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ZONAL POLYNOMIALS AND QUANTUM ANTISYMMETRIC MATRICES

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Abstract

We study the quantum symmetric spaces for quantum general linear groups modulo symplectic groups. We first determine the structure of the quotient quantum group and completely determine the quantum invariants. We then derive the characteristic property for quantum Phaffian as well as its role in the quantum invariant sub-ring. The spherical functions, viewed as Macdonald polynomials, are also studied as the quantum analog of zonal spherical polynomials.

1. Introduction

The regular representation of $GL(n,\mathbb{C})$ can be realized on the ring

$$A(X)=\mathbb{C}[x_{11},x_{12},\dots,x_{nn}] \tag{1}$$

where regular functions are polynomials of the matrix elements of the $n\times n$ matrices. It is well known that $A(X)$ is a completely reducible $GL(n,\mathbb{C})$-module and the associated irreducible polynomial sub-representations are parametrized by the set of partitions

$$P_n=\{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n\ ;\ \lambda_1\geq\cdots\geq\lambda_n\geq 0\}\,. \tag{2}$$

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For a given $\lambda \in P_n$, there is a unique (up to isomorphism) irreducible representation $V(\lambda)$ with highest weight λ. Similarly, one considers the modules $A(Sym(n))$ of the polynomials in the coordinates of the $n \times n$ symmetric matrix, and the module $A(Skew(2n))$ of the polynomials in the coordinates of the $2n \times 2n$ skew symmetric matrix. These representations decompose into the multiplicity free sums [4, 3]:

$$A(Sym(n)) \simeq \bigoplus_{\lambda \in P_n} V(2\lambda) \tag{3}$$

$$A(Skew(2n)) \simeq \bigoplus_{\lambda \in P_{2n}} V(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots, \lambda_n, \lambda_n), \tag{4}$$

which are invariant under the action of $O(n, \mathbb{C})$ and $Sp(2n, \mathbb{C})$ respectively. As L. Hua first noticed and A. James later formulated that the O_n and Sp_{2n} invariants are one dimensional and the zonal spherical functions enjoy similar properties of Schur symmetric functions [4, 6, 13].

In the case of quantum analog of the symmetric pair of general linear groups and symplectic groups, Noumi and Letzter [15, 12] showed that the quantum spherical functions are indeed certain Macdonald symmetric functions by working on the quantum algebra of the enveloping algebras. We will study directly the quantum invariant ring as a subring of the quantum general linear group. As in [8], we compute the Hopf ideal of quantum invariants for the symplectic case using certain quadratic polynomials of matrix coefficients of quantum general linear groups.

A new feature in current work on quantum invariants is that we will study the important role played by Pfaffian as in the classical symplectic case. In the quantum case, the quantum Phaffian played an important role in the invariant theory as well [18]. We first give a closed form definition for the quantum Phaffian and study its representation-theoretic meaning in the quantum setting. Through this we are able to give an appropriate quantum analog of its relations with quantum determinant. As expected, quantum Phaffians enjoy similar properties as quantum determinant in the orthogonal case.

This paper is organized as follows. In Section 2 we first recall some basic facts of certain quantum algebras, in particular, we discuss a quantum deformation of $A(X)$ and the associated quantum version of $GL(n, \mathbb{C})$ as presented in Noumi, Yamada, and Mimachi [16] and we recall the quantized universal enveloping algebra $U_q(\mathfrak{gl}(n,\mathbb{C}))$. In Section 3 we describe a quantum symplectic group, $Sp_q(2n,\mathbb{C})$. Since there does not seem to be a natural embedding of $Sp_q(2n,\mathbb{C})$ in $GL_q(2n,\mathbb{C})$ we define $Sp_q(2n,\mathbb{C})$ invariants (left and right) in an infinitesimal manner, similar to an earlier construction by Jing and Yamada [8] of polynomial invariants for a quantum orthogonal group. These quantum symplectic invariants give us a quantum version of the regular functions of the antisymmetric matrices. In addition to defining the generators of these functions, we describe their relations and we discuss a construction of a quantum analog to the Pfaffian function.

We then describe a complete reduction of the $Sp_q(2n,\mathbb{C})$ invariant spaces (left and right) into irreducible modules and we follow with a construction and characterization of the associated bi-invariant space and its basis of zonal polynomials. In the last section, a connection between the zonal polynomials and certain Macdonald polynomials is discussed.

2. Quantum Groups

Quantum groups are defined as certain one-parameter deformations of the algebra of algebraic functions on simple Lie groups [17]. In other words, we will describe $A_q(X)$ to be like the classical algebra $A(X)$, except with noncommuting relations imposed upon its generators. Throughout the paper we will let q be a complex number and for $q \neq 1$ we require that q not be a root of unity.

2.1. $A_q(X)$, $A(G)$ and $GL_q(n,\mathbb{C})$

We first define the algebra of functions $A_q(X)$ on $X = Mat_q(n,\mathbb{C})$ as a noncommutative $\mathbb{C}$-algebra

$$A_q(X) = \mathbb{C}_q\left[x_{11}, x_{12}, \ldots, x_{nn}\right]. \tag{5}$$

generated by $x_{11}, x_{12}, \ldots, x_{n,n}$ and with relations

$$\begin{aligned}
&x_{ik}x_{jk} = qx_{jk}x_{ik}, \quad x_{ki}x_{kj} = qx_{kj}x_{ki},\\
&x_{il}x_{jk} = x_{jk}x_{il},\\
&x_{ik}x_{jl} - x_{jl}x_{ik} = \left(q - q^{-1}\right)x_{il}x_{jk},
\end{aligned}$$

where $i < j$ and $k < l$. The relations can be visualized by the diagram (see Figure 1) with a "square" of generators.

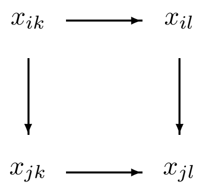

Figure 1: $A_q(X)$ Relations, $x \to y$ implies $xy = qyx$.

$A_q(X)$ is a bialgebra using the same coproduct and counit maps as defined on $A(X)$, see [14].

Let I and J be two subsets of $\{1, 2, \ldots, n\}$ with $\#I = \#J = r$ with ordered elements, i.e. $i_1 < i_2 < \ldots < i_r \in I$ and $j_1 < j_2 < \ldots j_r \in J$. The quantum r-minor determinants are defined as

$$\xi_J^I = \xi_{j_1,\ldots,j_r}^{i_1,\ldots,i_r} = \sum_{\sigma \in \mathfrak{S}_r} (-q)^{l(\sigma)} x_{i_1 j_{\sigma(1)}} x_{i_2 j_{\sigma(2)}} \ldots x_{i_r j_{\sigma(r)}} \tag{6}$$

where $l(\sigma)$ denotes the number of pairs (i, j) with $i < j$ and $\sigma(i) > \sigma(j)$. There is a unique quantum n-minor determinant, and it is denoted by det_q [8]. We define the algebra of regular functions $A(G)$ on the quantum group $GL_q(n, \mathbb{C})$ by adjoining det_q^{-1} to $A_q(X)$

$$A(G) = \left[x_{11}, x_{12}, x_{13}, \ldots, x_{nn}, det_q^{-1}\right] \tag{7}$$

Then, $GL_q(n, \mathbb{C})$ is defined as the spectrum of the Hopf algebra algebra $A(G)$, i.e.

$$GL_q(n, \mathbb{C}) = Spec(A(G)) \tag{8}$$

one usually refers to $GL(n, \mathbb{C})$ simply as $A(G)$.

In addition to the relations of $A_q(X)$, $A(G)$ also has the following relations [16]

$$x_{ij} \cdot det_q^{-1} = det_q^{-1} \cdot x_{ij} \tag{9}$$

$$det_q^{-1} \cdot det_q = det_q \cdot det_q^{-1} = 1. \tag{10}$$

This allows us to define the algebra morphism $S : A(G) \to A(G)$ by

$$S(x_{ij}) = (-q)^{i-j} \xi_{\hat{i}}^{\hat{j}} \cdot det_q^{-1} \quad 1 \leq i, j \leq n, \tag{11}$$

where $\hat{k} = \{1, \ldots, k-1, k+1, \ldots, n\}$. S is the antipode for $A(G)$ and makes $A(G)$ a Hopf algebra.

2.2. Additional Quantum Groups

In addition to the above mentioned quantum groups, we need some additional subgroups of $G = GL_q(n, \mathbb{C})$.

The diagonal subgroup H_n of $GL_q(n, \mathbb{C})$ is defined by its regular functions

$$A(H_n) = \mathbb{C}\left[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}\right]. \tag{12}$$

Associated with this commutative Hopf algebra, we have the restriction map $\pi_H : A(G) \to A(H_n)$ defined by

$$\pi_H(x_{ij}) = \delta_{i,j} t_i \tag{13}$$

The Borel subgroups B_+ and B_- of $GL_q(n, \mathbb{C})$ consist of the upper and lower triangular matrices and are defined in terms of their associated Hopf algebras

$$A(B_+) = \mathbb{C}\left[b_{ij}\right], \quad i \leq j, \tag{14}$$

$$A(B_-) = \mathbb{C}\left[b_{ij}\right], \quad i \geq j. \tag{15}$$

These algebras have relations induced from $A(G)$ and we note that the diagonal elements $b_{11}, \ldots, b_{nn}$ commute with each other, [16]. With each of these Hopf algebras we define the restrictions maps $\pi_{B_+} : A(G) \to A(B_+)$ and $\pi_{B_-} : A(G) \to A(B_-)$ respectively by

$$\pi_{B_+}(x_{ij}) = \begin{cases} b_{ij}, & (1 \leq i \leq j \leq n) \\ 0, & (i > j) \end{cases}, \tag{16}$$

$$\pi_{B_-}(x_{ij}) = \begin{cases} b_{ij}, & (1 \leq j \leq i \leq n) \\ 0, & (j > i) \end{cases}. \tag{17}$$

