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# LUSZTIG'S $A$-FUNCTION FOR COXETER GROUPS WITH COMPLETE GRAPHS

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**Abstract**

We show that Lusztig's $a$-function of a Coxeter group is bounded if the Coxeter group has a complete graph (i.e. any two vertices are joined) and the cardinalities of finite parabolic subgroups of the Coxeter group have a common upper bound.

## 0. Introduction

Lusztig's $a$-function for a Coxeter group is defined in [5] and is a very useful tool for studying cells in Coxeter groups and other related topics such as representations of Hecke algebras. For an affine Weyl group, Lusztig showed in [5] that the $a$-function is bounded by the length of the longest element of the corresponding Weyl group. It might be true that for any Coxeter group of finite rank, the $a$-function is bounded by the length of the longest element of certain finite parabolic subgroups of the Coxeter group. In this paper we first show that this property implies that the Coxeter group has a lowest two-sided cell (Theorem 1.5). Secondly, presenting Lusztig's $a$-function of a Coxeter group has this property (Theorem 2.1) if the Coxeter group has a complete graph (i.e. any two different simple reflections of the Coxeter group are not commutative) and the cardinalities of finite parabolic subgroups of the Coxeter group have a common upper bound. For Coxeter groups of rank 3, Peipei Zhou [12] showed an analogue result by using the

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approach stated in the paper. These facts support part (iv) of Question 1.13 in [11].

## 1. Preliminaries

**1.1.** Let $(W,S)$ be a Coxeter group. We use $l$ for the length function and $\leq$ for the Bruhat order of $W$. The rank of the Coxeter group $(W,S)$ is defined to be the cardinality $|S|$ of $S$, which may be infinite. A subgroup $P$ of $W$ is called a parabolic subgroup if it is generated by a subset $S'$ of $S$. So $(P,S')$ is also a Coxeter group with the cardinality of $S'$ being the rank of $P$. The neutral element of $W$ will be denoted by $e$.

Let $q$ be an indeterminate. The Hecke algebra $H$ of $(W,S)$ is a free $\mathcal{A}=\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}]$-module with a basis $T_w,\ w\in W$ and the multiplication relations are $(T_s-q)(T_s+1)=0$ if $s$ is in $S$, $T_wT_u=T_{wu}$ if $l(wu)=l(w)+l(u)$.

For any $w\in W$ set $\tilde{T}_w=q^{-\frac{l(w)}{2}}T_w$. For any $w,u\in W$, write

$$\tilde{T}_w\tilde{T}_u=\sum_{v\in W}f_{w,u,v}\tilde{T}_v,\qquad f_{w,u,v}\in\mathcal{A}.$$

The following fact is known and implicit in [5, 8.3].

(a) For any $w,u,v\in W$, $f_{w,u,v}\in\mathcal{A}$ is a polynomial in $q^{\frac{1}{2}}-q^{-\frac{1}{2}}$ with non-negative coefficients and $f_{w,u,v}=f_{u,v^{-1},w^{-1}}=f_{v^{-1},w,u^{-1}}$. Its degree is less than or equal to $\min\{l(w),l(u),l(v)\}$.

***Proof.*** Note that $f_{x,y,e}=0$ if $xy\neq e$ and $f_{x,x^{-1},e}=1$ for any $x,y\in W$. Then it is easy to verify

$$f_{w,u,v}f_{v,v^{-1},e}=f_{w,w^{-1},e}f_{u,v^{-1},w^{-1}}.$$

So we have $f_{w,u,v}=f_{u,v^{-1},w^{-1}}=f_{v^{-1},w,u^{-1}}$. It is clear that $f_{w,u,v}$ is a polynomial in $q^{\frac{1}{2}}-q^{-\frac{1}{2}}$ with non-negative coefficients and $\deg f_{w,u,v}$ is less than or equal to $\min\{l(w),l(u)\}$. The second assertion follows. □

For any $w,u,v$ in $W$, we shall regard $f_{w,u,v}$ as a polynomial in $\xi=q^{\frac{1}{2}}-q^{-\frac{1}{2}}$. The following fact is noted by Lusztig [6, 1.1 (c)].

(b) For any $w,u,v$ in $W$ we have $f_{w,u,v}=f_{u^{-1},w^{-1},v^{-1}}$.

**Lemma 1.2.** *Let $(W,S)$ be a Coxeter group and $I$ be a subset of $S$. The following conditions are equivalent.*

(a) *The subgroup $W_I$ of $W$ generated by $I$ is finite.*

(b) *There exists an element $w$ of $W$ such that $sw \leq w$ for all $s$ in $I$.*

(c) *There exists an element $w$ of $W$ such that $w \leq ws$ for all $s$ in $I$.*

***Proof.*** Clear. □

We set $L(w) = \{s \in S \mid sw \leq w\}$ and $R(w) = \{s \in S \mid ws \leq w\}$ for any $w \in W$.

**Lemma 1.3.** *Let $w$ be in $W$ and $I$ be a subset of $L(w)$ (resp. $R(w)$). Then $l(w_I w) + l(w_I) = l(w)$ (resp. $l(ww_I) + l(w_I) = l(w)$), here $w_I$ is the longest element of $W_I$.*

***Proof.*** Clear. □

**1.4.** For any $y, w \in W$, let $P_{y,w}$ be the Kazhdan-Lusztig polynomial. Then all the elements $C_w = q^{-\frac{l(w)}{2}} \sum_{y \leq w} P_{y,w} T_y$, $w \in W$, form a Kazhdan-Lusztig basis of $H$. It is known that $P_{y,w} = \mu(y,w) q^{\frac{1}{2}(l(w)-l(y)-1)}$ +lower degree terms if $y < w$ and $P_{w,w} = 1$.

For any $w, u$ in $W$, Write

$$C_w C_u = \sum_{v \in W} h_{w,u,v} C_v,\ \ h_{w,u,v} \in \mathcal{A}.$$

Following [5], for any $v \in W$ we define

$$a(v) = \max\{i \in \mathbb{N} \mid i = \deg h_{w,u,v},\ \ w, u \in W\},$$

here the degree is in terms of $q^{\frac{1}{2}}$. Since $h_{w,u,v}$ is a polynomial in $q^{\frac{1}{2}} + q^{-\frac{1}{2}}$, we have $a(v) \geq 0$.

We are interested in the bound of the function $a : W \to \mathbb{N}$. Clearly, $a$ is bounded if $W$ is finite. The following fact is known (see [6]), at the same time easy to verify.

(a) The $a$-function is bounded by a constant $c$ if and only if $\deg f_{w,u,v} \leq c$ for any $w, u, v \in W$.

Lusztig showed that for an affine Weyl group the $a$-function is bounded by the length of the longest element of the corresponding Weyl group. This fact is important in studying cells in affine Weyl groups. One consequence is that an affine Weyl group has a lowest two-sided cell [8]. We will show that the boundness of $a$-function is also interesting in general.

