_m = \xi_m' = \tilde{b}\}$ where $\xi_1 = sb$, $\xi_3 = tstb$, $\xi_5 = ststsb, \ldots$; $\xi_1' = tb$, $\xi_3' = stsb$, $\xi_5' = tststb, \ldots$; $l(\xi_{2i+1}) = l(\xi_{2i+1}') = \beta + 2i + 1$, $\xi_m = \xi_m' = b\mathbf{s}_m^* = b\mathbf{t}_m^* = \mathbf{s}_m b = \mathbf{t}_m b$, $l(\xi_m) = l(\xi_m') = \beta + m$.

(vi) $sb = bt^*$, $tb = bs^*$, $J = K$, m odd, $\Omega \cap \mathbf{I}_* = \{\xi_0 = \xi_0' = b, \xi_{2i}, \xi_{2i}'(i \in [0, (m-1)/2]), \xi_m = \xi_m' = \tilde{b}\}$ where $\xi_2 = stb$, $\xi_4 = tstsb$, $\xi_6 = ststFor$ $ststb$...

(vii) $sb = bt^*$, $tb = bs^*$, $J = K$, m even, $\Omega \cap \mathbf{I}_* = \{\xi_0 = \xi_0' = b, \xi_{2i}, \xi_{2i}'(i \in [0, m/2]), \xi_m = \xi_m' = \tilde{b}\}$ where $\xi_2 = stb$, $\xi_4 = tstsb$, $\xi_6 = ststkstb, \ldots$; $\xi_2' = tsb$, $\xi_4' = ststb$, $\xi_6' = tststsb, \ldots$; $l(\xi_{2i}) = l(\xi_{2i}') = \beta + 2i$.

2. Proof of Theorem 0.1

2.1. Let $\dot{M} = \mathbf{Q}(u) \otimes_{\mathcal{A}} M$ (a $\mathbf{Q}(u)$-vector space with basis $\{a_w, w \in \mathbf{I}_*\}$). Let $\dot{\mathfrak{H}} = \mathbf{Q}(u) \otimes_{\mathcal{A}} \mathfrak{H}$ (a $\mathbf{Q}(u)$-algebra with basis $\{T_w; w \in W\}$ defined by the relations 0.0(i),(ii)). The product of a sequence $\xi_1, \xi_2, \ldots$ of k elements of $\dot{\mathfrak{H}}$ is sometimes denoted by $(\xi_1 \xi_2 \cdots)_k$. It is well known that $\dot{\mathfrak{H}}$ is the associative $\mathbf{Q}(u)$-algebra (with 1) with generators $T_s (s \in S)$ and relations 0.0(ii) and

$(T_s T_t T_s \cdots)_m = (T_t T_s T_t \cdots)_m$ for any $s \ne t$ in S such that $m := m_{s,t} < \infty$.

For $s \in S$ we set $\overset{\circ}{T}_s = (u+1)^{-1}(T_s - u) \in \dot{\mathfrak{H}}$. Note that $T_s, \overset{\circ}{T}_s$ are invertible in $\dot{\mathfrak{H}}$: we have $\overset{\circ}{T}_s{}^{-1} = (u^2 - u)^{-1}(T_s + 1 + u - u^2)$.

2.2. For any $s \in S$ we define a $\mathbf{Q}(u)$-linear map $T_s : \dot{M} \to \dot{M}$ by the formulas in 0.1(i)-(iv). For $s \in S$ we also define a $\mathbf{Q}(u)$-linear map $\overset{\circ}{T}_s : \dot{M} \to \dot{M}$ by $\overset{\circ}{T}_s = (u+1)^{-1}(T_s - u)$. For $w \in \mathbf{I}_*$ we have:

(i) $a_{sw} = \overset{\circ}{T}_s a_w$ if $sw = ws^* > w$; $a_{sws} = T_s a_w$ if $sw \neq ws^* > w$.

2.3. To prove Theorem 0.1 it is enough to show that the formulas 0.1(i)-(iv) define an $\dot{\mathfrak{H}}$-module structure on $\dot{M}$.

Let $s \in S$. To verify that $(T_s + 1)(T_s - u^2) = 0$ on $\dot{M}$ it is enough to note that the 2×2 matrices with entries in $\mathbf{Q}(u)$

$$\begin{pmatrix} u & u+1 \\ u^2-u & u^2-u-1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ u^2 & u^2-1 \end{pmatrix}$$

which represent T_s on the subspace of $\dot{M}$ spanned by a_w, a_{sw} (with $w \in \mathbf{I}, sw = ws^* > w$) or by a_w, a_{sws^*} (with $w \in \mathbf{I}, sw \neq ws^* > w$) have eigenvalues $-1, u^2$.

Assume now that $s \neq t$ in S are such that $m := m_{s,t} < \infty$. It remains to verify the equality $(T_s T_t T_s \cdots)_m = (T_t T_s T_t \cdots)_m : \dot{M} \to \dot{M}$. We must show that $(T_s T_t T_s \cdots)_m a_w = (T_t T_s T_t \cdots)_m a_w$ for any $w \in \mathbf{I}_*$. We will do this by reducing the general case to calculations in a dihedral group.

Let $K = \{s, t\}$, so that $K^* = \{s^*, t^*\}$. Let Ω be the (W_K, W_{K^*})-double coset in W that contains w. From the definitions it is clear that the subspace $\dot{M}_\Omega$ of $\dot{M}$ spanned by $\{a_{w'}; w' \in \Omega \cap \mathbf{I}_*\}$ is stable under T_s and T_t. Hence it is enough to show that

(a) $(T_s T_t T_s \cdots)_m \mu = (T_t T_s T_t \cdots)_m \mu$ for any $\mu \in \dot{M}_\Omega$.

Since $w^{*-1} = w$ we see that $w' \mapsto w'^{*-1}$ maps Ω into itself. Thus Ω is as in 1.2 and we are in one of the cases (i)-(vii) in 1.4. The proof of (a) in the various cases is given in 2.4-2.10. Let $b \in \Omega$, $J \subset K$ be as in 1.2. Let $\mathbf{s}_i, \mathbf{t}_i$ be as in 1.4.

Let $\dot{\mathfrak{H}}_K$ be the subspace of $\dot{\mathfrak{H}}$ spanned by $\{T_y; y \in W_K\}$; note that $\dot{\mathfrak{H}}_K$ is a $\mathbf{Q}(u)$-subalgebra of $\dot{\mathfrak{H}}$.

2.4. Assume that we are in case 1.4(i). We define an isomorphism of vector spaces $\Phi : \dot{\mathfrak{H}}_K \to \dot{M}_\Omega$ by $T_c \mapsto a_{cbc^{*-1}}$ ($c \in W_K$). From definitions we have $T_s \Phi(T_c) = \Phi(T_s T_c)$, $T_t \Phi(T_c) = \Phi(T_t T_c)$ for any $c \in W_K$. It follows that for any $x \in \dot{\mathfrak{H}}_K$ we have $T_s \Phi(x) = \Phi(T_s x)$, $T_t \Phi(x) = \Phi(T_t x)$, hence $(T_s T_t T_s \cdots)_m \Phi(x) - (T_t T_s T_t \cdots)_m \Phi(x) = \Phi((T_s T_t T_s \cdots)_m x -$

$(T_tT_sT_t\cdots)_m x) = 0$. (We use that $(T_sT_tT_s\cdots)_m = (T_tT_sT_t\cdots)_m$ in $\dot{\mathfrak{H}}_K$.) Since Φ is an isomorphism we deduce that 2.3(a) holds in our case.

Assume that we are in case 1.4(ii). We define r, r' by $r = s$, $r' = t$ if m is odd, $r = t$, $r' = s$ if m is even. We have

$$a_{\xi_0} \xrightarrow{T_t} a_{\xi_2} \xrightarrow{T_s} a_{\xi_4} \xrightarrow{T_t} \dots \xrightarrow{T_r} a_{\xi_{2m-2}},$$

$$a_{\xi_1} \xrightarrow{T_t} a_{\xi_3} \xrightarrow{T_s} a_{\xi_5} \xrightarrow{T_s} \dots \xrightarrow{T_r} a_{\xi_{2m-1}}.$$

We have $s\xi_0 = \xi_0 s^* = \xi_1$ hence $a_{\xi_0} \xrightarrow{T_s} ua_{\xi_0} + (u+1)a_{\xi_1}$. We show that

$$r'\xi_{2m-2} = \xi_{2m-2}r'^* = \xi_{2m-1}$$

We have $r'\xi_{2m-2} = \cdots tstbt^*s^*t^*\cdots$ where the product to the left (resp. right) of b has m (resp. $m-1$) factors). Using the definition of m and the identity $sb = bs^*$ we deduce $r'\xi_{2m-2} = \cdots stsbt^*s^*t^*\cdots = \cdots stbs^*t^*s^*\cdots$ (in the last expression the product to the left (resp. right) of b has $m-1$ (resp. m) factors). Thus $r'\xi_{2m-2} = \xi_{2m-1}$. Using again the definition of m we have $\xi_{2m-1} = \cdots stbt^*s^*t^*\cdots$ where the product to the left (resp. right) of b has $m-1$ (resp. m) factors. Thus $\xi_{2m-1} = \xi_{2m-2}r'^*$ as required.

We deduce that

$$a_{\xi_{2m-2}} \xrightarrow{T_{r'}} ua_{\xi_{2m-2}} + (u+1)a_{\xi_{2m-1}}.$$

We set $a'_{\xi_1} = ua_{\xi_0} + (u+1)a_{\xi_1}$, $a'_{\xi_3} = ua_{\xi_2} + (u+1)a_{\xi_3}$, $\dots$, $a'_{\xi_{2m-1}} = ua_{\xi_{2m-2}} + (u+1)a_{\xi_{2m-1}}$. Note that $a_{\xi_0}, a_{\xi_2}, a_{\xi_4}, \dots, a_{\xi_{2m-2}}$ together with $a'_{\xi_1}, a'_{\xi_3}, \dots, a'_{\xi_{2m-1}}$ form a basis of $\dot{M}_\Omega$ and we have

$$a_{\xi_0} \xrightarrow{T_t} a_{\xi_2} \xrightarrow{T_s} a_{\xi_4} \xrightarrow{T_t} \dots \xrightarrow{T_r} a_{\xi_{2m-2}} \xrightarrow{T_{r'}} a'_{\xi_{2m-1}}$$

$$a_{\xi_0} \xrightarrow{T_s} a'_{\xi_1} \xrightarrow{T_t} a'_{\xi_3} \xrightarrow{T_s} a'_{\xi_5} \xrightarrow{T_s} \dots \xrightarrow{T_r} a'_{\xi_{2m-1}}.$$

We define an isomorphism of vector spaces $\Phi : \dot{\mathfrak{H}}_K \to \dot{M}_\Omega$ by $1 \mapsto a_{\xi_0}$, $T_t \mapsto a_{\xi_2}$, $T_sT_t \mapsto a_{\xi_4}$, $\dots$, $T_r\cdots T_sT_t \mapsto a_{\xi_{2m-2}}$ (the product has $m-1$ factors), $T_s \mapsto a'_{\xi_1}$, $T_tT_s \mapsto a'_{\xi_3}$, $\dots$, $T_r\cdots T_tT_s \mapsto a'_{\xi_{2m-1}}$ (the product has m factors). From definitions for any $c \in W_K$ we have

(a) $T_s\Phi(T_c) = \Phi(T_sT_c)$ if $sc > c$, $T_t\Phi(T_c) = \Phi(T_tT_c)$ if $tc > c$,

(b) $T_s^{-1}\Phi(T_c) = \Phi(T_s^{-1}T_c)$ if $sc < c$, $T_t^{-1}\Phi(T_c) = \Phi(T_t^{-1}T_c)$ if $tc < c$.

Since $T_s = u^2T_s^{-1} + (u^2-1)$ both as endomorphisms of $\dot{M}$ and as elements of $\dot{\mathfrak{H}}$ we see that (b) implies that $T_s\Phi(T_c) = \Phi(T_sT_c)$ if $sc < c$.

Thus $T_s\Phi(T_c) = \Phi(T_sT_c)$ for any $c \in W_K$. Similarly, $T_t\Phi(T_c) = \Phi(T_tT_c)$ for any $c \in W_K$. It follows that for any $x \in \dot{\mathfrak{H}}_K$ we have $T_s\Phi(x) = \Phi(T_sx)$, $T_t\Phi(x) = \Phi(T_tx)$, hence $(T_sT_tT_s\cdots)\Phi(x) - (T_tT_sT_t\cdots)\Phi(x) = \Phi((T_sT_tT_s\cdots)x-(T_tT_sT_t\cdots)x) = 0$ where the products $T_sT_tT_s\cdots, T_tT_sT_t\cdots$ have m factors. (We use that $T_sT_tT_s\cdots = T_tT_sT_t\cdots$ in $\dot{\mathfrak{H}}_K$.) Since Φ is an isomorphism we deduce that $(T_sT_tT_s\cdots)\mu - (T_tT_sT_t\cdots)\mu = 0$ for any $\mu \in \dot{M}_\Omega$. Hence 2.3(a) holds in our case.

2.5. Assume that we are in case 1.4(iii). By the argument in case 1.4(ii) with s, t interchanged we see that (a) holds in our case.

2.6. Assume that we are in one of the cases 1.4(iv)-(vii). We have $J = K$ that is, $K = bK^*b^{-1}$. We have $\Omega = W_Kb = bW_{K^*}$. Define $m' \geq 1$ by $m = 2m'+1$ if m is odd, $m = 2m'$ if m is even. Define s', t' by $s' = s, t' = t$ if m' is even, $s' = t, t' = s$ if m' is odd.

2.7. Assume that we are in case 1.4(iv). We define some elements of $\dot{\mathfrak{H}}_K$ as follows:

$$\begin{aligned}
\eta_0 &= T_{\mathbf{s}_{m'}} + T_{\mathbf{t}_{m'}} + (1+u-u^2)(T_{\mathbf{s}_{m'-1}} + T_{\mathbf{t}_{m'-1}}) \\
&\quad +(1+u-u^2-u^3+u^4)(T_{\mathbf{s}_{m'-2}} + T_{\mathbf{t}_{m'-2}}) + \cdots \\
&\quad +(1+u-u^2-u^3+u^4+u^5-\cdots+(-1)^{m'-2}u^{2m'-4} \\
&\quad +(-1)^{m'-2}u^{2m'-3}+(-1)^{m'-1}u^{2m'-2})(T_{\mathbf{s}_1}+T_{\mathbf{t}_1}) \\
&\quad +(1+u-u^2-u^3+u^4+u^5-\cdots+(-1)^{m'-1}u^{2m'-2} \\
&\quad +(-1)^{m'-1}u^{2m'-1}+(-1)^{m'}u^{2m'}), \\
\eta_1 &= \overset{\circ}{T}_s\eta_0, \eta_3 = T_t\eta_1, \ldots, \eta_{2m'-1} = T_{t'}\eta_{2m'-3}, \eta_{2m'+1} = T_{s'}\eta_{2m'-1}, \\
\eta'_1 &= \overset{\circ}{T}_t\eta_0, \eta'_3 = T_s\eta'_1, \ldots, \eta'_{2m'-1} = T_{s'}\eta'_{2m'-3}, \eta'_{2m'+1} = T_{t'}\eta'_{2m'-1}.
\end{aligned}$$

For example if $m = 7$ we have

$$\begin{aligned}
\eta_0 &= T_{sts} + T_{tst} + (1+u-u^2)T_{ts} + (1+u-u^2)T_{st} \\
&\quad +(1+u-u^2-u^3+u^4)T_s + (1+u-u^2-u^3+u^4)T_t \\
&\quad +(1+u-u^2-u^3+u^4+u^5-u^6), \\
\eta_1 &= (u+1)^{-1}(T_{stst} - uT_{tst} + (-u+u^3)T_{ts} + (-u+u^3)T_{st} \\
&\quad +(-u+2u^3-u^5)T_s + (-u+2u^3-u^5)T_t + (-u+2u^3-2u^5+u^7)), \\
\eta'_1 &= (u+1)^{-1}(T_{tsts} - uT_{sts} + (-u+u^3)T_{ts} + (-u+u^3)T_{st}
\end{aligned}$$

$$\begin{aligned}
&\quad +(-u+2u^3-u^5)T_s+(-u+2u^3-u^5)T_t+(-u+2u^3-2u^5+u^7)),\\
\eta_3 &= (u+1)^{-1}(T_{tstst}-u^3T_{st}+(-u^3+u^5)T_s+(-u^3+u^5)T_t\\
&\quad +(-u^3+2u^5-u^7)),\\
\eta'_3 &= (u+1)^{-1}(T_{ststs}-u^3T_{ts}+(-u^3+u^5)T_s+(-u^3+u^5)T_t\\
&\quad +(-u^3+2u^5-u^7)),\\
\eta_5 &= (u+1)^{-1}(T_{ststst}-u^5T_t+(-u^5+u^7)),\\
\eta'_5 &= (u+1)^{-1}(T_{tststs}-u^5T_s+(-u^5+u^7)),\\
\eta_7 &= \eta'_7=(u+1)^{-1}(T_{stststs}-u^7).
\end{aligned}$$

One checks by direct computation in $\dot{\mathfrak{H}}_K$ that

(a) $$\eta_m=\eta'_m=(u+1)^{-1}(T_{\mathbf{s}_m}-u^m)$$

and that the elements $\eta_0,\eta_1,\eta'_1,\eta_3,\eta'_3,\dots\eta_{2m'-1},\eta'_{2m'-1},\eta_m$ are linearly independent in $\dot{\mathfrak{H}}_K$; they span a subspace of $\dot{\mathfrak{H}}_K$ denoted by $\dot{\mathfrak{H}}_K^+$. From (a) we deduce:

(b) $$(T_{s'}T_{t'}T_{s'}\cdots T_tT_sT_t\overset{\circ}{T}_s)_{m'+1}\eta_0=(T_{t'}T_{s'}T_{t'}\cdots T_sT_tT_s\overset{\circ}{T}_t)_{m'+1}\eta_0.$$

We have

$$\overset{\circ}{T}{}_s^{-1}\eta_1=\eta_0, T_t^{-1}\eta_3=\eta_1,\dots,T_{t'}^{-1}\eta_{2m'-1}=\eta_{2m'-3},T_{s'}^{-1}\eta_{2m'+1}=\eta_{2m'-1},$$

$$\overset{\circ}{T}{}_t^{-1}\eta'_1=\eta_0, T_s^{-1}\eta'_3=\eta'_1,\dots,T_{s'}^{-1}\eta'_{2m'-1}=\eta'_{2m'-3},T_{t'}^{-1}\eta'_{2m'+1}=\eta'_{2m'-1}.$$

It follows that $\dot{\mathfrak{H}}_K^+$ is stable under left multiplication by T_s and T_t hence it is a left ideal of $\dot{\mathfrak{H}}_K$. From the definitions we have

$$a_{\xi_1}=\overset{\circ}{T}_sa_{\xi_0},a_{\xi_3}=T_ta_{\xi_1},\dots,a_{\xi_{2m'-1}}=T_{t'}a_{\xi_{2m'-3}},a_{\xi_{2m'+1}}=T_{s'}a_{\xi_{2m'-1}},$$

$$a_{\xi'_1}=\overset{\circ}{T}_ta_{\xi_0},a_{\xi'_3}=T_sa_{\xi'_1},\dots,a_{\xi'_{2m'-1}}=T_{s'}a_{\xi'_{2m'-3}},a_{\xi'_{2m'+1}}=T_{t'}a_{\xi'_{2m'-1}},$$

$$\overset{\circ}{T}{}_s^{-1}a_{\xi_1}=a_{\xi_0},T_t^{-1}a_{\xi_3}=a_{\xi_1},\dots,T_{t'}^{-1}a_{\xi_{2m'-1}}=a_{\xi_{2m'-3}},T_{s'}^{-1}a_{\xi_{2m'+1}}=a_{\xi_{2m'-1}},$$

$$\overset{\circ}{T}{}_t^{-1}a_{\xi'_1}=a_{\xi_0},T_s^{-1}a_{\xi'_3}=a_{\xi'_1},\dots,T_{s'}^{-1}a_{\xi'_{2m'-1}}=a_{\xi'_{2m'-3}},T_{t'}^{-1}a_{\xi'_{2m'+1}}=a_{\xi'_{2m'-1}}.$$

Hence the vector space isomorphism $\Phi : \dot{\mathfrak{H}}^+_K \xrightarrow{\sim} \dot{M}_\Omega$ given by $\eta_{2i+1} \mapsto a_{\xi_{2i+1}}$, $\eta'_{2i+1} \mapsto a_{\xi'_{2i+1}}$ $(i \in [0,(m-1)/2])$, $\eta_0 \mapsto a_{\xi_0}$ satisfies $\Phi(T_s h) = T_s\Phi(h)$, $\Phi(T_t h) = T_t\Phi(h)$ for any $h \in \dot{\mathfrak{H}}^+_K$. Since $(T_sT_tT_s\cdots)_m h = (T_tT_sT_t\cdots)_m h$ for $h \in \dot{\mathfrak{H}}^+_K$, we deduce that 2.3(a) holds in our case.

