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ON CERTAIN VARIETIES ATTACHED TO A WEYL GROUP ELEMENT

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0. Introduction and Statement of Results

0.1. Let $\mathbf{k}$ be an algebraically closed field. Let G be a connected reductive algebraic group over $\mathbf{k}$. We assume that we are in one of the following two cases.

(1): G is the identity component of a reductive group $\hat{G}$ with a fixed connected component D.

(2): $\mathbf{k}$ is an algebraic closure of a finite field F_q and G has a fixed F_q-rational structure with Frobenius map $F : G \to G$.
In case (1) we set $q = 1$ and denote by $F : G \to G$ the identity map of G so that $G^F = G$. Thus when $q = 1$ we are in case (1) and when $q > 1$ we are in case (2).

Let $\mathcal{B}$ be the variety of Borel subgroups of G. Let $\mathbf{W}$ be an indexing set for the set of G-orbits on $\mathcal{B} \times \mathcal{B}$ for the diagonal G-action. Let $\mathcal{O}_w$ be the G-orbit corresponding to $w \in \mathbf{W}$. Note that $\mathbf{W}$ is naturally a Coxeter group with length function $l(w) = \dim \mathcal{O}_w - \dim \mathcal{B}$.

Let I be an indexing set for the set S of simple reflections of $\mathbf{W}$. Let $s_i \in S$ be the simple reflection corresponding to $i \in I$. For $B \in \mathcal{B}$ we have $gBg^{-1} \in \mathcal{B}$ for any $g \in D$ (if $q = 1$) and $F(B) \in \mathcal{B}$ (if $q > 1$). There is a unique automorphism of $\mathbf{W}$ (denoted by $\bullet$ or by $w \mapsto w^\bullet$) such that

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$\mathcal{O}_{w^\bullet} = g\mathcal{O}_w g^{-1}$ for all $w \in \mathbf{W}, g \in D$ (if $q = 1$) and $\mathcal{O}_{w^\bullet} = F(\mathcal{O}_w)$ for all $w \in \mathbf{W}$ (if $q > 1$). We have $l(w^\bullet) = l(w)$ for all $w \in \mathbf{W}$. Hence there is a unique bijection $i \mapsto i^\bullet$ of I such that $s_i^\bullet = s_{i^\bullet}$ for all $i \in I$.

Two elements $w, w' \in \mathbf{W}$ are said to be $\bullet$-conjugate if $w' = a^{-1}wa^\bullet$ for some $a \in \mathbf{W}$. The relation of $\bullet$-conjugacy is an equivalence relation on $\mathbf{W}$; the equivalence classes are said to be $\bullet$-conjugacy classes. A $\bullet$-conjugacy class C in $\mathbf{W}$ (or an element of it) is said to be $\bullet$-elliptic if C does not meet any $\bullet$-stable proper parabolic subgroup of $\mathbf{W}$ (see [9]). (In the case where $\bullet = 1$ we say "elliptic, conjugacy class" instead of "$\bullet$-elliptic, $\bullet$-conjugacy class".) For $w \in \mathbf{W}$ let

$$\begin{aligned} \mathfrak{B}_w &= \{(g, B) \in D \times \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\} \text{ (if } q = 1) \\ X_w &= \{B \in \mathcal{B}; (B, F(B)) \in \mathcal{O}_w\} \text{ (if } q > 1). \end{aligned}$$

This is naturally an algebraic variety over $\mathbf{k}$. (The variety X_w is defined in [4]. The variety $\mathfrak{B}_w$ appears in [13] assuming that $D = G$ and in [14] in general.) We shall use the notation $\mathbf{X}_w$ for either $\mathfrak{B}_w$ or X_w. Let $\rho : \mathfrak{B}_w \to D$ be the first projection.

Now G^F acts on $\mathbf{X}_w$ by $x : (g, B) \mapsto (xgx^{-1}, xBx^{-1})$ (if $q = 1$) and by $x : B \mapsto xBx^{-1}$ (if $q > 1$).

One of the themes of this paper is the analogy between X_w and $\mathfrak{B}_w$. It seems that $\mathfrak{B}_w$ is a limit case of X_w as $q \to 1$. For example it is likely that for any i, the multiplicities of various unipotent character sheaves on D in a Jordan-Hölder series of the $(i + \dim G)$-th perverse cohomology sheaf of $\rho_!\bar{\mathbf{Q}}_l$ (with $q = 1$) are the same as the multiplicities of various irreducible unipotent representations of G^F in the G^F-module $H_c^i(X_w, \bar{\mathbf{Q}}_l)$ (with $q > 1$). Here l is a fixed prime number invertible in $\mathbf{k}$.

0.2. From [4, 1.11] it is known that if $w \in \mathbf{W}$, X_w has a natural finite covering $\tilde{X}_w$. We now show that (at least if w is $\bullet$-elliptic and G is semisimple), $\mathfrak{B}_w$ has a natural finite covering $\tilde{\mathfrak{B}}_w$.

Let $B^* \in \mathcal{B}$ and let T^* be a maximal torus of B^*; if $q > 1$ we assume in addition that B^*, T^* are defined over F_q. Let U^* be the unipotent radical of B^*. If $q = 1$ let $d \in D$ be such that $dT^*d^{-1} = T^*, dB^*d^{-1} = B^*$. Let $N = \{n \in G; nT^*n^{-1} = T^*\}$. We identify $N/T^* = \mathbf{W}$ by $nT^* \leftrightarrow w$, $(B^*, nB^*n^{-1}) \in \mathcal{O}_w$. According to Tits, for each $w \in \mathbf{W}$ we can choose a representative $\dot{w} \in N$ in such a way that $\dot{w} = \dot{w}_1\dot{w}_2$ whenever w, w_1, w_2 in $\mathbf{W}$ satisfy $w = w_1w_2$, $l(w) = l(w_1) + l(w_2)$. We can also assume that, if $w' = w^\bullet$, then $\dot{w}' = d\dot{w}d^{-1}$ (if $q = 1$) and $\dot{w}' = F(\dot{w})$ (if $q > 1$). For $w \in \mathbf{W}$ let $U^*_w = U^* \cap \dot{w}U^*\dot{w}^{-1}$ and let $T^*_w = \{t_1 \in T^*; \dot{w}^{-1}t\dot{w} = dtd^{-1}\}$ (if $q = 1$), $T^*_w = \{t \in T^*; \dot{w}^{-1}t\dot{w} = F(t)\}$ (if $q > 1$); let

$$\begin{aligned} \tilde{\mathfrak{B}}_w &= \{(g, g'U^*_w) \in D \times G/U^*_w; g'^{-1}gg' \in \dot{w}U^*d\} \text{ (if } q = 1), \\ \tilde{X}_w &= \{g'U^*_w \in G/U^*_w; g'^{-1}F(g') \in \dot{w}U^*\} \text{ (if } q > 1). \end{aligned}$$

We shall use the notation $\tilde{\mathbf{X}}_w$ for either $\tilde{\mathfrak{B}}_w$ or $\tilde{X}_w$. Now G^F acts on $\tilde{\mathbf{X}}_w$ by $x : (g, g'U^*_w) \mapsto (xgx^{-1}, xg'U^*_w)$ (if $q = 1$) and by $x : g'U^*_w \mapsto xg'U^*_w$ (if $q > 1$). Also T^*_w acts (freely) on $\tilde{\mathbf{X}}_w$ by $t : (g, g'U^*_w) \mapsto (g, g't^{-1}U^*_w)$ (if $q = 1$) and by $t : g'U^*_w \mapsto g't^{-1}U^*_w$ (if $q > 1$); this action commutes with the G^F-action. Define $\pi_w : \tilde{\mathbf{X}}_w \to \mathbf{X}_w$ by $(g, g'U^*_w) \mapsto (g, g'B^*g'^{-1})$ (if $q = 1$) and by $g'U^*_w \mapsto g'B^*g'^{-1}$ (if $q > 1$). Note that π_w is compatible with the T^*_w action where T^*_w acts on $\mathbf{X}_w$ trivially. Let $\mathfrak{F}$ be the fibre of π_w at a point of $\mathbf{X}_w$. Then for some $t_0 \in T^*$, $\mathfrak{F}$ can be identified with $\{t \in T^*; \mathrm{Ad}(d)(t^{-1})\mathrm{Ad}(\dot{w}^{-1})(t) = t_0\}$ (if $q = 1$) and with $\{t \in T^*; F(t)^{-1}\mathrm{Ad}(\dot{w}^{-1})(t) = t_0\}$ (if $q > 1$); hence it is either empty or a principal homogeneous space for T^*_w. Now if $q > 1$, T^*_w is finite, hence the homomorphism $T^* \to T^*$, $t \mapsto F(t)^{-1}\mathrm{Ad}(\dot{w}^{-1})(t)$ is surjective and $\mathfrak{F}$ is a principal homogeneous space for T^*_w so that in this case, π_w is a principal T^*_w-bundle. If for $q = 1$ we assume that G is semisimple and w is $\bullet$-elliptic then T^*_w is finite, hence the homomorphism $T^* \to T^*$, $t \mapsto \mathrm{Ad}(d)(t^{-1})\mathrm{Ad}(\dot{w}^{-1})(t)$ is surjective and $\mathfrak{F}$ is a principal homogeneous space for T^*_w so that in this case, π_w is again a principal T^*_w-bundle.

Here is another reason why $\mathfrak{B}_w$ looks like X_w when $q \to 1$: assuming that w is $\bullet$-elliptic and G is semisimple, the number $|T^*_w|$ (in the case $q = 1$) is obtained from the number $|T^*_w|$ (in the case $q > 1$) viewed as a polynomial in q by substituting $q = 1$.