Assume now that the $a$-function is bounded and its maximal value is $c$. Let $w,u,v$ be elements in $W$. We shall regard $h_{w,u,v}$ as a polynomial in $\eta=q^{\frac{1}{2}}+q^{-\frac{1}{2}}$. Following Lusztig [5], write $h_{w,u,v}=\gamma_{w,u,v}\eta^{a(v)}+\delta_{w,u,v}\eta^{a(v)-1}+$lower degree terms. Then $\gamma_{w,u,v}$ and $\delta_{w,u,v}$ are integers. Let $\Omega$ be the subset of $W$ consisting of all elements $w$ with $a(w)=c$.

Assume that $v\in\Omega$. For $w,u\in W$, we have $f_{w,u,v}=\gamma_{w,u,v}\xi^{c}+$lower degree terms. Using 1.1(a) and 1.1 (b) we get

(b) $\gamma_{w,u,v}=\gamma_{u,v^{-1},w^{-1}}=\gamma_{v^{-1},w,u^{-1}}=\gamma_{u^{-1},w^{-1},v^{-1}}$ for $w,u,v\in\Omega$.

(c) Let $w,u\in W$ and $v\in\Omega$. If $\gamma_{w,u,v}\neq 0$, then $w,u$ are in $\Omega$ and $\gamma_{w,u,v}=\gamma_{u,v^{-1},w^{-1}}=\gamma_{v^{-1},w,u^{-1}}=\gamma_{u^{-1},w^{-1},v^{-1}}$ is positive.

Since $h_{w,u,v}\neq 0$ implies that $w\underset{R}{\leq}v,\ u\underset{L}{\leq}v$, by (c) we obtain

(d) Let $w,u\in W$ and $v\in\Omega$. If $\gamma_{w,u,v}\neq 0$, then $w\underset{R}{\sim}v$, $w\underset{L}{\sim}u^{-1}$ and $u\underset{L}{\sim}v$. In particular, $w,u,v$ are in the same two-sided cell.

**Theorem 1.5.** *Let $(W,S)$ be a Coxeter group. Assume that the $a$-function is bounded by the length of the longest element $w_0$ of a finite parabolic subgroup $P$ of $W$. Then the two-sided cell of $W$ containing $w_0$ is the lowest two-sided cell of $W$. Moreover, the lowest two-sided cell contains all elements $w$ in $W$ with $a(w)=l(w_0)$.*

***Proof.*** We first show that $x\underset{LR}{\leq}w_0$ for any $x\in W$. (We refer to [3] for the definitions of the preorders $\underset{LR}{\leq},\underset{R}{\leq},\underset{L}{\leq}$ and the equivalences $\underset{L}{\sim},\underset{LR}{\sim}$ on $W$.)

Let $x\in W$ be such that $l(xw_0)=l(x)-l(w_0)$. We first show that $x$ and $w_0$ are in the same left cell. Clearly $x\underset{L}{\leq}w_0$. Let $y=xw_0$. Then

$$\tilde T_{x^{-1}}\tilde T_x=\tilde T_{w_0}(\sum_{z\in W}f_{y^{-1},y,z}\tilde T_z)\tilde T_{w_0}.$$

Since $f_{y^{-1},y,e} = 1$, $f_{w_0,w_0,w_0}$ has degree $l(w_0)$ as a polynomial in $\xi = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and $f_{w,u,v}$ has non-negative coefficients as a polynomial in $\xi$ for any $w, u, v \in W$. By 1.4(a) we conclude that $f_{x^{-1},x,w_0}$ has degree $l(w_0)$. Thus $h_{x^{-1},x,w_0}$ has degree $l(w_0)$ as a Laurent polynomial in $q^{\frac{1}{2}}$. In particular, $h_{x^{-1},x,w_0}$ is non-zero, so $w_0 \underset{L}{\leq} x$. Hence $x$ and $w_0$ are in the same left cell.

Now assume that $x$ is an arbitrary element in $W$. Let $w \in P$ be such that $xw$ is the longest element in $xP$. Then $l(xw) = l(x) + l(w)$ and $l(xww_0) = l(xw) - l(w_0)$. So we have $w_0 \underset{L}{\sim} xw \underset{R}{\leq} x$. Hence, the two-sided cell containing $w_0$ is the lowest one among the two-sided cells of $W$ (with respect to the partial order $\underset{LR}{\leq}$ on the set of two-sided cells of $W$).

Now we show that the lowest two-sided cell contain all elements $w$ in $W$ with $a(w) = l(w_0)$.

Assume that $a(w) = l(w_0)$. Then there exists $x, y \in W$ such that $\gamma_{x,y,w} \neq 0$ and $x, y, w$ are in the same two-sided cell. By 1.4 (c), $a(x) = a(y) = l(w_0)$. Choose $u \in P$ such that $l(yu) = l(y) + l(u) = l(yuw_0) + l(w_0)$. It is easy to see that $l(wu) = l(w) + l(u) = l(wuw_0) + l(w_0)$. Since $\tilde{T}_x\tilde{T}_{yu} = (\tilde{T}_x\tilde{T}_y)\tilde{T}_u$, we have $\gamma_{x,yu,wu} \geq \gamma_{x,y,w}$. Thus $x, yu, wu$ are in the same two-sided cell. But we have seen that $yu$ and $w_0$ are in the same two-sided cell.

The theorem is proved. □

**Corollary 1.6.** *Let $(W, S)$ be a Coxeter group. Assume that the $a$-function is bounded by the length of the longest element $w_0$ of a finite parabolic subgroup $P$ of $W$. Then*

$$\{x \in W \mid l(xw_0) = l(x) - l(w_0)\}$$

*is a left cell of $W$.*

***Proof.*** It follows from the proof of Theorem 1.5. □

**Remark.** For affine Weyl groups, this result is due to Lusztig [5].

**1.7.** Let $(W, S)$ be a Coxeter group. Assume that the $a$-function is bounded by the length of the longest element $w_0$ of a finite parabolic subgroup $P$ of $W$. Denote the left cell containing $w_0$ by $\Gamma$. Then $\Gamma = \{w \in W \mid l(w) = l(ww_0)+$

$l(w_0)\}$. Let $J_{\Gamma\cap\Gamma^{-1}}$ be the free $\mathbb{Z}$-module with a basis $\{t_w\,|\,w\in\Gamma\cap\Gamma^{-1}\}$. Define $t_wt_u=\sum_{v\in\Gamma\cap\Gamma^{-1}}\gamma_{w,u,v}t_v$. Then $J_{\Gamma\cap\Gamma^{-1}}$ is an associative ring with unit $1=t_{w_0}$.

Let $\Omega$ be the subset of $W$ consisting of all elements $w$ with $a(w)=l(w_0)$. We can define $J_\Omega$ and similarly the multiplication in $J_\Omega$. The multiplication is associative. However, $J_\Omega$ has no unit in general, since in general $\Omega$ contains infinite left cells, as shown in [1, 2], see also Proposition 3.2.