2.3. Enveloping Algebra $U_q(\mathfrak{g})$

We recall the quantum universal enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ or rather $\mathfrak{sl}(n, \mathbb{C})$[9]. Let L_n be the free $\mathbb{Z}$-module of rank n with the canonical basis $\{\epsilon_1, \ldots, \epsilon_n\}$, i.e. $L_n = \bigoplus_{k=1}^{n} \mathbb{Z}\epsilon_k$, endowed with the symmetric bilinear form $\langle \epsilon_i, \epsilon_j \rangle = \delta_{ij}$. We will define $\alpha_k = \epsilon_k - \epsilon_{k+1}$. Additionally, we will identify a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_n$ with $\lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n \in L_n$. We will refer to such an element of L_n as a dominant integral weight. The fundamental weights are defined by $\Lambda_k = \epsilon_1 + \cdots + \epsilon_k$ (see [8]). Now we define $U_q(\mathfrak{g})$ as the $\mathbb{C}$-algebra with generators e_k, f_k $(1 \le k < n)$ and q^λ $\left(\lambda \in \frac{1}{2}L_n\right)$ with the following relations [16]:

$$q^0 = 1, \quad q^\lambda q^\mu = q^{\lambda+\mu}, \tag{18}$$

$$q^\lambda e_k q^{-\lambda} = q^{\langle \lambda, \alpha_k \rangle} e_k \quad (1 \le k < n), \tag{19}$$

$$q^\lambda f_k q^{-\lambda} = q^{-\langle \lambda, \alpha_k \rangle} f_k \quad (1 \le k < n), \tag{20}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}} \quad (1 \le i, j < n), \tag{21}$$

$$e_i^2 e_j - \left(q + q^{-1}\right) e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1), \tag{22}$$

$$f_i^2 f_j - \left(q + q^{-1}\right) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1), \tag{23}$$

$$e_i e_j = e_j e_i, \quad f_i f_j = f_j f_i \quad (|i - j| > 1). \tag{24}$$

We define a coproduct, Δ_U, and a counit, ε_U, on the generators by

$$\Delta_U(q^\lambda) = q^\lambda \otimes q^\lambda, \quad \varepsilon(q^\lambda) = 1, \tag{25}$$

$$\Delta_U(e_k) = e_k \otimes q^{-\alpha_k/2} + q^{\alpha_k/2} \otimes e_k, \quad \varepsilon(e_k) = 0, \tag{26}$$

$$\Delta_U(e_k) = f_k \otimes q^{-\alpha_k/2} + q^{\alpha_k/2} \otimes f_k, \quad \varepsilon(f_k) = 0, \tag{27}$$

making $U_q(\mathfrak{g})$ a bialgebra. Additionally, with the antipode S_U defined by

$$S_U(q^\lambda) = q^{-\lambda}, \tag{28}$$

$$S_U(e_k) = -q^{-1} e_k, \tag{29}$$

$$S_U(f_k) = -q f_k. \tag{30}$$

$U_q(\mathfrak{g})$ becomes a Hopf algebra.

2.5. $A(G), U_q(\mathfrak{g})$ Duality

There exists a well-known dual pairing of Hopf algebras $U_q(\mathfrak{g})$ and $A(G)$

$$a(\varphi) \in \mathbb{C}, \qquad a \in U_q(\mathfrak{g}), \varphi \in A(G) \tag{31}$$

satisfying the following relations:

$$\begin{aligned}
q^\lambda(x_{ij}) =& \ \delta_{i,j} q^{\langle \lambda, \varepsilon_i \rangle}, \quad \lambda \in \frac{1}{2} L_n, \quad 1 \leq i, j \leq n & (32)\\
e_k(x_{ij}) =& \ \delta_{i,k}\delta_{j,k+1}, \quad 1 \leq i, j \leq n & (33)\\
f_k(x_{ij}) =& \ \delta_{i,k+1}\delta_{j,k}, \quad 1 \leq i, j \leq n & (34)\\
q^\lambda(det_q^m) =& \ q^{m\langle \lambda, \varepsilon_1, \ldots, \varepsilon_n \rangle} \quad m \in \mathbb{Z} & (35)\\
e_k(det_q^m) =& \ f_k(det_q^m) = 0 \quad m \in \mathbb{Z} & (36)
\end{aligned}$$

We extend these to the rest of $U_q(\mathfrak{g})$ and $A(G)$ by

$$\begin{aligned}
a(\varphi\psi) =& \ \Delta_U(a)(\varphi \otimes \psi) & (37)\\
a(1) =& \ \varepsilon_U(a) & (38)\\
(ab)(\varphi) =& \ (a \otimes b)\Delta(\varphi) & (39)\\
1(\varphi) =& \ \varepsilon(\varphi) \qquad (a, b \in U_q(\mathfrak{g}), \quad \varphi, \psi \in A(G)) & (40)
\end{aligned}$$

Additionally, we have

$$S_U(a).\psi = a.S(\psi) \quad a \in U_q(\mathfrak{g}), \psi \in A(G) \tag{41}$$

These relations realize a duality between the two Hopf algebras and allows us to regard the elements of $U_q(\mathfrak{g})$ as linear functionals on $A(G)$ (see [16]). This duality allows any right $A(G)$-comodule V (resp. left $A(G)$-comodule W) with structure map $R_G : V \to V \otimes A_q(G)$ (resp. $L_G : W \to A(G) \otimes W$) to become a left (resp. right) $U_q(\mathfrak{g})$-module with the following defined action

$$\begin{aligned}
a.v = (id \otimes a)R_G(v), \quad a \in U_q(\mathfrak{g}), v \in V, & \quad (42)\\
w.a = (a \otimes id)L_G(v), \quad a \in U_q(\mathfrak{g}), w \in W. & \quad (43)
\end{aligned}$$

More specifically, we already know $A_q(X)$ is a completely reducible two-sided $A(G)$-comodule using the comultiplication, Δ, as the comodule structure map. As such, it becomes a completely reducible left and right $U_q(\mathfrak{g})$-module

[16, 8]. We can describe the left module action of the generators of $U_q(\mathfrak{g})$ on the generators of $A_q(X)$ by

$$q^{\lambda}.x_{ij} = x_{ij}q^{\langle\lambda,\varepsilon_j\rangle}, \quad (44)$$
$$e_k.x_{ij} = x_{i,j-1}\delta_{j,k+1}, \quad (45)$$
$$f_k.x_{ij} = x_{i,j+1}\delta_{j,k}. \quad (46)$$

and the right module action as

$$x_{ij}\cdot q^{\lambda} = x_{ij}q^{\langle\lambda,\varepsilon_i\rangle}, \quad (47)$$
$$x_{ij}\cdot e_k = x_{i+1,j}\delta_{k,i}, \quad (48)$$
$$x_{ij}\cdot f_k = x_{i-1,j}\delta_{k+1,i}. \quad (49)$$

2.6. Relative Invariants

For an element $\lambda = \sum_{k=1}^{n}\lambda_k\epsilon_k \in L_n$, let $z^{\lambda} = \prod_{k=1}^{n} z_{kk}^{\lambda_k} \in A(B_{\pm})$ and $t^{\lambda} = \prod_{k=1}^{n} t_k^{\lambda_k} \in A(H)$, we define the spaces of relative invariants with respect to the subgroups $B_{\pm}$ by (see [16, 8])

$$A(G/B_+;z^{\lambda}) = \left\{\varphi \in A(G); (id\otimes\pi_{B_+})\Delta(\varphi) = \varphi\otimes z^{\lambda}\right\}, \quad (50)$$
$$A(B_-\backslash G;z^{\lambda}) = \left\{\varphi \in A(G); (\pi_{B_-}\otimes id)\Delta(\varphi) = z^{\lambda}\otimes\varphi\right\}, \quad (51)$$

where the restrictions maps $\pi_{\pm} : A(G) \to A_q(B_{\pm})$ are defined by $\pi_{B_+}(x_{ij}) = z_{i,j}$ $(1 \le i \le j \le n)$, $\pi_{B_+}(x_{ij}) = 0$ $(i > j)$, and $\pi_{B_-}(x_{ij}) = z_{i,j}$ $(1 \le j \le i \le n)$, $\pi_{B_-}(x_{ij}) = 0$ $(i < j)$.

$A(G/B_+;z^{\lambda})$ (resp. $A(B_-\backslash G;z^{\lambda})$) is a left (resp. right) $A(G)$-subcomodule of $A(G)$ with structure mapping Δ. It is proved in [16] that, for a dominant integral weight $\lambda \in P_n$, the space $A(G/B_+;z^{\lambda})$ (resp. $A(B_-\backslash G;z^{\lambda})$) gives a realization of the irreducible left (resp. right) $A(G)$-subcomodule $V_q^L(\lambda)$ (resp. $V_q^R(\lambda)$) of $A_q(X)$, with highest weight λ.

3. Spaces of q-Symplectic Invariants

3.1. $U_q(\mathfrak{sp}(2n,\mathbb{C}))$

Here we describe a subalgebra of $U_q(\mathfrak{g})$ that is a quantum deformation of $U(\mathfrak{sp}(2n,\mathbb{C}))$. Relative to the standard n dimensional representation of $U_q(\mathfrak{g})$, we identify the generators e_k of $U_q(\mathfrak{g})$ with $E_{k,k+1}$ and f_k with $E_{k+1,k}$. If we let $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_{2n}\epsilon_{2n} \in \frac{1}{2}L_{2n}$, then q^λ is represented by

$$q^{\lambda_1}E_{11} + q^{\lambda_2}E_{22} + \cdots + q^{\lambda_{2n}}E_{2n,2n} \tag{52}$$

We may then inductively generate the other elements, $E_{i,j}$, where $|i-j| > 1$, by

$$E_{i,j} = E_{i,k}E_{k,j} - E_{k,j}E_{i,k} \tag{53}$$

where $i < k < j$ or $j < k < i$ and $E_{i,j}$ and $E_{j,i}$ are independent of our choice of k, see [8].