The following result gives another instance of analogous behaviour of $X_w, \mathfrak{B}_w$.

Theorem 0.3. *Assume that $w \in \mathbf{W}$ is $\bullet$-elliptic and that w has minimal length in its $\bullet$-conjugacy class. If $q = 1$ assume further that G is semisimple.*

(a) *If $q = 1$ (resp. $q > 1$), any isotropy group of the G^F action on $\tilde{\mathfrak{B}}_w$ (resp. $\tilde{X}_w$) is $\{1\}$.*

(b) *If $q = 1$ (resp. $q > 1$), any isotropy group of the G^F action on $\mathfrak{B}_w$ (resp. X_w) is isomorphic to a subgroup of T_w^*; hence it is a finite diagonalizable group.*

(c) *If $q = 1$, the varieties $\mathfrak{B}_w$ and $\tilde{\mathfrak{B}}_w$ are affine.*

Note that (c) has the following known analogue (see [4] for sufficiently large q and [16], [9] for any q):

(d) *If $q > 1$, the varieties X_w and $\tilde{X}_w$ are affine.*

The proof of the theorem (given in §3) extends the proof of a weaker form of (b) given in [15, 5.2].

Let $G\backslash\tilde{\mathfrak{B}}_w$ (resp. $G^F\backslash\tilde{X}_w$) be the set of orbits of the G-action on $\tilde{\mathfrak{B}}_w$ (resp. of the G^F-action on $\tilde{X}_w$). Let $G\backslash\mathfrak{B}_w$ (resp. $G^F\backslash X_w$) be the set of orbits of the G-action on $\mathfrak{B}_w$ (resp. of the G^F-action on X_w). By (a)-(c) above. $G\backslash\tilde{\mathfrak{B}}_w$ and $G\backslash\mathfrak{B}_w$ are naturally affine varieties (they are the set of orbits of an action of a reductive group on an affine variety with all orbits being of the same dimension hence closed). Similarly, by (d) above, $G^F\backslash\tilde{X}_w$ and $G^F\backslash X_w$ are naturally affine varieties.

The affineness properties (c),(d) can be strengthened in certain cases as follows.

Theorem 0.4. *Assume that G is almost simple of type A_n, B_n, C_n or D_n. We assume also that $\bullet = 1$. Let $w \in \mathbf{W}$ be a $\bullet$-elliptic element of minimal length in its $\bullet$-conjugacy class.*

(a) *If $q = 1$, then $G\backslash\tilde{\mathfrak{B}}_w$ is isomorphic to $\mathbf{k}^{l(w)}$ and $G\backslash\mathfrak{B}_w$ is isomorphic to $T_w^*\backslash\mathbf{k}^{l(w)}$ for a T_w^*-action on $\mathbf{k}^{l(w)}$.*

(b) *If $q > 1$, then $G^F\backslash\tilde{X}_w$ is quasi-isomorphic (see 1.1) to $\mathbf{k}^{l(w)}$ and $G^F\backslash X_w$ is quasi-isomorphic (see 1.1) to $T_w^*\backslash\mathbf{k}^{l(w)}$ for a T_w^*-action on $\mathbf{k}^{l(w)}$.*

This is proved in Section 4. In a sequel to this paper it is shown that (a),(b) continue to hold without the assumption that $\bullet = 1$. We conjecture that (a),(b) hold for G of any type.

0.5. Let $w \in \mathbf{W}$ and let δ be the smallest integer ≥ 1 such that $\bullet^\delta = 1$. If $q > 1$, $F : X_w \to X_{w^\bullet}$ and $F^\delta : X_w \to X_w$ are well defined. We propose an extension of these maps to the case of $\mathfrak{B}_w$ namely $\Psi : \mathfrak{B}_w \to \mathfrak{B}_{w^\bullet}$,

$(g, B) \mapsto (g, gBg^{-1})$, see 1.2; we then have $\Psi^\delta : \mathfrak{B}_w \to \mathfrak{B}_w$. In some respects Ψ, Ψ^δ can be viewed as analogues for $q = 1$ of the Frobenius maps F, F^δ. Assume for example that w is $\bullet$-elliptic of minimal possible length in $\mathbf{W}$. There is some evidence that, for any i, the $(i + \dim G)$-th perverse cohomology sheaf of $\rho_! \bar{\mathbf{Q}}_l$ is direct sum of mutually nonisomorphic simple character sheaves stable under the map induced by Ψ^δ and Ψ^δ acts on each of these summands as multiplication by a root of 1 which is obtained from an eigenvalue of F^δ on $H^i_c(X_w, \bar{\mathbf{Q}}_l)$ (described in [11]) by $q \to 1$.

0.6. Let $w \in \mathbf{W}$ and let $i \in I$ be such that $l(w) = l(s_i w) + 1 = l(s_i w s_{i^\bullet})$. When $q > 1$ a quasi-isomorphism (see 1.1) $\sigma_i : X_w \to X_{s_i w s_{i^\bullet}}$ was defined in [4]. In the late 1970's and early 1980's I observed (unpublished but mentioned in [2, 5A] and [3]) that by taking compositions of various σ_i one can obtain nontrivial quasi-automorphisms of X_w corresponding to elements in the stabilizer of w for the $\bullet$-conjugacy action (see 1.3, 1.4). Further examples of this phenomenon were later found by Digne and Michel [5]. Additional examples are given in Section 1, 2. These examples are valid not only for X_w but also for $\tilde{X}_w$, $\mathfrak{B}_w$ or $\tilde{\mathfrak{B}}_w$ since in 2.3 and 2.6 we define quasi-isomorphisms analogous to σ_i in the case when X_w is replaced by $\tilde{X}_w$, $\mathfrak{B}_w$ or $\tilde{\mathfrak{B}}_w$.

0.7. In Section 5 we give another example of the close relation between the varieties $\mathfrak{B}_w, X_w$ by proving (under the assumption that $\mathbf{k}$ is as in case 2) a formula relating the number of rational points over a finite field of $\mathfrak{B}_w \times_D \mathfrak{B}_{w'}$ and of $G^F \backslash (X_w \times X_{w'})$.

0.8. *Notation.* For any $w \in \mathbf{W}$ we set $\mathcal{L}(w) = \{i \in I; l(s_i w) < l(w)\}$, $\mathcal{R}(w) = \{i \in I; l(w s_i) < l(w)\}$. For $k \in \mathbf{Z}$ let $w \mapsto w^{\bullet^k}$ be the k-th power of $\bullet$. Let w_0 be the longest element of $\mathbf{W}$. Let $\hat{\mathbf{W}}$ be the braid group of $\mathbf{W}$ with generators $\hat{s}_i$ corresponding to s_i. If X is a set and $f : X \to X$ is a map we write X^f instead of $\{x \in X; f(x) = x\}$. If X is finite we write $|X|$ for the cardinal of X.

1. Paths

1.1. Let C be a $\bullet$-elliptic $\bullet$-conjugacy class in $\mathbf{W}$. Let $C_{\min}$ be the set of elements of minimal length of C. If $w \in C_{\min}$ and $i \in \mathcal{L}(w)$ then $w' := s_i w s_i^\bullet \in C_{\min}$ and $i^* \in \mathcal{R}(w')$; we then write $w \xrightarrow{i^+} w'$. Conversely if $v \in C_{\min}$

and $j^\bullet \in \mathcal{R}(v)$ then $v' := s_j v s_j^\bullet \in C_{\min}$ and $j \in \mathcal{L}(v')$; we then write $v \xrightarrow{j^-} v'$. Note that if $w, w' \in \mathbf{W}$ then the conditions $w \xrightarrow{i^+} w'$ and $w' \xrightarrow{i^-} w$ are equivalent. Let Γ_C be the graph whose vertices are the elements of $C_{\min}$ and whose edges are the triples $w \overset{i}{\smile} w'$ with w, w' in $C_{\min}$ unordered and $i \in I$ such that either $w \xrightarrow{i^+} w'$ or $w' \xrightarrow{i^+} w$. The graph Γ_C has a canonical orientation in which an edge $w \overset{i}{\smile} w'$ is oriented from w to w' if $w \xrightarrow{i^+} w'$ and is oriented from w' to w if $w' \xrightarrow{i^+} w$. A path in Γ_C is by definition a sequence $\mathbf{i}$ of edges of Γ_C of the form $w_1 \overset{i_1}{\smile} w_2 \overset{i_2}{\smile}, \dots, \overset{i_{t-1}}{\smile} w_t$. For such $\mathbf{i}$ we must have $w_t = z_{\mathbf{i}}^{-1} w_1 z_{\mathbf{i}}^\bullet$ where $z_{\mathbf{i}} = s_{i_1} s_{i_2} \dots s_{i_{t-1}} \in \mathbf{W}$; we shall also set

(a) $$\tilde{z}_{\mathbf{i}} = \hat{s}_{i_1}^{\epsilon_1} \hat{s}_{i_2}^{\epsilon_2} \dots \hat{s}_{i_{t-1}}^{\epsilon_{t-1}} \in \hat{\mathbf{W}}$$

where $\epsilon_r = 1$ if $w_r \xrightarrow{i_r^+} w_{r+1}$, $\epsilon_r = -1$ if $w_r \xrightarrow{i_r^-} w_{r+1}$. We shall sometime specify $\mathbf{i}$ by the symbol $[w_1; *_1, *_2, \dots, *_{t-1}]$ where $*_k = i_k$ if $\epsilon_k = 1$ and $*_k = \overline{i}_k$ if $\epsilon_k = -1$ (ϵ_k as in (a).) Note that $w_2, \dots, w_t$ are uniquely determined by $w_1, i_1, i_2, \dots, i_{t-1}$).