**1.8. Remark.** Keep the assumption of Theorem 1.5. Motivated by the work of Shi [8, 9], we will pose some conjectures.

It is likely that the lowest two-sided cell is exactly the set of elements $w$ in $W$ with $a(w)=l(w_0)$. Furthermore, it is likely that the lowest two-sided cell coincides with the set of elements of $W$ of the form $xwy$ such that $l(xwy)=l(x)+l(w)+l(y)$, $l(w)=l(w_0)$ with $w$ being the longest element of a finite parabolic subgroup of $W$.

Let $D'$ be the set consisting of all elements $x\in W$ such that

(1) $x=wy$ for some $w$ in a finite parabolic subgroup of $W$ with length $l(w_0)$ and $y\in W$ and $l(x)=l(w)+l(y)$,

(2) for any $s$ in $L(w)$, there are no $z,z',u\in W$ such that $sx=zuz'$, $l(sx)=l(z)+l(u)+l(z')$ and $u$ is in a finite parabolic subgroup of $W$ with length $l(w_0)$.

For any $x\in D'$, let $\Gamma_x$ be the subset of $W$ consisting of all elements $zx$ satisfying $l(zx)=l(z)+l(x)$. It is likely that $\Gamma_x$ is a left cell in the lowest-sided cell of $W$ and the map $x\to\Gamma_x$ is a bijection between the set $D'$ and the set of left cells in the lowest two-sided cell. Also, the set $D=\{y^{-1}wy\,|\,wy\in D'\}$ should be the set of distinguished involutions in the lowest two-sided cell, here $wy$ satisfies the above (1) and (2). When the Coxeter graph of $W$ is connected we also conjecture that the set $D$ is finite if and only if $W$ is finite or is an affine Weyl group or $st$ has infinite order for any different simple reflections $s,t\in S$.

Assume that $wy$ satisfies (1) and (2). Let $zw\in W$ be such that $l(zw)=l(z)+l(w)$. Then we should have $C_{zw}C_{wy}=h_{w,w,w}C_{zwy}$. Also we should have $\mu(z'wy,zwy)=\mu(z'w,zw)$ if $l(z'w)=l(z')+l(w)$. For affine Weyl groups, these equalities are true, see [10, 7].

If $(W,S)$ is crystallographic, then the function $a$ is constant on a two-sided cell [5]. Since $a(w_0)=l(w_0)$ (see [5]), we see that the lowest two-sided cell is exactly the set $\{w\in W\mid a(w)=l(w_0)\}$.

For an affine Weyl group $W$, thanks to [8, 9], we know that (a) the lowest two-sided cell of $W$ coincides with the set of elements of $W$ of the form $xwy$ such that $l(xwy)=l(x)+l(w)+l(y)$, $l(w)=l(w_0)$ and $w$ is the longest element of a finite parabolic of $W$; (b) $D$ is the set of distinguished involutions in the lowest two-sided cell.

In Section 3 we will show that the above conjectures are true for certain Coxeter groups with complete graphs.

## 2. Coxeter Groups with Complete Graphs

Throughout this section $(W,S)$ is a Coxeter group and any two simple reflections in $S$ are not commutative. In other words, the Coxeter graph of $(W,S)$ is a complete graph. Another main result of this article is the following.

**Theorem 2.1.** *Let $(W,S)$ be a Coxeter group. Assume that any two different simple reflections are not commutative and the cardinalities of finite parabolic subgroups of $W$ have a common upper bound. Then Lusztig's a-function on $W$ is bounded by the length of the longest element of certain finite parabolic subgroups of $W$.*

Note that in the theorem the set $S$ is not assumed to be finite. If $S$ is finite, the set of cardinalities of finite parabolic subgroups of $W$ is finite, hence an upper bound exists. The remaining of this section is devoted to a proof of the theorem.

**Lemma 2.2.** *Let $r,s,t$ be simple reflections such that the orders of $rs,rt,st$ are all greater than 2. Then there is no element $w$ in $W$ such that $w=w_1r=w_2st$ and $l(w)=l(w_1)+1=l(w_2)+2$.*

***Proof.*** We use induction on $l(w)$. When $l(w)=0,1,2,3$, the lemma is clear. Now assume that the lemma is true for $u$ with length $l(w)-1$. Since $r,t\in R(w)$, by Lemma 1.2, we know that the subgroup $W_{rt}$ of $W$ generated by $r,t$ is finite. Let $w_{rt}$ be the longest element in $W_{rt}$. By Lemma 1.3, $w=w_3w_{rt}=w_4trt$ for some $w_3,w_4\in W$ and $l(w)=l(w_3)+l(w_{rt})$, $l(w)=l(w_4)+3$. So we get $w_4tr = w_2s$. Clearly we have $l(w_4rt) = l(w)-1{=}l(w_4)+2 = l(w_2)+1$. By induction hypothesis, $w_2s$ does not exist, hence $w$ does not exist. The lemma is proved. □

**Lemma 2.3.** *Keep the assumption of Theorem* 2.1. *Let $x \in W$ and $t_1 \cdots t_m$ $(m \geq 2)$ be a reduced expression of an element in $W$. Assume that $xt_1 \leq x$, $xt_2 \cdots t_{m-1}t_m \leq xt_2 \cdots t_{m-1}$, and $l(xt_2 \cdots t_{m-1}) = l(x) + m - 2$. If for any reduced expression $s_1s_2 \cdots s_m$ of $t_1t_2 \cdots t_m$ with $xs_1 \leq x$ we have $l(xs_2 \cdots s_{m-1}){=}l(x)+m-2$, then $t_1t_2 \cdots t_m$ is in a finite parabolic subgroup of $W$ generated by two simple reflections.*

***Proof.*** If $m = 2$, by Lemma 1.2, the result is clear. Now assume that $m \geq 3$. Let $s = t_{m-1}$, $t = t_m$, and $y = xt_2 \cdots t_{m-1}$. Then $s, t \in R(y)$. By Lemma 1.3, $y = y_1s^a(ts)^b$ and $l(y) = l(y_1) + a + 2b$, here $a = 0$ or 1 and $s^a(ts)^b$ is the longest element of the subgroup $W_{st}$ of $W$ generated by $s, t$. Write $t_1 \cdots t_{m-1} = t_1 \cdots t_is^d(ts)^c$, $t_i \neq t, s$, $d = 0$ or 1, $c \geq 0$ and $d+2c+i = m-1$. We understand that $t_1 \cdots t_i$ is the neutral element $e$ of $W$ if $i = 0$.

We need to show that $i = 0$. Since $t_m = t$ and $t_1t_2 \cdots t_m$ is a reduced expression, we must have $d + 2c < a + 2b$.