We define the subalgebra $U_q(\mathfrak{sp}(2n,\mathbb{C}))$ of $U_q(\mathfrak{g})$ as the subalgebra generated by the following elements:

$$sp_e(i,j) = E_{2i-1,2j} + q^{2(i-j)}E_{2j-1,2i} \qquad 1 \le i \ne j \le n \tag{54}$$
$$sp_e(i,i) = E_{2i-1,2i} \qquad 1 \le i \le n \tag{55}$$
$$sp_f(i,j) = E_{2i,2j-1} + q^{2(i-j)}E_{2j,2i-1} \qquad 1 \le i \ne j \le n \tag{56}$$
$$sp_f(i,i) = E_{2i,2i-1} \qquad 1 \le i \le n \tag{57}$$
$$sp_h(i,j) = E_{2i-1,2j-1} - q^{2(i-j)}E_{2j,2i} \qquad 1 \le i,j \le n \tag{58}$$

with $i,j \le n$. It can be directly shown that the elements of the form

$$sp_e(j,j),\ sp_f(j,j), \quad \text{where } 1 \le j \le n, \tag{59}$$
$$sp_e(i,i+1),\ sp_f(i,i+1), \quad 1 \le i \le n-1 \tag{60}$$

generate $U_q(\mathfrak{sp}(2n,\mathbb{C}))$.

3.2. q-Symplectic Invariants

For a given left (resp. right) $U_q(\mathfrak{g})$-module V (resp. W) we define the

q-symplectic invariants by

$$V^K = \{v \in V; sp_e(i,j).v = 0, sp_f(i,j).v = 0 \quad 1 \le i,j \le n\} \tag{61}$$
$$^K W = \{w \in W; w.sp_e(i,j) = 0, w.sp_f(i,j) = 0 \quad 1 \le i,j \le n\} \tag{62}$$

Using the fact that $A_q(X)$ is a two-sided $U_q(\mathfrak{g})$-module (see 42, 43) we define the left and right quantum symplectic invariants in $A_q(X)$ as

$$A_q(X)^K = \{\varphi \in A_q(X); sp_e(i,j).\varphi = 0, sp_f(i,j).\varphi = 0 \quad 1 \le i,j \le n\} \tag{63}$$
$$^K A_q(X) = \{\varphi \in A_q(X); \varphi.sp_e(i,j) = 0, \varphi.sp_f(i,j) = 0 \quad 1 \le i,j \le n\} \tag{64}$$

The spaces $A_q(X)^K$ and $^K A_q(X)$ are subalgebras of $A_q(X)$. Additionally, we see that $A_q(X)^K$ is a left $A(G)$-subcomodule of $A_q(X)$ (similarly $^K A_q(X)$ is a right $A(G)$-subcomodule of $A_q(X)$). Equivalently, $A_q(X)^K$ is a right $U_q(\mathfrak{g})$-submodule of $A_q(X)$ and $^K A_q(X)$ is a left $U_q(\mathfrak{g})$-submodule of $A_q(X)$.

Definition 3.1. For $n \in \mathbb{Z}_+$ even, the following quadratic elements of $A_q(X)$ may be defined

$$\begin{aligned} z^L_{i,j} &= \sum_{k=1}^{n} q^{(i+j+1-4k)/2} \left(x_{i,2k-1}x_{j,2k} - q x_{i,2k}x_{j,2k-1}\right) \\ &= \sum_{k=1}^{n/2} q^{(i+j+1-4k)/2} \xi^{i,j}_{2k-1,2k}, \end{aligned} \tag{65}$$

$$\begin{aligned} z^R_{i,j} &= \sum_{k=1}^{n} q^{-(i+j+1-4k)/2} \left(x_{2k-1,i}x_{2k,j} - q x_{2k,i}x_{2k-1,j}\right) \\ &= \sum_{k=1}^{n/2} q^{-(i+j+1-4k)/2} \xi^{2k-1,2k}_{i,j}. \end{aligned} \tag{66}$$

Using the fact

$$e_k.\xi^{i,j}_{r,s} = \delta_{k,r-1}\xi^{i,j}_{r-1,s} + \delta_{k,s-1}\xi^{i,j}_{r,s-1} \tag{67}$$
$$f_k.\xi^{i,j}_{r,s} = \delta_{k,r}\xi^{i,j}_{r+1,s} + \delta_{k,s}\xi^{i,j}_{r,s+1} \tag{68}$$

it can be shown that $z^L_{i,j}$ (resp. $z^R_{i,j}$) are annihilated by $sp_e(k,k)$, $sp_e(k,k+1)$, $sp_f(k,k)$ and $sp_f(k,k+1)$, which is sufficient to show they are annihilated by all $sp_e(k,l)$ and $sp_f(k,l)$ and therefore $z^L_{i,j} \in A_q(X)^K$ (resp. $z^R_{i,j} \in {}^K A_q(X)$)

We denote the subalgebra of $A_q(X)^K$ (resp. ${}^K A_q(X)$) by $A^L_q(\mathcal{A})$ (resp. $A^R_q(\mathcal{A})$) generated by $z^L_{i,j}$ (resp. $z^R_{i,j}$). $A^L_q(\mathcal{A})$ is a left $A(G)$-subcomodule of $A_q(X)^K$ and $A^R_q(\mathcal{A})$ is a right $A(G)$-subcomodule of $A_q(X)^K$.

Theorem 3.2. *The algebras $A^L_q(\mathcal{A})$ and $A^R_q(\mathcal{A})$ are isomorphic to the algebra $A_q(\mathcal{A})$ generated by $z_{i,j}$ $(1 \leq i, 1 \leq j)$ with the following relations:*

$$z_{i,j} = -q^{-1} z_{j,i}, \tag{69}$$

$$z_{i,l} z_{j,k} = z_{j,k} z_{i,l}, \tag{70}$$

$$z_{i,j} z_{i,k} = q z_{i,k} z_{i,j}, \tag{71}$$

$$z_{i,k} z_{j,l} - z_{j,l} z_{i,k} = \left(q - q^{-1}\right) z_{i,l} z_{j,k}, \tag{72}$$

$$z_{i,j} z_{k,l} - z_{k,l} z_{i,j} = \left(q - q^{-1}\right) z_{i,k} z_{j,l} - q\left(q - q^{-1}\right) z_{i,l} z_{j,k}, \tag{73}$$

where $i < j < k < l$.

Using Eq. (72) we may rewrite Eq. (73) as

$$z_{i,j} z_{k,l} - z_{k,l} z_{i,j} = q z_{j,l} z_{i,k} - q^{-1} z_{i,k} z_{j,l} \tag{74}$$

□

The definitions of these generators also imply

$$z_{i,i} = 0 \tag{75}$$

3.3. Quantum Antisymmetric Matrices

If we denote by $\mathcal{A}$, the vector space of $n \times n$ antisymmetric matrices with basis

$$B_{\mathcal{A}} = \{E_{i,j} - E_{j,i} \,|1 < i < j \leq n\} \tag{76}$$

then $dim(\mathcal{A}) = n(n-1)/2$. We observe that $Hom_{Alg}(A_q(\mathcal{A}), \mathbb{C})$ is the set of $n \times n$ matrices with restrictions imposed by the relations Eq. (69) and Eq.

(75). If we denote $Hom_{Alg}(A_q(\mathcal{A}), \mathbb{C})$ by $\mathcal{A}_q$, and treat it as a vector space (in other words we are ignoring multiplication) we see its basis is

$$B_{\mathcal{A}_q} = \{E_{i,j} - qE_{j,i} \,|1 < i < j \leq n\} \tag{77}$$

where $dim(\mathcal{A}_q) = n(n-1)/2$ and we have $\mathcal{A}_q \simeq \mathcal{A}$ as vector spaces. We may think of $\mathcal{A}_q$ as the quantum analog of the antisymmetric matrices.

3.4. Quantum Pfaffian

If $A = (a_{i,j}) \in Mat(2n, \mathbb{C})$ is an antisymmetric matrix, it can be written as

$$A = \begin{bmatrix} 0 & a_{1,2} & \cdots & a_{1,2n} \\ -a_{1,2} & 0 & \cdots & a_{2,2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1,2n} & -a_{2,2n} & \cdots & 0 \end{bmatrix} \tag{78}$$

and there exists a polynomial f in $\mathbb{Z}[x_{ij}]$ such that $f^2(A) = det(A)$, [5]. This polynomial is called the Pfaffian, denoted Pf, and we write

$$Pf^2(A) = det(A) \tag{79}$$

Moreover, if $B = (b_{i,j}) \in Mat(2n, \mathbb{C})$ and we define A by

$$a_{i,j} = det \begin{bmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{bmatrix} + det \begin{bmatrix} b_{i,3} & b_{i,4} \\ b_{j,3} & b_{j,4} \end{bmatrix} + \cdots + det \begin{bmatrix} b_{i,2n-1} & b_{i,2n} \\ b_{j,2n-1} & b_{j,2n} \end{bmatrix} \tag{80}$$

then A is antisymmetric and we have $Pf(A) = det(B)$, [5].