For $w, w' \in C_{\min}$ let $\mathcal{P}_{w,w'}$ be the set of paths in Γ_C such that the corresponding sequence $w_1, w_2, \dots, w_t$ satisfies $w_1 = w, w_t = w'$. For example if $w = s_{i_1} s_{i_2} \dots s_{i_r}$ is a reduced expression in $\mathbf{W}$ then $[w; i_1, i_2, \dots, i_r] \in \mathcal{P}_{w,w^\bullet}$.

The following result is due to Geck-Pfeiffer [7, 3.2.7] (in the case where $\bullet = 1$) and to Geck-Kim-Pfeiffer [8] and He [9] in the remaining cases.

(b) *For any $w, w' \in C_{\min}$, the set $\mathcal{P}_{w,w'}$ is nonempty.*
For $w, w' \in C_{\min}$ we identify a path $[w; *_1, *_2, \dots, *_{t-1}] \in \mathcal{P}_{w,w'}$ with the path $[w; *'_1, *'_2, \dots, *'_{t'-1}] \in \mathcal{P}_{w,w'}$ in the following cases:

(i) $t' = t - 2$, $*_k = i, *_{k+1} = \overline{i}$ (for some $i \in I$ and some k), and $*'_1, *'_2, \dots, *'_{t'-1}$ is obtained from $*_1, *_2, \dots, *_{t-1}$ by dropping $*_k, *_{k+1}$;

(ii) $t' = t - 2$, $*_k = \overline{i}, *_{k+1} = i$ (for some $i \in I$ and some k), and $*'_1, *'_2, \dots, *'_{t'-1}$ is obtained from $*_1, *_2, \dots, *_{t-1}$ by dropping $*_k, *_{k+1}$;

(iii) $t' = t$, $*_k = i, *_{k+1} = j, *_{k+2} = i, \dots$ (m terms), $*'_k = j, *'_{k+1} = i, *'_{k+2} = j, \dots$ (m terms), (for some $i \neq j$ in I with $s_i s_j$ of order m and some k) and $*'_u = *_u$ for all other indices;

(iv) $t' = t$, $*_k = \overline{i}, *_{k+1} = \overline{j}, *_{k+2} = \overline{i}, \dots$ (m terms), $*'_k = \overline{j}, *'_{k+1} = \overline{i}, *'_{k+2} = \overline{j}, \dots$ (m terms), (for some $i \neq j$ in I with $s_i s_j$ of order m and some k) and $*'_u = *_u$ for all other indices.

This generates an equivalence relation on $\mathcal{P}_{w,w'}$; we denote by $\bar{\mathcal{P}}_{w,w'}$ the set of equivalence classes. For $w, w', w'' \in C_{\min}$, concatenation $\mathcal{P}_{w,w'} \times \mathcal{P}_{w',w''} \to \mathcal{P}_{w,w''}$ induces a map $\bar{\mathcal{P}}_{w,w'} \times \bar{\mathcal{P}}_{w',w''} \to \bar{\mathcal{P}}_{w,w''}$ which makes $\sqcup_{w,w' \in C_{\min}} \bar{\mathcal{P}}_{w,w'}$ into a groupoid. In particular for $w \in C_{\min}$, $\bar{\mathcal{P}}_{w,w}$ has a natural group structure. Now $\mathbf{i} \mapsto z_{\mathbf{i}}$ induces a group homomorphism

$$\tau_w : \bar{\mathcal{P}}_{w,w} \to \mathbf{W}_w := \{z \in \mathbf{W}; z^{-1} w z^{\bullet} = w\}$$

and $\mathbf{i} \mapsto \tilde{z}_{\mathbf{i}}$ induces a group homomorphism $\tilde{\tau}_w : \bar{\mathcal{P}}_{w,w} \to \hat{\mathbf{W}}$.

1.2. Let C be a $\bullet$-elliptic $\bullet$-conjugacy class in $\mathbf{W}$ and let $w \in C_{\min}$. We state the following conjecture.

(a) *The homomorphism $\tau_w : \bar{\mathcal{P}}_{w,w} \to \mathbf{W}_w$ is surjective.*
In 1.5, 1.6 we sketch a proof of (a) assuming that $\mathbf{W}$ is of classical type and $\bullet = 1$; in 1.4 we consider in more detail a case arising from D_4.

In any case, if $w^{\bullet} = w$ then w is in the image of τ_w. In particular, if $\mathbf{W}_w$ is generated by w then (a) holds for w. Also from 1.1(b) we see that if (a) holds for some $w \in C_{\min}$ then it holds for any $w \in C_{\min}$. We say that (a) holds for C if it holds for some (or equivalently any) $w \in C_{\min}$.

1.3. Assume that $w = w_0$ and $y^{\bullet} = wyw^{-1}$ for any $y \in \mathbf{W}$. Then the $\bullet$-conjugacy class of w is $C = \{w\}$ and is $\bullet$-elliptic. For any $y \in \mathbf{W}$ and any reduced expression $y = s_{i_1} s_{i_2} \dots s_{i_k}$ for y, we have $\mathbf{i} := [y; i_1, i_2, \dots, i_k] \in \mathcal{P}_{w,w}$, $z_{\mathbf{i}} = y$. Thus the image of τ_w is $\mathbf{W}_w = \mathbf{W}$ and 1.2(a) holds in this case.

1.4. In the remainder of this section we assume that $\bullet = 1$ on $\mathbf{W}$. We will often denote an element $s_{i_1} s_{i_2} s_{i_3} \dots s_{i_k}$ of $\mathbf{W}$ as $i_1 i_2 i_3 \dots i_k$.

The following example appeared in the author's work (1982, unpublished). Assume that $\mathbf{W}$ is of type D_4. Let $S = \{s_0, s_1, s_2, s_3\}$ with s_1, s_2, s_3 commuting. Let C be the conjugacy class of $\mathbf{W}$ consisting of the twelve elements (of length six) $0i0j0k$ and $i0j0k0$ (where i, j, k is a permutation of $1, 2, 3$). Note that $C = C_{\min}$ is elliptic and any $w \in C$ has order 4. We have $\mathcal{L}(0i0j0k) = \{0, i\}$, $\mathcal{R}(0i0j0k) = \{j, k\}$, $\mathcal{L}(i0j0k0) = \{i, j\}$, $\mathcal{R}(i0j0k0) = \{0, k\}$. We have $0i0j0k \xrightarrow{0^+} i0j0k0$, $0i0j0k \xrightarrow{i^+} 0j0i0k$, $i0j0k0 \xrightarrow{i^+} 0j0k0i$, $i0j0k0 \xrightarrow{j^+} i0k0j0$ for any i, j, k.

Let $w = i0j0k0 \in C = C_{\min}$. Now $\mathbf{W}_w$ is a nonabelian group of order 16 generated by three elements $\alpha = 0ij0$, $\beta = jk$, $\gamma = i0ki0i$ satisfying

(a) $$\gamma\alpha\beta = \alpha\beta\gamma = \beta\gamma\alpha = w.$$

Note that β (resp. α) is the unique element of length 2 (resp. 4) in $\mathbf{W}_w$: if n_i is the number of elements of length i in $\mathbf{W}_w$ and t is an indeterminate, then $\sum_{i\ge 0} n_i t^i = 1 + t^2 + t^4 + 10t^6 + t^8 + t^{10} + t^{12}$. We have

$$\mathbf{i} := [w; \overline{0}, i, j, 0] \in \mathcal{P}_{w,w}, \mathbf{i}' := [w; j, k] \in \mathcal{P}_{w,w}, \mathbf{i}'' := [w; i, 0, k, i, \overline{0}, \overline{i}] \in \mathcal{P}_{w,w},$$

and $z_{\mathbf{i}} = \alpha$, $z_{\mathbf{i}'} = \beta$, $z_{\mathbf{i}''} = \gamma$. Thus the image of τ_w contains the generators α, β, γ of $\mathbf{W}_w$ hence it is equal to $\mathbf{W}_w$ and 1.2(a) holds for C. Note that a relation like (a) also holds in the group $\bar{\mathcal{P}}_{w,w}$:

(b) $$\mathbf{i}''\mathbf{i}\mathbf{i}' = \mathbf{i}\mathbf{i}'\mathbf{i}'' = \mathbf{i}'\mathbf{i}''\mathbf{i} = [w; i, 0, j, 0, k, 0].$$

For example,

$$\begin{aligned} \mathbf{i}''\mathbf{i}\mathbf{i}' &= [w; i, 0, k, i, \overline{0}, \overline{i}, \overline{0}, i, j, 0, j, k] \\ &= [w; i, 0, k, i, \overline{i}, \overline{0}, \overline{i}, i, j, 0, j, k] = [w; i, 0, k, j, 0, k] = [w; i, 0, j, 0, k, 0]. \end{aligned}$$

Also $\mathbf{i}, \mathbf{i}', \mathbf{i}''$ commute with $[w; i, 0, j, 0, k, 0]$ in $\mathcal{P}_{w,w}$. It follows that $\tilde{z}_{\mathbf{i}''}, \tilde{z}_{\mathbf{i}}, \tilde{z}_{\mathbf{i}'}$ satisfy a relation like (b) in $\hat{\mathbf{W}}$.