2.8. Assume that we are in case 1.4(v). We define some elements of $\dot{\mathfrak{H}}_K$ as follows:

$$\begin{aligned}\eta_0 \;=\; & T_{\mathbf{s}_{m'-1}} + T_{\mathbf{t}_{m'-1}} + (1-u^2)(T_{\mathbf{s}_{m'-2}} + T_{\mathbf{t}_{m'-2}}) \\ & +(1-u^2+u^4)(T_{\mathbf{s}_{m'-3}} + T_{\mathbf{t}_{m'-3}}) + \cdots \\ & +(1-u^2+u^4-\cdots+(-1)^{m'-2}u^{2(m'-2)})(T_{\mathbf{s}_1}+T_{\mathbf{t}_1}) \\ & +(1-u^2+u^4-\cdots+(-1)^{m'-1}u^{2(m'-1)}),\end{aligned}$$

(if $m \geq 4$), $\eta_0 = 1$ (if $m = 2$),

$$\begin{aligned}\eta_1 &= \mathring{T}_s\eta_0, \eta_3 = T_t\eta_1, \ldots, \eta_{2m'-1} = T_{t'}\eta_{2m'-3}, \eta_{2m'} = \mathring{T}_{s'}\eta_{2m'-1}, \\ \eta'_1 &= \mathring{T}_t\eta_0, \eta'_3 = T_s\eta'_1, \ldots, \eta'_{2m'-1} = T_{s'}\eta'_{2m'-3}, \eta'_{2m'} = \mathring{T}_{t'}\eta'_{2m'-1}.\end{aligned}$$

For example if $m = 4$ we have

$$\begin{aligned}\eta_0 &= T_s + T_t + (1-u^2), \\ \eta_1 &= (u+1)^{-1}(T_{st} - uT_s - uT_t + (-u+u^2+u^3)), \\ \eta'_1 &= (u+1)^{-1}(T_{ts} - uT_s - uT_t + (-u+u^2+u^3)), \\ \eta_3 &= (u+1)^{-1}(T_{tst} - uT_{ts} + u^2T_s - u^3), \\ \eta'_3 &= (u+1)^{-1}(T_{sts} - uT_{st} + u^2T_s - u^3), \\ \eta_4 &= \eta'_4 = (u+1)^{-2}(T_{stst} - uT_{sts} - uT_{tst} + u^2T_{st} + u^2T_{ts} - u^3T_s - u^3T_t + u^4).\end{aligned}$$

If $m = 6$ we have

$$\begin{aligned}\eta_0 &= T_{st} + T_{ts} + (1-u^2)T_s + (1-u^2)T_t + (1-u^2+u^4), \\ \eta_1 &= (u+1)^{-1}(T_{sts} - uT_{st} - uT_{ts} + (-u+u^2+u^3)T_s \\ &\quad +(-u+u^2+u^3)T_t + (-u+u^2+u^3-u^4-u^5)), \\ \eta'_1 &= (u+1)^{-1}(T_{tst} - uT_{st} - uT_{ts} + (-u+u^2+u^3)T_s \\ &\quad +(-u+u^2+u^3)T_t + (-u+u^2+u^3-u^4-u^5)), \\ \eta_3 &= (u+1)^{-1}(T_{tsts} - uT_{tst} - u^2T_{ts} - u^3T_s - u^3T_t + (-u^3+u^4+u^5)),\end{aligned}$$

$$\begin{aligned}
\eta'_3 &= (u+1)^{-1}(T_{stst} - uT_{sts} - u^2T_{st} - u^3T_s - u^3T_t + (-u^3+u^4+u^5)),\\
\eta_5 &= (u+1)^{-1}(T_{ststs} - uT_{stst} - u^2T_{sts} - u^3T_{st} + u^4T_s - u^5),\\
\eta'_5 &= (u+1)^{-1}(T_{tstst} - uT_{tsts} - u^2T_{tst} - u^3T_{ts} + u^4T_t - u^5),\\
\eta_6 &= \eta'_6 = (u+1)^{-2}(T_{ststst} - uT_{ststs} - uT_{tstst} + u^2T_{stst} + u^2T_{tsts}\\
&\quad -u^3T_{sts} - u^3T_{tst} + u^4T_{st} + u^4T_{ts} - u^5T_s - u^5T_t + u^6).
\end{aligned}$$

If $m = 8$ we have

$$\begin{aligned}
\eta_0 &= T_{sts} + T_{tst} + (1-u^2)T_{st} + (1-u^2)T_{ts} + (1-u^2+u^4)T_s\\
&\quad +(1-u^2+u^4)T_t + (1-u^2+u^4-u^6),\\
\eta_1 &= (u+1)^{-1}(T_{stst} - uT_{sts} - uT_{tst} + (-u+u^2+u^3)T_{st} + (-u+u^2+u^3)T_{ts}\\
&\quad +(-u+u^2+u^3-u^4-u^5)T_s + (-u+u^2+u^3-u^4-u^5)T_t\\
&\quad +(-u+u^2+u^3-u^4-u^5+u^6+u^7)),\\
\eta'_1 &= (u+1)^{-1}(T_{tsts} - uT_{sts} - uT_{tst} + (-u+u^2+u^3)T_{st} + (-u+u^2+u^3)T_{ts}\\
&\quad +(-u+u^2+u^3-u^4-u^5)T_s + (-u+u^2+u^3-u^4-u^5)T_t\\
&\quad +(-u+u^2+u^3-u^4-u^5+u^6+u^7)),\\
\eta_3 &= (u+1)^{-1}(T_{tstst} - uT_{tsts} + u^2T_{tst} - u^3T_{st} - u^3T_{ts} + (-u^3+u^4+u^5)T_s\\
&\quad +(-u^3+u^4+u^5)T_t + (-u^3+u^4+u^5-u^6-u^7)),\\
\eta'_3 &= (u+1)^{-1}(T_{ststs} - uT_{stst} + u^2T_{sts} - u^3T_{st} - u^3T_{ts} + (-u^3+u^4+u^5)T_s\\
&\quad +(-u^3+u^4+u^5)T_t + (-u^3+u^4+u^5-u^6-u^7)),\\
\eta_5 &= (u+1)^{-1}(T_{ststst} - uT_{ststs} + u^2T_{stst} - u^3T_{sts} + u^4T_{st} - u^5T_s - u^5T_t\\
&\quad +(-u^5+u^6+u^7)),\\
\eta'_5 &= (u+1)^{-1}(T_{tststs} - uT_{tstst} + u^2T_{tsts} - u^3T_{tst} + u^4T_{ts} - u^5T_s - u^5T_t\\
&\quad +(-u^5+u^6+u^7)),\\
\eta_7 &= (u+1)^{-1}(T_{tststst} - uT_{tststs} + u^2T_{tstst} - u^3T_{tsts} + u^4T_{tst} - u^5T_{ts}\\
&\quad +u^6T_t - u^7),\\
\eta'_7 &= (u+1)^{-1}(T_{stststs} - uT_{ststst} + u^2T_{ststs} - u^3T_{stst} + u^4T_{sts} - u^5T_{st}\\
&\quad +u^6T_s - u^7),\\
\eta_8 &= \eta'_8 = (u+1)^{-2}(T_{stststst} - uT_{stststs} - uT_{tststst} + u^2T_{tststs}\\
&\quad +u^2T_{ststst} - u^3T_{ststs} - u^3T_{tstst} + u^4T_{stst} + u^4T_{tsts} - u^5T_{sts} - u^5T_{tst}\\
&\quad +u^6T_{st} + u^6T_{ts} - u^7T_s - u^tT_t + u^8).
\end{aligned}$$

One checks by direct computation in $\dot{\mathfrak{H}}_K$ that

$$\text{(a)} \qquad \eta_m = \eta'_m = (u+1)^{-2} \sum_{y \in W_K} (-u)^{m-l(y)} T_y$$

and that the elements $\eta_0, \eta_1, \eta'_1, \eta_3, \eta'_3, \dots \eta_{2m'-1}, \eta'_{2m'-1}, \eta_m$ are linearly independent in $\dot{\mathfrak{H}}_K$; they span a subspace of $\dot{\mathfrak{H}}_K$ denoted by $\dot{\mathfrak{H}}_K^+$. From (a) we deduce:

$$\text{(b)} \qquad (\overset{\circ}{T}_{s'} T_{t'} T_{s'} \cdots T_t T_s T_t \overset{\circ}{T}_s)_{m'+1} \eta_0 = (\overset{\circ}{T}_{t'} T_{s'} T_{t'} \cdots T_s T_t T_s \overset{\circ}{T}_t)_{m'+1} \eta_0.$$

We have

$$\begin{aligned} \overset{\circ}{T}{}_s^{-1} \eta_1 &= \eta_0, T_t^{-1}\eta_3 = \eta_1, \dots, T_{t'}^{-1}\eta_{2m'-1} = \eta_{2m'-3}, \overset{\circ}{T}{}_{s'}^{-1} \eta_{2m'} = \eta_{2m'-1}, \\ \overset{\circ}{T}{}_t^{-1} \eta'_1 &= \eta_0, T_s^{-1}\eta'_3 = \eta'_1, \dots, T_{s'}^{-1}\eta'_{2m'-1} = \eta'_{2m'-3}, \overset{\circ}{T}{}_{t'}^{-1} \eta'_{2m'} = \eta'_{2m'-1}. \end{aligned}$$

It follows that $\dot{\mathfrak{H}}_K^+$ is stable under left multiplication by T_s and T_t hence it is a left ideal of $\dot{\mathfrak{H}}_K$. From the definitions we have

$$\begin{aligned} &a_{\xi_1} = \overset{\circ}{T}_s a_{\xi_0}, a_{\xi_3} = T_t a_{\xi_1}, \dots, a_{\xi_{2m'-1}} = T_{t'} a_{\xi_{2m'-3}}, a_{\xi_{2m'}} = \overset{\circ}{T}_{s'} a_{\xi_{2m'-1}}, \\ &a_{\xi'_1} = \overset{\circ}{T}_t a_{\xi_0}, a_{\xi'_3} = T_s a_{\xi'_1}, \dots, a_{\xi'_{2m'-1}} = T_{s'} a_{\xi'_{2m'-3}}, a_{\xi'_{2m'}} = \overset{\circ}{T}_{t'} a_{\xi'_{2m'-1}}, \\ &\overset{\circ}{T}{}_s^{-1} a_{\xi_1} = a_{\xi_0}, T_t^{-1} a_{\xi_3} = a_{\xi_1}, \dots, T_{t'}^{-1} a_{\xi_{2m'-1}} = a_{\xi_{2m'-3}}, \overset{\circ}{T}{}_{s'}^{-1} a_{\xi_{2m'}} = a_{\xi_{2m'-1}}, \\ &\overset{\circ}{T}{}_t^{-1} a_{\xi'_1} = a_{\xi_0}, T_s^{-1} a_{\xi'_3} = a_{\xi'_1}, \dots, T_{s'}^{-1} a_{\xi'_{2m'-1}} = a_{\xi'_{2m'-3}}, \overset{\circ}{T}{}_{t'}^{-1} a_{\xi'_{2m'}} = a_{\xi'_{2m'-1}}. \end{aligned}$$

Hence the vector space isomorphism $\Phi : \dot{\mathfrak{H}}_K^+ \xrightarrow{\sim} \dot{M}_\Omega$ given by $\eta_{2i+1} \mapsto a_{\xi_{2i+1}}$, $\eta'_{2i+1} \mapsto a_{\xi'_{2i+1}}$ $(i \in [0, (m-2)/2])$, $\eta_0 \mapsto a_{\xi_0}$, $\eta_m \mapsto a_{\xi_m}$ satisfies $\Phi(T_s h) = T_s \Phi(h)$, $\Phi(T_t h) = T_t \Phi(h)$ for any $h \in \dot{\mathfrak{H}}_K^+$. Since $(T_s T_t T_s \cdots)_m h = (T_t T_s T_t \cdots)_m h$ for $h \in \dot{\mathfrak{H}}_K^+$, we deduce that 2.3(a) holds in our case.

2.9. Assume that we are in case 1.4(vi). We define some elements of $\dot{\mathfrak{H}}_K$ as follows:

$$\begin{aligned} \eta_0 &= T_{\mathbf{s}_{m'}} + T_{\mathbf{t}_{m'}} + (1-u-u^2)(T_{\mathbf{s}_{m'-1}} + T_{\mathbf{t}_{m'-1}}) \\ &\quad + (1-u-u^2+u^3+u^4)(T_{\mathbf{s}_{m'-2}} + T_{\mathbf{t}_{m'-2}}) + \cdots \\ &\quad + (1-u-u^2+u^3+u^4-u^5-\cdots+(-1)^{m'-2}u^{2m'-4} \end{aligned}$$

$$+(-1)^{m'-1}u^{2m'-3}+(-1)^{m'-1}u^{2m'-2})(T_{\mathbf{s}_1}+T_{\mathbf{t}_1})$$
$$+(1+u-u^2-u^3+u^4+u^5-\cdots+(-1)^{m'-1}u^{2m'-2}$$
$$+(-1)^{m'}u^{2m'-1}+(-1)^{m'}u^{2m'}),$$

$$\begin{aligned}\eta_2 &= T_s\eta_0, \eta_4 = T_t\eta_2, \ldots, \eta_{2m'} = T_{s'}\eta_{2m'-2}, \eta_{2m'+1} = \mathring{T}_{t'}\eta_{2m'},\\ \eta'_2 &= T_t\eta_0, \eta'_4 = T_s\eta'_2, \ldots, \eta'_{2m'} = T_{t'}\eta'_{2m'-2}, \eta'_{2m'+1} = \mathring{T}_{s'}\eta'_{2m'}.\end{aligned}$$

For example if $m=7$ we have

$$\begin{aligned}\eta_0 &= T_{sts}+T_{tst}+(1-u-u^2)T_{ts}+(1-u-u^2)T_{st}\\ &\quad +(1-u-u^2+u^3+u^4)T_s+(1-u-u^2+u^3+u^4)T_t\\ &\quad +(1-u-u^2+u^3+u^4-u^5-u^6),\\ \eta_2 &= T_{stst}-uT_{sts}+u^2T_{st}+(u^2-u^3-u^4)T_s+(u^2-u^3-u^4)T_t\\ &\quad +(u^2-u^3-u^4+u^5+u^6),\\ \eta'_2 &= T_{tsts}-uT_{tst}+u^2T_{st}+(u^2-u^3-u^4)T_s+(u^2-u^3-u^4)T_t\\ &\quad +(u^2-u^3-u^4+u^5+u^6),\\ \eta_4 &= T_{ststs}-uT_{stst}+u^2T_{sts}-u^3T_{st}+u^4T_s+u^4T_t+(u^4-u^5-u^6),\\ \eta'_4 &= T_{tstst}-uT_{tsts}+u^2T_{tst}-u^3T_{ts}+u^4T_s+u^4T_t+(u^4-u^5-u^6),\\ \eta_6 &= T_{ststst}-uT_{ststs}+u^2T_{stst}-u^3T_{sts}+u^4T_{st}-u^5T_s+u^6,\\ \eta'_6 &= T_{tststs}-uT_{tstst}+u^2T_{tsts}-u^3T_{tst}+u^4T_{ts}-u^5T_t+u^6,\\ \eta_7 &= \eta'_7=(u+1)^{-1}(T_{stststs}-uT_{ststst}-uT_{tststs}+u^2T_{ststs}+u^2T_{tstst}-u^3T_{stst}\\ &\quad -u^3T_{tsts}+u^4T_{sts}+u^4T_{tst}-u^5T_{st}-u^5T_{ts}+u^6T_s+u^6T_t-u^7).\end{aligned}$$

One checks by direct computation in $\dot{\mathfrak{H}}_K$ that

(a) $$\eta_m=\eta'_m=(u+1)^{-1}\sum_{y\in W_K}(-u)^{m-l(y)}T_y$$

and that the elements $\eta_0,\eta_2,\eta'_2,\eta_4,\eta'_4,\ldots\eta_{2m'},\eta'_{2m'},\eta_m$ are linearly independent in $\dot{\mathfrak{H}}_K$; they span a subspace of $\dot{\mathfrak{H}}_K$ denoted by $\dot{\mathfrak{H}}_K^+$. From (a) we deduce:

(b) $$(\mathring{T}_{s'}T_{t'}T_{s'}\cdots T_tT_s)_{m'+1}\eta_0=(\mathring{T}_{t'}T_{s'}T_{t'}\cdots T_sT_t)_{m'+1}\eta_0.$$

We have

$$\begin{aligned}
\eta_0 &= T_s^{-1}\eta_2, \eta_2 = T_t^{-1}\eta_4, \dots, \eta_{2m'-2} = T_{s'}^{-1}\eta_{2m'}, \eta_{2m'} = \overset{\circ}{T}{}_{t'}^{-1}\eta_{2m'+1},\\
\eta_0 &= T_t^{-1}\eta'_2, \eta'_2 = T_s^{-1}\eta'_4, \dots, \eta'_{2m'-2} = T_{t'}^{-1}\eta'_{2m'}, \eta'_{2m'} = \overset{\circ}{T}{}_{s'}^{-1}\eta_{2m'+1}.
\end{aligned}$$

It follows that $\dot{\mathfrak{H}}_K^+$ is stable under left multiplication by T_s and T_t hence it is a left ideal of $\dot{\mathfrak{H}}_K$. From the definitions we have

$$\begin{aligned}
&a_{\xi_2} = T_s a_{\xi_0}, a_{\xi_4} = T_t a_{\xi_2}, \dots, a_{\xi_{2m'}} = T_{s'} a_{\xi_{2m'-2}}, a_{\xi_{2m'+1}} = \overset{\circ}{T}_{t'} a_{\xi_{2m'}},\\
&a_{\xi'_2} = T_t a_{\xi_0}, a_{\xi'_4} = T_s a_{\xi'_2}, \dots, a_{\xi'_{2m'}} = T_{t'} a_{\xi'_{2m'-2}}, a_{\xi'_{2m'+1}} = \overset{\circ}{T}_{s'} a_{\xi_{2m'}},\\
&a_{\xi_0} = T_s^{-1} a_{\xi_2}, a_{\xi_2} = T_t^{-1} a_{\xi_4}, \dots, a_{\xi_{2m'-2}} = T_{s'}^{-1} a_{\xi_{2m'}}, a_{\xi_{2m'}} = \overset{\circ}{T}{}_{t'}^{-1} a_{\xi_{2m'+1}},\\
&a_{\xi_0} = T_t^{-1} a_{\xi'_2}, a_{\xi'_2} = T_s^{-1} a_{\xi'_4}, \dots, a_{\xi'_{2m'-2}} = T_{t'}^{-1} a_{\xi'_{2m'}}, a_{\xi'_{2m'}} = \overset{\circ}{T}{}_{s'}^{-1} a_{\xi_{2m'+1}}.
\end{aligned}$$

Hence the vector space isomorphism $\Phi : \dot{\mathfrak{H}}_K^+ \xrightarrow{\sim} \dot{M}_\Omega$ given by $\eta_{2i} \mapsto a_{\xi_{2i}}$, $\eta'_{2i} \mapsto a_{\xi'_{2i}}$ ($i \in [0,(m-1)/2]$), $\eta_m \mapsto a_{\xi_m}$ satisfies $\Phi(T_s h) = T_s\Phi(h)$, $\Phi(T_t h) = T_t\Phi(h)$ for any $h \in \dot{\mathfrak{H}}_K^+$. Since $(T_sT_tT_s\cdots)_m h = (T_tT_sT_t\cdots)_m h$ for $h \in \dot{\mathfrak{H}}_K^+$, we deduce that 2.3(a) holds in our case.