To construct an explicit formula for Pf we can define an index set Π, consisting of all ordered, 2-partitions of $2n$. In other words,

$$\Pi = \{(i_1, j_1)(i_2, j_2) \ldots (i_n, j_n) \; ; \; i_k < j_k \text{ and } i_k < i_{k+1}\} \tag{81}$$

For example, if $2n = 4$ we have

$$\Pi = \{(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \tag{82}$$

We can associate the elements of Π with elements of the symmetric group $\mathfrak{S}_{2n}$ in the following manner

$$\pi \sim \begin{bmatrix} 1 & 2 & 3 & 4 & \cdots & 2n \\ i_1 & j_1 & i_2 & j_2 & \cdots & j_n \end{bmatrix} \in \mathfrak{S}_{2n} \tag{83}$$

for $\pi = \{(i_1, j_1)(i_2, j_2)\dots(i_n, j_n)\}$. This allows us to define $sgn(\pi)$ and $l(\pi)$. If $A = (a_{i,j})$ is an antisymmetric matrix we can then write

$$Pf(A) = \sum_{\pi\in\Pi} sgn(\pi)a_\pi = \sum_{\pi\in\Pi} sgn(\pi)a_{i_1j_1}a_{i_2j_2}\cdots a_{i_n,j_n} \tag{84}$$

Example 3.3. As an example, when $2n = 4$

$$Pf(A) = a_{1,2}a_{3,4} - a_{1,3}a_{2,4} + a_{1,4}a_{2,3} \tag{85}$$

Before we construct a quantum analog of the Pfaffian, we note that the quantum antisymmetric generators $z^L_{i,j}$ (resp. $z^R_{i,j}$), defined by Eq. (65) (resp. Eq. (66)), are in fact quantum analogs of Eq. (80). Additionally, we have already noted that $Z = (z^L_{i,j})$ is a quantum antisymmetric matrix with the relation $z^L_{i,j} = -\frac{1}{q}z^L_{j,i}$ for $i < j$. We now use the same index set Π, to define the quantum Pfaffian as

$$Pf_q(Z) = \sum_{\pi\in\Pi}(-q)^{l(\pi)}z^L_\pi = \sum_{\pi\in\Pi}(-q)^{l(\pi)}z^L_{i_1j_1}z^L_{i_2j_2}\cdots z^L_{i_n,j_n}. \tag{86}$$

Remark 3.4. An inductive definition of quantum Phaffian was given in [18]. One can show that our definition matches with Strickland's.

Example 3.5. As an example, when $2n = 4$

$$Pf_q(Z) = z^L_{1,2}z^L_{3,4} - qz^L_{1,3}z^L_{2,4} + q^2z^L_{1,4}z^L_{2,3} \tag{87}$$

Theorem 3.6. *For every positive even $2n$, $Pf_q(Z) = det_q(X)$.*

Proof. To show this equality, we will prove that Pf_q is simultaneously a highest and lowest weight vector for the right action of $U_q(\mathfrak{g})$. This will show Pf_q to be a scalar multiple of $(det_q)^c$ for some $c \in \mathbb{Z}_+$.

To begin, we let k be a positive integer such that $1 \leq k < 2n$. Since the right action of generators of $U_q(\mathfrak{g})$ on products of elements of $A_q(X)$ can be described by [8],

$$\phi\psi.e_k = (\phi\otimes\psi).(e_k\otimes q^{-a_k/2} + q^{a_k/2}\otimes e_k) \tag{88}$$
$$\phi\psi.f_k = (\phi\otimes\psi).(f_k\otimes q^{-a_k/2} + q^{a_k/2}\otimes f_k) \tag{89}$$

we may expand this notation to describe the following right action of e_k on the components of Pf_q as

$$\begin{aligned} & z^L_{a_1b_1}z^L_{a_2b_2}\cdots z^L_{a_{n/2}b_{n/2}}\cdot e_k \\ = \;& z^L_{a_1b_1}\cdot e_k\otimes z^L_{a_2b_2}\cdot q^{-\alpha_k/2}\otimes\cdots\otimes z^L_{a_nb_n}\cdot q^{-\alpha_k/2} \\ & + z^L_{a_1b_1}\cdot q^{\alpha_k/2}\otimes z^L_{a_2b_2}\cdot e_k\otimes\cdots\otimes z^L_{a_nb_n}\cdot q^{-\alpha_k/2} \\ & \vdots \\ & + z^L_{a_1b_1}\cdot q^{\alpha_k/2}\otimes z^L_{a_2b_2}\cdot q^{\alpha_k/2}\otimes\cdots\otimes z^L_{a_nb_n}\cdot e_k \end{aligned} \tag{90}$$

and

$$\begin{aligned} & z^L_{a_1b_1}z^L_{a_2b_2}\cdots z^L_{a_{n/2}b_{n/2}}\cdot f_k \\ = \;& z^L_{a_1b_1}\cdot f_k\otimes z^L_{a_2b_2}\cdot q^{-\alpha_k/2}\otimes\cdots\otimes z^L_{a_nb_n}\cdot q^{-\alpha_k/2} \\ & + z^L_{a_1b_1}\cdot q^{\alpha_k/2}\otimes z^L_{a_2b_2}\cdot f_k\otimes\cdots\otimes z^L_{a_nb_n}\cdot q^{-\alpha_k/2} \\ & \vdots \\ & + z^L_{a_1b_1}\cdot q^{\alpha_k/2}\otimes z^L_{a_2b_2}\cdot q^{\alpha_k/2}\otimes\cdots\otimes z^L_{a_nb_n}\cdot f_k \end{aligned} \tag{91}$$

Additionally, each of these $A^L_q(\mathcal{A})$ generators is a sum of quantum 2-minor determinants (see Eq. (65)) in which the indices i and j of z^L_{ij} define the rows for each of these quantum 2-minor determinants . As such, the right action of e_k and f_k on these generators can be described by the following,

$$z^L_{i,j}\cdot e_k = q^{-1/2}\left(\delta_{i,k}z^L_{k+1,j} + \delta_{j,k}z^L_{j,k+1}\right), \tag{92}$$
$$z^L_{i,j}\cdot f_k = q^{1/2}\left(\delta_{i,k+1}z^L_{k,j} + \delta_{j,k+1}z^L_{j,k}\right) \tag{93}$$

and the right action of $q^{\alpha/2}$ and $q^{-\alpha/2}$ are described by

$$z^L_{i,j}\cdot q^{\alpha_k/2} = q^{1/2(\delta_{i,k}-\delta_{i,k+1}+\delta_{j,k}-\delta_{j,k+1})}z^L_{i,j}, \tag{94}$$
$$z^L_{i,j}\cdot q^{-\alpha_k/2} = q^{1/2(-\delta_{i,k}+\delta_{i,k+1}-\delta_{j,k}+\delta_{j,k+1})}z^L_{i,j}. \tag{95}$$

For example

$$z^L_{3,4} \cdot q^{\alpha_4/2} = q^{1/2} z^L_{3,4} \tag{96}$$

Before we give a detailed description of the action of e_k on Pf_q, we show how the components of Π may be paired, relative to the value of k. Since the components of Pf_q are indexed by all ordered 2-partitions, this will allow us to group the components of Pf_q in a way that the right action of e_k (and f_k) will annihilate the pairs.

We first fix $k \in \mathbb{Z}$ such that $1 \leq k < 2n$. Now if we choose any of the ordered 2-partitions, say $\pi = (a_1, b_1)(a_2, b_2) \cdots (a_n, b_n)$, it must have an index r, containing k and an index s containing $k+1$. In other words, there exist r and s such that

$$k \in (a_r, b_r) \quad \text{and} \quad k+1 \in (a_s, b_s) \tag{97}$$

This fixes r and s. Also contained in the (a_r, b_r) and (a_s, b_s) pairs are two other integers, u and v such that $u < v$. If it happens that $r = s$, in other words, there exists (a_r, b_r) such that $(a_r, b_r) = (k, k+1)$ then we will not pair it with another 2-partition. We will show later how the right action of e_k and f_k already annihilate it.

Example 3.7. Suppose $2n = 8$ and we fix $k = 5$. One of the ordered 2-partitions of Π is $(1,3)(2,6)(4,8)(5,7)$. In this case we see that $r = 4$ and $s = 2$. We then designate $u = 2$ and $v = 7$.

Now, with r and s still fixed, and for the designated u and v, there are precisely three possibilities describing how k, $k+1$, u and v can be ordered. These are:

$$k < k+1 < u < v \tag{98}$$

$$u < k < k+1 < v \tag{99}$$

$$u < v < k < k+1 \tag{100}$$

For each of these possibilities we have the following,

- $k < k+1 < u < v$

In this case, if $r \neq s$, there is another 2-partition, $\hat{\pi}$ identical to π except in the r^{th} and s^{th} pairs, u and v are switched.

$$\pi = (a_1, b_1) \cdots (k, u)(k+1, v) \cdots (a_n, b_n) \tag{101}$$

$$\hat{\pi} = (a_1, b_1) \cdots (k, v)(k+1, u) \cdots (a_n, b_n) \tag{102}$$

If $r = s$ then we have

$$\pi = (a_1, b_1) \cdots (k, k+1) \cdots (u, v) \cdots (a_n, b_n) \tag{103}$$

- $u < k < k+1 < v$

  In this case, if $r \neq s$, there is a second partion $\hat{\pi}$ identical to π except in the r^{th} and s^{th} pairs, k and $k+1$ are switched.

$$\pi = (a_1, b_1) \cdots (u, k) \cdots (k+1, v) \cdots (a_n, b_n) \tag{104}$$

$$\hat{\pi} = (a_1, b_1) \cdots (u, k+1) \cdots (k, v) \cdots (a_n, b_n) \tag{105}$$

  If $r = s$ then we have

$$\pi = (a_1, b_1) \cdots (u, v) \cdots (k, k+1) \cdots (a_n, b_n) \tag{106}$$

- $u < v < k < k+1$

  In this case, if $r \neq s$, there is a second partion $\hat{\pi}$ identical to π except in the r^{th} and s^{th} pairs, k and $k+1$ are switched.

$$\pi = (a_1, b_1) \cdots (u, k) \cdots (v, k+1) \cdots (a_n, b_n) \tag{107}$$

$$\hat{\pi} = (a_1, b_1) \cdots (u, k+1) \cdots (v, k) \cdots (a_n, b_n) \tag{108}$$

  If $r = s$ then we have

$$\pi = (a_1, b_1) \cdots (u, v) \cdots (k, k+1) \cdots (a_n, b_n) \tag{109}$$

Example 3.8. Continuing with the previous example (Example 3.7), with $2n = 8$, $k = 5$ and 2-partition $(1,3)(2,6)(4,8)(5,7)$, the other 2-partition with which this would be paired is $(1,3)(2,5)(4,8)(6,7)$.