1.5. Let $\mathbf{n}$ be an integer ≥ 3. Define $n \in \mathbf{N}$ by $\mathbf{n} = 2n$ if $\mathbf{n}$ is even , $\mathbf{n} = 2n+1$ if $\mathbf{n}$ is odd. Let W be the group of all permutations of $[1, \mathbf{n}]$ which commute with the involution $i \mapsto \mathbf{n} - i + 1$ of $[1, \mathbf{n}]$. For $i \in [1, n-1]$ define $s_i \in W$ as a product of two transpositions $i \leftrightarrow i+1$, $\mathbf{n} + 1 - i \leftrightarrow \mathbf{n} - i$; define $s_n \in W$ to be the transposition $n \leftrightarrow \mathbf{n} - n + 1$. Then $(W, \{s_i; i \in [1, n]\})$ is a Weyl group of type B_n. In this subsection we assume that G is almost simple of type C_n (or B_n) and we identify $\mathbf{W}$ with W with $\mathbf{n} = 2n$ (or $\mathbf{n} = 2n+1$) as Coxeter groups in the standard way.

Let $p_* = (p_1 \ge p_2 \ge \dots \ge p_\sigma)$ be a sequence in $\mathbf{Z}_{>0}$ such that $p_1 + \dots + p_\sigma = n$. Define a partition $m_1 + m_2 + \dots + m_e = \sigma$ by

$$p_1 = p_2 = \dots = p_{m_1} > p_{m_1+1} = p_{m_1+2} = \dots = p_{m_1+m_2} > \dots.$$

For any $r \in [1,\sigma]$ we define a permutation w_r in W by

$$p_{<r}+1 \mapsto p_{<r}+2 \mapsto \ldots \mapsto p_{<r}+p_r \mapsto \mathbf{n}-p_{<r} \mapsto \mathbf{n}-p_{<r}-1 \mapsto \ldots \mapsto \mathbf{n}-p_{<r}-p_r+1 \mapsto p_{<r}+1,$$

where $p_{<r}=\sum_{r'\in[1,r-1]} p_{r'}$ and all unspecified elements are mapped to themselves. Note that w_r is a $2p_r$-cycle and that $w_1,w_2,\ldots,w_\sigma$ are commuting with each other. Let $w=w_1w_2\ldots w_\sigma$ and let C be the conjugacy class of w. Note that C is elliptic and $w\in C_{\min}$. For every $r\in[1,\sigma-1]$ such that $p_r=p_{r+1}$ we define an involutive permutation $h_r\in W$ by

$$\begin{aligned} p_{<r}+j &\mapsto p_{<r+1}+j \mapsto p_{<r}+j, \mathbf{n}-p_{<r}-j \\ &\mapsto \mathbf{n}-p_{<r+1}-j \mapsto \mathbf{n}-p_{<r}-j \text{ for } j\in[1,p_r] \end{aligned}$$

(all unspecified elements are mapped to themselves). Note that $h_rw_{r+1}h_r=w_r$ and $h_rw_t=w_th_r$ for all $t\{r,r+1\}$. Hence $h_rw=wh_r$. The following result is easily verified:

(a) *The group* $\mathbf{W}_w$ *is generated by the elements* w_σ, w_r $(r\in[1,\sigma-1], p_{r+1}>p_r)$ *and* h_r $(r\in[1,\sigma-1], p_r=p_{r+1})$.
These generators satisfy the "braid group relations" of a complex reflection group of type

$$B_{m_1}^{(2p_{m_1})}\times B_{m_2}^{(2p_{m_1+m_2})}\times\ldots\times B_{m_e}^{(2p_{m_1+\cdots+m_e})}$$

(described in [2, 3A]); the factor $B_{m_k}^{(2p_{m_1+\cdots+m_k})}$ is generated by $h_{m_1+\cdots+m_{k-1}+u}$ $(u\in[1,m_k-1])$ and by $w_{m_1+\cdots+m_k}$.

It is immediate that for $r\in[1,\sigma]$ we have (setting $a=n-(p_\sigma+p_{\sigma-1}+\cdots+p_{r+1})$):

$$\mathbf{i}_r := [w; a, a+1,\ldots,n-1,n,n-1,\ldots,a-p_r+2,a-p_r+1]\in\mathcal{P}_{w,w}.$$

Note that $z_{\mathbf{i}_r}=w_r$.

One can verify that for $r\in[1,\sigma-1]$ such that $p_r=p_{r+1}=p$ we have (setting $a=n-(p_\sigma+p_{\sigma-1}+\cdots+p_{r+1})$):

$$\begin{aligned} \mathbf{i}'_r := {} & [w; a,a+1,\ldots,a+p-2,a-1,a,a+1\ldots,a+p-4,a-2,a-1, \\ & a,\ldots,a+p-6,\ldots,a-p+2,a+p-1,a+p-3,\ldots,a-p+1, \end{aligned}$$

$$\overline{a-p+2},\dots,\overline{a+p-6},\dots,\overline{a},\overline{a-1},\overline{a-2},\overline{a+p-4},\dots,\overline{a+1},$$
$$\overline{a},\overline{a-1},\overline{a+p-2},\dots,\overline{a+1},\overline{a}]\in\mathcal{P}_{w,w},$$

For example if $p=1$ we have $z_{\mathbf{i}'_r}=[w;a]$; if $p=2$ we have $z_{\mathbf{i}'_r}=[w;a,a+1,a-1,\overline{a}]$; if $p=3$ we have

$$z_{\mathbf{i}'_r}=[w;a,a+1,a-1,a+2,a,a-2,\overline{a-1},\overline{a+1},\overline{a}].$$

Note that $z_{\mathbf{i}'_r}=h_r$. Using (a), we see that the image of τ_w contains a set of generators of $\mathbf{W}_w$ hence 1.2(a) holds for C. (In the case where $p_1=p_2=\dots=p_\sigma$, this result is due to Digne and Michel [5].) Note that any elliptic conjugacy class in $\mathbf{W}$ is of the form C as above. We conjecture that

(b) *the braid group relations satisfied by the generators in* (a) *remain valid as equations in* $\bar{\mathcal{P}}_{w,w}$ *if* w_r *is replaced by* $\mathbf{i}_r$ *and* h_r *is replaced by* $\mathbf{i}'_r$.
Appplying $\tilde{\tau}_w$ we would get corresponding braid group relations in $\hat{\mathbf{W}}$ which actually can be verified.

1.6. In this subsection we assume that G is almost simple of type D_n. Let W,s_i be as in 1.5 (with $\mathbf{n}=2n\ge 8$). Le W' be the group of even permutations in W (a subgroup of index 2 of W). If $i\in[1,n-1]$ we have $s_i\in W'$ and we set $s_{(n-1)'}=s_ns_{n-1}s_n\in W'$. Then $(W',\{s_1,s_2,\dots,s_{n-1},s_{(n-1)'}\})$ is a Weyl group of type D_n. We identify $\mathbf{W}$ with W' as Coxeter groups as in [15, 1.5]. Let $p_*=(p_1\ge p_2\ge\dots\ge p_\sigma)$, w_r,w,h_r be as in 1.5; we assume that σ is even. Then $w\in W'$. Let C' be the conjugacy class of w in W'. Then C' is elliptic and $w\in C'_{\min}$. For any $r\in[1,\sigma]$ we have $w'_r:=w_rw_\sigma\in W'$. For any $r\in[1,\sigma-1]$ such that $r\in[1,\sigma-1],p_r=p_{r+1}$ we have $h_r\in W'$. If $p_{\sigma-1}=p_\sigma$ we set $h'_{\sigma-1}=w_\sigma^{-1}h_{\sigma-1}w_\sigma$. The following result is easily verified:

(a) *If* $p_{\sigma-1}>p_\sigma$ *then* $\mathbf{W}_w$ *is generated by the elements* w'_σ, w'_r *(*$r\in[1,\sigma-1],p_{r+1}>p_r$*) and* h_r *(*$r\in[1,\sigma-2],p_r=p_{r+1}$*). If* $p_{\sigma-1}=p_\sigma$ *then* $\mathbf{W}_w$ *is generated by the elements* w'_σ, w'_r *(*$r\in[1,\sigma-2],p_{r+1}>p_r$*),* $h'_{\sigma-1}$ *and* h_r *(*$r\in[1,\sigma-1],p_r=p_{r+1}$*).*
These generators satisfy the "braid group relations" of a complex reflection group of type

$$B_{m_1}^{(2p_{m_1})}\times B_{m_2}^{(2p_{m_1+m_2})}\times\ldots\times B_{m_{e-1}}^{(2p_{m_1+\dots+m_{e-1}})}\times D_{m_e}^{(2p_{m_1+\dots+m_e})}$$

(described in [2, 3A]); the factor $B_{m_k}^{(2p_{m_1+\dots+m_k})}$ (with $k < m$) is generated by $h_{m_1+\dots+m_{k-1}+u}$ ($u \in [1, m_k - 1]$) and by $w'_{m_1+\dots+m_k}$; if $m_e > 1$ then the factor $D_{m_e}^{(2p_{m_1+\dots+m_e})}$ is generated by $h_{m_1+\dots+m_{e-1}+u}$ ($u \in [1, m_e - 1]$), by $h'_{m_1+\dots+m_e-1}$ and by by $w'_{m_1+\dots+m_k}$; if $m_e = 1$ the factor $D_{m_e}^{(2p_{m_1+\dots+m_e})}$ is taken to be a cyclic group of order p_σ.For example the "braid group relation"

$$h_{\sigma-1}w'_\sigma h'_{\sigma-1} = w'_\sigma h'_{\sigma-1}h_{\sigma-1} = h'_{\sigma-1}h_{\sigma-1}w'_\sigma$$

holds if $m_e > 1$. (Compare with 1.4(a).)