2.10. Assume that we are in case 1.4(vii). We define some elements of $\dot{\mathfrak{H}}_K$ as follows:

$$\begin{aligned}
\eta_0 &= T_{\mathbf{s}_{m'}} + T_{\mathbf{t}_{m'}} + (1-u^2)(T_{\mathbf{s}_{m'-1}} + T_{\mathbf{t}_{m'-1}})\\
&\quad +(1-2u^2+u^4)(T_{\mathbf{s}_{m'-3}} + T_{\mathbf{t}_{m'-3}}) + \cdots\\
&\quad +(1-2u^2+2u^4-\cdots+(-1)^{m'-2}2u^{2(m'-2)}\\
&\quad +(-1)^{m'-1}u^{2(m'-1)})(T_{\mathbf{s}_1} + T_{\mathbf{t}_1})\\
&\quad +(1-2u^2+2u^4-\cdots+(-1)^{m'-1}2u^{2(m'-1)} + (-1)^{m'}u^{2m'}),\\
\eta_2 &= T_s\eta_0, \eta_4 = T_t\eta_2, \dots, \eta_{2m'} = T_{s'}\eta_{2m'-2},\\
\eta'_2 &= T_t\eta_0, \eta'_4 = T_s\eta'_2, \dots, \eta'_{2m'} = T_{t'}\eta'_{2m'-2}.
\end{aligned}$$

For example if $m = 8$ we have

$$\begin{aligned}
\eta_0 &= T_{stst} + T_{tsts} + (1-u^2)T_{sts} + (1-u^2)T_{tst} + (1-2u^2+u^4)T_{st}\\
&\quad +(1-2u^2+u^4)T_{ts} + (1-2u^2+2u^4-u^6)T_s + (1-2u^2+2u^4-u^6)T_t\\
&\quad +(1-2u^2+2u^4-2u^6+u^8),
\end{aligned}$$

$$\begin{aligned}
\eta_2 &= T_{ststs} + u^2 T_{tst} + (u^2-u^4)T_{st} + (u^2-u^4)T_{ts} + (u^2-2u^4+u^6)T_s \\
&\quad +(u^2-2u^4+u^6)T_t + (u^2-2u^4+2u^6-u^8), \\
\eta'_2 &= T_{tstst} + u^2 T_{sts} + (u^2-u^4)T_{st} + (u^2-u^4)T_{ts} + (u^2-2u^4+u^6)T_s \\
&\quad +(u^2-2u^4+u^6)T_t + (u^2-2u^4+2u^6-u^8), \\
\eta_4 &= T_{tststs} + u^4 T_{st} + (u^4-u^6)T_s + (u^4-u^6)T_t + (u^4-2u^6+u^8), \\
\eta'_4 &= T_{ststst} + u^4 T_{ts} + (u^4-u^6)T_s + (u^4-u^6)T_t + (u^4-2u^6+u^8), \\
\eta_6 &= T_{stststs} + u^6 T_t + (u^6-u^8), \\
\eta'_6 &= T_{tststst} + u^6 T_s + (u^6-u^8), \\
\eta_8 &= \eta'_8 = T_{stststst} + u^8.
\end{aligned}$$

One checks by direct computation in $\dot{\mathfrak{H}}_K$ that

(a) $$\eta_m = \eta'_m = T_{\mathbf{s}_m} + u^m$$

and that the elements $\eta_0, \eta_2, \eta'_2, \eta_4, \eta'_4, \dots \eta_{2m'}, \eta'_{2m'}, \eta_m$ are linearly independent in $\dot{\mathfrak{H}}_K$; they span a subspace of $\dot{\mathfrak{H}}_K$ denoted by $\dot{\mathfrak{H}}^+_K$. From (a) we deduce:

(c) $$(T_{t'}T_{s'}\cdots T_tT_s)_{m'}\eta_0 = (T_{s'}T_{t'}\cdots T_sT_t)_{m'}\eta_0.$$

We have

$$\begin{aligned}
\eta_0 &= T_s^{-1}\eta_2, \eta_2 = T_t^{-1}\eta_4, \dots, \eta_{2m'-2} = T_{s'}^{-1}\eta_{2m'}, \\
\eta_0 &= T_t^{-1}\eta'_2, \eta'_2 = T_s^{-1}\eta'_4, \dots, \eta'_{2m'-2} = T_{t'}^{-1}\eta'_{2m'}.
\end{aligned}$$

It follows that $\dot{\mathfrak{H}}^+_K$ is stable under left multiplication by T_s and T_t hence it is a left ideal of $\dot{\mathfrak{H}}_K$. From the definitions we have

$$\begin{aligned}
a_{\xi_2} &= T_s a_{\xi_0}, a_{\xi_4} = T_t a_{\xi_2}, \dots, a_{\xi_{2m'}} = T_{s'} a_{\xi_{2m'-2}}, \\
a_{\xi'_2} &= T_t a_{\xi_0}, a_{\xi'_4} = T_s a_{\xi'_2}, \dots, a_{\xi'_{2m'}} = T_{t'} a_{\xi'_{2m'-2}}, \\
a_{\xi_0} &= T_s^{-1} a_{\xi_2}, a_{\xi_2} = T_t^{-1} a_{\xi_4}, \dots, a_{\xi_{2m'-2}} = T_{s'}^{-1} a_{\xi_{2m'}}, \\
a_{\xi_0} &= T_t^{-1} a_{\xi'_2}, a_{\xi'_2} = T_s^{-1} a_{\xi'_4}, \dots, a_{\xi'_{2m'-2}} = T_{t'}^{-1} a_{\xi'_{2m'}}.
\end{aligned}$$

Hence the vector space isomorphism $\Phi : \dot{\mathfrak{H}}^+_K \xrightarrow{\sim} \dot{M}_\Omega$ given by $\eta_{2i} \mapsto a_{\xi_{2i}}$, $\eta'_{2i} \mapsto a_{\xi'_{2i}}$ ($i \in [0, m/2]$) satisfies $\Phi(T_s h) = T_s\Phi(h)$, $\Phi(T_t h) = T_t\Phi(h)$ for any

$h \in \dot{\mathfrak{H}}_K^+$. Since $(T_sT_tT_s\cdots)_m h = (T_tT_sT_t\cdots)_m h$ for $h \in \dot{\mathfrak{H}}_K^+$, we deduce that 2.3(a) holds in our case. This completes the proof of Theorem 0.1. $\square$

2.11. We show that the $\dot{\mathfrak{H}}$-module $\dot{M}$ is generated by a_1. Indeed, from 2.2(i) we see by induction on $l(w)$ that for any $w \in \mathbf{I}_*$, a_w belongs to the $\dot{\mathfrak{H}}$-submodule of $\dot{M}$ generated by a_1.

3. Proof of Theorem 0.2

3.1. We define a $\mathbf{Z}$-linear map $B : M \to M$ by $B(u^n a_w) = \epsilon_w u^{-n} T_{w^*}^{-1} a_{w^*}$ for any $w \in \mathbf{I}_*, n \in \mathbf{Z}$. Note that $B(a_1) = a_1$.

For any $w \in \mathbf{I}_*, s \in S$ we show:

(a) $B(T_s a_w) = T_s^{-1} B(a_w)$.

Assume first that $sw = ws^* > w$. We must show that $B(ua_w + (u+1)a_{sw}) = T_s^{-1} B(a_w)$ or that

$$u^{-1}\epsilon_w T_{w^*}^{-1} a_{w^*} - (u^{-1}+1)\epsilon_w T_{s^*w^*}^{-1} a_{s^*w^*} = T_s^{-1}\epsilon_w T_{w^*}^{-1} a_{w^*}$$

or that

$$T_{w^*}^{-1} a_{w^*} - (u+1) T_{w^*}^{-1} T_{s^*}^{-1} a_{s^*w^*} = u T_{w^*}^{-1} T_{s^*}^{-1} a_{w^*}$$

or that

$$T_{s^*} a_{w^*} - (u+1) a_{s^*w^*} = u a_{w^*}.$$

This follows from 0.1(i) with s, w replaced by s^*, w^*.

Assume next $sw = ws^* < w$. We set $y = sw \in \mathbf{I}_*$ so that $sy > y$. We must show that $B((u^2-u-1)a_{sy} + (u^2-u)a_y) = T_s^{-1} B(a_{sy})$ or that

$$-(u^{-2}-u^{-1}-1)\epsilon_y T_{s^*y^*}^{-1} a_{s^*y^*} + (u^{-2}-u^{-1})\epsilon_y T_{y^*}^{-1} a_{y^*} = -T_s^{-1}\epsilon_y T_{s^*y^*}^{-1} a_{s^*y^*}$$

or that

$$-(u^{-2}-u^{-1}-1) T_{y^*}^{-1} T_{s^*}^{-1} a_{s^*y^*} + (u^{-2}-u^{-1}) T_{y^*}^{-1} a_{y^*} = -T_{y^*}^{-1} T_{s^*}^{-2} a_{s^*y^*}$$

or that

$$-(u^{-2}-u^{-1}-1) T_{s^*}^{-1} a_{s^*y^*} + (u^{-2}-u^{-1}) a_{y^*} = -T_{s^*}^{-2} a_{s^*y^*}$$

or that

$$-(1-u-u^2)a_{s^*y^*}+(1-u)T_{s^*}a_{y^*}=-(T_{s^*}+1-u^2)a_{s^*y^*}.$$

Using 0.1(i),(ii) with w,s replaced by y^*,s^* we see that it is enough to show that

$$\begin{aligned}&-(1-u-u^2)a_{s^*y^*}+(1-u)(ua_{y^*}+(u+1)a_{s^*y^*})\\&=-(u^2-u-1)a_{s^*y^*}-(u^2-u)a_{y^*}-(1-u^2)a_{s^*y^*}\end{aligned}$$

which is obvious.

Assume next that $sw\neq ws^*>w$. We must show that $B(a_{sws^*})=T_s^{-1}B(a_w)$ or that

$$\epsilon_w T_{s^*w^*s}^{-1}a_{s^*w^*s}=T_s^{-1}\epsilon_w T_{w^*}^{-1}a_{w^*}$$

or that

$$T_s^{-1}T_{w^*}^{-1}T_{s^*}^{-1}a_{s^*w^*s}=T_s^{-1}T_{w^*}^{-1}a_{w^*}$$

or that

$$a_{s^*w^*s}=T_{s^*}a_{w^*}.$$

This follows from 0.1(iii) with s,w replaced by s^*,w^*.

Finally assume that $sw\neq ws^*>w$. We set $y=sws^*\in\mathbf{I}_*$ so that $sy>y$. We must show that $B((u^2-1)a_{sys^*}+u^2a_y)=T_s^{-1}B(a_{sys^*})$ or that

$$(u^{-2}-1)\epsilon_y T_{s^*y^*s}^{-1}a_{s^*y^*s}+u^{-2}\epsilon_y T_{y^*}^{-1}a_{y^*}=T_s^{-1}\epsilon_y T_{s^*y^*s}^{-1}a_{s^*y^*s}$$

or that

$$(u^{-2}-1)T_s^{-1}T_{y^*}^{-1}T_{s^*}^{-1}a_{s^*y^*s}+u^{-2}T_{y^*}^{-1}a_{y^*}=T_s^{-1}T_s^{-1}T_{y^*}^{-1}T_{s^*}^{-1}a_{s^*y^*s}$$

or (using 0.1(iii) with w,s replaced by y^*,s^*) that

$$(u^{-2}-1)T_s^{-1}T_{y^*}^{-1}a_{y^*}+u^{-2}T_{y^*}^{-1}a_{y^*}=T_s^{-1}T_s^{-1}T_{y^*}^{-1}a_{y^*}$$

or that

$$(u^{-2}-1)T_s^{-1}+u^{-2}=T_s^{-1}T_s^{-1}$$

which is obvious. $\square$

This completes the proof of (a). Since the elements T_s generate the algebra $\mathfrak{H}$, from (a) we deduce that $B(hm) = \bar{h}B(m)$ for any $h \in \mathfrak{H}, m \in M$. This proves the existence part of 0.2(a).

For $n \in \mathbf{Z}, w \in \mathbf{I}_*$ we have

$$B(B(u^n a_w)) = \epsilon_w B(u^{-n} T_{w^*}^{-1} a_{w^*}) = \epsilon_w \epsilon_{w^*} u^n T_{w^{*-1}} T_w^{-1} a_w = u^n a_w.$$

Thus $B^2 = 1$. The uniqueness part of 0.2(a) is proved as in [6, 2.9]. This completes the proof of 0.2(a). Now 0.2(b) follows from the proof of 0.2(a).

4. Proof of Theorem 0.4

4.1. For $w \in \mathbf{I}_*$ we have

$$\overline{a'_w} = \sum_{y \in \mathbf{I}_*} \overline{r_{y,w}} a'_y$$

where $r_{y,w} \in \underline{\mathcal{A}}$ is zero for all but finitely many y. (This $r_{y,w}$ differs from that in [6, 0.2(b)].)

For $s \in S$ we set $T'_s = u^{-1}T_s$. We rewrite the formulas 0.1(i)-(iv) as follows.

(i) $T'_s a'_w = a'_w + (v + v^{-1}) a'_{sw}$ if $sw = ws^* > w$;
(ii) $T'_s a'_w = (u - 1 - u^{-1}) a'_w + (v - v^{-1}) a'_{sw}$ if $sw = ws^* < w$;
(iii) $T'_s a'_w = a'_{sws^*}$ if $sw \neq ws^* > w$;
(iv) $T'_s a'_w = (u - u^{-1}) a'_w + a'_{sws^*}$ if $sw \neq ws^* < w$.

4.2. Now assume that $y \in \mathbf{I}_*, sy > y$. From the equality $\overline{T'_s a'_y} = \overline{T'_s}(\overline{a'_y})$ (where $\overline{T'_s} = T'_s + u^{-1} - u$) we see that

$$\sum_x \overline{r_{x,y}} a'_x + (v + v^{-1}) \sum_x \overline{r_{x,sy}} a'_x \text{ (if } sy = ys^*) \text{ or } \sum_x \overline{r_{x,sys^*}} a'_x \text{ (if } sy \neq ys^*)$$

is equal to

$$\sum_{x; sx = xs^*, sx > x} \overline{r_{x,y}} a'_x + \sum_{x; sx = xs^*, sx > x} \overline{r_{x,y}} (v + v^{-1}) a'_{sx}$$

$$
\begin{aligned}
&+\sum_{x;sx=xs^*,sx<x}\overline{r_{x,y}}(u-1-u^{-1})a'_x+\sum_{x;sx=xs^*,sx<x}\overline{r_{x,y}}(v-v^{-1})a'_{sx}\\
&+\sum_{x;sx\ne xs^*,sx>x}\overline{r_{x,y}}a'_{sxs^*}+\sum_{x;sx\ne xs^*,sx<x}\overline{r_{x,y}}(u-u^{-1})a'_x\\
&+\sum_{x;sx\ne xs^*,sx<x}\overline{r_{x,y}}a'_{sxs^*}+(u^{-1}-u)\sum_x\overline{r_{x,y}}a'_x\\
=&\sum_{x;sx=xs^*,sx>x}\overline{r_{x,y}}a'_x+\sum_{x;sx=xs^*,sx<x}\overline{r_{sx,y}}(v+v^{-1})a'_x\\
&+\sum_{x;sx=xs^*,sx<x}\overline{r_{x,y}}(u-1-u^{-1})a'_x+\sum_{x;sx=xs^*,sx>x}\overline{r_{sx,y}}(v-v^{-1})a'_x\\
&+\sum_{x;sx\ne xs^*,sx<x}\overline{r_{sxs^*,y}}a'_x+\sum_{x;sx\ne xs^*,sx<x}\overline{r_{x,y}}(u-u^{-1})a'_x\\
&+\sum_{x;sx\ne xs^*,sx>x}\overline{r_{sxs^*,y}}a'_x+(u^{-1}-u)\sum_x\overline{r_{x,y}}a'_x.
\end{aligned}
$$

Hence when $sy=ys^*>y$ and $x\in\mathbf{I}_*$, we have

$$
\begin{aligned}
(v+v^{-1})\overline{r_{x,sy}}&=\overline{r_{sx,y}}(v-v^{-1})+(u^{-1}-u)\overline{r_{x,y}} \text{ if } sx=xs^*>x,\\
(v+v^{-1})\overline{r_{x,sy}}&=-2\overline{r_{x,y}}+\overline{r_{sx,y}}(v+v^{-1}) \text{ if } sx=xs^*<x,\\
(v+v^{-1})\overline{r_{x,sy}}&=\overline{r_{sxs^*,y}}+(u^{-1}-1-u)\overline{r_{x,y}} \text{ if } sx\ne xs^*>x,\\
(v+v^{-1})\overline{r_{x,sy}}&=-\overline{r_{x,y}}+\overline{r_{sxs^*,y}} \text{ if } sx\ne xs^*<x;
\end{aligned}
$$

when $sy\ne ys^*>y$ and $x\in\mathbf{I}_*$, we have

$$
\begin{aligned}
\overline{r_{x,sys^*}}&=\overline{r_{sx,y}}(v-v^{-1})+(u^{-1}+1-u)\overline{r_{x,y}} \text{ if } sx=xs^*>x,\\
\overline{r_{x,sys^*}}&=\overline{r_{sx,y}}(v+v^{-1})-\overline{r_{x,y}} \text{ if } sx=xs^*<x,\\
\overline{r_{x,sys^*}}&=\overline{r_{sxs^*,y}}+(u^{-1}-u)\overline{r_{x,y}} \text{ if } sx\ne xs^*>x,\\
\overline{r_{x,sys^*}}&=\overline{r_{sxs^*,y}} \text{ if } sx\ne xs^*<x.
\end{aligned}
$$

Applying $\bar{}$ we see that when $sy=ys^*>y$ and $x\in\mathbf{I}_*$, we have

(a)
$$
\begin{aligned}
(v+v^{-1})r_{x,sy}&=r_{sx,y}(v^{-1}-v)+(u-u^{-1})r_{x,y} \text{ if } sx=xs^*>x,\\
(v+v^{-1})r_{x,sy}&=-2r_{x,y}+r_{sx,y}(v+v^{-1}) \text{ if } sx=xs^*<x,\\
(v+v^{-1})r_{x,sy}&=r_{sxs^*,y}+(u-1-u^{-1})r_{x,y} \text{ if } sx\ne xs^*>x,\\
(v+v^{-1})r_{x,sy}&=-r_{x,y}+r_{sxs^*,y} \text{ if } sx\ne xs^*<x;
\end{aligned}
$$

when $sy \neq ys^* > y$ and $x \in \mathbf{I}_*$, we have

$$\text{(b)}\qquad \begin{aligned} r_{x,sys^*} &= r_{sx,y}(v^{-1}-v)+(u+1-u^{-1})r_{x,y} \text{ if } sx=xs^*>x,\\ r_{x,sys^*} &= r_{sx,y}(v+v^{-1})-r_{x,y} \text{ if } sx=xs^*<x,\\ r_{x,sys^*} &= r_{sxs^*,y}+(u-u^{-1})r_{x,y} \text{ if } sx\neq xs^*>x,\\ r_{x,sys^*} &= r_{sxs^*,y} \text{ if } sx\neq xs^*<x. \end{aligned}$$

4.3. Setting $r'_{x,w}=v^{-l(w)+l(x)}r_{x,w}$, $r''_{x,w}=v^{-l(w)+l(x)}\overline{r_{x,w}}$ for $x,w\in\mathbf{I}_*$ we can rewrite the last formulas in 4.2 as follows.