Assume $a+2b = d+2c+1$. If $i \geq 1$, then $t_1t_2 \cdots t_{m-1}t_m = t_1 \cdots t_is^a(ts)^b$ and $R(xt_2 \cdots t_i) \cap \{s, t\}$ contains exactly one element, denoted by $r$. (We understand that $t_2 \cdots t_i = e$ if $i = 1$.) Then $t_1t_2 \cdots t_m$ has a reduced expression in the form $t_1t_2 \cdots t_ir \cdots$. Since $xt_1 \leq x$ and $i \leq m - 2$, by the assumptions of the lemma, we know that $xt_2 \cdots t_ir$ has length $l(x) + i$, which contradicts that $xt_2 \cdots t_ir \leq xt_2 \cdots t_i$. So $i = 0$ in this case.

Assume $a + 2b > d + 2c + 1$, then $b > c$ since $0 \leq a, d \leq 1$. We have $y_1s^a(ts)^b = (xt_1)t_1t_2 \cdots t_is^d(ts)^c$. So $y_1s^a(ts)^{b-c}s^d = (xt_1)t_1t_2 \cdots t_i$ and $l(s^a(ts)^{b-c}s^d) \geq 2$. Then $y_1s^a(ts)^{b-c}s^d = y_3st$ or $y_3ts$ for some $y_3$ with $l(y_1s^a(ts)^{b-c}s^d) = l(y_3)+2$. If $i \geq 1$, we have $y_1s^a(ts)^{b-c}s^dt_i \leq y_1s^a(ts)^{b-c}s^d$. Since $t_i \neq s, t$, by Lemma 2.2, the above is impossible. The contradiction leads $i = 0$. That is, all $t_1, t_2, \ldots, t_m$ are in $\{s, t\}$. Hence the lemma is proved. □

**Remark.** The lemma is not true in general. For instance, let $(W, S)$ be of type $A_3$, $s_1, s_2, s_3$ are simple reflections such that $s_1s_3 = s_3s_1$. Consider $x = t_1t_2t_3t_4 = s_2s_1s_3s_2$.

**Lemma 2.4.** *Let $x \in W$ and $t_1 \cdots t_m \cdots t_n$ $(1 < m < n)$ be a reduced expression of an element in $W$. Assume that (1) $l(xt_2 \cdots t_{m-1}t_{m+1} \cdots t_{n-1})$ has length $l(x) + n - 3$, (2) $xt_1 \le x$, (3) $xt_2 \cdots t_{m-1}t_m \le xt_2 \cdots t_{m-1}$, and (4) $xt_2 \cdots t_{m-1}t_{m+1} \cdots t_{n-1}t_n \le xt_2 \cdots t_{m-1}t_{m+1} \cdots t_{n-1}$. Further, assume that $t_1t_2 \cdots t_m$ (resp. $t_m \cdots t_n$) is in a parabolic subgroup $P$ (resp. $Q$) of $W$ with rank 2. Then $P = Q$ is finite and $n = m + 1$. In particular, $t_1 \cdots t_n$ is in a finite parabolic subgroup of $W$ generated by two simple reflections.*

***Proof.*** Let $t_m = s$ and $t_{m-1} = r$. Then $R(xt_2 \cdots t_{m-1})$ contains $r, s$. Since the graph of $W$ is complete, any parabolic subgroup of $W$ generated by more than two simple reflections is infinite. By Lemma 1.2 we know that $R(xt_2 \cdots t_{m-1})$ is exactly $\{r, s\}$ and $P =< r, s >$ (the subgroup of $W$ generated by $r, s$) is finite. Assume that $Q$ is generated by $s, t$. Clearly $t_{m+1} \ne r, s$, so $t_{m+1} = t$. Let $xt_2 \cdots t_{m-1} = yt_m$. Then $l(yt_m) = l(y) + 1$ and $R(y)$ does not contain $s$. We must have $t \in R(y)$. Otherwise, $R(y) \cap \{s, t\}$ is empty and $xt_2 \cdots t_{m-1}t_{m+1} \cdots t_n = yt_mt_{m+1} \cdots t_n$ has length $l(x) + n - 2$. It contradicts the assumption $xt_2 \cdots t_{m-1}t_{m+1} \cdots t_{n-1}t_n \le xt_2 \cdots t_{m-1}t_{m+1} \cdots t_{n-1}$. Therefore $xt_2 \cdots t_{m-1} = yt_m = y_1ts = y_2srs$ has length $l(y_1) + 2 = l(y_2) + 3$. So $y_1t = y_2sr$ has length $l(y_1) + 1) = l(y_3) + 2$. By Lemma 2.2 we must have $t = r$ and then $n = m + 1$. The lemma is proved. □

**Lemma 2.5.** *Let $x, w, y$ be elements in $W$. Assume that $w$ is in a parabolic subgroup generated by two simple reflections $r, s \in S$, $l(w) \ge 3$ and $r, s$ are not in $R(x) \cup L(y)$. Then $l(xwy) = l(x) + l(w) + l(y)$.*

***Proof.*** By Lemma 2.2, $R(xw) = R(w)$. Let $t_1 \cdots t_n$ be a reduced expression of $y$. Assume that $l(xwt_1 \cdots t_{m-1}) = l(x) + l(w) + m - 1$, $xwt_1 \cdots t_{m-1}t_m \le xwt_1 \cdots t_{m-1}$, and $m \le n$ is minimal for all reduced expressions of $y$, then $m \ge 2$. By Lemma 2.3, there exists $t_0 \in R(w)$ such that $t_0t_1 \cdots t_{m-1}$ is in the finite parabolic subgroup of $W$ generated by $t_0, t_1$. Since $l(w) \ge 3$ and $r, s$ are not in $R(x) \cup L(y)$, by Lemma 2.2, $R(xwt_0)$ does not contain $t_1 \in L(y)$. Thus $t_0t_1 \cdots t_{m-1}$ is the longest element of the parabolic subgroup of $W$ generated by $t_0, t_1$ and $R(xwt_1 \cdots t_{m-1}) = \{t_0, t_1\}$. So $t_0t_1 \cdots t_{m-1} = t_1 \cdots t_m$. Thus $t_0 \in L(y)$. This contradicts that $r, s \notin R(x) \cup L(y)$. The lemma is proved. □

**Corollary 2.6.** *Let $r, s$ be simple reflections and $x, y, z \in W$ such that $x = yrs$ with $l(x) = l(y) + 2$, $R(x) = \{s\}$, $R(yr) = \{r\}$, $r, s \notin L(z)$. Then $l(xz) = l(x) + l(z)$.*

***Proof.*** It follows from the proof of the above lemma. □

**Lemma 2.7.** *Assume that $w,u$ are elements of a finite parabolic subgroup $P$ of $W$ generated by two simple reflections. Then $\deg f_{w,u,v} \le l(v)$ for $v \in P$ and $f_{w,u,v} = 0$ if $v \notin P$. (Recall that $f_{w,u,v}$ is a polynomial in $q^{\frac{1}{2}} - q^{-\frac{1}{2}}$.)*