One can verify that for $r \in [1, \sigma]$ we have (setting $a = n - (p_\sigma + p_{\sigma-1} + \dots + p_{r+1})$):

$$\begin{aligned}\mathbf{i}''_r &= [w; a, a+1, \dots, n-1, (n-1)', n-2, \dots, a-p_r+2, a-p_r+1, n-1,\\ &\quad n-2, \dots, n-p_\sigma+1] \in \mathcal{P}_{w,w}.\end{aligned}$$

Note that $z_{\mathbf{i}''_r} = w'_r$. On the other hand for $r \in [1, \sigma - 1], p_r = p_{r+1}$ we have $h_r = z_{\mathbf{i}'_r}$ where $\mathbf{i}'_r$ is given by the same formula as in 1.5 (but viewed in W'); we have $\mathbf{i}'_r \in \mathcal{P}_{w,w}$. If $p_{\sigma-1} = p_\sigma = p$ then

$$\begin{aligned}\tilde{\mathbf{i}} &= [w; (n-1)', n-2, n-3, \dots, p+1, p, p+1, \dots, n-2, p-1, p, \dots,\\ &\quad n-4, \dots, 3, 4, 2, (n-1)', n-3, \dots, 5, 3, 1, \overline{2}, \overline{4}, \overline{3}, \dots, \overline{n-4}, \dots, \overline{p},\\ &\quad \overline{p-1}, \overline{n-2}, \dots, \overline{p+1}, \overline{p}, \overline{p+1}, \dots, \overline{n-3}, \overline{n-2}, \overline{(n-1)'}] \in \mathcal{P}_{w,w}.\end{aligned}$$

For example if $n = 10$, $p = 5$ then

$$\begin{aligned}\tilde{\mathbf{i}} &:= [w; 9', 8, 7, 6, 5, 6, 7, 8, 4, 5, 6, 3, 4, 2, 9', 7, 5, 3, 1, \overline{2}, \overline{4}, \overline{3}, \overline{6}, \overline{5}, \overline{4}, \overline{8}, \overline{7}, \overline{6},\\ &\quad \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9'}].\end{aligned}$$

Note that $z_{\tilde{\mathbf{i}}} = h'_{\sigma-1}$. Using (a), we see that the image of τ_w contains a set of generators of $\mathbf{W}_w$ hence 1.2(a) holds for C. Note that any elliptic conjugacy class in $\mathbf{W}$ is of the form C' as above. We conjecture that

(b) *the braid group relations satisfied by the generators in* (a) *remain valid as equations in* $\bar{\mathcal{P}}_{w,w}$ *if* w'_r *is replaced by* $\mathbf{i}''_r$, h_r *is replaced by* $\mathbf{i}'_r$ *and* $h'_{\sigma-1}$ *is replaced by* $\tilde{\mathbf{i}}$.

Appplying $\tilde{\tau}_w$ we would get corresponding braid group relations in $\hat{\mathbf{W}}$ which actually can be verified.

1.7. In this subsection we assume that C is an elliptic conjugacy class in $\mathbf{W}$ such that for some $w \in C_{\min}$ we have $w = w_1 w_2 \dots w_r$ where $w_1, \dots, w_r$ commute with each other, $l(w) = l(w_1)+l(w_2)+\dots+l(w_r)$ and the centralizer of w is generated by $w_1, \dots, w_r$. (An example of this situation is the case of w in 1.5 with $p_1 > p_2 > \dots > p_\sigma$.) In this case it is immediate that w_i is in the image of τ_w hence 1.2(a) holds for C.

Another example arises for $\mathbf{W}$ of type E_8 (with the elements of I labelled as in [7]) and with C consisting of elements whose characteristic polynomial in the reflection representation is $(X+1)(X^7+1)$. The element $w = 213423454234565768$ belongs to $C_{\min}$ and $l(w) = 18$. We have $w = s_2 x = x s_2$ for some x such that $l(x) = 17$ and $s_2 x^7 = w_0$. (This equation holds also in $\hat{\mathbf{W}}$.) The centralizer of w is a product of a cyclic group of order 2 generated by s_2 and a cyclic group of order 14 generated by x. We see that 1.2(a) holds for C.

1.8. Assume that $\mathbf{W}$ is of type E_8. Let C be the elliptic conjugacy class in $\mathbf{W}$ consisting of the elements of order 15. We can find $w \in C_{\min}$ such that $w = u^2$ where $u = 12345678$ so that $l(u) = 8$, $l(w) = 16$. Then the centralizer of w consists of the powers of u. We have $\mathbf{i} := [w; 1, 2, 3, 4, 5, 6, 7, 8] \in \mathcal{P}_{w,w}$, $z_{\mathbf{i}} = u$ and we see that 1.2(a) holds for C.

2. The Morphisms $\sigma_i, \tilde{\sigma}_i$

2.1. If V, V' are algebraic varieties over $\mathbf{k}$, we say that a map of sets $f : V \to V'$ is a quasi-morphism if:

(for $q = 1$) f is a morphism, or

(for $q > 1$) f is composition $V = V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \dots \xrightarrow{f_{t-1}} V_t = V'$ where for each $i \in [1, t-1]$, $f_i : V_i \to V_{i+1}$ is either a morphism of algebraic varieties or $V_i = V_{i+1}$ and f_i is the inverse of the Frobenius map on V_i for a rational structure over a finite subfield of $\mathbf{k}$.

We say that f is a quasi-isomorphism if it is a quasi-morphism and has an inverse which is a quasi-morphism. If in addition we have $V = V'$ we say that f is a quasi-automorphism.

2.2. Let $w \in \mathbf{W}$. We define a morphism $\Psi : \mathbf{X}_w \to \mathbf{X}_{w^\bullet}$ by

$$(g, B) \mapsto (g, gBg^{-1}) \text{ if } q = 1 \text{ and } B \mapsto F(B) \text{ if } q > 1.$$

We define a morphism $\Psi : \tilde{\mathbf{X}}_w \to \tilde{\mathbf{X}}_{w^\bullet}$ by

$$(g, g'U_w^*) \mapsto (g, gg'd^{-1}U_{w^\bullet}^*) \text{ (if } q = 1) \text{ and } g'U_w^* \mapsto F(g')U_{w^\bullet}^* \text{ (if } q > 1).$$

Note that each of the morphisms Ψ is a quasi-isomorphism.

2.3. For any $w, w', a, b \in \mathbf{W}$ such that $w = ab, w' = ba^\bullet$, $l(w) = l(a)+l(b) = l(w')$ we define a morphism $\sigma(a) : \mathbf{X}_w \to \mathbf{X}_{w'}$ by

$(g, B) \mapsto (g, B')$ where $B' \in \mathcal{B}$ is determined by the conditions $(B, B') \in \mathcal{O}_a$,
$(B', gBg^{-1}) \in \mathcal{O}_b$ (if $q = 1$);
$B \mapsto B'$ where $B' \in \mathcal{B}$ is determined by the conditions $(B, B') \in \mathcal{O}_a$,
$(B', F(B)) \in \mathcal{O}_b$ (if $q > 1$).

(If $q > 1$, the map $\sigma(a)$ is defined in [4, p.107,108].) We have a commutative diagram

$$\begin{array}{ccc} \mathbf{X}_w & \xrightarrow{\sigma(a)} & \mathbf{X}_{w'} \\ {\scriptstyle\Psi}\downarrow & & {\scriptstyle\Psi}\downarrow \\ \mathbf{X}_{w^\bullet} & \xrightarrow{\sigma(a^\bullet)} & \mathbf{X}_{w'^\bullet} \end{array}$$

Note that for any $w \in \mathbf{W}$ we have $\sigma(w) = \Psi : \mathbf{X}_w \to \mathbf{X}_{w^\bullet}$.

If w, w', a, b are as above then $\sigma(b) : \mathbf{X}_{w'} \to \mathbf{X}_{w^\bullet}$ is defined and $\sigma(b)\sigma(a) : \mathbf{X}_w \to \mathbf{X}_{w^\bullet}$ is equal to Ψ. Interchanging (a, b) with $(b^{\bullet^{-1}}, a)$ we see that
$\sigma(a)\sigma(b^{\bullet^{-1}}) : X_{w'^{\bullet-1}} \to X_{w'}$ is equal to Ψ. Thus $\sigma(a) : \mathbf{X}_w \to \mathbf{X}_{w'}$ is a quasi-isomorphism.

Let $w \in \mathbf{W}$ and let $i \in \mathcal{L}(w)$ be such that, setting $w' = s_i w s_i^\bullet$, we have $l(w) = l(w')$. Then $\sigma(s_i) : \mathbf{X}_w \to \mathbf{X}_{w'}$ is a well defined quasi-isomorphism; we shall often write σ_i instead of $\sigma(s_i)$.

2.4. Assume that $w \in \mathbf{W}$ and i, j are distinct elements of $\mathcal{L}(w)$. Let m be the order of $s_i s_j$ and let $v = s_i s_j s_i \dots = s_j s_i s_j \dots$ (both products have m factors). Let $w' = vwv^\bullet$ and assume that

$$l(w) = l(s_i w s_i^\bullet) = l(s_j s_i w s_i^\bullet s_j^\bullet) = \dots = l(vwv^\bullet),$$
$$l(w) = l(s_j w s_j^\bullet) = l(s_i s_j w s_j^\bullet s_i^\bullet) = \dots = l(vwv^\bullet)$$

so that the sequences of m maps

$$\mathbf{X}_w \xrightarrow{s_i} \mathbf{X}_{s_i w s_i^\bullet} \xrightarrow{s_j} \mathbf{X}_{s_j s_i w s_i^\bullet s_j^\bullet} \xrightarrow{s_i} \dots \to \mathbf{X}_{vwv^\bullet}$$
$$\mathbf{X}_w \xrightarrow{s_j} \mathbf{X}_{s_j w s_j^\bullet} \xrightarrow{s_i} \mathbf{X}_{s_i s_j w s_j^\bullet s_i^\bullet} \xrightarrow{s_j} \dots \to \mathbf{X}_{vwv^\bullet}$$

are defined. We show that both compositions are equal to $\sigma_v : \mathbf{X}_w \to \mathbf{X}_{vwv^\bullet}$.