Using this construction, we see that after fixing k, we may exhaustively list all of the ordered 2-partitions of Π, identifying each 2-partition as containing a pair $(k, k+1)$ or as being one of the pairs just described.

This allows us to write Pf_q as a sum of components of the form

$$(-q)^* z^L_{a_1,b_1} \cdots z^L_{k,k+1} \cdots z^L_{a_n,b_n} \tag{110}$$

or which appear in pairs such as

$$\begin{aligned} &(-q)^* z^L_{a_1,b_1} \cdots z^L_{k,u} z^L_{k+1,v} \cdots z^L_{a_n,b_n} \\ &(-q)^{*+1} z^L_{a_1,b_1} \cdots z^L_{k,v} z^L_{k+1,u} \cdots z^L_{a_n,b_n} \end{aligned} \tag{111}$$

or

$$\begin{aligned} &(-q)^* z^L_{a_1,b_1} \cdots z^L_{u,k} \cdots z^L_{k+1,v} \cdots z^L_{a_n,b_n} \\ &(-q)^{*+1} z^L_{a_1,b_1} \cdots z^L_{u,k+1} \cdots z^L_{k,v} \cdots z^L_{a_n,b_n} \end{aligned} \tag{112}$$

or

$$\begin{aligned} &(-q)^* z^L_{a_1,b_1} \cdots z^L_{u,k} \cdots z^L_{v,k+1} \cdots z^L_{a_n,b_n} \\ &(-q)^{*+1} z^L_{a_1,b_1} \cdots z^L_{u,k+1} \cdots z^L_{v,k} \cdots z^L_{a_n,b_n}, \end{aligned} \tag{113}$$

where $(-q)^*$ represents an appropriate power of $(-q)$ determined by (a_1b_1) $(a_2b_2)\cdots(a_nb_n)$. The right action of e_k can now be calculated. In the first case, we have the index that contains $(k, k+1)$ and we have

$$\begin{aligned} q^* z^L_{a_1,b_1} &\cdots z^L_{k,k+1} \cdots z^L_{a_n,b_n}.e_k \\ &= (z^L_{a_1,b_1} \cdot e_k) \cdots (z^L_{k,k+1} \cdot q^{-\alpha_k/2}) \cdots (z^L_{a_n,b_n} \cdot q^{-\alpha_k/2}) \\ &\quad + \ (z^L_{a_1,b_1} \cdot q^{\alpha_k/2}) \cdots (z^L_{k,k+1} \cdot e_k) \cdots (z^L_{a_n,b_n} \cdot q^{-\alpha_k/2}) \\ &\quad + \ (z^L_{a_1,b_1} \cdot q^{\alpha_k/2}) \cdots (z^L_{k,k+1} \cdot q^{\alpha_k/2}) \cdots (z^L_{a_n,b_n} \cdot e_k) \\ &= (0) \cdots (z^L_{k,k+1} \cdot q^{-\alpha_k/2}) \cdots (z^L_{a_n,b_n} \cdot q^{-\alpha_k/2}) \\ &\quad + \ (z^L_{a_1,b_1} \cdot q^{\alpha_k/2}) \cdots (0) \cdots (z^L_{a_n,b_n} \cdot q^{-\alpha_k/2}) \\ &\quad + \ (z^L_{a_1,b_1} \cdot q^{\alpha_k/2}) \cdots (z^L_{k,k+1} \cdot q^{\alpha_k/2}) \cdots (0) \\ &= 0 \end{aligned} \tag{114}$$

In the next case, with the indexes of the paired 2-partitions containing $(k, u)(k+1, v)$ and $(k, v)(k+1, u)$, the right action of e_k can be seen to be zero as well, by using Eq. (92), Eq. (94), and Eq. (95). In fact all remaining cases are treated similarly, and we get that

$$Pf_q.e_k = 0 \tag{115}$$

A similar argument shows

$$Pf_q.f_k = 0 \tag{116}$$

Since Pf_q is an element of $A_q(X)$ annihilated by the right action of all e_k and f_k, $1 \leq k < n$, Pf_q must be generated by det_q. By comparing degree and coefficients, we see $Pf_q(Z) = det_q(X)$. □

We extend the notation slightly and define

$$Pf_q(Z)^I = \sum_{\pi \in \Pi^I} (-q)^{l(\pi)} z_\pi^L \tag{117}$$

where $I = \{1, 2, \ldots, r\}$, $r < 2n$, r is even and Π^I is the set of ordered 2-partitions of I. The proof above also shows that $Pf_q(Z)^I$ is annihilated on the right by all f_k $(k < 2n)$ and by all e_k except for $r < k < 2n$. As such $Pf_q(Z)^I$ is still a highest weight vector under the right action of $U_q(\mathfrak{g})$ and, because it is an element constructed from left symplectic invariant generators, it provides a realization of an element in $A_q(X)^K \cap A(B_-\backslash G; z^{\Lambda_r})$.

3.5. Decomposition of ${}^K A_q(X))$ and $A_q(X)^K$

We show the decomposition of ${}^K A_q(X))$ as a right $A(G)$-comodule (resp. left $U_q(\mathfrak{g})$-module) and the decomposition of $A_q(X))^K$ as a left $A(G)$-comodule (resp. right $U_q(\mathfrak{g})$-module. To perform this decomposition, several preliminary propositions are presented, along with the introduction of some notational conventions. First some notation:

We define the map ϕ from the power set of $\{1, 2, 3, \ldots, n\}$ into the power set of $\{1, 2, 3, \ldots, 2n\}$ by

$$\phi(A) = \bigcup_{\alpha \in A} \{2\alpha - 1, 2\alpha\} \tag{118}$$

for example $\phi(\{1, 3, 4, 5\}) = \{1, 2, 5, 6, 7, 8, 9, 10\}$. We will use ϕ to construct indices for the rows and columns of quantum minor determinants used in q-symplectic invariants and then to describe a specific set of dominant weights as

$$P_{2n}^{\mathcal{A}} = \{\lambda \in P_{2n} \; ; \; \lambda = (\mu_1, \mu_1, \mu_2, \mu_2, \dots, \mu_n, \mu_n), \; \mu \in P_n\} \tag{119}$$

For example $(4,4,4,4,3,3,2,2,2,2,1,1) \in P_{12}^{\mathcal{A}}$.

One of the key ideas used in the decomposition of ${}^K A_q(X)$ and $A_q(X)^K$ is presented in the following proposition (cf. [8]).

Proposition 3.9. *Let $\mu \in P_n$ be a dominant integral weight and $V_q^R(\mu)$ be the irreducible left $U_q(\mathfrak{g})$ submodule with highest weight μ. Then the space of the q-symplectic invariants in V_q^R has the dimension equal to the multiplicity of V in ${}^K A_q(X)$).*

Proof. To decompose the algebra ${}^K A_q(X)$) as a right $A(G)$-comodule (or left $U_q(\mathfrak{g})$ module), it suffices to find the singular weight vectors, i.e. the weight vectors $\phi \in {}^K A_q(X)$ such that $e_k.\phi = 0$ for $k = 1, \dots, n-1$. Since such a singular vector φ is contained in the space ${}^K A_q(X) \cap A(X/B^+; z^\lambda)$ for some dominant integral weigh $\lambda \in L_n$, and generates an irreducible right $A(G)$-comodule with highest weight λ. Thus if there are m_λ singular weight vectors of weight λ in ${}^K A_q(X)$, then the irreducible right $A(G)$-comodule isomorphic to $V_q^R(\lambda)$ occurs m_λ times in the decomposition of ${}^K A_q(X)$. On the other hand, a singular vector φ in ${}^K A_q(X) \cap A(X/B^+; z^\lambda)$ is regarded as a left q-symplectic invariant in V_q^L (i.e. annihilated on left). Since $V_q^L(\lambda)$ and $V_q^R(\lambda)$ are dual to each other, the dimension of the space of q-symplectic invariants coincides. □

Next, we show by construction, the existence of a left invariant in the left $U_q(\mathfrak{g})$-module $A(B_-\backslash G; z^\lambda)$. We build this left invariant from elements of the following form

$$a_r^R = \sum_J q^{-2|J|} \xi_{\phi(J)}^{1,\dots,2r} \tag{120}$$

where the sum is over all J such that $\#J = r$ and $J \subseteq \{1, 2, \dots, n\}$. $|J|$ represents the sum of the elements of J. As such $a_r^R \in A_q(X)^K \cap A(B_-\backslash G; z^{\lambda_r})$.