***Proof.*** The first assertion follows from 1.1 (a) and the second assertion is clear.

**Lemma 2.8.** *Let $r,s,t$ be simple reflections and $x,y,z \in W$. Assume that $x = yrs$, $R(yr) = \{r,t\}$, $R(x) = \{s\}$, $R(y) = \{t\}$. If $r,s \notin L(z)$, then deg $f_{x,z,w} \le 1$ for all $w$ in $W$.*

***Proof.*** If $l(xz) = l(x) + l(z)$, nothing needs to be proved. Assume that $l(xz) < l(x) + l(z)$. Let $t_1 t_2 \cdots t_n$ be a reduced expression of $z$. Then we can find a positive integer $m$ such that $xt_1 \cdots t_{m-1}t_m \le xt_1 \cdots t_{m-1}$. By assumptions of the lemma, clearly we have $m \ge 2$. We choose the reduced expression of $z$ so that $m$ is minimal in all possibilities. According to Lemma 2.3, $st_1 \cdots t_{m-1}$ is in the parabolic subgroup of $W$ generated by $s, t_1$.

We claim that $t_1 = t$. Otherwise, for $r,s \notin L(z)$, Lemma 2.2 implies that the element $st_1 \cdots t_{m-1} = t_1 \cdots t_m$ is the longest element of the subgroup $< s, t_1 >$ of $W$ generated by $s, t_1$. This contradicts that $s \notin L(z)$.

Let $y_1 \in W$ be such that $x = y_1 trts$. Then $l(x) = l(y_1) + 4$. By Lemma 2.2 we know that $R(y_1 tr) = \{r\}$. So $tst_1 \cdots t_{m-1}$ is the longest element $w_{st}$ in $< s,t >$. Then (recall that $\xi = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$)

$$\tilde{T}_x \tilde{T}_z = \xi \tilde{T}_{y_1 trw_{st}} \tilde{T}_{t_{m+1} \cdots t_k} + \tilde{T}_{y_1 trw_{st}t_m} \tilde{T}_{t_{m+1} \cdots t_k}.$$

We must have $s,t \notin L(t_{m+1} \cdots t_n)$. Otherwise $t_1 \cdots t_{m+1}$ is the longest element of $< s,t >$ and $s \in L(z)$, which contradicts our assumptions.

Since $l(w_{st}) \ge 3$, by Lemma 2.5, we have

$$\tilde{T}_{y_1 trw_{st}} \tilde{T}_{t_{m+1} \cdots t_k} = \tilde{T}_{y_1 trw_{st} t_{m+1} \cdots t_k}.$$

If $l(w_{st} t_m) \ge 3$, using Lemma 2.5, we get

$$\tilde{T}_{y_1 trw_{st} t_m} \tilde{T}_{t_{m+1} \cdots t_k} = \tilde{T}_{y_1 trw_{st} t_m t_{m+1} \cdots t_k}.$$

We are done in this case.

Assume now $l(w_{st}t_m) = 2$, then $w_{st} = tst$, $m = 2$, $t_1 = t$, $t_2 = s$. So $y_1trw_{st}t_m = y_1trst$. If the longest element $w_{rt}$ of $< r, t >$ is at least 4, then $w_{rt}t$ has the length of at least 3. Since $s, t \notin L(t_3 \cdots t_n)$ and $r, s \notin L(t_1 \cdots t_n)$ we see that $r, t \notin L(stt_3 \cdots t_n)$. Write $y_1trw_{st}t_m = y_2w_{rt}tst$, then $l(y_2w_{rt}tst) = l(y_2) + l(w_{rt}t) + 2$. By Lemma 2.5, we know that

$$\tilde{T}_{y_1trw_{st}t_m}\tilde{T}_{t_{m+1}\cdots t_k} = \tilde{T}_{y_1trw_{st}t_mt_{m+1}\cdots t_k}.$$

We are done in this case.

Assume now $w_{st} = sts$ and $w_{rt} = rtr$. Let $u = y_1trw_{st}t_m = y_1trst$. Assume that $s_1 \cdots s_{n-2}$ is a reduced expression of $t_3 \cdots t_n$ and $us_1 \cdots s_{i-1}s_i \leq us_1 \cdots s_{i-1}$ and $i$ is minimal in all possibilities. Note that $R(u) = \{t\}$. By Lemma 2.3, $ts_1 \cdots s_{i-1}$ is in a parabolic subgroup of $W$ of rank 2. Since $s, t \notin L(t_3 \cdots t_n)$, we have $i \geq 2$. Also, we have $s_1 = r$ and $R(ut) = \{r, s\}$. Otherwise, Lemma 2.2 implies that $ts_1 \cdots s_{i-1} = s_1 \cdots s_i$ is the longest element in $< t, s_1 >$, so $t \in L(t_3 \cdots t_n)$, is a contradiction. Now we have $R(u) = \{t\}$, $R(ut) = \{r, s\}$, $R(uts) = \{r\}$ and $s, t \notin L(t_3 \cdots t_n)$. So one can use induction on $l(z)$ to see the lemma is true in this case.

The lemma is proved. $\square$

**2.9.** Now we prove Theorem 2.1. Let $x, y \in W$ and consider

$$\tilde{T}_x\tilde{T}_y = \sum_{z \in W} f_{x,y,z}\tilde{T}_z.$$

We will prove that $\deg f_{x,y,z} \leq a_0$, here $a_0$ is the maximal number among the lengths of the longest elements of all finite parabolic subgroups of $W$. Let $t_1t_2 \cdots t_k$ be a reduced expression of $y$. We may assume that $xt_1 \leq x$, otherwise $x$ is replace by $xt_1$. We may further assume that $xs_1 \leq x$ for any reduced expression $s_1 \cdots s_m$ of $y$.

We use induction on $k$. For $k = 0, 1$, the result is clear. Now assume that $k > 1$.

If $xt_2 \cdots t_k$ has length $l(x) + k - 1$, then we have

$$\tilde{T}_x\tilde{T}_y = \xi\tilde{T}_{xt_2\cdots t_k} + \tilde{T}_{xt_1}\tilde{T}_{t_1y},$$

where $\xi = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Using induction hypothesis we see the theorem is true in this case.

Now assume that $xt_2 \cdots t_{m-1}t_m \leq xt_2 \cdots t_{m-1}$ for some $2 \leq m \leq k$. We may require that $m' \geq m > 1$ if $s_1s_2 \cdots s_k$ is another reduced expression of $y$ and $xs_2 \cdots s_{m'-1}s_{m'} \leq xs_2 \cdots s_{m'-1}$. By Lemma 2.3, $t_1 \cdots t_m$ is in the parabolic subgroup $P$ generated by $t_1, t_2$.