Let $(g, B) \in \mathfrak{B}_w$ (resp. $B \in X_w$). We can find a unique sequence $B_0, B_1, \dots, B_m$ in $\mathcal{B}$ such that $B_0 = B$, $(B_0, B_1) \in \mathcal{O}_{s_i}$, $(B_1, B_2) \in \mathcal{O}_{s_j}$, $(B_2, B_3) \in \mathcal{O}_{s_i}$, ... and $(B_m, gBg^{-1}) \in \mathcal{O}_{vw}$ (if $q = 1$), $(B_m, F(B)) \in \mathcal{O}_{vw}$ (if $q > 1$). If $q = 1$ we have $\sigma_i(g, B) = (g, B_1)$, $\sigma_j(g, B_1) = (g, B_2)$, ... and $\sigma_v(g, B) = (g, B_m)$; thus $\sigma_v(g, B) = \dots \sigma_i \sigma_j \sigma_i(g, B)$ (the product has m factors); similarly we have $\sigma_v(g, B) = \dots \sigma_j \sigma_i \sigma_j(g, B)$ (the product has m factors). If $q > 1$ we have $\sigma_i(B) = B_1$, $\sigma_j(B_1) = B_2$, ... and $\sigma_v(B) = B_m$; thus $\sigma_v(B) = \dots \sigma_i \sigma_j \sigma_i(B)$ (the product has m factors); similarly we have $\sigma_v(B) = \dots \sigma_j \sigma_i \sigma_j(B)$ (the product has m factors). If $q = 1$, it follows that $\dots \sigma_i \sigma_j \sigma_i(g, B) = \dots \sigma_j \sigma_i \sigma_j(g, B)$ as required. If $q > 1$, it follows that $\dots \sigma_i \sigma_j \sigma_i(B) = \dots \sigma_j \sigma_i \sigma_j(B)$ as required.

2.5. Assume that $w \in \mathbf{W}$ and $w = s_{i_1} s_{i_2} \dots s_{i_k}$ is a reduced decomposition of w. Let

$$w_1 = w, w_2 = s_{i_2} \dots s_{i_k} s_{i_1^\bullet}, \dots, w_{k+1} = s_{i_1^\bullet} \sigma_{i_2^\bullet} \dots s_{i_k^\bullet} = w^\bullet.$$

Assume that $l(w_1) = l(w_2) = \dots = l(w_{k+1})$. Then the sequence of maps

$$\mathbf{X}_{w_1} \xrightarrow{\sigma_{i_1}} \mathbf{X}_{w_2} \xrightarrow{\sigma_{i_2}} \dots \xrightarrow{\sigma_{i_k}} \mathbf{X}_{w_{k+1}}$$

is defined. We show that the composition is equal to $\Psi : \mathbf{X}_w \to \mathbf{X}_{w^\bullet}$.

Let $(g, B) \in \mathfrak{B}_w$ (resp. $B \in X_w$). We can find a unique sequence $B_0, B_1, \dots, B_k$ in $\mathcal{B}$ such that $B_0 = B$, $(B_0, B_1) \in \mathcal{O}_{s_{i_1}}$, $(B_1, B_2) \in \mathcal{O}_{s_{i_2}}$,

$\dots,(B_{k-1},B_k)\in\mathcal{O}_{s_{i_k}}$, and $B_k=gBg^{-1}$ (if $q=1$), $B_k=F(B)$ (if $q>1$). From the definitions we have $\sigma_{i_1}(g,B)=(g,B_1)$, $\sigma_{i_2}(g,B_1)=(g,B_2)$, $\dots$, $\sigma_{i_k}(g,B_{k-1})=(g,B_k)$ if $q=1$ and $\sigma_{i_1}(B)=B_1$, $\sigma_{i_2}(B_1)=B_2$, $\dots$, $\sigma_{i_k}(B_{k-1})=B_k$ if $q>1$. The desired result follows.

2.6. Let $i\in I$. Let U^*_i be the unique root subgroup of U^* such that $(\dot{s}^\bullet)^{-1}U^*_i\dot{s}^\bullet\not\subset U^*$ where $s=s_i$. Let $U^{*!}=\{u\in U^*;(\dot{s}^\bullet)^{-1}u\dot{s}^\bullet\in U^*\}$. Note that any $u\in U^*$ can be written uniquely in the form $u=u_!u^!$ where $u_!\in U^*_i$, $u^!\in U^{*!}$ and that $u\mapsto u_!$, $U^*\to U^*_i$ is a homomorphism.

Now assume that $w,w',b\in\mathbf{W}$ are such that $w=sb,w'=bs^\bullet$, $l(w)=l(b)+1=l(w')$. Note that

(a) $\dot{b}U^*_i\dot{b}^{-1}\subset U^*$.

If $q=1$ we fix $g\in D$. Let $g'\in G$ be such that $g'^{-1}gg'=\dot{w}ud,u\in U^*$ (if $q=1$) and $g'^{-1}F(g')=\dot{w}u,u\in U^*$ (if $q>1$). We set $g'_1=g'\dot{w}u_!\dot{b}^{-1}$. Using (a) and the definition, we have

$$\begin{aligned}g_1'^{-1}gg'_1&=\dot{b}u_!^{-1}\dot{w}^{-1}g'^{-1}gg'\dot{w}u_!\dot{b}^{-1}=\dot{b}u_!^{-1}\dot{w}^{-1}\dot{w}ud\dot{w}u_!\dot{b}^{-1}\\&=\dot{b}u_!^{-1}ud\dot{s}\dot{b}u_!\dot{b}^{-1}=\dot{b}\dot{s}^\bullet((\dot{s}^\bullet)^{-1}u^!\dot{s}^\bullet)d(\dot{b}u_!\dot{b}^{-1})\in\dot{w}'U^*dU^*=\dot{w}'U^*d\end{aligned}$$

(if $q=1$),

$$\begin{aligned}g_1'^{-1}F(g'_1)&=\dot{b}u_!^{-1}\dot{w}^{-1}g'^{-1}F(g')F(\dot{w})F(u_!)F(\dot{b}^{-1})\\&=\dot{b}u_!^{-1}\dot{w}^{-1}\dot{w}uF(\dot{w})F(u_!)F(\dot{b}^{-1})\\&=\dot{b}u^!F(\dot{s})F(\dot{b}u_!\dot{b}^{-1})=\dot{b}F(\dot{s})F(\dot{s}^{-1})u^!F(\dot{s})F(\dot{b}u_!\dot{b}^{-1})\in\dot{w}'U^*\end{aligned}$$

(if $q>1$).

Now let $v\in U^*_w$. We have $v'=\dot{w}^{-1}v\dot{w}\in U^*$. Using this and $w=sb,l(w)=l(b)+1$, we deduce

(b) $\dot{s}^{-1}v\dot{s}\in U^*$; hence $(\dot{s}^\bullet)^{-1}dvd^{-1}\dot{s}^\bullet\in U^*$, $(dvd^{-1})_!=1$ (if $q=1$) and $F(\dot{s}^{-1})F(v)F(\dot{s})\in U^*$, $F(v)_!=1$ (if $q>1$).

We have

$$(g'v)^{-1}gg'v=v^{-1}g'^{-1}gg'v=v^{-1}\dot{w}udv=\dot{w}v'^{-1}udv\in\dot{w}U^*d$$

(if $q=1$),

$$(g'v)^{-1}F(g'v)=v^{-1}g'^{-1}F(g')F(v)=v^{-1}\dot{w}uF(v)=\dot{w}v'^{-1}uF(v)\in\dot{w}U^*$$

(if $q > 1$). We define $(g'v)_1$ in terms of $g'v$ in the same way as g'_1 was defined in terms of g'. Thus we have

$$(g'v)_1 = g'v\dot{w}v'^{-1}_! u_! (dvd^{-1})_! \dot{b}^{-1} = g'v\dot{w}v'^{-1}_! u_! \dot{b}^{-1} \qquad \text{(if } q = 1\text{)},$$
$$(g'v)_1 = g'v\dot{w}v'^{-1}_! u_! F(v)_! \dot{b}^{-1} = g'v\dot{w}v'^{-1}_! u_! \dot{b}^{-1} \qquad \text{(if } q > 1\text{)};$$

we have used that $(dvd^{-1})_! = 1$ if $q = 1$ and $F(v)_! = 1$ if $q > 1$, see (b). We have $(g'v)_1 = g'_1 v_1$ where

$$v_1 = (g'_1)^{-1}(g'v)_1 = \dot{b}u_!^{-1}\dot{w}^{-1}g'^{-1}g'v\dot{w}v'^{-1}_! u_! \dot{b}^{-1} = \dot{b}u_!^{-1}v'v'^{-1}_! u_! \dot{b}^{-1}.$$

We show that $v_1 \in U^*_{w'}$. We have

$$v_1 = (\dot{b}u_!\dot{b}^{-1})(\dot{s}^{-1}v\dot{s})(\dot{b}v'^{-1}_! u_! \dot{b}^{-1})$$

and this belongs to U^* by (a),(b). We have

$$\dot{w}'^{-1}v_1\dot{w}' = (\dot{s}^\bullet)^{-1}z\dot{s}^\bullet$$

where $z = u_!^{-1}v'v'^{-1}_! u_! \in U^*$. To show that $\dot{w}'^{-1}v_1\dot{w}' \in U^*$ it is enough to observe that $z_! = u_!^{-1}v'_!v'^{-1}_! u_! = 1$ so that $z \in U^{*!}$.