When $x,y\in\mathbf{I}_*$, $sy=ys^*>y$ we have

$$\begin{aligned} (v+v^{-1})vr'_{x,sy} &= v^{-1}r'_{sx,y}(v^{-1}-v)+(u-u^{-1})r'_{x,y} \text{ if } sx=xs^*>x,\\ (v+v^{-1})vr'_{x,sy} &= -2r'_{x,y}+r'_{sx,y}v(v+v^{-1}) \text{ if } sx=xs^*<x,\\ (v+v^{-1})vr'_{x,sy} &= v^{-2}r'_{sxs^*,y}+(u-1-u^{-1})r'_{x,y} \text{ if } sx\neq xs^*>x,\\ (v+v^{-1})vr'_{x,sy} &= -r'_{x,y}+v^2r'_{sxs^*,y} \text{ if } sx\neq xs^*<x. \end{aligned}$$

When $x,y\in\mathbf{I}_*$, $sy\neq ys^*>y$, we have

$$\begin{aligned} v^2r'_{x,sys^*} &= r'_{sx,y}v^{-1}(v^{-1}-v)+(u+1-u^{-1})r'_{x,y} \text{ if } sx=xs^*>x,\\ v^2r'_{x,sys^*} &= r'_{sx,y}v(v+v^{-1})-r'_{x,y} \text{ if } sx=xs^*<x,\\ v^2r'_{x,sys^*} &= v^{-2}r'_{sxs^*,y}+(u-u^{-1})r'_{x,y} \text{ if } sx\neq xs^*>x,\\ v^2r'_{x,sys^*} &= v^2r'_{sxs^*,y} \text{ if } sx\neq xs^*<x. \end{aligned}$$

When $x,y\in\mathbf{I}_*$, $sy=ys^*>y$ we have

$$\begin{aligned} (v+v^{-1})vr''_{x,sy} &= v^{-1}r''_{sx,y}(v-v^{-1})+(u^{-1}-u)r''_{x,y} \text{ if } sx=xs^*>x,\\ (v+v^{-1})vr''_{x,sy} &= -2r''_{x,y}+r''_{sx,y}v(v+v^{-1}) \text{ if } sx=xs^*<x,\\ (v+v^{-1})vr''_{x,sy} &= v^{-2}r''_{sxs^*,y}+(u^{-1}-1-u)r''_{x,y} \text{ if } sx\neq xs^*>x,\\ (v+v^{-1})vr''_{x,sy} &= -r''_{x,y}+v^2r''_{sxs^*,y} \text{ if } sx\neq xs^*<x. \end{aligned}$$

When $x,y\in\mathbf{I}_*$, $sy\neq ys^*>y$, we have

$$\begin{aligned} v^2r''_{x,sys^*} &= r''_{sx,y}v^{-1}(v-v^{-1})+(u^{-1}+1-u)r''_{x,y} \text{ if } sx=xs^*>x,\\ v^2r''_{x,sys^*} &= r''_{sx,y}v(v+v^{-1})-r''_{x,y} \text{ if } sx=xs^*<x,\\ v^2r''_{x,sys^*} &= v^{-2}r''_{sxs^*,y}+(u^{-1}-u)r''_{x,y} \text{ if } sx\neq xs^*>x, \end{aligned}$$

$$v^2 r''_{x,sys^*} = v^2 r''_{sxs^*,y} \text{ if } sx \neq xs^* < x.$$

Proposition 4.4. *Let $w \in \mathbf{I}_*$.*

(a) *If $x \in \mathbf{I}_*, r_{x,w} \neq 0$ then $x \leq w$.*

(b) *If $x \in \mathbf{I}_*, x \leq w$ we have $r'_{x,w} \in \mathbf{Z}[v^{-2}]$, $r''_{x,w} \in \mathbf{Z}[v^{-2}]$.*

We argue by induction on $l(w)$. If $w = 1$ then $r_{x,w} = \delta_{x,1}$ so that the result holds. Now assume that $l(w) \geq 1$. We can find $s \in S$ such that $sw < w$. Let $y = s \bullet w \in \mathbf{I}_*$ (see 0.6). We have $y < w$. In the setup of (a) we have $r_{x,s\bullet y} \neq 0$. From the formulas in 4.3 we deduce the following.

If $sx = xs^*$ then $r'_{sx,y} \neq 0$ or $r'_{x,y} \neq 0$ hence (by the induction hypothesis) $sx \leq y$ or $x \leq y$; if $x \leq y$ then $x \leq w$ while if $sx \leq y$ we have $sx \leq w$ hence by [5, 2.5] we have $x \leq w$.

If $sx \neq xs^*$ then $r'_{sxs^*,y} \neq 0$ or $r'_{x,y} \neq 0$ hence (by the induction hypothesis) $sxs^* \leq y$ or $x \leq y$; if $x \leq y$ then $x \leq w$ while if $sxs^* \leq y$ we have $sxs^* \leq w$ hence by [5, 2.5] we have $x \leq w$.

We see that $x \leq w$ and (a) is proved.

In the remainder of the proof we assume that $x \leq w$. Assume that $sy = ys^*$. Using the formulas in 4.3 and the induction hypothesis we see that $v(v + v^{-1})r'_{x,w} \in v^2\mathbf{Z}[v^{-2}]$, $v(v + v^{-1})r''_{x,w} \in v^2\mathbf{Z}[v^{-2}]$; hence $r'_{x,w} \in \mathbf{Z}[[v^{-2}]]$, $r''_{x,w} \in \mathbf{Z}[[v^{-2}]]$. Since $r'_{x,w} \in \mathbf{Z}[v, v^{-1}]$, $r''_{x,w} \in \mathbf{Z}[v, v^{-1}]$, it follows that $r'_{x,w} \in \mathbf{Z}[v^{-2}]$, $r''_{x,w} \in \mathbf{Z}[v^{-2}]$.

Assume now that $sy \neq ys^*$. Using the formulas in 4.3 and the induction hypothesis we see that $v^2 r'_{x,w} \in v^2\mathbf{Z}[v^{-2}]$, $v^2 r''_{x,w} \in v^2\mathbf{Z}[v^{-2}]$; hence $r'_{x,w} \in \mathbf{Z}[v^{-2}]$, $r''_{x,w} \in \mathbf{Z}[v^{-2}]$. This completes the proof. $\square$

Proposition 4.5. (a) *There is a unique function $\phi : \mathbf{I}_* \to \mathbf{N}$ such that $\phi(1) = 0$ and for any $w \in \mathbf{I}_*$ and any $s \in S$ with $sw < w$ we have $\phi(w) = \phi(sw) + 1$ (if $sw = ws^*$) and $\phi(w) = \phi(sws^*)$ (if $sw \neq ws^*$). For any $w \in \mathbf{I}_*$ we have $l(w) = \phi(w) \bmod 2$. Hence, setting $\kappa(w) = (-1)^{(l(w)+\phi(w))/2}$ for $w \in \mathbf{I}_*$ we have $\kappa(1) = 1$ and $\kappa(w) = -\kappa(s \bullet w)$ (see 0.6) for any $s \in S, w \in \mathbf{I}_*$ such that $sw < w$.*

(b) *If $x, w \in \mathbf{I}_*, x \leq w$ then the constant term of $r'_{x,w}$ is 1 and the constant term of $r''_{x,w}$ is $\kappa(x)\kappa(w)$ (see 4.4(b)).*

We prove (a). Assume first that $*$ is the identity map. For $w \in \mathbf{I}_*$ let $\phi(w)$ be the dimension of the -1 eigenspace of w on the reflection representation of W. This function has the required properties. If $*$ is not the identity map, the proof is similar: for $w \in \mathbf{I}_*$, $\phi(w)$ is the dimension of the -1 eigenspace of wT minus the dimension of the -1 eigenspace of T where T is an automorphism of the reflection representation of W induced by $*$.

We prove (b). Let $n'_{x,w}$ (resp. $n''_{x,w}$) be the constant term of $r'_{x,w}$ (resp. $r''_{x,w}$). We shall prove for any $w \in \mathbf{I}_*$ the following statement:

(c) *If* $x \in \mathbf{I}_*, x \le w$ *then* $n'_{x,w} = 1$ *and* $n''_{x,w} = n''_{1,x} n''_{1,w} \in \{1, -1\}$.

We argue by induction on $l(w)$. If $w = 1$ we have $r'_{w,w} = r''_{w,w} = 1$ and (c) is obvious. We assume that $w \in \mathbf{I}_*, w \ne 1$. We can find $s \in S$ such that $sw < w$. We set $y = s \bullet w$. Taking the coefficients of v^2 in the formulas in 4.3 and using 4.4(b) we see that the following holds for any $x \in \mathbf{I}_*$ such that $x \le w$:

$$n'_{x,w} = n'_{x,y}, n''_{x,w} = -n''_{x,y} \text{ if } sx > x,$$

(by [5, 2.5(b)], we must have $x \le y$) and

$$n'_{x,w} = n'_{s\bullet x,y}, n''_{x,w} = n''_{s\bullet x,y} \text{ if } sx < x$$

(by [5, 2.5(b)], we must have $s \bullet x \le y$).

Using the induction hypothesis we see that $n'_{x,w} = 1$ and

$$\begin{aligned} n''_{x,w} &= -n''_{1,x} n''_{1,y} \text{ if } sx > x, \\ n''_{x,w} &= n''_{1,s\bullet x} n''_{1,y} \text{ if } sx < x. \end{aligned}$$

Also, taking $x = 1$ we see that

(d) $$n''_{1,w} = -n''_{1,y}.$$

Returning to a general x we deduce

$$\begin{aligned} n''_{x,w} &= n''_{1,x} n''_{1,w} \text{ if } sx > x, \\ n''_{x,w} &= -n''_{1,s\bullet x} n''_{1,w} \text{ if } sx < x. \end{aligned}$$

Applying (d) with w replaced by x we see that $n''_{1,x} = -n''_{1,s\bullet x}$ if $sx < x$. This shows by induction on $l(x)$ that $n''_{1,x} = \kappa(x)$ for any $x \in \mathbf{I}_*$. Thus we have

$n''_{x,w} = n''_{1,x}n''_{1,w} = \kappa(x)\kappa(w)$ for any $x \le w$. This completes the inductive proof of (c) and that of (b). The proposition is proved. $\square$

4.6. We show:

(a) *For any $x, z \in \mathbf{I}_*$ such that $x \le z$ we have* $\sum_{y\in\mathbf{I}_*;x\le y\le z} \overline{r_{x,y}}r_{y,z} = \delta_{x,z}$.

Using the fact that $^- : uM \to \underline{M}$ is an involution we have

$$a'_z = \overline{\overline{a'_z}} = \overline{\sum_{y\in\mathbf{I}_*} \overline{r_{y,z}a'_y}} = \sum_{y\in\mathbf{I}_*} r_{y,z}\overline{a'_y} = \sum_{y\in\mathbf{I}_*}\sum_{x\in\mathbf{I}_*} r_{y,z}\overline{r_{x,y}}a'_x.$$

We now compare the coefficients of a'_x on both sides and use 4.4(a); (a) follows.

The following result provides the Möbius function for the partially ordered set $(\mathbf{I}_*, \le)$.

Proposition 4.7. *Let $x, z \in \mathbf{I}_*, x \le z$. Then* $\sum_{y\in\mathbf{I}_*;x\le y\le z} \kappa(x)\kappa(y) = \delta_{x,z}$.

We can assume that $x < z$. By 4.4(b), 4.5(b) for any $y \in \mathbf{I}_*$ such that $x \le y \le z$ we have

$$\overline{r_{x,y}}r_{y,z} = v^{l(y)-l(x)}v^{l(z)-l(x)}r''_{x,y}r'_{y,z} \in v^{l(z)-l(x)}(\kappa(x)\kappa(y) + v^{-2}\mathbf{Z}[v^{-2}]).$$

Hence the identity 4.6(a) implies that

$$\sum_{y\in\mathbf{I}_*;x\le y\le z} v^{l(z)-l(x)}\kappa(x)\kappa(y) + \text{ strictly lower powers of } v \text{ is } 0.$$

In particular, $\sum_{y\in\mathbf{I}_*;x\le y\le z} \kappa(x)\kappa(y) = 0$. The proposition is proved.

4.8. For any $w \in \mathbf{I}_*$ we have

(a) $$r_{w,w} = 1.$$

Indeed by 4.4(b) we have $r_{w,w} \in \mathbf{Z}[v^{-2}]$, $\overline{r_{w,w}} \in \mathbf{Z}[v^{-2}]$ hence $r_{w,w}$ is a constant. By 4.5(b) this constant is 1.

4.9. Let $w \in \mathbf{I}_*$. We will construct for any $x \in \mathbf{I}_*$ such that $x \le w$ an element $u_x \in \underline{\mathcal{A}}_{\le 0}$ such that

(a) $u_x = 1$,

(b) $u_x \in \underline{\mathcal{A}}_{<0}$, $\overline{u_x} - u_x = \sum_{y\in\mathbf{I}_*; x<y\le w} r_{x,y}u_y$ for any $x < w$.

The argument is almost a copy of one in [5, 5.2]. We argue by induction on $l(w) - l(x)$. If $l(w) - l(x) = 0$ then $x = w$ and we set $u_x = 1$. Assume now that $l(w) - l(x) > 0$ and that u_z is already defined whenever $z \le w$, $l(w) - l(z) < l(w) - l(x)$ so that (a) holds and (b) holds if x is replaced by any such z. Then the right hand side of the equality in (b) is defined. We denote it by $\alpha_x \in \underline{\mathcal{A}}$. We have

$$\begin{aligned}\alpha_x + \bar\alpha_x &= \sum_{y\in\mathbf{I}_*;x<y\le w} r_{x,y}u_y + \sum_{y\in\mathbf{I}_*;x<y\le w} \overline{r_{x,y}}\bar u_y \\ &= \sum_{y\in\mathbf{I}_*;x<y\le w} r_{x,y}u_y + \sum_{y\in\mathbf{I}_*;x<y\le w} \overline{r_{x,y}}(u_y + \sum_{z\in\mathbf{I}_*;y<z\le w} r_{y,z}u_z) \\ &= \sum_{y\in\mathbf{I}_*;x<y\le w} r_{x,y}u_y + \sum_{z\in\mathbf{I}_*;x<z\le w} \overline{r_{x,z}}u_z \\ &\quad + \sum_{z\in\mathbf{I}_*;x<z\le w}\sum_{y\in\mathbf{I}_*;x<y<z} \overline{r_{x,y}}r_{y,z}u_z \\ &= \sum_{z\in\mathbf{I}_*;x<z\le w}\sum_{y\in\mathbf{I}_*;x\le y<z} \overline{r_{x,y}}r_{y,z}u_z = \sum_{z\in\mathbf{I}_*;x<z\le w} \delta_{x,z}u_z = 0.\end{aligned}$$

(We have used 4.6(a), 4.8(a).) Since $\alpha_x + \bar\alpha_x = 0$ we have $\alpha_x = \sum_{n\in\mathbf{Z}} \gamma_n v^n$ (finite sum) where $\gamma_n \in \mathbf{Z}$ satisfy $\gamma_n + \gamma_{-n} = 0$ for all n and in particular $\gamma_0 = 0$. Then $u_x = -\sum_{n<0} \gamma_n v^n \in \underline{\mathcal{A}}_{<0}$ satisfies $\bar u_x - u_x = \alpha_x$. This completes the inductive construction of the elements u_x.

We set $A_w = \sum_{y\in\mathbf{I}_*;y\le w} u_y a'_y \in \underline{M}_{\le 0}$. We have

$$\begin{aligned}\overline{A_w} &= \sum_{y\in\mathbf{I}_*;y\le w} \bar u_y \overline{a'_y} = \sum_{y\in\mathbf{I}_*;y\le w} \bar u_y \sum_{x\in\mathbf{I}_*;x\le y} \overline{r_{x,y}}a'_x \\ &= \sum_{x\in\mathbf{I}_*;x\le w} (\sum_{y\in\mathbf{I}_*;x\le y\le w} \overline{r_{x,y}}\bar u_y)a'_x = \sum_{x\in\mathbf{I}_*;x\le w} u_x a'_x = A_w.\end{aligned}$$

We will also write $u_y = \pi_{y,w} \in \underline{\mathcal{A}}_{\le 0}$ so that

$$A_w = \sum_{y\in\mathbf{I}_*;y\le w} \pi_{y,w}a'_y.$$

Note that $\pi_{w,w}=1$, $\pi_{y,w}\in\underline{\mathcal{A}}_{<0}$ if $y<w$ and

$$\overline{\pi_{y,w}}=\sum_{z\in\mathbf{I}_*;y\le z\le w}r_{y,z}\pi_{z,w}.$$

We show that for any $x\in\mathbf{I}_*$ such that $x\le w$ we have:

(c) $v^{l(w)-l(x)}\pi_{x,w}\in\mathbf{Z}[v]$ *and has constant term* 1.

We argue by induction on $l(w)-l(x)$. If $l(w)-l(x)=0$ then $x=w$, $\pi_{x,w}=1$ and the result is obvious. Assume now that $l(w)-l(x)>0$. Using 4.4(b) and 4.5(b) and the induction hypothesis we see that

$$\sum_{y\in\mathbf{I}_*;x<y\le w}r_{x,y}\pi_{y,w}=\sum_{y\in\mathbf{I}_*;x<y\le w}v^{-l(y)+l(x)}\overline{r''_{x,y}}\pi_{y,w}$$

is equal to

$$\sum_{y\in\mathbf{I}_*;x<y\le w}v^{-l(y)+l(x)}\kappa(x)\kappa(y)v^{-l(w)+l(y)}=v^{-l(w)+l(x)}\sum_{y\in\mathbf{I}_*;x<y\le w}\kappa(x)\kappa(y)$$

plus strictly higher powers of v. Using 4.7, this is $-v^{-l(w)+l(x)}$ plus strictly higher powers of v. Thus,

$$\overline{\pi_{x,w}}-\pi_{x,w}=-v^{-l(w)+l(x)}+\text{ plus strictly higher powers of }v.$$

Since $\overline{\pi_{x,w}}\in v\mathbf{Z}[v]$, it is in particular a $\mathbf{Z}$-linear combination of powers of v strictly higher than $-l(w)+l(x)$. Hence

$$-\pi_{x,w}=-v^{-l(w)+l(x)}+\text{ plus strictly higher powers of }v.$$

This proves (c).

We now show that for any $x\in\mathbf{I}_*$ such that $x\le w$ we have:

(d) $$v^{l(w)-l(x)}\pi_{x,w}\in\mathbf{Z}[u,u^{-1}].$$

We argue by induction on $l(w)-l(x)$. If $l(w)-l(x)=0$ then $x=w$, $\pi_{x,w}=1$ and the result is obvious. Assume now that $l(w)-l(x)>0$. Using 4.4(b) and the induction hypothesis we see that

$$\sum_{y\in\mathbf{I}_*;x<y\le w}r_{x,y}\pi_{y,w}=\sum_{y\in\mathbf{I}_*;x<y\le w}v^{-l(y)+l(x)}\overline{r''_{x,y}}\pi_{y,w}$$

belongs to

$$\sum_{y \in \mathbf{I}_*; x < y \leq w} v^{-l(y)+l(x)} v^{-l(w)+l(y)} \mathbf{Z}[v^2, v^{-2}]$$

hence to $v^{-l(w)+l(x)}\mathbf{Z}[v^2, v^{-2}]$. Thus,

$$\overline{\pi_{x,w}} - \pi_{x,w} \in v^{-l(w)+l(x)} \mathbf{Z}[v^2, v^{-2}].$$

It follows that both $\overline{\pi_{x,w}}$ and $\pi_{x,w}$ belong to $v^{-l(w)+l(x)}\mathbf{Z}[v^2, v^{-2}]$. This proves (d).