Let $x_1$ (resp. $y_1$) be the element in the coset $xP$ (resp. $Py$) with minimal length. Let $u, v \in P$ be such that $x = x_1w$ and $y = uy_1$. Then we have

$$\tilde{T}_x\tilde{T}_y = \sum_{v \in P} f_{w,u,v}\tilde{T}_{x_1v}\tilde{T}_{y_1}.$$

By Lemma 2.7, $\deg f_{w,u,v} \leq l(v)$ and $v \in P$ if $f_{w,u,v} \neq 0$. If $l(v) \geq 3$, by Lemma 2.5, $l(x_1vy_1) = l(x_1v) + l(y_1)$. Hence $\tilde{T}_{x_1w}\tilde{T}_{y_1} = \tilde{T}_{x_1wy_1}$. If $l(v) = 2$, using Corollary 2.6 and Lemma 2.8 we see that $\deg f_{x_1v,y_1,z} \leq 1$ for any $z$. If $l(v) = 0$, by induction hypothesis, we see that the degrees of $f_{x_1,y_1,z}$ are not greater than $a_0$ for any $z \in W$.

Now consider the case $l(v) = 1$. In this case $v$ is a simple reflection. We have $l(x_1v) = l(x_1) + 1$ and $l(vy_1) = l(y_1) + 1 < l(y)$ since $m \geq 2$. Applying induction hypothesis to the equality

$$\tilde{T}_{x_1v}\tilde{T}_{vy_1} = \xi\tilde{T}_{x_1v}\tilde{T}_{y_1} + \tilde{T}_{x_1}\tilde{T}_{y_1},$$

we see that $\deg f_{x_1v,y_1,z} \leq a_0 - 1$ for any $z \in W$.

Theorem 2.1 is proved. $\square$

**Corollary 2.10.** *Keep the assumption of Theorem* 2.1. *Let $a_0$ be the maximal number among the lengths of the longest elements of all finite parabolic subgroups of $W$. Then $a(w) = a_0$ if and only if $w = xuy$ for some $x, u, y \in W$ with $u$ being the longest element of a finite parabolic subgroup, $l(w) = l(x) + l(y) + l(u)$ and $l(u) = a_0$.*

This is clear from the proof of Theorem 2.1.

## 3. Some Consequences I - the Lowest Two-Sided Cell

In this section $(W, S)$ is a Coxeter group with completed graph. The set $S$ may be infinite, but the cardinalities of finite parabolic subgroups of $W$ are required to have a common upper bound. The lowest two-sided cell of W is discussed. Let $a_0$ be the maximal value of the lengths of the longest elements of finite parabolic subgroups of $W$ and let $\Lambda$ be the set consisting of all the longest elements of finite parabolic subgroups of the maximal cardinality (which is $2a_0$). Let $D'$, $D$ and $\Gamma_x$ $(x \in D')$ be as in subsection 1.8.

**Proposition 3.1.** *Keep the assumptions and notations above. We have*

(a) *The lowest two-sided cell of $W$ coincides with the set $\{w \in W \mid a(w) = a_0\}$. So for any $x$ in the lowest two-sided cell, there exists $y, z \in W$ and $u \in \Lambda$ such that $x = zuy$ and $l(x) = l(z) + l(u) + l(y)$.*

(b) *The map $x \to \Gamma_x$ defines a bijection between the set $D'$ and the set of left cells in the lowest two-sided cell $c_0$.*

(c) *The set $D = \{y^{-1}uy \mid u \in \Lambda,\ uy \in D',\ l(uy) = l(u) + l(y)\}$ is the set of distinguished involutions in the lowest two-sided cell.*

(d) *Let $z, z' \in W$ and $uy \in D'$ be such that $u \in \Lambda$, $l(zuy) = l(z) + l(u) + l(y)$ and $l(z'uy) = l(z') + l(u) + l(y)$. Then $C_{zu}C_{uy} = h_{u,u,u}C_{zuy}$ and $\mu(z'uy, zuy) = \mu(z'u, zu)$.*

***Proof.*** Let

$$\Omega = \{zuy \in W \mid y, z \in W,\ u \in \Lambda,\ \text{and}\ l(zuy) = l(z) + l(u) + l(y)\}.$$

We claim that $\Omega$ is the lowest two-sided cell. Since $\Lambda \subset \Omega$, it suffices to prove that $\Omega$ is a two-sided cell.

Let $x \in \Omega$. Then there exist $y, z \in W$ and $u \in \Lambda$ such that $x = zuy$ and $l(x) = l(z) + l(u) + l(y)$. It is no harm to assume that $uy$ is in $D'$. By computing $\tilde{T}_{zu}\tilde{T}_{uy}$, it is easily seen $\gamma_{zu,uy,w} \neq 0$ if and only if $w = x$ and $\gamma_{zu,uy,x} = 1$. This implies that $C_{zu}C_{uy} = h_{u,u,u}C_{zuy}$. The first part of (d) is proved.

Let $w \in W$ and $w \underset{LR}{\sim} x$. There exists $w = w_1, w_2, \ldots, w_n = x$ such that $\mu(w_i, w_{i+1}) \neq 0$ or $\mu(w_{i+1}, w_i) \neq 0$, and $L(w_i) \not\subset L(w_{i+1})$ or $R(w_i) \not\subset R(w_{i+1})$ for all $i = 1, 2, \ldots, n-1$. We show that all $w_i$ are in $\Omega$. It is no harm to assume that $n=2$ and $L(w)\not\subset L(x)$. Let $s$ be the simple reflection in $L(w)-L(x)$. Then $C_w$ appears in $C_sC_x$ with coefficient $\mu(w,x)$. Using the identity $C_{zu}C_{uy}=h_{u,u,u}C_{zuy}$, we see that there exists $z_1\in W$ such that $l(z_1u)=l(z_1)+l(u)$, $C_{z_1u}$ appears in $C_sC_{zu}$ and $\gamma_{z_1u,uy,w}\neq 0$. We must have $w=z_1uy$ and $\mu(z_1u,zu)=\mu(w,x)$ or $\mu(zu,z_1u)=\mu(x,w)$. So $w\in\Omega$. Part (a) is proved.

In addition, we have shown that $\Gamma_{uy}=\{zuy\,|\,z\in W,\ l(zuy)=l(z)+l(u)+l(y)\}$ is a left cell of $W$. Let $u_1y_1\in D'$, $l(u_1y_1)=l(u_1)+l(y_1)$, $u_1$ has the length $a_0$ and is the longest element of a finite parabolic subgroup of $W$. If $u_1y_1\in\Gamma_{uz}$, then $u_1y_1=zuy$ for some $z\in W$ and $l(zuy)=l(z)+l(u)+l(y)$. By the definition of $D'$ we see that $z=e$. So $u_1y_1=uy$. Part (b) is proved.

Comparing the coefficients of $\tilde{T}_e$ in both sides of the equality $C_{zu}C_{uy}=h_{u,u,u}C_{zuy}$, we see that the $l(w)-2\deg P_{e,zuy}-a(u)\geq 0$. Moreover, $l(w)-2\deg P_{e,zuy}-a(u)=0$ if and only if $z=y^{-1}$. In this case, the coefficient of the term $q^{l(y)}$ is 1. Part (c) is proved.