Summarizing, we see that there is a well defined morphism $\tilde{\sigma}_i : \tilde{\mathbf{X}}_w \to \tilde{\mathbf{X}}_{w'}$ such that (if $q = 1$) $(g, g'U^*_w) \mapsto (g, g'\dot{w}u_!\dot{b}^{-1}U^*_{w'})$ with $u \in U^*$ given by $g'^{-1}gg' = \dot{w}ud$ and (if $q > 1$) $g'U^*_w \mapsto g'\dot{w}u_!\dot{b}^{-1}U^*_{w'}$ with $u \in U^*$ given by $g'^{-1}F(g') = \dot{w}u$. The map $\tilde{\sigma}_i$ commutes with the G^F-actions, is compatible with the T^*_w and $T^*_{w'}$ actions via the isomorphism $T^*_w \to T^*_{w'}$, $t \mapsto \dot{s}^{-1}t\dot{s}$ and is compatible with the map σ_i (see 2.3) via the maps π_w, $\pi_{w'}$. In the case where T^*_w (hence $T^*_{w'}$) is finite so that π_w (resp. $\pi_{w'}$) is a principal T^*_w-(resp. $T^*_{w'}$-) bundle over $\mathbf{X}_w$ (resp. $\mathbf{X}_{w'}$) we deduce (using the fact that $\sigma_i : \mathbf{X}_w \to \mathbf{X}_{w'}$ is bijective) that $\tilde{\sigma}_i$ is bijective; it is easy to see that in this case, $\tilde{\sigma}_i$ is a quasi-isomorphism.

2.7. Assume that $c \in \mathbf{W}$ and $i_1, i_2, \dots, i_k \in I$ are such that each of

$$w_1 = s_{i_1}s_{i_2}\dots s_{i_k}c, w_2 = s_{i_2}s_{i_3}\dots s_{i_k}cs^\bullet_{i_1}, \dots, w_{k+1} = cs^\bullet_{i_1}s^\bullet_{i_2}\dots s^\bullet_{i_k}$$

has length $k + l(c)$.

Let $(g, g'U^*_{w_1}) \in \tilde{\mathbf{X}}_{w_1}$ (if $q = 1$), $g'U^*_{w_1} \in \tilde{\mathbf{X}}_{w_1}$ (if $q > 1$). Using the definitions repeatedly we see that

(a) $\tilde{\sigma}_{i_k} \dots \tilde{\sigma}_{i_2}\tilde{\sigma}_{i_1}(g, g'U^*_{w_1}) = (g, \hat{g}'U^*_{w_{k+1}})$ (if $q = 1$)

(b) $\tilde{\sigma}_{i_k} \dots \tilde{\sigma}_{i_2}\tilde{\sigma}_{i_1}(g'U^*_{w_1}) = \hat{g}'U^*_{w_{k+1}}$ (if $q > 1$)

where

$$\hat{g}' = g'\dot{s}_{i_1}\dot{s}_{i_2} \dots \dot{s}_{i_k}\dot{c}\xi\dot{c}^{-1},$$

$$\xi = u_{i_1}(\dot{s}^\bullet_{i_1}u_{i_2}(\dot{s}^\bullet_{i_1})^{-1}) \dots (\dot{s}^\bullet_{i_1}\dot{s}^\bullet_{i_2} \dots \dot{s}^\bullet_{i_{k-1}}u_{i_k}(\dot{s}^\bullet_{i_{k-1}})^{-1} \dots (\dot{s}^\bullet_{i_2})^{-1}(\dot{s}^\bullet_{i_1})^{-1}),$$

with $u_{i_s} \in U^*_{i_s}$ for $s \in [1, k]$.

2.8. In the setup of 2.7 we assume that $c = 1$ so that $w_1 = s_{i_1}s_{i_2} \dots s_{i_k}$, $w_{k+1} = w_1^\bullet$, $l(w_1) = k$. We show that

(a) $\tilde{\sigma}_{i_k} \dots \tilde{\sigma}_{i_2}\tilde{\sigma}_{i_1}(g, g'U^*_{w_1}) = \Psi(g, g'U^*_{w_1})$ (if $q = 1$)

(b) $\tilde{\sigma}_{i_k} \dots \tilde{\sigma}_{i_2}\tilde{\sigma}_{i_1}(g'U^*_{w_1}) = \Psi(g'U^*_{w_1})$ (if $q > 1$).

From 2.5 we see that

$$\begin{aligned} \sigma_{i_k} \dots \sigma_{i_2}\sigma_{i_1}(g, g'B^*g'^{-1}) &= (g, gg'B^*g'^{-1}g^{-1}) \text{ (if } q = 1) \\ \sigma_{i_k} \dots \sigma_{i_2}\sigma_{i_1}(g'B^*g'^{-1})) &= F(g')B^*F(g')^{-1} \text{ (if } q > 1). \end{aligned}$$

hence

$$\begin{aligned} \tilde{\sigma}_{i_k} \dots \tilde{\sigma}_{i_2}\tilde{\sigma}_{i_1}(g, g'U^*_{w_1}) &= (g, gg'd^{-1}t^{-1}U^*_{w_1^\bullet}) \text{ (if } q = 1) \\ \tilde{\sigma}_{i_k} \dots \tilde{\sigma}_{i_2}\tilde{\sigma}_{i_1}(g'U^*_{w_1}) &= (F(g')t^{-1}U^*_{w_1^\bullet}) \text{ (if } q > 1). \end{aligned}$$

for some $t \in T^*_{w_1^\bullet}$. Let $\hat{g}', \xi$ be as in 2.7(a),(b). We have $\hat{g}' = g'\dot{w}_1\dot{c}\xi\dot{c}^{-1}$. If $q = 1$ we have $g'^{-1}gg' = \dot{w}_1ud$ with $u \in U^*$ hence

$$gg'd^{-1}t^{-1}U^*_{w_1^\bullet} = \hat{g}'U^*_{w_1^\bullet} = g'\dot{w}_1\dot{c}\xi\dot{c}^{-1}U^*_{w_1^\bullet} = gg'd^{-1}u^{-1}\dot{c}\xi\dot{c}^{-1}U^*_{w_1^\bullet}.$$

If $q > 1$ we have $g'^{-1}F(g') = \dot{w}_1u$ with $u \in U^*$ hence

$$F(g')t^{-1}U^*_{w_1^\bullet} = \hat{g}'U^*_{w_1^\bullet} = g'\dot{w}_1\dot{c}\xi\dot{c}^{-1}U^*_{w_1^\bullet} = F(g')u^{-1}\dot{c}\xi\dot{c}^{-1}U^*_{w_1^\bullet}.$$

We see that in both cases, $t^{-1} \in u^{-1}\dot{c}\xi\dot{c}^{-1}U^*_{w_1^\bullet} \subset U^*$. Since t is semisimple it follows that $t = 1$. This proves (a), (b).

2.9. Next we assume that w, w', i, j, m, v are as in 2.4. We have $w = vc, w' = cv^\bullet$ where $c \in \mathbf{W}$, $l(c) + m = l(w) = l(w')$ and the sequences of m maps

$$\tilde{\mathbf{X}}_w \xrightarrow{\tilde{\sigma}_i} \tilde{\mathbf{X}}_{s_i w s_i^\bullet} \xrightarrow{\tilde{\sigma}_j} \tilde{\mathbf{X}}_{s_j s_i w s_i^\bullet s_j^\bullet} \xrightarrow{\tilde{\sigma}_i} \cdots \to \tilde{\mathbf{X}}_{vwv^\bullet},$$
$$\tilde{\mathbf{X}}_w \xrightarrow{\tilde{\sigma}_j} \tilde{\mathbf{X}}_{s_j w s_j^\bullet} \xrightarrow{\tilde{\sigma}_i} \tilde{\mathbf{X}}_{s_i s_j w s_j^\bullet s_i^\bullet} \xrightarrow{\tilde{\sigma}_j} \cdots \to \tilde{\mathbf{X}}_{vwv^\bullet},$$

are defined. We show:

(a) the two compositions are equal.

We apply 2.7(a), (b) with $k = m$, $i_1, i_2, \dots, i_k$ equal to $i, j, i, j, \dots$ and with $w_1 = w, w_{k+1} = w'$. Let U_v^* be the subgroup of U^* generated by U_i^* and U_j^*. Le $(g, g'U_w^*) \in \tilde{\mathbf{X}}_w$ (if $q = 1$), $g'U_w^* \in \tilde{\mathbf{X}}_w$ (if $q > 1$). We have $g'^{-1}gg' = \dot{v}\dot{c}u'u''d$ (if $q = 1$), $g'^{-1}F(g') = \dot{v}\dot{c}u'u''$ (if $q > 1$) where $u' \in U_v^*$ and $u'' \in U^* \cap (\dot{v}^\bullet U^*(\dot{v}^\bullet)^{-1})$ are uniquely determined.