Lemma 3.10. *(Existence) For $\lambda = \sum_{r=1}^{n} m_{2r}\Lambda_{2r}$, i.e, $\lambda \in P_{2n}^{\mathcal{A}}$, $A(B_-\backslash G; z^\lambda)$ contains a left q-symplectic invariant.*

Proof. Suppose $\lambda = \sum_{r=1}^{n} m_{2r}\Lambda_{2r}$, then we define

$$a_\lambda^R = \prod_{r=1}^{n} \left(a_r^R\right)^{m_{2r}} \tag{121}$$

where a_r^R is defined by Eq. (120). We see by its construction, $a_\lambda^R \in A_q(X)^K \cap A(B_-\backslash G; z^\lambda)$. As such, each right $A(G)$-comodule $V_q^R(\lambda)$ has a q-symplectic invariant. □

Lemma 3.11. *(Nonexistence) There does not exist a left q-symplectic invariant in the irreducible right $U_q(\mathfrak{g})$-submodule $V_q^L(\lambda)$ if $\lambda \notin P_{2n}^{\mathcal{A}}$.*

Proof. $A_q(X)^K$ is a right $U_q(\mathfrak{g})$-submodule of $A_q(X)$. As such, it has its own decomposition into irreducible right $U_q(\mathfrak{g})$-submodules indexed by $\lambda \in P_n$, where λ is a dominant integral weight

$$A_q(X)^K = \bigoplus_{\lambda} V_q^L(\lambda) \tag{122}$$

Each $V_q^L(\lambda)$ is a highest weight module [16]. Each of these highest weight modules has a realization of $A(G/B^+; z^\lambda)$ with highest weight vector of the form

$$v_\lambda = \left(\xi_{1,\dots,s}^{1,\dots,s}\right)^{m_s} \left(\xi_{1,\dots,s-1}^{1,\dots,s-1}\right)^{m_{s-1}} \cdots \left(\xi_1^1\right)^{m_1} \tag{123}$$

However, because, the elements of $A_q(X)^K$ are annihilated on the left by all e_k and f_k where k is odd, then for the highest weight vector v_λ, it must be true that $\lambda = \sum_{r=1}^{n} m_r\Lambda_r$ where $m_r = 0$ when r is odd. Thus,

$$A_q(X)^K = \bigoplus_{\lambda \in P_{2n}^{\mathcal{A}}} V_q^L(\lambda) \tag{124}$$

□

Lemma 3.12. *(Uniqueness) The multiplicity of $V_q^R(\lambda)$ an irreducible right $U_q(\mathfrak{g})$-module, in the decomposition of $A_q(X)^K$ is exactly one.*

Proof. As mentioned earlier, by Proposition 3.9, the multiplicity of $V_q^R(\lambda)$ in the decomposition of $A_q(X)^K$ is equal to the number of left q-symplectic invariants in $A(B_-\backslash G; z^\lambda)$. Let v^K be a non zero left invariant in $A(B_-\backslash G; z^\lambda)$, as such, it can be written as a linear combination of weight vectors from the standard basis of $A(B_-\backslash G; z^\lambda)$ [16].

However, since $A(B_-\backslash G; z^\lambda)$ is a highest weight vector space, there must be at least one basis (weight) vector, η, in the composition of v^K for which there are no higher weight vectors in v^K. In other words

$$v^K = \eta \oplus v_1 \oplus \cdots v_j \tag{125}$$

where the weights of $v_1, \ldots, v_j$ are less than or equal to that of η. As such, η must be annihilated by all e_k, where $k < 2n$ and k is odd. Additionally, the elements of the form

$$\begin{aligned} sp_f(i, i+1) = e_{2i} + q^{-2}\,(&f_{2i-1}f_{2i}f_{2i+1} - f_{2i}f_{2i-1}f_{2i+1} \\ &-f_{2i+1}f_{2i-1}f_{2i} + f_{2i+1}f_{2i}f_{2i-1}) \end{aligned} \tag{126}$$

where $1 \leq i < n$, also annihilate v^K, and this in turn requires that η also be annihilated by all e_k, where $k < 2n$ and k is even. Therefore η must be a highest weight vector of $A(B_-\backslash G; z^\lambda)$, but this vector is unique up to constant multiple, because $A(B_-\backslash G; z^\lambda)$ is a highest weight module. So

$$\eta = cv_\lambda, \quad c \in \mathbb{C} \tag{127}$$

where v_λ is defined by Eq. (123). This tells us that any non-zero left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$ must be written as

$$cv_\lambda \oplus w_1 \oplus \cdots w_j, \quad c \in \mathbb{C}, c \neq 0 \tag{128}$$

where $w_1, \ldots, w_j$ are lower weight vectors of $A(B_-\backslash G; z^\lambda)$.

Now assume there is more than one left quantum q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$, say v^K and w^K. Each of these may be written as a sum of standard basis elements, each including a non-zero term for the highest weight vector v_λ. In other words, they may be written as

$$v^K = c_0v_\lambda + c_1v_1 + c_2v_2 + \cdots + c_iv_i, \quad c_0 \neq 0 \tag{129}$$

$$w^K = k_0v_\lambda + d_1v_1 + d_2v_2 + \cdots + d_jv_j, \quad k_0 \neq 0 \tag{130}$$

Since the linear combination of any left q-symplectic invariant is also a left q-symplectic invariant then it must be true that $k_0v^K - c_0w^K$ is also a left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$. If $k_0v^K - c_0w^K \neq 0$ then we have a contradiction to the requirement that any left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$ has a nonzero v_λ component. On the other hand, if $k_0v^K - c_0w^K = 0$ then w^K is a constant multiple of v^K. Therefore, any left q-symplectic invariant in $A(B_-\backslash G; z^\lambda)$ is unique up to a constant multiple. □

The following proposition summarizes Lemmas 3.10, 3.11, and 3.12.

Proposition 3.13. *The space of q-symplectic invariants in the right $A(G)$-comodule $V_q^R(\mu)$ is one dimensional if and only if $\mu = \sum_{r=1}^{n} m_{2r}\Lambda_{2r}$, in other words, $\mu \in P_{2n}^{\mathcal{A}}$. Otherwise there are no q-symplectic invariants in V_q^R*

By Proposition 3.9 we may then summarize our results with the following theorem

Theorem 3.14. *The irreducible decomposition of $A_q(X)^K$ as a right $U_q(\mathfrak{g})$-module is given by*

$$A_q(X)^K = \bigoplus_{\lambda \in P_{2n}^{\mathcal{A}}} V_q^L(\lambda) \tag{131}$$

similarly ${}^K A_q(X)$, as a left $U_q(\mathfrak{g})$-module has the irreducible decomposition

$${}^K A_q(X) = \bigoplus_{\lambda \in P_{2n}^{\mathcal{A}}} V_q^R(\lambda) \tag{132}$$

Where $P_{2n}^{\mathcal{A}}$ is defined by Eq. (119).

Proposition 3.15. *The space $A_q^L(\mathcal{A}) = A_q(X)^K$, (resp. ${}^K A_q(X) = A_q^R(\mathcal{A})$). As such, $A_q^L(\mathcal{A})$ (resp. $A_q^R(\mathcal{A})$) also have the decompositions as a right (resp. left) $U_q(\mathfrak{g})$-modules,*

$$A_q^L(\mathcal{A}) = \bigoplus_{\lambda \in P_{2n}^{\mathcal{A}}} V_q^L(\lambda) \tag{133}$$

$$A_q^R(\mathcal{A}) = \bigoplus_{\lambda \in P_{2n}^{\mathcal{A}}} V_q^R(\lambda) \tag{134}$$

Proof. From its definition, we already have $A_q^L(\mathcal{A}) \subseteq A_q(X)^K$. The elements, ${Pf_q}^I$ described by Eq. (117) provide a formula for explicitly constructing a left $U_q(\mathfrak{sp}(2n,\mathbb{C}))$ invariant in $A(B_-\backslash G; z^\lambda)$ for any λ. As such, $A_q(X)^K \subseteq A_q^L(\mathcal{A})$, and we have $A_q^L(\mathcal{A}) = A_q(X)^K$. □

4. Bi-invariants

In this section we define a subalgebra of $A_q(X)$ by the intersection of $A_q^R(\mathcal{A})$ and $A_q^L(\mathcal{A})$. Defined in this way, this space is annihilated on the left and right by $U_q(\mathfrak{sp}(2n,\mathbb{C}))$. We then proceed to show that this algebra is really $\mathbb{C}[s_1,\ldots,s_n]^{\mathfrak{S}_n}$, the symmetric algebra of n variables. To start, we define A_{ZP}, as

$$A_{ZP} = A_q^R(\mathcal{A}) \cap A_q^L(\mathcal{A}) = \bigoplus_{m=0}^{\infty} A_{ZP,2m}, \tag{135}$$

Recall, the polynomials of $A_q^R(\mathcal{A})$ and $A_q^L(\mathcal{A})$ have even degree so it has the natural grading into the subspaces $A_{ZP,2m}$.

Now we define

$$E_r = \sum_{I,J} q^{2(|I|-|J|)} \xi^{\phi(I)}_{\phi(J)}, \quad 1 \le r \le n \tag{136}$$

where the summation runs over all subsets I and J of $\{1,\ldots,n\}$ and $\#I = \#J = r$. Here, $|I|$ and $|J|$ are the sums of the elements of I and J respectively.

Lemma 4.16. $E_r \in A_{ZP,2r}$

Proof. If we examine the component of E_r that is obtained by holding I fixed at $I = \{1,2,\ldots,r\}$, we see that this component is precisely a_r^R, defined in Eq. (120). As such, this component is invariant under the left action of $U_q(\mathfrak{sp}(2n,\mathbb{C}))$. The remaining components of E_r (the components obtained by fixing I at other values) can be obtained by the right action of $U_q(\mathfrak{g})$ on a_r^R. Since $A_q(X)^K$ is a right submodule of $A_q(X)$ these other components of E_r must also be left invariant. Thus, $E_r \in A_q(X)^K$. Similarly, we see that the component of E_r associated with the fixed $J = \{1,2,\ldots,n\}$ is in ${}^K A_q(X)$ and likewise the other components of E_r can be obtained by the left action of $U_q(\mathfrak{g})$. Thus, $E_r \in {}^K A_q(X)$. Since E_r has degree $2r$ (by its construction) and $E_r \in A_q^R(\mathcal{A}) \cap A_q^L(\mathcal{A})$, it follows that $E_r \in A_{ZP,2r}$. □

Theorem 4.17. *The algebra A_{ZP} is generated by $E_r (1 \leq r \leq n)$ and the algebra A_{ZP} is isomorphic to the algebra of symmetric polynomials of n variables;*

$$\pi : A_{ZP} \tilde{\to} \mathbb{C}[s_1, \ldots, s_n]^{\mathfrak{S}_n} \tag{137}$$

Proof. Because of the decomposition given in Proposition 3.15, the dimension of the bi-invariant space associated with each $\lambda \in P_{2n}^{\mathcal{A}}$ must be exactly one. Since the degree of the polynomial in each of these bi-invariant spaces is $\sum_{k=1}^{n} \lambda_k$, the dimension of $A_{ZP,2m}$ can then be calculated as the number of partitions in $P_{2n}^{\mathcal{A}}$ of $2m$. As these partitions are in $P_{2n}^{\mathcal{A}}$ we may also consider this as the number of partitions of m whose number of parts is less than or equal to n. Adopting the notation of Jing and Yamada [8] we denote this by $p_n(m)$.