Now to prove the second part of (d), let $E_y,F_y\in H$ be such that $C_uF_y=C_{uy}$ and $E_yC_u=C_{y^{-1}u}$. Then $C_{zu}F_y=C_{zuy}$ and $E_yC_{uz^{-1}}=C_{(zuy)^{-1}}$. Thus $h_{uz^{-1},z'u,w}=h_{y^{-1}uz^{-1},z'uy,y^{-1}wy}$. Assume that $z'uy<zuy$, comparing the coefficients of $\tilde{T}_e$ on both sides of the equality $C_{(zuy)^{-1}}C_{z'uy}=\sum_w h_{y^{-1}uz^{-1},z'uy,y^{-1}wy}C_{y^{-1}wy}=\sum_w h_{uz^{-1},z'u,w}C_{y^{-1}wy}$, as in [7, 2.2], we see that the second part of (d) is true.

The proposition is proved. □

**Proposition 3.2.** *Let $(W,S)$ be a Coxeter group with complete Coxeter graph and the cardinalities of finite parabolic subgroups of $W$ have a common upper bound. Assume the cardinality of $S$ is greater than 2 and the order of $st$ is finite for some simple reflections $s,t$ in $S$. Then the number of left cells in the lowest two-sided cell of $W$ is finite if and only if $W$ is an affine Weyl group of type $\tilde{A}_2$.*

***Proof.*** The if part is clear (see [5]). Now assume that $W$ is not of type $\tilde{A}_2$. Let $s,t$ be simple reflections such that the order $st$ is finite and maximal in all possibilities. Let $w$ be the longest element of the subgroup $<s,t>$ of $W$ generated by $s,t$. Then $w$ is in the lowest two-sided cell of $W$. If $w$ has length of at least 4, then $w(rst)^k$ is in $D'$ (see 1.8 for the definition of $D'$) for any positive integer $k$, here $r$ is a simple reflection in $S-\{s,t\}$. By

Proposition 3.1 (b), we know that the number of left cells in the lowest when two sided cell of $W$ is infinite.

If $w$ has the length 3, then either $|S| \geq 4$ or one of $rs$, $rt$ has infinite order for $r \in S - \{s,t\}$ since $W$ is not of type $\tilde{A}_2$ and the length of $w$ is maximal among the longest elements of finite parabolic subgroups of $W$. In the first case, we can find two different simple reflections $r, v$ in $S - \{s,t\}$. Hence $w(rvst)^k$ is in $D'$ for any positive integer $k$. For the second case, let $r \in S$ be different from $s,t$. It is no harm to assume that $rs$ has infinite order. Hence $w(rs)^k$ is in $D'$ for any positive integer $k$. By Proposition 3.1 (b), in both cases the number of left cells in the lowest two-sided cell of $W$ is infinite.

The proposition is proved. □

## 4. Some Consequences II - Other Results

In this section, $(W,S)$ is a Coxeter group such that any two simple reflections are not commutative, except when other specifications are given. We shall give some other consequences of Theorem 2.1.

In [4], Lusztig showed that the elements in $W$ with the unique reduced expressions form a two-sided cell of $W$. If the order of $st$ is $\infty$ for any two different simple reflections $s,t$ of the Coxeter group $(W,S)$, then $W$ has only two two-sided cells: $\{e\}$, $W - \{e\}$, see [6].

**Proposition 4.1.** *Let $m \geq 3$ be a positive integer. Assume that the order of $st$ is either $m$ or $\infty$ for any two different simple reflections $s,t$ of the Coxeter group $(W,S)$ and the order of some $st$ is $m$. Then $W$ has only three two-sided cells.*

***Proof.*** If $w \in W$ has different reduced expressions, then there exists simple reflections $s,t$ in $W$ and $x,y \in W$ such that $st$ has order $m$ and $w = xuy$, $l(w) = l(x) + l(u) + l(y)$, where $u$ is the longest element in the subgroup of $W$ generated by $s,t$. By Theorem 2.1, $m$ is the maximal value of the $a$-function on $W$. According to Proposition 3.1, $w$ is in the lowest two-sided cell of $W$. Therefore, $W$ has only the following three two-sided cells: $\{e\}$, {elements in $W$ with unique reduced expression}, {elements in $W$ having different reduced expressions}. The proposition is proved. □

**4.2.** Assume that any two simple reflections in $W$ are not commutative. Let $O$ be the set of isomorphism classes of finite parabolic subgroups of $W$ with rank 2. It is likely that the number of the two-sided cells of $W$ is $|O|+2$, here $|O|$ denotes the cardinality of $O$. Proposition 4.1 supports this conjecture. Below we will see that when $W$ is crystallographic, the conjecture is also true. We first establish some lemmas.

**Lemma 4.3.** *Let $P$ and $Q$ be two different finite parabolic subgroups of $W$ with rank 2. Denote their longest elements by $w$ and $u$ respectively. Assume $l(w) \leq l(u)$. Let $x, y \in Q$ be such that $l(wx) = l(w) + l(x)$, $l(wx) = l(wxu) + l(u)$, $l(ywx) = l(y) + l(wx)$ and $l(y) = l(wx) - l(u) - 1$. Then $\mu(u, ywx) = 1$.*

***Proof.*** The existence of $x$ is clear. Since $l(w) \geq 3$, by Lemma 2.5, $y$ exists. Using the formulas (2.2.c) and (2.3.g) in [3], we can prove this lemma by a direct computation. □

**Corollary 4.4.** *Let $P$ and $Q$ be two finite parabolic subgroups of $W$ with rank 2. Denote their longest elements by $w$ and $u$ respectively. Then $u \underset{LR}{\leq} w$ if $l(u) \geq l(w)$. In particular, $w$ and $u$ are in the same two-sided cell if $l(w) = l(u)$.*

***Proof.*** Let $y, x$ be as in Lemma 4.3. Since $l(w) \leq (u)$, we have $l(y) < l(x)$ and $L(ywx)$ is a proper subset of $L(u)$. Lemma 4.3 then implies that $u \underset{L}{\leq} ywx$. Clearly, $ywx \underset{LR}{\leq} w$. So $u \underset{LR}{\leq} w$. The lemma is proved. □

**Proposition 4.5.** *Let $(W, S)$ be a crystallographic Coxeter group with complete Coxeter graph and $O$ be the set of isomorphism classes of finite parabolic subgroups of $W$ with rank 2. Then the number of the two-sided cells of $W$ is $|O| + 2$.*

***Proof.*** By Theorem 2.1, the maximal value of the $a$-function on $W$ is at most 6. For $i = 3, 4, 6$, let $W_i$ be the set of elements $x$ of $W$ with the properties below: (1) $x = zuy$ for some $z, y \in W$, $u$ has the length $i$ and is the longest element of a parabolic subgroup of $W$, (2) $l(x) = l(z)+l(u)+l(y)$.