Combining (c), (d) we see that for any $x \in \mathbf{I}_*$ such that $x \leq w$ we have:

(e) $v^{l(w)-l(x)}\pi_{x,w} = P^{\sigma}_{x,w}$ *where* $P^{\sigma}_{x,w} \in \mathbf{Z}[u]$ *has constant term* 1*.*

We have

$$A_w = v^{-l(w)} \sum_{y \in \mathbf{I}_*; y \leq w} P^{\sigma}_{y,w} a_y.$$

Also, $P^{\sigma}_{w,w} = 1$ and for any $y \in \mathbf{I}_*$, $y < w$, we have $\deg P^{\sigma}_{y,w} \leq (l(w) - l(y) - 1)/2$ (since $\pi_{y,w} \in \underline{\mathcal{A}}_{<0}$). Thus the existence statement in 0.4(a) is established. To prove the uniqueness statement in 0.4(a) it is enough to prove the following statement:

(f) *Let* $m, m' \in \underline{M}$ *be such that* $\bar{m} = m$, $\bar{m}' = m'$, $m - m' \in \underline{M}_{>0}$. *Then* $m = m'$.

The proof is entirely similar to that in [6, 3.2] (or that of [5, 5.2(e)]). The proof of 0.4(b) is immediate. This completes the proof of Theorem 0.4. □

The following result is a restatement of (e).

Proposition 4.10. *Let* $y, w \in \mathbf{I}_*$ *be such that* $y \leq w$. *The constant term of* $P^{\sigma}_{y,w} \in \mathbf{Z}[u]$ *is equal to* 1*.*

5. The Submodule $\underline{M}^K$ of $\underline{M}$

5.1. Let K be a subset of S which generates a finite subgroup W_K of W and let K^* be the image of K under $*$. For any (W_K, W_{K^*})-double coset Ω in W we denote by d_Ω (resp. b_Ω) the unique element of maximal (resp. minimal) length of Ω. Now $w \mapsto w^{*-1}$ maps any (W_K, W_{K^*})-double coset in W to a (W_K, W_{K^*})-double coset in W; let $\mathbf{I}_*^K$ be the set of (W_K, W_{K^*})-double cosets Ω in W such that Ω is stable under this map, or equivalently, such that $d_\Omega \in \mathbf{I}_*$, or such that $b_\Omega \in \mathbf{I}_*$. We set

$$\mathbf{P}_K = \sum_{x \in W_K} u^{l(x)} \in \mathbf{N}[u].$$

If in addition K is $*$-stable we set

$$\mathbf{P}_{H,*} = \sum_{x \in W_K, x^* = x} u^{l(x)} \in \mathbf{N}[u].$$

Lemma 5.2. *Let $\Omega \in \mathbf{I}_*^K$. Let $x \in \mathbf{I}_* \cap \Omega$ and let $b = b_\Omega$. Then there exists a sequence $x = x_0, x_1, \dots, x_n = b$ in $\mathbf{I}_* \cap \Omega$ and a sequence $s_1, s_2, \dots, s_n$ in S such that for any $i \in [1, n]$ we have $x_i = s_i \bullet x_{i-1}$.*

We argue by induction on $l(x)$ (which is $\geq l(b)$). If $l(x) = l(b)$ then $x = b$ and the result is obvious (with $n = 0$). Now assume that $l(x) > l(b)$. Let $H = K \cap (bK^*b^{-1})$. By 1.2(a) we have $x = cbzc^{*-1}$ where $c \in W_K$, $z \in W_{H^*}$ satisfies $bz = z^*b$ and $l(x) = l(c) + l(b) + l(z) + l(c)$. If $c \neq 1$ we write $c = sc'$, $s \in K, c' \in W_K$, $c' < c$ and we set $x_1 = c'bzc'^{*-1}$. We have $x_1 = sxs^* \in \Omega$, $l(x_1) < l(x)$. Using the induction hypothesis for x_1 we see that the desired result holds for x. Thus we can assume that $c = 1$ so that $x = bz$. Let $\tau : W_{H^*} \to W_{H^*}$ be the automorphism $y \mapsto b^{-1}y^*b$; note that $\tau(H^*) = H^*$ and $\tau^2 = 1$. We have $z \in \mathbf{I}_\tau$ where $\mathbf{I}_\tau := \{y \in W_{H^*}; \tau(y)^{-1} = y\}$.

Since $l(bz) > l(b)$ we have $z \neq 1$. We can find $s \in H^*$ such that $sz < z$.

If $sz = z\tau(s)$ then $sz \in \mathbf{I}_\tau$, $bsz \in \Omega$, $l(bsz) < l(bz)$. Using the induction hypothesis for bsz instead of x we see that the desired result holds for $x = bz$. (We have $bsz = tbz = bzt^*$ where $t = (\tau(s))^* \in H$.)

If $sz \neq z\tau(s)$ then $sz\tau(s) \in \mathbf{I}_t$, $bsz\tau(s) \in \Omega$, $l(bsz\tau(s)) < l(bz)$. Using the induction hypothesis for $bsz\tau(s)$ instead of x we see that the desired result holds for $x = bz$. (We have $bsz\tau(s) = tbzt^*$ where $t = (\tau(s))^* \in H$.) The lemma is proved. □

5.3. For any $\Omega \in \mathbf{I}_*^K$ we set

$$a_\Omega = \sum_{w \in \mathbf{I}_* \cap \Omega} a_w \in \underline{M}.$$

Let $\underline{M}^K$ be the $\underline{\mathcal{A}}$-submodule of $\underline{M}$ spanned by the elements $a_\Omega(\Omega \in \mathbf{I}_*^K)$. In other words, $\underline{M}^K$ consists of all $m = \sum_{w\in\mathbf{I}_*} m_w a_w \in \underline{M}$ such that the function $\mathbf{I}_* \to \underline{\mathcal{A}}$ given by $w \mapsto m_w$ is constant on $\mathbf{I}_* \cap \Omega$ for any $\Omega \in \mathbf{I}_*$.

Lemma 5.4. (a) *We have* $\underline{M}^K = \cap_{s\in K}\underline{M}^{\{s\}}$.

(b) *The* $\underline{\mathcal{A}}$*-submodule* $\underline{M}^K$ *is stable under* ${}^-\colon \underline{M} \to \underline{M}$.

(c) *Let* $\mathbf{S} = \sum_{x\in W_K} T_x \in \underline{\mathfrak{H}}$ *and let* $m \in \underline{M}$. *We have* $\mathbf{S}m \in \underline{M}^K$.

We prove (a). The fact that $\underline{M}^K \subset \underline{M}^{\{s\}}$ (for $s \in K$) follows from the fact that any (W_K, W_{K^*})-double coset in W is a union of $(W_{\{s\}}, W_{\{s^*\}})$-double cosets in W. Thus we have $\underline{M}^K \subset \cap_{s\in K}\underline{M}^{\{s\}}$. Conversely let $m \in \cap_{s\in K}\underline{M}^{\{s\}}$. We have $m = \sum_{w\in\mathbf{I}_*} m_w a_w \in \underline{M}$ where $m_w \in \underline{\mathcal{A}}$ is zero for all but finitely many w and we have $m_w = m_{s\bullet w}$ if $w \in \mathbf{I}_*, s \in K$. Using 5.2 we see that $m_x = m_{b_\Omega} = m_{x'}$ whenever $x, x' \in \mathbf{I}_*$ are in the same (W_K, W_{K^*})-double coset Ω in W. Thus, $m \in \underline{M}^K$. This proves (a).

We prove (b). Using (a), we can assume that $K = \{s\}$ with $s \in S$. By 1.3, if $\Omega \in \mathbf{I}_*^{\{s\}}$, then we have $\Omega = \{w, s \bullet w\}$ for some $w \in \mathbf{I}_*$ such that $sw > w$. Hence it is enough to show that for such w we have $\overline{a_w + a_{s\bullet w}} \in \underline{M}^{\{s\}}$. We have $\overline{a_w + a_{s\bullet w}} = \sum_{x\in\mathbf{I}_*} m_x a_x$ with $m_x \in \underline{\mathcal{A}}$ and we must show that $m_x = m_{s\bullet x}$ for any $x \in \mathbf{I}_*$. If we can show that $f\overline{a_w + a_{s\bullet w}} \in \underline{M}^{\{s\}}$ for some $f \in \underline{\mathcal{A}} - \{0\}$ then it would follow that for any $x \in \mathbf{I}_*$ we have $fm_x = fm_{s\bullet x}$ hence $m_x = m_{s\bullet x}$ as desired. Thus it is enough to show that

(d) $(u^{-1}+1)\overline{a_w + a_{sw}} \in \underline{M}^{\{s\}}$ if $w \in \mathbf{I}_*$ is such that $sw = ws^* > w$,

(e) $\overline{a_w + a_{sws^*}} \in \underline{M}^{\{s\}}$ if $w \in \mathbf{I}_*$ is such that $sw \neq ws^* > w$.

In the setup of (d) we have

$$\begin{aligned}(u^{-1}+1)\overline{a_w + a_{sw}} &= \overline{(u+1)(a_w + a_{sw})} = \overline{(T_s+1)a_w} = \overline{T_s+1}(\overline{a_w})\\ &= u^{-2}(T_s+1)\overline{a_w}\end{aligned}$$

(see 0.1(i)); in the setup of (e) we have

$$\overline{a_w + a_{sws^*}} = \overline{(T_s+1)a_w} = \overline{T_s+1}(\overline{a_w}) = u^{-2}(T_s+1)(\overline{a_w})$$

(see 0.1(iii)). Thus it is enough show that $(T_s+1)(\overline{a_w}) \in \underline{M}^{\{s\}}$ for any $w \in \mathbf{I}_*$. Since $\overline{a_w}$ is an $\underline{\mathcal{A}}$-linear combination of elements $a_x, x \in \mathbf{I}_*$ it is enough to show that $(T_s + 1)a_x \in \underline{M}^{\{s\}}$. This follows immediately from 0.1(i)-(iv).

We prove (c). Let $m' = \mathbf{S}m = \sum_{w \in \mathbf{I}_*} m'_w a_w$, $m'_w \in \underline{\mathcal{A}}$. For any $s \in K$ we have $\mathbf{S} = (T_s + 1)h$ for some $h \in \mathfrak{H}$ hence $m' \in (T_s + 1)\underline{M}$. This implies by the formulas 0.1(i)-(iv) that $m'_w = w'_{s \bullet w}$ for any $w \in \mathbf{I}_*$; in other words we have $m' \in \underline{M}^{\{s\}}$. Since this holds for any $s \in K$ we see, using (a), that $m' \in \underline{M}^K$. The lemma is proved. $\square$

5.5. For $\Omega, \Omega' \in \mathbf{I}_*^K$ we write $\Omega \le \Omega'$ when $d_\Omega \le d_{\Omega'}$. This is a partial order on $\mathbf{I}_*^K$. For any $\Omega \in \mathbf{I}_*^K$ we set

$$a'_\Omega = v^{-l(d_\Omega)} a_\Omega = \sum_{x \in \Omega \cap \mathbf{I}_*^K} v^{l(x) - l(d_\Omega)} a'_x.$$

Clearly, $\{a'_{\Omega'}; \Omega' \in \mathbf{I}_*^K\}$ is an $\underline{\mathcal{A}}$-basis of $\underline{M}^K$. Hence from 5.4(b) we see that

$$\overline{a'_\Omega} = \sum_{\Omega' \in \mathbf{I}_*^K} \overline{r_{\Omega',\Omega}} a'_{\Omega'}$$

where $r_{\Omega',\Omega} \in \underline{\mathcal{A}}$ is zero for all but finitely many Ω'. On the other hand we have

(a) $$\overline{a'_\Omega} = \sum_{x \in \Omega \cap \mathbf{I}_*, y \in \mathbf{I}_*; y \le x} v^{-l(x) + l(d_\Omega)} \overline{r_{y,x}} a'_y$$

hence

$$r_{\Omega',\Omega} = \sum_{x \in \Omega \cap \mathbf{I}_*; d_{\Omega'} \le x} v^{l(x) - l(d_\Omega)} r_{d_{\Omega'},x}$$

It follows that

(b) $$r_{\Omega,\Omega} = 1$$

(we use that $r_{d_\Omega,d_\Omega} = 1$) and

(c) $$r_{\Omega',\Omega} \ne 0 \implies \Omega' \le \Omega.$$

Indeed, if for some $x \in \Omega \cap \mathbf{I}_*$ we have $d_{\Omega'} \le x$, then $d_{\Omega'} \le d_\Omega$. We have

$$a'_\Omega = \overline{\overline{a'_\Omega}} = \overline{\sum_{\Omega' \in \mathbf{I}_*^K} \overline{r_{\Omega',\Omega}} a'_{\Omega'}} = \sum_{\Omega' \in \mathbf{I}_*^K} r_{\Omega',\Omega} \sum_{\Omega'' \in \mathbf{I}_*^K} \overline{r_{\Omega'',\Omega'}} a'_{\Omega''}.$$

Hence

(d) $$\sum_{\Omega' \in \mathbf{I}_*^K} \overline{r_{\Omega'',\Omega'}} r_{\Omega',\Omega} = \delta_{\Omega,\Omega''}$$

for any Ω, Ω'' in $\mathbf{I}_*^K$.

Note that

(e) $$a'_\Omega = a'_{d_\Omega} \operatorname{mod} \underline{M}_{<0}.$$

Indeed, if $x \in \Omega \cap \mathbf{I}_*^K$, $x \ne d_\Omega$ then $l(x) - l(d_\Omega) < 0$.

5.6. Let $\Omega \in \mathbf{I}_*^K$. We will construct for any $\Omega' \in \mathbf{I}_*^K$ such that $\Omega' \le \Omega$ an element $u_{\Omega'} \in \underline{\mathcal{A}}_{\le 0}$ such that

(a) $u_\Omega = 1$,

(b) $u_{\Omega'} \in \underline{\mathcal{A}}_{<0}$, $\overline{u_{\Omega'}} - u_{\Omega'} = \sum_{\Omega'' \in \mathbf{I}_*^K; \Omega' < \Omega'' \le \Omega} r_{\Omega',\Omega''} u_{\Omega''}$ for any $\Omega' < \Omega$.

The proof follows closely that in 4.9. We argue by induction on $l(d_\Omega) - l(d_{\Omega'})$. If $l(d_\Omega) - l(d_{\Omega'}) = 0$ then $\Omega = \Omega'$ and we set $u_{\Omega'} = 1$. Assume now that $l(d_\Omega) - l(d_{\Omega'}) > 0$ and that u_{Ω_1} is already defined whenever $\Omega_1 \le \Omega$, $l(d_\Omega) - l(d_{\Omega_1}) < l(d_\Omega) - l(d_{\Omega'})$ so that (a) holds and (b) holds if Ω' is replaced by any such Ω_1. Then the right hand side of the equality in (b) is defined. We denote it by $\alpha_{\Omega'} \in \underline{\mathcal{A}}$. We have $\alpha_{\Omega'} + \overline{\alpha_{\Omega'}} = 0$ by a computation like that in 4.9, but using 5.5(b),(c),(d). From this we see that $\alpha_{\Omega'} = \sum_{n \in \mathbf{Z}} \gamma_n v^n$ (finite sum) where $\gamma_n \in \mathbf{Z}$ satisfy $\gamma_n + \gamma_{-n} = 0$ for all n and in particular $\gamma_0 = 0$. Then $u_{\Omega'} = -\sum_{n<0} \gamma_n v^n \in \underline{\mathcal{A}}_{<0}$ satisfies $\overline{u_{\Omega'}} - u_{\Omega'} = \alpha_{\Omega'}$. This completes the inductive construction of the elements $u_{\Omega'}$.

We set $A_\Omega = \sum_{\Omega' \in \mathbf{I}_*^K; \Omega' \le \Omega} u_{\Omega'} a'_{\Omega'} \in \underline{M}_{\le 0} \cap \underline{M}^K$. We have

(c) $$\overline{A_\Omega} = A_\Omega.$$

(This follows from (b) as in the proof of the analogous equality $\overline{A_w} = A_w$ in 4.9.) We will also write $u_{\Omega'} = \pi_{\Omega',\Omega} \in \underline{\mathcal{A}}_{\le 0}$ so that

$$A_\Omega = \sum_{\Omega' \in \mathbf{I}_*^K; \Omega' \le \Omega} \pi_{\Omega',\Omega} a'_{\Omega'}.$$

We show

(d) $$A_\Omega - A_{d_\Omega} \in \underline{M}_{<0}.$$

Using 5.5(a) and $\pi_{\Omega',\Omega} \in \underline{\mathcal{A}}_{<0}$ (for $\Omega' < \Omega$) we see that $A_\Omega = a'_{d_\Omega} \operatorname{mod} \underline{M}_{<0}$; it remains to use that $A_{d_\Omega} = a'_{d_\Omega} \operatorname{mod} \underline{M}_{<0}$.

Applying 4.9(f) to $m = A_\Omega$, $m' = A_{d_\Omega}$ (we use (c),(d)) we deduce:

(e) $$A_\Omega = A_{d_\Omega}.$$

In particular,

(f) *For any* $\Omega \in \mathbf{I}_*^K$ *we have* $A_{d_\Omega} \in \underline{M}^K$.

5.7. We define an $\mathcal{A}$-linear map $\zeta : M \to \mathbf{Q}(u)$ by $\zeta(a_w) = u^{l(w)}(\frac{u-1}{u+1})^{\phi(w)}$ (see 4.5(a)) for $w \in \mathbf{I}_*$. We show:

(a) *For any* $x \in W, m \in M$ *we have* $\zeta(T_x m) = u^{2l(x)}\zeta(m)$.

We can assume that $x = s, m = a_w$ where $s \in S, w \in \mathbf{I}_*$. Then we are in one of the four cases (i)-(iv) in 0.1. We set $n = l(w)$, $d = \phi(w)$, $\lambda = \frac{u-1}{u+1}$. The identities to be checked in the cases 0.1(i)-(iv) are:

$$u^2 u^n \lambda^d = u u^n \lambda^d + (u+1) u^{n+1} \lambda^{d+1},$$

$$u^2 u^n \lambda^d = (u^2 - u - 1) u^n \lambda^d + (u^2 - u) u^{n-1} \lambda^{d-1},$$

$$u^2 u^n \lambda^d = u^{n+2} \lambda^d,$$

$$u^2 u^n \lambda^d = (u^2 - 1) u^n \lambda^d + u^2 u^{n-2} \lambda^d,$$

respectively. These are easily verified.

5.8. Assuming that $K^*=K$, we set

$$\mathcal{R}_{K,*}=\sum_{y\in W_K;y^*=y^{-1}} u^{l(y)}(\frac{u-1}{u+1})^{\phi(y)}\in\mathbf{Q}(u).$$

Let $\Omega\in\mathbf{I}_*^K$. Define b,H,τ as in 5.2. Let

$$W_K^H=\{c\in W_K;l(w)\le l(wr)\text{ for any }r\in W_H\}.$$

Using 1.2(a) we have $\sum_{w\in\Omega\cap\mathbf{I}_*}\zeta(a_w)=\sum_{c\in W_K^H}u^{2l(c)}\zeta(a_b)\mathcal{R}_{H^*,\tau}(u)$ hence

(a) $$\sum_{w\in\Omega\cap\mathbf{I}_*}\zeta(a_w)=\mathbf{P}_K(u^2)\mathbf{P}_H(u^2)^{-1}\zeta(a_b)\mathcal{R}_{H^*,\tau}(u).$$

We have the following result.