Consider the restriction of the projection map π to A_{ZP}

$$\pi'_H : A_{ZP} \to A_+(H), \tag{138}$$

where $A_+(H) = \mathbb{C}[t_1, \ldots, t_n]$. Then $Ker(\pi'_H) = \bigoplus_{r=0}^{\infty} Ker(\pi'_{H,2r})$, where

$$\pi'_{H,2r} : A_{ZP,2r} \to A_{2r}(H). \tag{139}$$

Similar to the proof by [8], the monomials $E_{r_1} E_{r_2} \ldots E_{r_k}$ $(r_1 \leq r_2 \leq \ldots \leq r_k)$ have the degree $2(r_1 + r_2 + \ldots + r_k)$ and are linearly independent over $\mathbb{C}$. As such the space of degree $2m$ spanned by these monomials has dimension $p_n(m)$. This shows that the space A_{ZP} is generated by E_r $(1 \leq r \leq n)$.

Additionally, the map $\pi'_{H,2r}$ acts on the generators of A_{ZP} in the following manner

$$\begin{aligned}
\pi'(E_r) &= \pi' \left(\sum_I \xi_{\phi(I)}^{\phi(I)} \right) \\
&= \sum_I (t_{2i_1-1} t_{2i_1}) (t_{2i_2-1} t_{2i_2}) \cdots (t_{2i_r-1} t_{2i_r}) \neq 0
\end{aligned}$$

where the sum runs over all subsets I of $\{1,2,\dots,n\}$ and $\#I = r$, thus $Ker(\pi'_{H,2r}) = (0)$. Another way of viewing this is that each of these E_r has monomials which are products of diagonal elements. As such, $\pi(E_r) \neq 0$ for $1 \leq r \leq n$. Thus we have the isomorphism

$$\begin{aligned} A_{ZP} &\cong \mathbb{C}[(t_1t_2),(t_3t_4),\dots,(t_{n-1}t_n)]^{\mathfrak{S}_n} \\ &\cong \mathbb{C}[s_1,s_2,\dots,s_n]^{\mathfrak{S}_n} \end{aligned}$$

where we let $s_i = t_{2i-1}t_{2i}$. □

5. Spherical Functions and Symmetric Polynomials

Through the isomorphism in Theorem (4.1) our q-zonal polynomials are basis elements in the ring of symmetric polynomials, and they are clearly q-deformations of the zonal polynomials defined on $GL(2n,\mathbb{C})/Sp(2n,\mathbb{C})$. We describe the relation with Macdonald polynomials [13].

Macdonald polynomials are special orthogonal basis of the commutative algebra
$\mathbb{Q}(q,t)[x_1,\dots,x_n]^{\mathfrak{S}_n}$, where q and t are two parameters. To describe them we consider the following shift operator T_{u,x_i} by

$$(T_{u,x_i}f)(x_1,\dots,x_n) = f(x_1,\dots,ux_i,\dots,x_n)$$

for each $f \in \mathbb{Q}(q,t)[x_1,\dots,x_n]$. Let X be another indeterminate and define

$$\begin{aligned} D(X,q,t) &= \Delta^{-1}\sum_{w\in\mathfrak{S}_n}\epsilon(w)z^{w\delta}\prod_{i=1}^{n}(X + t^{(w\delta)_i}T_{q,x_i}) \\ &= \sum_{r=0}^{n} D_rX^{n-r}, \end{aligned}$$

where $\delta = (n-1,n-2,\dots,1,0)$ and

$$\Delta = \prod_{1\leq i<j\leq n}(x_i - x_j)$$

is the Vandermonde determinant in $x_1,\dots,x_n$. It follows immediately that

$D_0 = 1$ and

$$D_1 = \sum_{i=1}^{n} (\prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j}) T_{q,x_i}.$$

Macdonald showed that for each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ there is a unique symmetric polynomial $P_\lambda(x; q, t)$ satisfying the two conditions (4.5 - 4.6):

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu \tag{140}$$

where $u_{\lambda\mu} \in \mathbb{Q}(q, t)$ and $m_\mu = x_1^{\mu_1} \ldots x_n^{\mu_n} + \ldots$ is the monomial symmetric polynomial;

$$D_1 P_\lambda = (\sum_{i=1}^{n} q^{\lambda_i} t^{n-i}) P_\lambda. \tag{141}$$

Moreover Macdonald proves that P_λ is also an eigenfunction for all the difference operators D_r, and

$$D(X; q, t) P_\lambda = \prod_{i=1}^{n} (X + t^{n-i} q^{\lambda_i}) P_\lambda. \tag{142}$$

The polynomial $P_\lambda(x; q, t)$ is called the Macdonald polynomial associated with the partition λ. In particular, $P_\lambda(x; q, q)$ is the famous Schur polynomial; $\lim_{t \to 1} P_\lambda(x; t^2, t)$ is the zonal polynomial.

Proposition 5.18. *Under the isomorphism* $\pi : A_{zp} \longrightarrow \mathbb{C}[z_1, \ldots, z_n]^{\mathfrak{S}_n}$, *the* q*-zonal polynomial in* $V_q(\lambda)$ *is a constant multiple of the Macdonald polynomial* $P_\lambda(z; q^2, q^{-4})$.

The general case of quantum spherical functions was studied by Noumi [15] using quantum groups and Letzter [12] using quantum enveloping algebras. In the following we will outline a different approach to understand the relationship between symmetric functions and quantum invariants. First of all let's study the q-difference operators on $V_q(2\lambda)$.

Recall the center of the quantized universal enveloping algebra $U_q(\mathfrak{sl}_{n-1})$

is generated by the following $n-1$ elements [17].

$$c_k = \sum_{\sigma,\sigma'\in\mathfrak{S}_n} (-q)^{l(\sigma)+l(\sigma')} l^{(+)}_{\sigma_1,\sigma'_1}\cdots l^{(+)}_{\sigma_k\sigma'_k} l^{(-)}_{\sigma_{k+1}\sigma'_{k+1}}\cdots l^{(-)}_{\sigma_n\sigma'_n}, \quad k=1,\cdots,n-1$$

where $L^{(\pm)} = (l^{(\pm)}_{ij})$ is the upper (lower) triangular defining matrix for the quantum algebra $U_q(sl_{n-1})$ in the FRT formulation [17] and $l(\sigma) = \#\{i < j|\sigma_i > \sigma_j\}$. We only remark that the elements $l^{(\pm)}_{ij}$ are analogs of Weyl-generators for $U_q(\mathfrak{sl}_{n-1})$. In particular

$$l^{(\pm)}_{ii} = q^{\pm\epsilon_i},$$

The algebra $U_q(\mathfrak{s}l_{n-1})$ acts on $GL_q(n,\mathbb{C})$ as q-difference operators, thus the center of $U_q(\mathfrak{sl}_{n-1})$ acts on modules $V_q(2\lambda)$ as scalar operators. In particular our q-zonal polynomials are simultaneous eigenfunctions of these q-difference operators.

Theorem 5.19. *For $1 \le k \le n-1$, the central element c_k acts on the irreducible $U_q(sl_n)$-module $V(\lambda)$ as a scalar multiplication by*

$$q^{2|\lambda|+\binom{n}{2}+k(n-1)}[k]![n-k]!\Big(\sum_{1\le i_1<\cdots<i_k\le n} q^{-2\lambda_{i_1}-\cdots-2\lambda_{i_k}+2(i_1-n)+\cdots+2(i_k-n)}\Big),$$

where $|\lambda| = \lambda_1+\cdots+\lambda_n$.

Proof. Pick a lowest weight vector v_0 in $V(\lambda)$ with the weight $-\lambda = -\lambda_1\epsilon_1 - \cdots - \lambda_n\epsilon_n$. Note that the generators $l^{(+)}_{ij}, l^{(-)}_{ji} (i<j)$ belong to the so-called strict upper and lower Borel subalgebra generated by e_i and f_i $(i = 1,\cdots n-1)$ respectively. The element $l^{(-)}_{\sigma_{k+1}\sigma'_{k+1}}\cdots l^{(-)}_{\sigma_n\sigma'_n}$ kills v_0 unless $\sigma_{k+1} = \sigma'_{k+1}$, $\ldots$, $\sigma_n = \sigma'_n$. But $\sigma_1 \le \sigma'_1$, $\ldots$, $\sigma_k \le \sigma'_k$, so one must have $\sigma = \sigma'$ in the action of c_{n-k} on v_0. We thus have

$$\begin{aligned} c_k v_0 &= \sum_{\sigma\in\mathfrak{S}_n} q^{2l(\sigma)} q^{-\lambda_{\sigma_1}-\cdots-\lambda_{\sigma_k}+\lambda_{\sigma_{k+1}}+\cdots+\lambda_{\sigma_n}} v_0 \\ &= q^{|\lambda|}\sum_{\sigma\in\mathfrak{S}_n} q^{2l(\sigma)-2\lambda_{\sigma_1}-\cdots-2\lambda_{\sigma_k}} v_0. \end{aligned}$$

Consider the Young subgroup $\mathfrak{S}_k\times\mathfrak{S}_{n-k}$ of $\mathfrak{S}_n$. We can choose its left coset representatives to be the elements τ such that $\tau_1 < \cdots < \tau_k$,

$\tau_{k+1} < \cdots < \tau_n$. Recall that an inversion of the permutation τ is a pair (ij) such that $i < j$ and $\tau_i > \tau_j$. By construction the inversions of τ may only take place among (ij) where $i \le k$ and $j \ge k+1$. For each $i(\le k)$, there are $\tau_i - 1$ natural numbers less than τ_i, and $i-1$ of them already appear before τ_i in the permutation. So there are $\tau_i - i$ inversions of τ in the form (ij), which implies that $l(\tau) = \sum_{i=1}^{k}(\tau_i - i)$.