We have two obvious two-sided cells: $\{e\}$ and $\{$ elements in $W$ with unique reduced expression$\}$. We claim that $W_6, W_4 - W_4 \cap W_6$, $W_3 - W_3 \cap (W_4 \cup W_6)$ are two-sided cells whenever they are not empty.

First, assume that $W_6$ is not empty. According to Proposition 3.1, $W_6$ is the lowest two-sided cell of $W$. We claim that $W_4 - W_4 \cap W_6$ is a two-sided cell of $W$ if it is not empty. Clearly, $a(x) \geq 4$ for any $x \in W_4$. From the argument of Theorem 2.1, it is easily seen that $a(x) \leq 4$ if $x$ is not in $W_6$. Since $W$ is crystallographic, by [6, Corollary 1.9], for $x \in W_4 - W_4 \cap W_6$ we have $x \underset{LR}{\sim} u$, here $x = zuy$ is as in (1). Using Corollary 4.4 we know that $W_4 - W_4 \cap W_6$ is in a two-sided cell.

Let $w$ be in $W_3$ but not in $W_4 \cup W_6$. According to [6, Proposition 1.4], $\gamma_{w^{-1},w,d} = 1$, here $d$ is the distinguished involution in the left cell containing $w$. Using the positivity for Kazhdan-Lusztig polynomials of $W$ and for $h_{w,w',w''}$ with $w, w', w''$ in $W$, we see that the argument for Theorem 2.1 implies that $v \in W_6$ if $\deg h_{w^{-1},w,v} \geq 4$. Since $a(w) = a(d) \geq 3$ and $d$ is not in $W_6$, we must have $a(w) = 3$ and $w$ is not in the two-sided cell containing $x$. Therefore, $W_4 - W_4 \cap W_6$ is a two-sided cell if it is not empty. We also showed that $W_3 - W_3 \cap (W_4 \cup W_6)$ is a two-sided cell if it is not empty.

If $W_6$ is empty, the discussion is similar and simpler. The proposition is proved. $\square$

## 5. Examples

In this section the left cells in the Coxeter groups considered in Proposition 4.1. Let $(W, S)$ be a Coxter group of rank greater than 2 and $m \geq 3$ be a positive integer. Assume that the order of $st$ is either $m$ or $\infty$ for any two different simple reflections $s, t$ in $S$ and the order of some $st$ is $m$. By Proposition 4.1 and its proof, $W$ has only three two-sided cells: the trivial two-sided cell $\{e\}$; $\Omega_1 = \{$elements in $W$ with unique reduced expression$\}$; $\Omega = \{$elements in $W$ having different reduced expressions$\}$. Below we describe left cells in $\Omega_1$ and $\Omega$.

For any simple reflection $s$ in $S$, define $\Gamma_{1,s}$ to be the subset of $\Omega_1$ consisting all $w \in \Omega_1$ satisfying $R(w) = \{s\}$. Then we have

(a) The map $s \to \Gamma_{1,s}$ defiens a bijection between $S$ and the set of all left cells in $\Omega_1$. See [4].

(b) Let $w = t_1t_2\cdots t_p$ be a reduced expression of $w \in W$. It is clearly that $w$ belongs to $\Omega_1$ if and only if $t_it_{i+1}\cdots t_{i+m-1}$ is not in a finite parabolic subgroup of $W$ (of rank 2) for any $i = 1, 2, \ldots, p - m + 1$.

To describe the left cells in the lowest two-sided cell $\Omega$ of $W$, we need the sets $\Lambda$ and $D'$, see Section 3 and 1.8 respectively for the definition. In current case, $\Lambda$ is the set consisting of all the longest elements of finite parabolic subgroups of $W$ of rank 2. The length of any element in $\Lambda$ is $m$. For each $w \in \Lambda$, let $D'_w$ be the set consisting of all elements $x \in W$ such that

(1) $x = wy$ for $y \in W$ and $l(x) = l(w) + l(y)$,
(2) for any $s$ in $L(w)$, there are no $z, z', u \in W$ such that $sx = zuz'$, $l(sx) = l(z) + l(u) + l(z')$ and $u \in \Lambda$

The set $D'$ is the union of all $D'_w, w \in \Lambda$.

For any $x \in D'$, recall that $\Gamma_x$ is the subset of $W$ consisting of all elements $zx$ satisfying $l(zx) = l(z) + l(x)$ (see 1.8). By Proposition 3.1, we have

(c) The set $\Gamma_x$ is a left cell in $\Omega$ and the map $x \to \Gamma_x$ is a bijection between the set $D'$ and the set of all left cells in $\Omega$.

The following statements (d) and (e), which are easy to check, show that the sets $D'_w$ and $\Gamma_x$ are computable.

(d) Let $w \in \Lambda$ and $y = t_1t_2\cdots t_p \in W$ be a reduced expression. Then $wy$ is $D'_w$ if and only if the following three conditions are satisfied: (1) $y$ is in $\Omega_1$ or $y = e$, (2) $L(y) \cap R(w)$ is empty, (3) when $p \geq m - 1$, for any simple reflection $s \in R(w)$, $t_1t_2\cdots t_{m-1}$ is not in the subgroup $< t_1, s >$ of $W$ generated by $t_1$ and $s$, as long as the order of $t_1s$ is $m$.

(e) For $w \in \Lambda$, $x \in D'_w$ and $z \in W$, the element $zx$ is in $\Gamma_x$ if and only if $R(z) \cap L(w)$ is empty.

Let $(W, S)$ be a crystallographic Coxeter group of rank 3 with complete graph. We assume that $(W, S)$ is not of type $\tilde{A}_2$ and any parabolic subgroup of $W$ with rank 2 is finite. Then there are nine such Coxeter groups, two of them are among the Coxeter groups discussed above: the order of $st$ is 4 (resp. 6) for any $s, t \in S$. The left V-cells of these nine Coxeter groups $(W, S)$ are described in [1].

It should not be difficult to describe the left cells in $(W, S)$ for other seven cases by using the results in section 3, section 4 and [5].

## 6. Some Comments

In this section two questions are proposed. Let $(W, S)$ be an arbitrary Coxeter group. In [5], $a(w) \leq l(w)$ for any $w$ in a Weyl group is shown. This result was extended to arbitrary crystallographic Coxeter groups by Springer, see [6] for the proof. It is natural to suggest that $a(w) \leq l(w)$ for $w$ in an arbitrary Coxeter group.

Assume that $(W, S)$ is connected (i.e. its Coxeter graph is connected). Let $P$ and $Q$ be two finite parabolic subgroups of $W$. It is likely that the longest elements of $P$ and $Q$ are in the same two-sided cell of $W$ if $P$ and $Q$ are isomorphic Coxeter groups.

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