Proposition 5.9. *Assume that W is finite. We have*

(a) $$\mathcal{R}_{S,*}(u)=\mathbf{P}_S(u^2)\mathbf{P}_{S,*}(u)^{-1}.$$

We can assume that W is irreducible. We prove (a) by induction on $|S|$. If $|S|\le 2$, (a) is easily checked. Now assume that $|S|\ge 3$. Taking sum over all $\Omega\in\mathbf{I}_*^K$ in 5.7(a) we obtain

$$\mathcal{R}_{S,*}(u)=\mathbf{P}_K(u^2)\sum_{\Omega\in\mathbf{I}_*^K}\mathbf{P}_H(u^2)^{-1}\zeta(a_b)\mathcal{R}_{H^*,\tau}(u)$$

where b,H,τ depend on Ω as in 5.2. Using the induction hypothesis we obtain

$$\mathcal{R}_{S,*}(u)=\mathbf{P}_K(u^2)\sum_{\Omega\in\mathbf{I}_*^K}\zeta(a_b)\mathbf{P}_{H^*,\tau}(u)^{-1}.$$

We now choose $K\subset S$ so that W_K is of type

$$A_{n-1},B_{n-1},D_{n-1},A_1,B_3,A_5,D_7,E_7,I_2(5),H_3$$

where W is of type

$$A_n,B_n,D_n,G_2,F_4,E_6,E_7,E_8,H_3,H_4$$

respectively. Then there are few (W_K, W_{K^*}) double cosets and the sum above can be computed in each case and gives the desired result. (In the case where W is a Weyl group, there is an alternative, uniform, proof of (a) using flag manifolds over a finite field.)

5.10. We return to the general case. Let $\Omega \in \mathbf{I}_*^K$ and let b, H, τ be as in 5.2. By 5.4(c) we have $\mathbf{S}a_b \in \underline{M}^K$. From 0.1(i)-(iv) we see that $\mathbf{S}a_b = \sum_{y\in\Omega\cap\mathbf{I}_*} f_y a_y$ where $f_y \in \mathbf{Z}[u]$ for all y. Hence we must have $\mathbf{S}a_b = f a_\Omega$ for some $f \in \mathbf{Z}[u]$. Appplying ζ to the last equality and using 5.7(a) we obtain $\mathbf{P}_K(u^2)\zeta(a_b) = f \sum_{y\in\Omega\cap\mathbf{I}_*} \zeta(a_y)$. From 5.8(a), 5.9(a) we have

$$\sum_{y\in\Omega\cap\mathbf{I}_*} \zeta(a_y) = \mathbf{P}_K(u^2)\zeta(a_b)\mathbf{P}_{H^*,\tau}(u)^{-1}$$

where b, H, τ depend on Ω as in 5.8. Thus $f = \mathbf{P}_{H^*,\tau}(u)$. We see that

(a) $$\mathbf{S}a_b = \mathbf{P}_{H^*,\tau}(u)a_\Omega.$$

5.11. In this subsection we assume that $K^* = K$. Then $\Omega := W_K \in \mathbf{I}_*^K$. We have the following result.

(a) $$A_\Omega = v^{-l(w_K)}a_\Omega.$$

By 5.6(f) we have $A_\Omega = f a_\Omega$ for some $f \in \underline{\mathcal{A}}$. Taking the coefficient of a_{w_K} in both sides we get $f = v^{-l(w_K)}$ proving (a).

Here is another proof of (a). It is enough to prove that $v^{-l(w_K)}a_\Omega$ is fixed by $\bar{}$. By 5.10(a) we have $u^{-l(w_K)}\mathbf{S}a_1 = u^{-l(w_K)}\mathbf{P}_{K,*}(u)a_\Omega$. The left hand side of this equality is fixed by $\bar{}$ since a_1 and $u^{-l(w_K)}\mathbf{S}$ are fixed by $\bar{}$. Hence $v^{-2l(w_K)}\mathbf{P}_{K,*}(u)a_\Omega$ is fixed by $\bar{}$. Since $v^{-l(w_K)}\mathbf{P}_{K,*}(u)$ is fixed by $\bar{}$ and is nonzero, it follows that $v^{-l(w_K)}a_\Omega$ is fixed by $\bar{}$, as desired.

6. The action of $u^{-1}(T_s + 1)$ in the basis (A_w)

6.1. In this section we fix $s \in S$.

Let $y, w \in \mathbf{I}_*$. When $y \le w$ we have as in 4.9, $\pi_{y,w} = v^{-l(w)+l(y)}P^\sigma_{y,w}$ so that $\pi_{y,w} \in \underline{\mathcal{A}}_{<0}$ if $y < w$ and $\pi_{w,w} = 1$; when $y \not\le w$ we set $\pi_{y,w} = 0$. In any case we set as in [6, 4.1]:

(a) $\pi_{y,w} = \delta_{y,w} + \mu'_{y,w} v^{-1} + \mu''_{y,w} v^{-2} \bmod v^{-3}\mathbf{Z}[v^{-1}]$

where $\mu'_{y,w} \in \mathbf{Z}, \mu''_{y,w} \in \mathbf{Z}$. Note that

(b) $\mu'_{y,w} \neq 0 \implies y < w, \epsilon_y = -\epsilon_w$,

(c) $\mu''_{y,w} \neq 0 \implies y < w, \epsilon_y = \epsilon_w$.

6.2. As in [6, 4.3], for any $y, w \in \mathbf{I}_*$ such that $sy < y < sw > w$ we define $\mathcal{M}^s_{y,w} \in \underline{\mathcal{A}}$ by:

$$\mathcal{M}^s_{y,w} = \mu''_{y,w} - \sum_{x \in \mathbf{I}_*; y < x < w, sx < x} \mu'_{y,x}\mu'_{x,w} - \delta_{sw,ws^*}\mu'_{y,sw} + \mu'_{sy,w}\delta_{sy,ys^*}$$

if $\epsilon_y = \epsilon_w$,

$$\mathcal{M}^s_{y,w} = \mu'_{y,w}(v + v^{-1})$$

if $\epsilon_y = -\epsilon_w$.

The following result was proved in [6, 4.4] assuming that W is a Weyl group or affine Weyl group. (We set $c_s = u^{-1}(T_s + 1) \in \underline{\mathfrak{H}}$.)

Theorem 6.3. *Let $w \in \mathbf{I}_*$.*

(a) *If $sw = ws^* > w$ then $c_s A_w = (v + v^{-1})A_{sw} + \sum_{z \in \mathbf{I}_*; sz < z < sw} \mathcal{M}^s_{z,w} A_z$.*

(b) *If $sw \neq ws^* > w$ then $c_s A_w = A_{sws^*} + \sum_{z \in \mathbf{I}_*; sz < z < sws^*} \mathcal{M}^s_{z,w} A_z$.*

(c) *If $sw < w$ then $c_s A_w = (u + u^{-1})A_w$.*

(In the case considered in [6, 4.4] the last sum in the formula which corresponds to (b) involves $sz < z < sw$ instead of $sz < z < sws^*$; but as shown in *loc.cit.* the two conditions are equivalent.)

We prove (c). We have $sw < w$. By 5.6(f) we have $A_w \in \underline{M}^{\{s\}}$. Hence it is enough to show that $c_s m = (u + u^{-1})m$ where m runs through a set of generators of the $\underline{\mathcal{A}}$-module $\underline{M}^{\{s\}}$. Thus it is enough to show that $c_s(a_x + a_{s\bullet x}) = (u + u^{-1})(a_x + a_{s\bullet x})$ for any $x \in \mathbf{I}_*$. This follows immediately from 0.1(i)-(iv).

Now the proof of (a),(b) (assuming (c)) is exactly as in [6, 4.4]. (Note that in [6, 3.3], (c) was proved (in the Weyl group case) by an argument (based on geometry via [6, 3.4]) which is not available in our case and which we have replaced by the analysis in §5.)

7. An inversion formula

7.1. In this section we assume that W is finite. Let $\underline{\hat{M}} = \mathrm{Hom}_{\underline{\mathcal{A}}}(\underline{M}, \underline{\mathcal{A}})$. For any $w \in \mathbf{I}_*$ we define $\hat{a}'_w \in \underline{\hat{M}}$ by $\hat{a}'_w(a'_y) = \delta_{y,w}$ for any $y \in \mathbf{I}_*$. Then $\{\hat{a}'_w; w \in \mathbf{I}_*\}$ is an $\underline{\mathcal{A}}$-basis of $\underline{\hat{M}}$. We define an $\underline{\mathfrak{H}}$-module structure on $\underline{\hat{M}}$ by $(hf)(m) = f(h^\flat m)$ (with $f \in \underline{\hat{M}}$, $m \in \underline{M}$, $h \in \underline{\mathfrak{H}}$) where $h \mapsto h^\flat$ is the algebra antiautomorphism of $\underline{\mathfrak{H}}$ such that $T'_s \mapsto T'_s$ for all $s \in S$. (Recall that $T'_s = u^{-1}T_s$.) We define a bar operator $^-\colon \underline{\hat{M}} \to \underline{\hat{M}}$ by $\bar{f}(m) = \overline{f(\bar{m})}$ (with $f \in \underline{\hat{M}}$, $m \in \underline{M}$); in $\overline{f(\bar{m})}$ the lower bar is that of $\underline{M}$ and the upper bar is that of $\underline{\mathcal{A}}$. We have $\overline{hf} = \bar{h}\bar{f}$ for $f \in \underline{\hat{M}}$, $h \in \underline{\mathfrak{H}}$.

Let $\diamond : W \to W$ be the involution $x \mapsto w_S x^* w_S = (w_S x w_S)^*$ which leaves S stable. We have $\mathbf{I}_\diamond = w_S \mathbf{I}_* = \mathbf{I}_* w_S$. We define the $\underline{\mathcal{A}}$-module $\underline{M}_\diamond$ and its basis $\{b'_z; z \in \mathbf{I}_\diamond\}$ in terms of $\diamond$ in the same way as $\underline{M}$ and its basis $\{a'_w; w \in \mathbf{I}_*\}$ were defined in terms of $*$. Note that $\underline{M}_\diamond$ has an $\underline{\mathfrak{H}}$-module structure and a bar operator $^-\colon \underline{M}_\diamond \to \underline{M}_\diamond$ analogous to those of $\underline{M}$.

We define an isomorphism of $\underline{\mathcal{A}}$-modules $\Phi : \underline{\hat{M}} \to \underline{M}_\diamond$ by $\Phi(\hat{a}'_w) = \kappa(w) b'_{ww_S}$. Here $\kappa(w)$ is as in 4.5(a). Let $h \mapsto h^\dagger$ be the algebra automorphism of $\underline{\mathfrak{H}}$ such that $T'_s \mapsto -T'^{-1}_s$ for any $s \in S$. We have the following result.

Lemma 7.2. *For any $f \in \underline{\hat{M}}$, $h \in \underline{\mathfrak{H}}$ we have $\Phi(hf) = h^\dagger \Phi(f)$.*

It is enough to show this when h runs through a set of algebra generators of $\underline{\mathfrak{H}}$ and f runs through a basis of $\underline{\hat{M}}$. Thus it is enough to show for any $w \in \mathbf{I}_*, s \in S$ that $\Phi(T_s \hat{a}'_w) = -T_s^{-1}\Phi(\hat{a}'_w)$ or that

(a) $\Phi(T_s \hat{a}'_w) = -\kappa(w) T_s^{-1} b'_{ww_S}$.

We write the formulas in 4.1 with $*$ replaced by $\diamond$ and a'_w replaced by b'_{ww_S}:

$$\begin{aligned}
T'_s b'_{ww_S} &= b'_{ww_S} + (v+v^{-1}) b'_{sww_S} \text{ if } sw = ws^* < w,\\
T'_s b'_{ww_S} &= (u-1-u^{-1}) b'_{ww_S} + (v-v^{-1}) b'_{sww_S} \text{ if } sw = ws^* > w,\\
T'_s b'_{ww_S} &= b'_{sws^*w_S} \text{ if } sw \neq ws^* < w,\\
T'_s b'_{ww_S} &= (u-u^{-1}) b'_{ww_S} + b'_{sws^*w_S} \text{ if } sw \neq ws^* > w.
\end{aligned}$$

Since $T_s'^{-1} = T_s' + u^{-1} - u$ we see that

$$
\begin{aligned}
\text{(b)}\ -T_s'^{-1}b'_{ww_S} &= -(u^{-1}+1-u)b'_{ww_S} - (v+v^{-1})b'_{sww_S} \text{ if } sw = ws^* < w\\
-T_s'^{-1}b'_{ww_S} &= b'_{ww_S} - (v-v^{-1})b'_{sww_S} \text{ if } sw = ws^* > w\\
-T_s'^{-1}b'_{ww_S} &= -(u^{-1}-u)b'_{ww_S} - b'_{sws^*w_S} \text{ if } sw \neq ws^* < w\\
-T_s'^{-1}b'_{ww_S} &= -b'_{sws^*w_S} \text{ if } sw \neq ws^* > w
\end{aligned}
$$

Using again the formulas in 4.1 for $T_s'a_y'$ we see that for $y, w \in \mathbf{I}_*$ we have

$$
\begin{aligned}
&(T_s'\hat{a}_w')(a_y) = \hat{a}_w'(T_s'a_y)\\
&= \delta_{sy=ys^*>y}\delta_{y,w} + \delta_{sy=ys^*>y}\delta_{sy,w}(v+v^{-1}) + \delta_{sy=ys^*<y}\delta_{y,w}(u-1-u^{-1})\\
&\quad +\delta_{sy=ys^*<y}\delta_{sy,w}(v-v^{-1}) + \delta_{sy\neq ys^*>y}\delta_{sys^*,w} + \delta_{sy\neq ys^*<y}\delta_{y,w}(u-u^{-1})\\
&\quad +\delta_{sy\neq ys^*<y}\delta_{sys^*,w}\\
&= \delta_{sw=ws^*>w}\delta_{y,w} + \delta_{sw=ws^*<w}\delta_{y,sw}(v+v^{-1}) + \delta_{sw=ws^*<w}\delta_{y,w}(u-1-u^{-1})\\
&\quad +\delta_{sw=ws^*>w}\delta_{y,sw}(v-v^{-1}) + \delta_{sw\neq ws^*<w}\delta_{y,sws^*}\\
&\quad +\delta_{sw\neq ws^*<w}\delta_{y,w}(u-u^{-1}) + \delta_{sw\neq ws^*>w}\delta_{y,sws^*}\\
&= (\delta_{sw=ws^*>w}\hat{a}_w' + \delta_{sw=ws^*<w}(v+v^{-1})\hat{a}_{sw}' + \delta_{sw=ws^*<w}(u-1-u^{-1})\hat{a}_w'\\
&\quad +\delta_{sw=ws^*>w}(v-v^{-1})\hat{a}_{sw}' + \delta_{sw\neq ws^*<w}\hat{a}_{sws^*}'\\
&\quad +\delta_{sw\neq ws^*<w}(u-u^{-1})\hat{a}_w' + \delta_{sw\neq ws^*>w}\hat{a}_{sws^*}')(a_y).
\end{aligned}
$$

Since this holds for any $y \in \mathbf{I}_*$ we see that

$$
\begin{aligned}
T_s'\hat{a}_w' &= \delta_{sw=ws^*>w}\hat{a}_w' + \delta_{sw=ws^*<w}(v+v^{-1})\hat{a}_{sw}'\\
&\quad +\delta_{sw=ws^*<w}(u-1-u^{-1})\hat{a}_w'\\
&\quad +\delta_{sw=ws^*>w}(v-v^{-1})\hat{a}_{sw}' + \delta_{sw\neq ws^*<w}\hat{a}_{sws^*}'\\
&\quad +\delta_{sw\neq ws^*<w}(u-u^{-1})\hat{a}_w' + \delta_{sw\neq ws^*>w}\hat{a}_{sws^*}'.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
T_s'\hat{a}_w' &= \hat{a}_w' + (v-v^{-1})\hat{a}_{sw}' \text{ if } sw = ws^* > w,\\
T_s'\hat{a}_w' &= (u-1-u^{-1})\hat{a}_w' + (v+v^{-1})\hat{a}_{sw}' \text{ if } sw = ws^* < w,\\
T_s'\hat{a}_w' &= \hat{a}_{sws^*}' \text{ if } sw \neq ws^* > w,\\
T_s'\hat{a}_w' &= (u-u^{-1})\hat{a}_w' + \hat{a}_{sws^*}' \text{ if } sw \neq ws^* < w.
\end{aligned}
$$

so that

(c) $\Phi(T'_s\hat{a}'_w) = \kappa(w)b'_{ww_S} + (v - v^{-1})\kappa(sw)b'_{sww_S}$ if $sw = ws^* > w$
$\Phi(T'_s\hat{a}'_w) = (u-1-u^{-1})\kappa(w)b'_{ww_S} + (v+v^{-1})\kappa(sw)b'_{sww_S}$ if $sw = ws^* < w$
$\Phi(T'_s\hat{a}'_w) = \kappa(sws^*)b'_{sws^*w_S}$ if $sw \neq ws^* > w$
$\Phi(T'_s\hat{a}'_w) = (u - u^{-1})\kappa(w)b'_w + \kappa(sws^*)b'_{sws^*w_S}$ if $sw \neq ws^* < w$.

From (b), (c) we see that to prove (a) we must show:

$$\kappa(w)b'_{ww_S} + (v - v^{-1})\kappa(sw)b'_{sww_S}$$
$$= \kappa(w)b'_{ww_S} - \kappa(w)(v - v^{-1})b'_{sww_S} \text{ if } sw = ws^* > w,$$
$$(u - 1 - u^{-1})\kappa(w)b'_{ww_S} + (v + v^{-1})\kappa(sw)b'_{sww_S}$$
$$= -\kappa(w)(u^{-1} + 1 - u)b'_{ww_S} - \kappa(w)(v + v^{-1})b'_{sww_S} \text{ if } sw = ws^* < w,$$
$$\kappa(sws^*)b'_{sws^*w_S} = -\kappa(w)b'_{sws^*w_S} \text{ if } sw \neq ws^* > w,$$
$$(u - u^{-1})\kappa(w)b'_w + \kappa(sws^*)b'_{sws^*w_S}$$
$$= -\kappa(w)(u^{-1} - u)b'_{ww_S} - \kappa(w)b'_{sws^*w_S} \text{ if } sw \neq ws^* < w.$$

This is obvious. The lemma is proved. □

Lemma 7.3. *We define a map $B : \underline{\hat{M}} \to \underline{\hat{M}}$ by $B(f) = \Phi^{-1}(\overline{\Phi(f)})$ where the bar refers to $\underline{M}_\diamond$. We have $B(f) = \bar{f}$ for all $f \in \underline{\hat{M}}$.*

We show that

(a) $B(hf) = \bar{h}B(f)$

for all $h \in \underline{\mathfrak{H}}, f \in \underline{\hat{M}}$. This is equivalent to $\Phi^{-1}(\overline{\Phi(hf)}) = \bar{h}\Phi^{-1}(\overline{\Phi(f)})$ or (using 7.2) to $\overline{h^\dagger\Phi(f)} = \Phi(\bar{h}\Phi^{-1}(\overline{\Phi(f)}))$ or (using 7.2) to $\overline{h^\dagger}(\overline{\Phi(f)}) = (\bar{h})^\dagger\Phi(\Phi^{-1}(\overline{\Phi(f)}))$; it remains to use that $\overline{h^\dagger} = (\bar{h})^\dagger$.

Next we show that

(b) $B(\hat{a}'_{w_S}) = \hat{a}'_{w_S}$.