Let $\tau\sigma$ be the general element in $\mathfrak{S}_n$ where $\sigma = \sigma_1\sigma_2 \in \mathfrak{S}_k \times \mathfrak{S}_{n-k}$. In the sequence $(\tau\sigma(1), \ldots, \tau\sigma(k), \tau\sigma(k+1), \ldots, \tau\sigma(n))$ we divide the inversions of $\tau\sigma$ into three parts: the inversions among the first k numbers, those among the last $n-k$ numbers, and the inversions between the first k numbers and the last $n-k$ numbers. The second part $(\tau\sigma(k+1), \ldots, \tau\sigma(n)) = (\tau\sigma_2(k+1), \ldots, \tau\sigma_2(n))$ has $l(\sigma_2)$ inversions as τ preserves the order of $k+1, \ldots, n$, similarly the first part $(\tau\sigma(1), \ldots, \tau\sigma(k)) = (\tau\sigma_1(1), \ldots, \tau\sigma_1(k))$ has $l(\sigma_1)$ inversions among them. Observe that we are free to switch the numbers in each part when considering the inversions between the first part and the second part, thus the number of inversions of this type are exactly $l(\tau)$. Therefore we have

$$\begin{aligned} l(\tau\sigma_1\sigma_2) &= l(\tau) + l(\sigma_1) + l(\sigma_2) \\ &= l(\sigma_1) + l(\sigma_2) + \sum_{i=1}^{k}(\tau_i - i), \end{aligned}$$

where $\sigma_1 \in \mathfrak{S}_k, \sigma_2 \in \mathfrak{S}_{n-k}, \tau \in \mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k})$.

Now let's return back to the action $c_k v_o$. Using the invariance of $\lambda_{\sigma(1)} + \cdots + \lambda_{\sigma(k)}$ under $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$, we have that

$$\begin{aligned} c_k v_0 &= q^{2|\lambda|} \sum_{\tau,\sigma_1,\sigma_2} q^{2l(\tau\sigma_1\sigma_2) - 2\lambda_{\tau\sigma_1\sigma_2(1)} - \cdots - 2\lambda_{\tau\sigma_1\sigma_2(k)}} v_0 \\ &= q^{2|\lambda|} \sum_{\tau,\sigma_1,\sigma_2} q^{2l(\tau\sigma_1\sigma_2) - 2\lambda_{\tau(1)} - \cdots - 2\lambda_{\tau(k)}} v_0 \\ &= q^{2|\lambda|} \sum_{\sigma_1 \in \mathfrak{S}_k} q^{2l(\sigma_1)} \sum_{\sigma_2 \in \mathfrak{S}_{n-k}} q^{2l(\sigma_2)} \sum_{\tau} q^{2l(\tau) - 2\lambda_{\tau(1)} - \cdots - 2\lambda_{\tau(k)}} v_0 \\ &= q^{2|\lambda| + \binom{k}{2} + \binom{n-k}{2}} [k]![n-k]! \\ &\qquad \cdot \Big(\sum_{\tau(1) < \cdots < \tau(k)} q^{-2\lambda_{\tau(1)} - \cdots - 2\lambda_{\tau(k)} + 2(\tau(1)-1) + \cdots + 2(\tau(k)-k)} \Big) v_0 \\ &= q^{2|\lambda| + \binom{n}{2} + k(n-1)} [k]![n-k]! \end{aligned}$$

$$\cdot\Big(\sum_{1\le\tau(1)<\cdots<\tau(k)\le n} q^{-2\lambda_{\tau(1)}-\cdots-2\lambda_{\tau(k)}+2(\tau(1)-n)+\cdots+2(\tau(k)-n)}\Big)v_0$$

where we have used the well-known identity $\sum_{\sigma\in\mathfrak{S}_n} q^{2l(\sigma)} = q^{\binom{n}{2}}[n]!$ (cf.[2]). □

Now we restrict ourselves to the case of irreducible highest $U_q(sl_{2n})$-module $V(\tilde\lambda)$ such that $\tilde\lambda = \tilde\lambda_1\epsilon_1 + \cdots + \tilde\lambda_{2n}\epsilon_{2n}$ and $\lambda_{2i-1} = \lambda_{2i} = \lambda_i$ for $i = 1, \ldots, n$. It is also a lowest weight module with the lowest weight $-\tilde\lambda$.

Theorem 5.20. *The bi-invariant function inside $V(\tilde\lambda)$, restricted to the ring of symmetric functions, is the Macdonald symmetric function $P_\lambda(q^2, q^4)$.*

Proof. It follows from the theorem in the case of $U_q(sl_{2n})$-module $V(\tilde\lambda)$ that

$$\begin{aligned} c_1v_0 &= q^{2|\tilde\lambda|+\binom{2n}{2}+2(2n-1)}[2]![2n-2]!\sum_{1\le i\le 2n} q^{-2\tilde\lambda_i+2(i-2n)}v_0\\ &= q^{4|\lambda|+\binom{2n}{2}+2(2n-1)-1}[2]^2[2n-2]!\sum_{1\le i\le n} q^{-2\lambda_i+4(i-n)}v_0. \end{aligned}$$

In other words, the quantum Casimir operator c_1 agrees with Macdonald operator $D_1(q^2, q^4)$ or $D_1(q^{-2}, q^{-4})$ on the space. We note that the leading term of the spherical functions, when restricted to the zonal part, is exactly the leading term of the Macdonald spherical function $P_\lambda(q^2, q^4)$ (which also agrees with Schur function s_λ). Hence the eigenfunction restricted to the ring of symmetric functions is the Macdonald symmetric function $P_\lambda(q^2, q^4)$. Similarly the action of the higher difference operators are given by

$$\begin{aligned} c_{2k}v_0 = {}& q^{2|\tilde\lambda|+\binom{2n}{2}+2k(2n-1)}[2k]![2n-2k]!\\ &\cdot\Big(\sum_{1\le\tau(1)<\cdots<\tau(2k)\le n} q^{-2\tilde\lambda_{\tau(1)}-\cdots-2\tilde\lambda_{\tau(2k)}+2(\tau(1)-2n)+\cdots+2(\tau(2k)-2n)}\Big)v_0. \end{aligned}$$ □

The last identity plus the same idea also gives that

Corollary 5.21. *The restriction of c_k to the ring $\mathbb{C}[z_1, \ldots, z_n]^{\mathfrak{S}_n}$ is exactly the difference operator $D_k(q^2, q^4)$, $k = 1, \ldots, n$ up to a constant.*

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References

1. E. Abe, *Hopf Algebras*. Cambridge Tracts in Mathematics, 74. Cambridge University Press, Cambridge-New York, 1980.

2. N. Bourbaki, *Groupes et algébres de Lie*, Ch. 4-6, Mason, Paris, 1981.

3. R. Howe, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), no. 2, 539-570.

4. L. K. Hua, *Harmonic analysis of functions of several complex variables in the classical domains*, AMS Translations **6**, Providence, RI, 1963.

5. N. Jacobson, *Basic Algebra*, 2nd ed., W. H. Freeman and Company, 1985.

6. A. T. James, Zonal polynomials of the real positive definite symmetric matrices, *Ann. Math.*, **74** (1961), 456-469.

7. M. Jimbo, A q-difference analogue of $U(g)$ and the Yang-Baxter equation, *Lett. Math. Phys.*, **10** (1985), 63-69.

8. N. Jing and H.-F. Yamada, Zonal polynomials on the quantum general linear groups, In: *Nankai Workshop on Quantum groups*, Edited by M. L. Ge, World Sci, Singapore, 1995, 66-72.

9. C. Kassel, *Quantum Groups*, Springer-Verlag, New York, 1995.

10. T. Kornwinder, Orthogonal polynomials in connections with quantum groups, In: *Orthogonal Polynomials*, Edited by P. Nevai, NATO ASI ser., Kluwer Acad. Publishers, 1990, 257-297.

11. G. I. Lehrer, H. Zhang and R. Zhang, A quantum analogue of the first fundamental theorem of classical invariant theory, *Comm. Math. Phys.*, **301** (2011), No. 1, 131-174.

12. G. Letzter, Quantum zonal spherical functions and Macdonald polynomials, *Adv. Math.*, **189** (2004), No. 1, 88-147.

13. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd ed., Clarendon Press, Oxford, 1995.

14. S. Montgomery, *Hopf algebras and their actions on rings*, CBMS ser. **82**, AMS, Providence, RI., 1993.

15. M. Noumi, Macdonald's symmetric polynomials as zonal symmetric functions on some quantum homogeneous spaces, *Adv. Math.*, **123**(1996), 16-77.

16. M. Noumi, H. Yamada and K. Mimachi, Finite dimensional representations of the quantum groups $gl_q(n;C)$ and the zonal spherical functions on $u_q(n-1)\backslash U_q(n)$, *Japanese J. of Math.*, (1993), 31-80.

17. N. Yu. Reshetikhin, L. A. Takhtajan and L. D. Faddeev, Quantization of Lie groups and Lie algebras, *Algebra and Analysis* **1** (1989), 178-206; English Transl. *Leningrad Math. J.* **1**(1990), 193-225.

18. E. Strickland, Classical invariant theory for the quantum symplectic group. *Adv. Math.*, **123** (1996), No. 1, 78-90.

19. K. Ueno and T. Takebayashi, Zonal spherical functions on quantum symmetric spaces and Macdonald's symmetric polynomials, In *Quantum Groups*, Edited by P. Kulish, Lect. Notes Math. vol. 1510, Springer-Verlag, 1992, 142-147.