Indeed the left hand side is

$$\Phi^{-1}(\overline{\Phi(\hat{a}'_{w_S})}) = \Phi^{-1}(\overline{\kappa(w_S)b'_1}) = \kappa(w_S)\Phi^{-1}(b'_1) = \hat{a}'_{w_S}$$

as required. (We have used that $\overline{b'_1} = b'_1$ in $\underline{M}_\diamond$.) Next we show:

(c) $\overline{\hat{a}'_{w_S}} = \hat{a}'_{w_S}$.

Indeed for $y \in \mathbf{I}_*$ we have

$$\overline{\hat{a}'_{w_S}}(a'_y) = \overline{\hat{a}'_{w_S}(\overline{a'_y})} = \overline{\hat{a}'_{w_S}(\sum_{x\in\mathbf{I}_*;x\le y} \bar{r}_{x,y}a'_x)} = \overline{\bar{r}_{w_S,w_S}\delta_{y,w_S}} = \delta_{y,w_S} = \hat{a}'_{w_S}(a'_y)$$

(we use that $r_{w_S,w_S} = 1$). This proves (c).

Since $\overline{hf} = \bar{h}\bar{f}$ for all $h \in \underline{\mathfrak{H}}, f \in \underline{\hat{M}}$ we see (using (a),(b),(c)) that the map $f \mapsto \overline{B(f)}$ from $\underline{\hat{M}}$ into itself is $\underline{\mathfrak{H}}$-linear and carries $\hat{a}'_{w_S}$ to itself. This implies that this map is the identity. (It is enough to show that $\hat{a}'_{w_S}$ generates the $\underline{\mathfrak{H}}$-module $\underline{\hat{M}}$ after extending scalars to $\mathbf{Q}(v)$. Using 7.2 it is enough to show that b'_1 generates the $\underline{\mathfrak{H}}$-module $\underline{M}_\diamond$ after extending scalars to $\mathbf{Q}(v)$. This is known from 2.11.) We see that $f = \overline{B(f)}$ for all $f \in \underline{\hat{M}}$. Applying $\bar{}$ to both sides (an involution of $\underline{\hat{M}}$) we deduce that $\bar{f} = B(f)$ for all $f \in \underline{\hat{M}}$. The lemma is proved.

7.4. Recall that $\overline{a'_w} = \sum_{y\in\mathbf{I}_*;y\le w} \overline{r_{y,w}} a'_y$ for $w \in \mathbf{I}_*$. The analogous equality in $\underline{M}_\diamond$ is

(a) $$\overline{b'_z} = \sum_{x\in\mathbf{I}_\diamond;x\le z} \overline{r^\diamond_{x,z}} b'_x \text{ for } x \in \mathbf{I}_\diamond.$$

Here $r^\diamond_{x,z} \in \underline{\mathcal{A}}$. We have the following result.

Proposition 7.5. *Let $y, w \in \mathbf{I}_*$ be such that $y \le w$. We have*

$$\overline{r_{y,w}} = \kappa(y)\kappa(w) r^\diamond_{ww_S,yw_S}.$$

We show that for any $y \in \mathbf{I}_*$ we have

(a) $$\overline{\hat{a}'_y} = \sum_{w\in\mathbf{I}_*;y\le w} r_{y,w}\hat{a}'_w.$$

Indeed for any $x \in \mathbf{I}_*$ we have

$$\begin{aligned}\overline{\hat{a}'_y}(a'_x) &= \overline{\hat{a}'_y(\overline{a'_x})} = \overline{\hat{a}'_y(\sum_{x'\in\mathbf{I}_*;x'\le x} \bar{r}_{x',x}a'_{x'})} = \overline{\delta_{y\le x}\bar{r}_{y,x}} = \delta_{y\le x} r_{y,x} \\ &= \sum_{w\in\mathbf{I}_*;y\le w} r_{y,w}\hat{a}'_w(a'_x).\end{aligned}$$

Using (a) and 7.3 we see that for any $y \in \mathbf{I}_*$ we have

$$\Phi^{-1}(\overline{\Phi(\hat{a}'_y)}) = \sum_{w \in \mathbf{I}_*; y \le w} r_{y,w} \hat{a}'_w.$$

It follows that $\overline{\Phi(\hat{a}'_y)} = \sum_{w \in \mathbf{I}_*; y \le w} r_{y,w} \Phi(\hat{a}'_w)$ that is,

$$\overline{\kappa(y) b'_{yw_S}} = \sum_{w \in \mathbf{I}_*; y \le w} r_{y,w} \kappa(w) b'_{ww_S}.$$

Using 7.4(a) to compute the left hand side we obtain

$$\kappa(y) \sum_{w \in \mathbf{I}_*; ww_S \le yw_S} \overline{r^\diamond_{ww_S, yw_S}} b'_{ww_S} = \sum_{w \in \mathbf{I}_*; y \le w} r_{y,w} \kappa(w) b'_{ww_S}.$$

Hence for any $w \in \mathbf{I}_*$ such that $y \le w$ we have $r_{y,w}\kappa(w) = \kappa(y)\overline{r^\diamond_{ww_S,yw_S}}$. The proposition follows.

7.6. Recall that for $y, w \in \mathbf{I}_*$, $y \le w$ we have $P^\sigma_{y,w} = v^{l(w)-l(y)} \pi_{y,w}$ where $\pi_{y,w} \in \underline{\mathcal{A}}$ satisfies $\pi_{w,w} = 1$, $\pi_{y,w} \in \underline{\mathcal{A}}_{<0}$ if $y < w$ and

(a)
$$\overline{\pi_{y,w}} = \sum_{t \in \mathbf{I}_*; y \le t \le w} r_{y,t} \pi_{t,w}.$$

Replacing $*$ by $\diamond$ in the definition of $P^\sigma_{y,w}$ we obtain polynomials $P^{\sigma,\diamond}_{x,z} \in \mathbf{Z}[u]$ ($x, z \in \mathbf{I}_\diamond, x \le z$) such that $P^{\sigma,\diamond}_{x,z} = v^{l(z)-l(x)} \pi^\diamond_{x,z}$ where $\pi^\diamond_{x,z} \in \underline{\mathcal{A}}$ satisfies $\pi^\diamond_{z,z} = 1$, $\pi^\diamond_{x,z} \in \underline{\mathcal{A}}_{<0}$ if $x < z$ and

(b)
$$\overline{\pi^\diamond_{x,z}} = \sum_{t' \in \mathbf{I}_\diamond; x \le t' \le z} r^\diamond_{x,t'} \pi^\diamond_{t',z}.$$

The following inversion formula (and its proof) is in the same spirit as [2, 3.1] (see also [7]).

Theorem 7.7. *For any $y, w \in \mathbf{I}_*$ such that $y \le w$ we have*

$$\sum_{t \in \mathbf{I}_*; y \le t \le w} \kappa(y)\kappa(t) P^\sigma_{y,t} P^{\sigma,\diamond}_{ww_S, tw_S} = \delta_{y,w}.$$

The last equality is equivalent to

$$\text{(a)} \qquad \sum_{t\in \mathbf{I}_*; y\le t\le w} \kappa(y)\kappa(t)\pi_{y,t}\pi^{\diamond}_{ww_S,tw_S} = \delta_{y,w}.$$

Let $M_{y,w}$ be the left hand side of (a). When $y = w$ we have $M_{y,w} = 1$. Thus, we may assume that $y < w$ and that $M_{y',w'} = 0$ for all $y', w' \in \mathbf{I}_*$ such that $y' < w'$, $l(w') - l(y') < l(w) - l(y)$. Using 7.6(a),(b) we have

$$\begin{aligned} M_{y,w} &= \sum_{t\in \mathbf{I}_*; y\le t\le w} \kappa(y)\kappa(t) \sum_{x,x'\in \mathbf{I}_*; y\le x\le t\le x'\le w} \overline{r_{y,x}p_{x,t}}\overline{r^{\diamond}_{ww_S,x'w_S}p^{\diamond}_{x'w_S,tw_S}} \\ &= \sum_{x,x'\in \mathbf{I}_*; y\le x\le x'\le w} \kappa(y)\kappa(x)\overline{r_{y,x}}\overline{r^{\diamond}_{ww_S,x'w_S}M_{x,x'}}. \end{aligned}$$

The only x, x' which can contribute to the last sum satisfy $x = x'$ or $x = y, x' = w$. Thus

$$M_{y,w} = \sum_{x\in \mathbf{I}_*; y\le x\le w} \kappa(y)\kappa(x)\overline{r_{y,x}}\overline{r^{\diamond}_{ww_S,xw_S}} + \overline{M_{y,w}}.$$

(We have used 4.8(a).) Using 7.5 we see that the last sum over x is equal to

$$\kappa(y)\kappa(w) \sum_{x\in \mathbf{I}_*; y\le x\le w} \overline{r_{y,x}}r_{x,w} = 0,$$

see 4.6(a). Thus we have $M_{y,w} = \overline{M_{y,w}}$. Since $M_{y,w} \in \underline{\mathcal{A}}_{<0}$, this forces $M_{y,w} = 0$. The theorem is proved.

8. A $(-u)$ Analogue of Weight Multiplicities?

8.1. In this section we assume that W is an irreducible affine Weyl group. An element $x \in W$ is said to be a translation if its W-conjugacy class is finite. The set of translations is a normal subgroup $\mathcal{T}$ of W of finite index. We fix an element $s_0 \in S$ such that, setting $K = S - \{s_0\}$, the obvious map $W_K \to W/\mathcal{T}$ is an isomorphism. (Such an s_0 exists.) We assume that $*$ is the automorphism of W such that $x \mapsto w_K x w_K$ for all $x \in W_K$ and $y \mapsto w_K y^{-1} w_K$ for any $y \in \mathcal{T}$ (this automorphism maps s_0 to s_0 hence it maps S onto itself). We have $K^* = K$.

Proposition 8.2. *If x is an element of W which has maximal length in its (W_K, W_K) double coset Ω then $x^* = x^{-1}$.*

Note that $\mathcal{T}_\Omega := \Omega \cap \mathcal{T}$ is a single W-conjugacy class. If $y \in \mathcal{T}_\Omega$ then $y^{*-1} = w_K y w_K \in \mathcal{T}_\Omega$. Thus $w \mapsto w^{*-1}$ maps some element of Ω to an element of Ω. Hence it maps Ω onto itself. Since it is length preserving it maps x to itself.

8.3. Let Ω, Ω' be two (W_K, W_K)-double cosets in W such that $\Omega' \le \Omega$. As in 5.1, let d_Ω (resp. $d_{\Omega'}$) be the longest element in Ω (resp. Ω'). Let $P_{d_{\Omega'}, d_\Omega} \in \mathbf{Z}[u]$ be the polynomial attached in [2] to the elements $d_{\Omega'}, d_\Omega$ of the Coxeter group W. Let G be a simple adjoint group over $\mathbf{C}$ for which W is the associated affine Weyl group so that $\mathcal{T}$ is the lattice of weights of a maximal torus of G. Let V_Ω be the (finite dimensional) irreducible rational representation of G whose extremal weights form the set $\mathcal{T}_\Omega$. Let $N_{\Omega',\Omega}$ be the multiplicity of a weight in $\mathcal{T}_{\Omega'}$ in the representation V_Ω. Now $P_{d_{\Omega'}, d_\Omega}$ is the u-analogue (in the sense of [4]) of the weight multiplicity $N_{\Omega',\Omega}$; in particular, according to [4], we have

$$N_{\Omega',\Omega} = P_{d_{\Omega'}, d_\Omega}|_{u=1}.$$

We have the following

Conjecture 8.4. $P^\sigma_{d_{\Omega'}, d_\Omega}(u) = P_{d_{\Omega'}, d_\Omega}(-u)$.

8.5. Now assume that Ω (resp. Ω') is the (W_K, W_K)-double coset that contains s_0 (resp. the unit element). Let $e_1 \le e_2 \le \cdots \le e_n$ be the exponents of W_K (recall that $e_1 = 1$). The following result supports the conjecture in 8.4.

Proposition 8.6. *In the setup of* 8.5*, assume that W_K is simply laced. We have:*

(a) $A_{d_\Omega} = v^{-l(d_\Omega)} a_\Omega + (-1)^{e_n} \sum_{j \in [1,n]} (-u)^{-e_j} v^{-l(d_{\Omega'})} a_{\Omega'}$;

(b) $P_{d_{\Omega'}, d_\Omega}(u) = \sum_{j \in [1,n]} u^{e_j - 1}$;

(c) $P^\sigma_{d_{\Omega'}, d_\Omega}(u) = \sum_{j \in [1,n]} (-u)^{e_j - 1}$.

We prove (a). It is enough to show that

$$v^{-l(d_\Omega)}a_\Omega+(-1)^{e_n}\sum_{j\in[1,n]}(-u)^{-e_j}v^{-l(d_{\Omega'})}a_{\Omega'}$$

is fixed by $\bar{}$. Let $H=K\cap s_0Ks_0$. We have $H=H^*$ and W_H is contained in the centralizer of s_0. Let $\tau:W_H\to W_H$ be the automorphism $y\mapsto s_0y^*s_0=y^*$. We have $d_{\Omega'}=w_K$, $d_\Omega=w_Kw_Hs_0w_K$, $l(d_\Omega)=2l(w_K)-l(w_H)+1$ and we must show that

(d) $$v^{-2l(w_K)+l(w_H)-1}a_\Omega+(-1)^{e_n}\sum_{j\in[1,n]}(-u)^{-e_j}v^{-l(w_K)}a_{\Omega'}\text{ is fixed by }\bar{}\,.$$

Let $\mathbf{S}=\sum_{x\in W_K}T_x\in\underline{\mathfrak{H}}$. Using 5.10(a) we see that

$$\mathbf{S}(a_{s_0}+a_1)=\mathbf{P}_{H,*}a_\Omega+\mathbf{P}_{K,*}a_{\Omega'}.$$

Hence

$$\begin{aligned}&v^{-2l(w_K)}\mathbf{S}(v^{-1}(a_{s_0}+a_\emptyset))\\&\quad=v^{-l(w_H)}\mathbf{P}_{H,*}v^{-2l(w_K)+l(w_H)-1}a_\Omega+v^{-l(w_K)-1}\mathbf{P}_{K,*}v^{-l(w_K)}a_{\Omega'}.\end{aligned}$$

Since $v^{-2l(w_K)}\mathbf{S}$ and $v^{-1}(a_{s_0}+a_1)$ are fixed by $\bar{}$, we see that that the left hand side of the last equality is fixed by $\bar{}$, hence

$$v^{-l(w_H)}\mathbf{P}_{H,*}v^{-2l(w_K)+l(w_H)-1}a_\Omega+v^{-l(w_K)-1}\mathbf{P}_{K,*}v^{-l(w_K)}a_{\Omega'}$$

is fixed by $\bar{}$. Since $v^{-l(w_H)}\mathbf{P}_{H,*}$ is fixed by $\bar{}$ and divides $\mathbf{P}_{K,*}$, we see that

$$v^{-2l(w_K)+l(w_H)-1}a_\Omega+v^{-l(w_K)+l(w_H)-1}\mathbf{P}_{K,*}\mathbf{P}_{H,*}^{-1}v^{-l(w_K)}a_{\Omega'}$$

is fixed by $\bar{}$. Hence to prove (d) it is enough to show that

$$v^{-l(w_K)+l(w_H)-1}\mathbf{P}_{K,*}\mathbf{P}_{H,*}^{-1}v^{-l(w_K)}a_{\Omega'}-(-1)^{e_n}\sum_{j\in[1,n]}(-u)^{-e_j}v^{-l(w_K)}a_{\Omega'}$$

is fixed by $\bar{}$. Now $v^{-l(w_K)}a_{\Omega'}$ is fixed by $\bar{}$, see 5.11(a). Hence it is enough to show that

$$v^{-l(w_K)+l(w_H)-1}\mathbf{P}_{K,*}\mathbf{P}_{H,*}^{-1}-(-1)^{e_n}\sum_{j\in[1,n]}(-u)^{-e_j}\text{ is fixed by }\bar{}\,.$$

This is verified by direct computation in each case. This completes the proof of (a). Now (c) follows from (a) using the equality $l(w_K w_H s_0 w_K) - l(w_K) = 2e_n$ and the known symmetry property of exponents; (b) follows from [4].

8.7. In this subsection we assume that W_K is of type A_2 with $K = \{s_1, s_2\}$. Note that $s_1^* = s_2$, $s_2^* = s_1$. We write $i_1 i_2 \cdots$ instead of $s_{i_1} s_{i_2} \cdots$ (the indices are in $\{0,1,2\}$). Let $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5$ be the (W_K, W_K) double coset of $01210, 0120, 0210, 0$ and unit element respectively. We have $d_{\Omega_1} = 1210120121$, $d_{\Omega_2} = 121012012$, $d_{\Omega_3} = 121021021$, $d_{\Omega_4} = 1210121$, $d_{\Omega_5} = 121$. A direct computation shows that

$$A_{d_{\Omega_1}} = v^{-11}(a_{\Omega_1} + a_{\Omega_2} + a_{\Omega_3} + (1-u)a_{\Omega_4} + (1-u+u^2)a_{\Omega_5}).$$

This provides further evidence for the conjecture in 8.4.

8.8. In this subsection we assume that $K = \{s_1, s_2\}$ with $s_1 s_2$ of order 4 and with $s_0 s_2 = s_2 s_0$, $s_0 s_1$ of order 4. Note that $x^* = x$ for all $x \in W$. Let $\Omega_1, \Omega_2, \Omega_3$ be the (W_K, W_K) double coset of $s_0 s_1 s_0, s_0$ and unit element respectively. We have $d_{\Omega_1} = 1212010212$, $d_{\Omega_2} = 12120121$, $d_{\Omega_3} = 1212$ (notation as in 8.7). A direct computation shows that

$$A_{d_{\Omega_1}} = v^{-10}(a_{\Omega_1} + a_{\Omega_2} + (1+u^2)a_{\Omega_3}).$$

This provides further evidence for the conjecture in 8.4.

9. Reduction Modulo 2

9.1. Let $\mathcal{A}_2 = \mathcal{A}/2\mathcal{A} = (\mathbf{Z}/2)[u, u^{-1}]$, $\underline{\mathcal{A}}_2 = \underline{\mathcal{A}}/2\underline{\mathcal{A}} = (\mathbf{Z}/2)[v, v^{-1}]$. We regard $\mathcal{A}_2$ as a subring of $\underline{\mathcal{A}}_2$ by setting $u = v^2$. Let $\mathfrak{H}_2 = \mathfrak{H}/2\mathfrak{H}$; this is naturally an $\mathcal{A}_2$-algebra with $\mathcal{A}_2$-basis $(T_x)_{x \in W}$ inherited from $\mathfrak{H}$ and with a bar operator $^-: \mathfrak{H}_2 \to \mathfrak{H}_2$ inherited from that $\mathfrak{H}$. Let $M_2 = \mathcal{A}_2 \otimes_{\mathcal{A}} M = M/2M$. This has a $\mathfrak{H}_2$-module structure and a bar operator $^-: M_2 \to M_2$ inherited from M. It has an $\mathcal{A}_2$-basis $(a_w)_{w \in \mathbf{I}_*}$ inherited from M. In this section we give an alternative construction of the $\mathfrak{H}_2$-module structure on M_2 and its bar operator.

Let $\mathcal{H}$ be the free $\underline{\mathcal{A}}$-module with basis $(t_w)_{w \in W}$ with the unique $\underline{\mathcal{A}}$-algebra structure with unit t_1 such that

$t_w t_{w'} = t_{ww'}$ if $l(ww') = l(w) + l(w')$ and

$(t_s+1)(t_s-v^2)=0$ for all $s\in S$.

Let $^- : \mathcal{H}\to\mathcal{H}$ be the unique ring involution such that $\overline{v^n t_x}=v^{-n}t_{x^{-1}}^{-1}$ for any $x\in W, n\in\mathbf{Z}$ (see [2]). Let $\mathcal{H}_2=\mathcal{H}/2\mathcal{H}$; this is naturally an $\underline{\mathcal{A}}_2$-algebra with $\underline{\mathcal{A}}_2$-basis $(t_x)_{x\in W}$ inherited from $\mathcal{H}$ and with a bar operator $^-: \mathcal{H}_2\to\mathcal{H}_2$ inherited from that of $\mathcal{H}$. Let $h\mapsto h^{\spadesuit}$ be the unique algebra antiautomorphism of $\mathcal{H}$ such that $t_w\mapsto t_{w^{*-1}}$. (It is an involution.)

We have $\mathcal{H}_2=\mathcal{H}'_2\oplus\mathcal{H}''_2$ where $\mathcal{H}'_2$ (resp. $\mathcal{H}''_2$) is the $\underline{\mathcal{A}}$-submodule of $\mathcal{H}_2$ spanned by $\{t_w; w\in\mathbf{I}_*\}$ (resp. $\{t_w; w\in W-\mathbf{I}_*\}$). Let $\pi:\mathcal{H}_2\to\mathcal{H}'_2$ be the projection on the first summand. Note that for $\xi'\in\mathcal{H}_2$ we have

(a) $\xi'^{\spadesuit}=\xi'$ if and only if $\xi'=\xi'_1+\xi'_2+\xi_2'^{\spadesuit}$ where $\xi'_1\in\mathcal{H}'_2, \xi'_2\in\mathcal{H}_2$.

(b) $\pi(\xi'^{\spadesuit})=\pi(\xi')$.

Lemma 9.2. *The map $\mathcal{H}_2\times\mathcal{H}'_2\to\mathcal{H}'_2$, $(h,\xi)\mapsto h\circ\xi=\pi(h\xi h^{\spadesuit})$ defines an $\mathcal{H}_2$-module structure on the abelian group $\mathcal{H}'_2$.*

Let $h,h'\in\mathcal{H}_2, \xi\in\mathcal{H}'_2$. We first show that $(h+h')\circ\xi=h\circ\xi+h'\circ\xi$ or that $\pi((h+h')\xi(h+h')^{\spadesuit})=\pi(h\xi h^{\spadesuit})+\pi(h'\xi h'^{\spadesuit})$. It is enough to show that $\pi(h\xi h'^{\spadesuit})=\pi(h'\xi h^{\spadesuit})$. This follows from 9.1(b) since $(h'\xi h^{\spadesuit})^{\spadesuit}=h\xi^{\spadesuit}h'^{\spadesuit}=h\xi h'^{\spadesuit}$.

We next show that $(hh')\circ\xi=h\circ(h'\circ\xi)$ or that $\pi(hh'\xi h'^{\spadesuit}h^{\spadesuit})=\pi(h\pi(h'\xi h'^{\spadesuit})h^{\spadesuit})$. Setting $\xi'=h'\xi h'^{\spadesuit}$ we see that we must show that $\pi(h\xi' h^{\spadesuit})=\pi(h\pi(\xi')h^{\spadesuit})$. Setting $\eta=\xi'-\pi(\xi')$ we are reduced to showing that $\pi(h\eta h^{\spadesuit})=0$. Since $\xi\in\mathcal{H}'_2$ we have $\xi^{\spadesuit}=\xi$. Hence $\xi'^{\spadesuit}=(h'^{\spadesuit})^{\spadesuit}\xi^{\spadesuit}h'^{\spadesuit}=h'\xi h'^{\spadesuit}$ so that $\xi'^{\spadesuit}=\xi'$. We write $\xi'=\xi'_1+\xi'_2+\xi_2'^{\spadesuit}$ as in 9.1(a). Then $\pi(\xi')=\xi'_1$ and $\eta=\xi'_2+\xi_2'^{\spadesuit}$. We have $h\eta h^{\spadesuit}=h\xi'_2h^{\spadesuit}+h\xi_2'^{\spadesuit}h^{\spadesuit}=\zeta+\zeta^{\spadesuit}$ where $\zeta=h\xi'_2h^{\spadesuit}$. Thus $\pi(h\eta h^{\spadesuit})=\pi(\zeta+\zeta^{\spadesuit})=0$ (see 9.1(b)). Clearly we have $1\circ\xi=\xi$. The lemma is proved.

9.3. Consider the group isomorphism $\psi:\mathcal{H}_2\xrightarrow{\sim}\mathfrak{H}_2$ such that $v^nt_w\mapsto u^nT_w$ for any $n\in\mathbf{Z}, w\in W$. This is a ring isomorphism satisfying $\psi(fh)=f^2\psi(h)$ for all $f\in\underline{\mathcal{A}}_2, h\in\mathcal{H}_2$ (we have $f^2\in\mathcal{A}_2$). Using now 9.2 we see that:

(a) *The map $\mathfrak{H}_2\times\mathcal{H}'_2\to\mathcal{H}'_2$, $(h,\xi)\mapsto h\odot\xi:=\pi(\psi^{-1}(h)\xi(\psi^{-1}(h))^{\spadesuit})$ defines an $\mathfrak{H}_2$-module structure on the abelian group $\mathcal{H}'_2$.*

Note that the $\mathfrak{H}_2$-module structure on $\mathcal{H}'_2$ given in (a) is compatible with the $\mathcal{A}$-module structure on $\mathcal{H}'_2$. Indeed if $f\in\mathcal{A}_2$ and $f'\in\underline{\mathcal{A}}_2$ is such that $f'^2=f$ then f acts in the $\mathfrak{H}_2$-module structure in (a) by $\xi\mapsto f'\xi f'=f'^2\xi=f\xi$.

9.4. Let $s \in S, w \in \mathbf{I}_*$. The equations in this subsection take place in $\mathcal{H}_2$. If $sw = ws^* > w$ we have

$$T_s \odot t_w = \pi(t_s t_w t_{s^*}) = \pi(t_{sw} t_{s^*}) = \pi((u-1)t_{sw} + ut_w) = ut_w + (u+1)t_{sw}.$$

If $sw = ws^* < w$ we have

$$\begin{aligned} T_s \odot t_w &= \pi(t_s t_w t_{s^*}) = \pi(((u-1)t_w + ut_{sw})t_{s^*}) \\ &= \pi((u-1)^2 t_w + (u-1)ut_{ws^*} + ut_w) \\ &= (u^2 - u - 1)t_w + (u^2 - u)t_{sw}. \end{aligned}$$

If $sw \neq ws^* > w$ we have

$$T_s \odot t_w = \pi(t_s t_w t_{s^*}) = \pi(t_{sws^*}) = t_{sws^*}.$$

If $sw \neq ws^* < w$ we have

$$\begin{aligned} T_s \odot t_w &= \pi(t_s t_w t_{s^*}) = \pi(((u-1)t_w + ut_{sw})t_{s^*}) \\ &= \pi((u-1)^2 t_w + (u-1)ut_{ws^*} + u(u-1)t_{sw} + u^2 t_{sws^*}) \\ &= (u^2 - 1)t_w + u^2 t_{sws^*}. \end{aligned}$$

(We have used that $\pi(t_{ws^*}) = \pi(t_{sw})$ which follows from 9.1(b).) From these formulas we see that

(a) *the isomorphism of $\mathcal{A}_2$-modules $\mathcal{H}'_2 \xrightarrow{\sim} M_2$ given by $t_w \mapsto a_w$ ($w \in \mathbf{I}_*$) is compatible with the $\mathfrak{H}_2$-module structures.*

9.5. For $w \in W$ we set $\overline{t_w} = \sum_{y \in W; y \leq w} \overline{\rho_{y,w}} v^{-l(w)-l(y)} t_y$ where $\rho_{y,w} \in \underline{\mathcal{A}}$ satisfies $\rho_{w,w} = 1$. For $y \in W, y \not\leq w$ we set $\rho_{y,w} = 0$.

For $x, y \in W, s \in S$ such that $sy > y$ we have

(i) $\rho_{x,sy} = \rho_{sx,y}$ if $sx < x$,

(ii) $\rho_{x,sy} = \rho_{sx,y} + (v - v^{-1})\rho_{x,y}$ if $sx > x$.
For $x, y \in W, s \in S$ such that $ys > y$ we have

(iii) $\rho_{x,ys} = \rho_{xs,y}$ if $xs < x$,

(iv) $\rho_{x,ys} = \rho_{xs,y} + (v - v^{-1})\rho_{x,y}$ if $xs > x$.
Note that (iii),(iv) follow from (i),(ii) using

(v) $\rho_{z,w} = \rho_{z^{*-1},w^{*-1}}$ for any $z, w \in W$.

9.6. If $f, f' \in \underline{\mathcal{A}}$ we write $f \equiv f'$ if f, f' have the same image under the obvious ring homomorphism $\underline{\mathcal{A}} \to \underline{\mathcal{A}}_2$. We have the following result.

Proposition 9.7. *For any $y, w \in \mathbf{I}_*$ we have $r_{y,w} \equiv \rho_{y,w}$.*

Since the formulas 4.2(a), (b) together with $r_{x,1} = \delta_{x,1}$ define uniquely $r_{x,y}$ for any $x, y \in \mathbf{I}_*$ and since $\rho_{x,1} = \delta_{x,1}$ for any x, it is enough to show that the equations 4.2(a),(b) remain valid if each r is replaced by ρ and each $=$ is replaced by $\equiv$.

Assume first that $sy = ys^* > y$ and $x \in \mathbf{I}_*$.

If $sx = xs^* > x$ we have

$$\begin{aligned}&(v+v^{-1})\rho_{x,sy} - (\rho_{sx,y}(v^{-1}-v) - (u-u^{-1})\rho_{x,y})\\ &\quad\equiv (v+v^{-1})(\rho_{x,sy} - \rho_{sx,y} - (v-v^{-1})\rho_{x,y}) = 0.\end{aligned}$$

(The $=$ follows from 9.5(ii).)

If $sx = xs^* < x$ we have

$$(v+v^{-1})\rho_{x,sy} - (-2\rho_{x,y} + \rho_{sx,y}(v+v^{-1})) \equiv (v+v^{-1})(\rho_{x,sy} - \rho_{sx,y}) = 0.$$

(The $=$ follows from 9.5(i).)

If $sx \neq xs^* > x$ we have

$$\begin{aligned}&(v+v^{-1})\rho_{x,sy} - (\rho_{sxs^*,y} + (u-1-u^{-1})\rho_{x,y})\\ &\quad= (v+v^{-1})\rho_{sx,y} + (u-u^{-1})\rho_{x,y} - \rho_{sxs^*,y} - (u-1-u^{-1})\rho_{x,y}\\ &\quad\equiv (v-v^{-1})\rho_{sx,y} - \rho_{x,y} + \rho_{sxs^*,y} = \rho_{sx,ys^*} - \rho_{x,y} = 0.\end{aligned}$$

(The first, second and third $=$ follow from 9.5(ii),(iv),(iii).)

If $sx \neq xs^* < x$ we have

$$\begin{aligned}(v+v^{-1})\rho_{x,sy} - (-\rho_{x,y} + \rho_{sxs^*,y}) &= (v+v^{-1})\rho_{sx,y} - (-\rho_{x,y} + \rho_{sxs^*,y})\\ &\equiv (v-v^{-1})\rho_{sx,y} + \rho_{x,y} - \rho_{sxs^*,y}\\ &= \rho_{sx,sy} - \rho_{sxs^*,y} = \rho_{sx,sy} - \rho_{sx,ys^*} = 0.\end{aligned}$$

(The first, second and third $=$ follow from 9.5(i),(ii),(iii).)

Next we assume that $sy \neq ys^* > y$ and $x \in \mathbf{I}_*$.

If $sx = xs^* > x$ we have

$$\begin{aligned}
&\rho_{x,sys^*} - (\rho_{sx,y}(v^{-1} - v) + (u + 1 - u^{-1})\rho_{x,y}) \\
&\quad = \rho_{sx,ys^*} + (v - v^{-1})\rho_{x,ys^*} - \rho_{sx,y}(v^{-1} - v) - (u + 1 - u^{-1})\rho_{x,y} \\
&\quad = \rho_{x,y} + (v - v^{-1})\rho_{x,ys^*} - \rho_{xs^*,y}(v^{-1} - v) - (u + 1 - u^{-1})\rho_{x,y} \\
&\quad = \rho_{x,y} + (v - v^{-1})\rho_{xs^*,y} + (v - v^{-1})^2\rho_{x,y} - \rho_{xs^*,y}(v^{-1} - v) \\
&\qquad -(u + 1 - u^{-1})\rho_{x,y} \equiv 0.
\end{aligned}$$

(The first, second and third = follow from 9.5(ii),(iv),(iv).)

If $sx = xs^* < x$ we have

$$\begin{aligned}
\rho_{x,sys^*} - (\rho_{sx,y}(v + v^{-1}) - \rho_{x,y}) &= \rho_{sx,sy} - (\rho_{sx,y}(v + v^{-1}) - \rho_{x,y}) \\
&\equiv \rho_{sx,sy} - (\rho_{sx,y}(v - v^{-1}) + \rho_{x,y}) = 0,
\end{aligned}$$

(The first and second = follow from 9.5(i),(ii.)

If $sx \neq xs^* > x$ we have

$$\begin{aligned}
&\rho_{x,sys^*} - (\rho_{sxs^*,y} + (u - u^{-1})\rho_{x,y}) \\
&\quad = \rho_{xs^*,sy} + (v - v^{-1})\rho_{x,sy} - \rho_{sxs^*,y} - (u - u^{-1})\rho_{x,y} \\
&\quad = \rho_{sxs^*,y} + (v - v^{-1})\rho_{xs^*,y} + (v - v^{-1})\rho_{sx,y} + (v - v^{-1})^2\rho_{x,y} \\
&\qquad -\rho_{sxs^*,y} - (u - u^{-1})\rho_{x,y} \equiv (v - v^{-1})(\rho_{xs^*,y} - \rho_{sx,y}) \\
&\quad = (v - v^{-1})(\rho_{(xs^*)^{*-1},y^{*-1}} - \rho_{sx,y}) = (v - v^{-1})(\rho_{sx,y} - \rho_{sx,y}) = 0.
\end{aligned}$$

(The first, second, and third = follow from 9.5(iv),(ii),(v).)

If $sx \neq xs^* < x$ we have

$$\rho_{x,sys^*} - \rho_{sxs^*,y} = \rho_{xs^*,ys^*} - \rho_{sxs^*,y} = 0.$$

(The first and second = follow from 9.5(iii),(i).)

Thus the equations 4.2(a),(b) with each r replaced by ρ and each = replaced by $\equiv$ are verified. The proposition is proved.

9.8. We define a group homomorphism $B : \mathcal{H}'_2 \to \mathcal{H}'_2$ by $\xi \mapsto \pi(\overline{\xi})$. From 9.7 we see that

(a) *under the isomorphism* 9.4(*a*) *the map* $B : \mathcal{H}'_2 \to \mathcal{H}'_2$ *corresponds to the map* $^-: M_2 \to M_2$.

We now give an alternative proof of (a). Using 0.2(b) and 9.4(a) we see that it is enough to show that for any $w \in \mathbf{I}_*$ we have $\pi(t_{w^{-1}}^{-1}) = T_{w^{-1}}^{-1} \odot t_{w^{-1}}$ in $\mathcal{H}'_2$. Since ψ in 9.3 is a ring isomorphism, we have $\psi(t_{w^{-1}}^{-1}) = T_{w^{-1}}^{-1}$ hence

$$\begin{aligned} T_{w^{-1}}^{-1} \odot t_{w^{-1}} &= \pi(\psi^{-1}(T_{w^{-1}}^{-1})t_{w^{-1}}(\psi^{-1}(T_{w^{-1}}^{-1}))^{\spadesuit}) \\ &= \pi(t_{w^{-1}}^{-1}t_{w^{-1}}(t_{w^{-1}}^{-1})^{\spadesuit}) = \pi(t_{w^{-1}}^{-1}t_{w^{-1}}t_{w^*}^{-1}) \\ &= \pi(t_{w^{-1}}^{-1}t_{w^{-1}}t_{w^{-1}}^{-1}) = \pi(t_{w^{-1}}), \end{aligned}$$

as required.

9.9. For $y, w \in W$ let $P_{y,w} \in \mathbf{Z}[u]$ be the polynomials defined in [2, 1.1]. (When $y \not\leq w$ we set $P_{y,w} = 0$.) We set $p_{y,w} = v^{-l(w)+l(y)}P_{y,w} \in \underline{\mathcal{A}}$. Note that $p_{w,w} = 1$ and $p_{y,w} = 0$ if $y \not\leq w$. We have $p_{y,w} \in \mathcal{A}_{<0}$ if $y < w$ and

(i) $\overline{p_{x,w}} = \sum_{y \in W; x \leq y \leq w} r_{x,y}p_{y,w}$ if $x \leq w$,

(ii) $p_{x^{*-1},w^{*-1}} = p_{x,w}$, if $x \leq w$.

We have the following result which, in the special case where W is a Weyl group or an affine Weyl group, can be deduced from the last sentence in the first paragraph of [6].

Theorem 9.10. *For any* $x, w \in \mathbf{I}_*$ *such that* $x \leq w$ *we have* $P^{\sigma}_{x,w} \equiv P_{x,w}$ (*with* $\equiv$ *as in* 9.6).

It is enough to show that $\pi_{x,w} \equiv p_{x,w}$. We can assume that $x < w$ and that the result is known when x is replaced by $x' \in \mathbf{I}_*$ with $x < x' \leq w$. Using 9.9(i) and the definition of $\pi_{x,w}$ we have

$$\overline{p_{x,w}} - \overline{\pi_{x,w}} = \sum_{y \in W; x \leq y \leq w} r_{x,y}p_{y,w} - \sum_{y \in \mathbf{I}_*; x \leq y \leq w} \rho_{x,y}\pi_{y,w}.$$

Using 9.7 and the induction hypothesis we see that the last sum is $\equiv$ to

$$\begin{aligned} &p_{x,w} - \pi_{x,w} + \sum_{y \in W; x < y \leq w} r_{x,y}p_{y,w} - \sum_{y \in \mathbf{I}_*; x < y \leq w} r_{x,y}p_{y,w} \\ &= p_{x,w} - \pi_{x,w} + \sum_{y \in W; y \neq y^{*-1}, x < y \leq w} r_{x,y}p_{y,w}. \end{aligned}$$

In the last sum the terms corresponding to y and y^{*-1} cancel out (after reduction mod 2) since

$$r_{x,y^{*-1}}p_{y^{*-1},w} = r_{x^{*-1},y}p_{y,w^{*-1}} = r_{x,y}p_{y,w}.$$

(We use 9.5(v), 9.9(ii).) We see that

$$\overline{p_{x,w}} - \overline{\pi_{x,w}} \equiv p_{x,w} - \pi_{x,w}.$$

After reduction mod 2 the right hand side is in $v^{-1}(\mathbf{Z}/2)[v^{-1}]$ and the left hand side is in $v(\mathbf{Z}/2)[v]$; hence both sides are zero in $(\mathbf{Z}/2)[v,v^{-1}]$. This completes the proof. □

9.11. For $x, w \in \mathbf{I}_*$ such that $x \leq w$ we set $P^+_{x,w} = (1/2)(P_{x,w} + P^\sigma_{x,w})$, $P^-_{x,w} = (1/2)(P_{x,w} - P^\sigma_{x,w})$. From 9.10 we see that $P^+_{x,w} \in \mathbf{Z}[u]$, $P^-_{x,w} \in \mathbf{Z}[u]$.

Conjecture 9.12. *We have* $P^+_{x,w} \in \mathbf{N}[u]$, $P^-_{x,w} \in \mathbf{N}[u]$.

This is a refinement of the conjecture in [2] that $P_{x,w} \in \mathbf{N}[u]$ for any $x \leq w$ in W. In the case where W is a Weyl group or an affine Weyl group, the (refined) conjecture holds by results of [6].

Note added July 25, 2012. Conjecture 8.4 is now proved, see G.Lusztig and Z.Yun, A $(-q)$-analogue of weight multiplicities, arxiv:1203.0521.

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