In 2.7(a),(b) we have $\hat{g}' = g'\dot{v}\dot{c}\xi\dot{c}^{-1}$ where $\xi \in U_v^*$. Since $(g, \hat{g}'U_{w'}^*) \in \tilde{\mathbf{X}}_{w'}$ (if $q = 1$) and $\hat{g}'U_{w'}^* \in \tilde{\mathbf{X}}_{w'}$ (if $q > 1$), we have $\hat{g}'^{-1}g\hat{g}' \in \dot{w}'U^*d$ (if $q = 1$) and $\hat{g}'^{-1}F(\hat{g}') \in \dot{w}'U^*$ (if $q > 1$). Thus $\dot{c}\xi^{-1}\dot{c}^{-1}\dot{v}^{-1}g'^{-1}gg'\dot{v}\dot{c}\xi\dot{c}^{-1} \in \dot{w}'U^*d$ if $q = 1$ and $\dot{c}\xi^{-1}\dot{c}^{-1}\dot{v}^{-1}g'^{-1}F(g')\dot{v}^\bullet F(\dot{c}\xi\dot{c}^{-1}) \in \dot{w}'U^*$ if $q > 1$. Hence $\dot{c}\xi^{-1}u'u''d\dot{v}\dot{c}\xi\dot{c}^{-1} \in \dot{w}'U^*d$ if $q = 1$ and $\dot{c}\xi^{-1}u'u''\dot{v}^\bullet F(\dot{c}\xi\dot{c}^{-1}) \in \dot{w}'U^*$ if $q > 1$. We have $\dot{c}\xi\dot{c}^{-1} \in U^*$ hence $\xi^{-1}u'u'' \in \dot{v}^\bullet U^*(\dot{v}^\bullet)^{-1}$ in both cases. Since $u'' \in \dot{v}^\bullet U^*(\dot{v}^\bullet)^{-1}$ we have $\xi^{-1}u' \in \dot{v}^\bullet U^*(\dot{v}^\bullet)^{-1}$. But we have also $\xi^{-1}u' \in U_v^*$ and $U_v^* \cap (\dot{v}^\bullet U^*(\dot{v}^\bullet)^{-1}) = \{1\}$ hence $\xi^{-1}u' = 1$ and $\xi = u'$.

If we now apply 2.7(a),(b) with $k = m$, $i_1, i_2, \dots, i_k$ equal to $j, i, j, j, \dots$ and with $w_1 = w, w_{k=1} = w'$ then $\hat{g}'$ is replaced by an element $\hat{g}'_1 = g'\dot{v}\dot{c}\xi_1\dot{c}^{-1}$ where $\xi_1 \in U_v^*$ and by the same argument as above we have $\xi_1 = u'$. Thus $\xi = \xi_1$ so that $\hat{g}'U_{w'}^* = \hat{g}'_1U_{w'}^*$. This proves (a).

2.10. Let C be a $\bullet$-elliptic $\bullet$-conjugacy class of $\mathbf{W}$ and let $w, w' \in C_{\min}$. For any $\mathbf{i} \in \mathcal{P}_{w,w'}$ given by

$$w = w_1 \overset{i_1}{\smile} w_2 \overset{i_2}{\smile} \dots \overset{i_{t-1}}{\smile} w_t = w'$$

we define a quasi-isomorphism $T_{\mathbf{i}} : \mathbf{X}_w \to \mathbf{X}_{w'}$ as the composition

$$\mathbf{X}_{w_1} \xrightarrow{\sigma_{i_1}^{\epsilon_1}} \mathbf{X}_{w_2} \xrightarrow{\sigma_{i_2}^{\epsilon_2}} \mathbf{X}_{w_3} \to \cdots \to \mathbf{X}_{w_{t-1}} \xrightarrow{\sigma_{i_{t-1}}^{\epsilon_{t-1}}} \mathbf{X}_{w_t}$$

and a quasi-isomorphism $\tilde{T}_{\mathbf{i}} : \tilde{\mathbf{X}}_w \to \tilde{\mathbf{X}}_{w'}$ as the composition

$$\tilde{\mathbf{X}}_{w_1} \xrightarrow{\tilde{\sigma}_{i_1}^{\epsilon_1}} \tilde{\mathbf{X}}_{w_2} \xrightarrow{\tilde{\sigma}_{i_2}^{\epsilon_2}} \tilde{\mathbf{X}}_{w_3} \to \dots \to \tilde{\mathbf{X}}_{w_{t-1}} \xrightarrow{\tilde{\sigma}_{i_{t-1}}^{\epsilon_{t-1}}} \tilde{\mathbf{X}}_{w_t};$$

here $\epsilon_1, \dots, \epsilon_{t-1}$ are as in 1.1(a). Note that $T_{\mathbf{i}}, \tilde{T}_{\mathbf{i}}$ commute with the G^F-actions. From the definitions we see that $\Psi T_{\mathbf{i}} = T_{\mathbf{i}^\bullet}\Psi$ as maps $\mathbf{X}_w \to \mathbf{X}_{w'^\bullet}$ and $\Psi \tilde{T}_{\mathbf{i}} = \tilde{T}_{\mathbf{i}^\bullet}\Psi$ as maps $\tilde{\mathbf{X}}_w \to \tilde{\mathbf{X}}_{w'^\bullet}$ where $\mathbf{i}^\bullet$ is given by

$$w^\bullet = w_1^\bullet \overset{i_1^b ul}{\smile} w_2^\bullet \overset{i_2^b ul}{\smile} \dots \overset{i_{t-1}^b ul}{\smile} w_t^\bullet = w'^\bullet.$$

If $w \in C_{\min}$ then $\mathbf{i} \mapsto T_{\mathbf{i}}$ (resp. $\mathbf{i} \mapsto \tilde{T}_{\mathbf{i}}$) defines a homomorphism of the group opposed to $\bar{\mathcal{P}}_{w,w}$ into the group $\mathcal{G}_w$ (resp. $\tilde{\mathcal{G}}_w$) of quasi-automorphisms of $\mathbf{X}_w$ (resp. of $\tilde{\mathbf{X}}_w$) which commute with the G^F-action. (We use 2.4, 2.9.) Hence if $q > 1$ and $i \in \mathbf{Z}$ we obtain a representation of $\bar{\mathcal{P}}_{w,w}^{opp}$ on $H_c^i(X_w, \bar{\mathbf{Q}}_l)$ and on $H_c^i(\tilde{X}_w, \bar{\mathbf{Q}}_l)$ which commutes with the G^F-action; if $q = 1$ and $i \in \mathbf{Z}$ we obtain a representation of $\bar{\mathcal{P}}_{w,w}^{opp}$ on the i-th perverse cohomology sheaf of $\rho_! \bar{\mathbf{Q}}_l$ (ρ as in 0.1).

2.11. Let us return to the setup of 1.4. The following relation in the group $\mathcal{G}_w$ (which I found in 1982 for X_w) follows from 1.4(b):

(a) $$T_{\mathbf{i}'} T_{\mathbf{i}} T_{\mathbf{i}''} = T_{\mathbf{i}''} T_{\mathbf{i}'} T_{\mathbf{i}} = T_{\mathbf{i}} T_{\mathbf{i}''} T_{\mathbf{i}'} = \Psi.$$

(An analogous relation holds for $\tilde{T}_{\dots}$ instead of $T_{\dots}$.) Similarly, assuming that 1.5(b), 1.6(b) hold we see that in the setup of 1.5, the quasi-automorphisms $T_{\mathbf{i}_r}, T_{\mathbf{i}'_r}$ corresponding to the generators w_r, h_r of $\mathbf{W}_w$ satisfy the braid group relations in 1.5 and that in the setup of 1.6, the quasi-automorphisms $T_{\mathbf{i}''_r}, T_{\mathbf{i}'_r}, T_{\tilde{\mathbf{i}}}$ corresponding to the generators $w'_r, h_r, h'_{\sigma-1}$ of $\mathbf{W}_w$ satisfy the braid group relations in 1.6. The apparition of braid group relations for quasi-automorphisms of X_w has been predicted (in the special case where w is regular) by Broué and Michel [3] (based on the example in 1.4, that in [12, p.24] and that for the Coxeter element in [11]) as a part of a stronger conjecture in which the cyclotomic Hecke algebras [2] enter; this stronger conjecture has been verified for C as in 1.5 with $p_1 = p_2 = \dots = p_\sigma$ in [5].

3. Proof of Theorem 0.3

3.1. We prove 0.3(a). Using 1.1(b) and the quasi-isomorphisms $\tilde{\sigma}_i$ we see that if 0.3(a) holds for some element $\bullet$-conjugate to w and of the same length as w then it will hold also for w. Let β^+ be the braid monoid attached to the Coxeter grop $\mathbf{W}$. Let $w_1 \mapsto \hat{w}_1$ be the canonical imbedding $\mathbf{W} \to \beta^+$, see [7, 4.1.1]. From the results on "good elements" of Geck-Michel [6], Geck-Kim-Pfeiffer [8], He [9], we see that, after replacing w by a $\bullet$-conjugate element of the same length as w, the following holds:

($*$) we can find an integer $e \geq 1$ and an element $z \in \beta^+$ such that

$ww^{\bullet}w^{\bullet^2} \dots w^{\bullet^{e-1}} = 1$ and $\hat{w}\hat{w}^{\bullet}\hat{w}^{\bullet^2} \dots \hat{w}^{\bullet^{e-1}} = \hat{w}_0 z$ in β^+.

Thus it is enough to prove 0.3(a) for w satisfying ($*$). Let $s_1 s_2 \dots s_k$ be a reduced expression of w. Let $s'_1 s'_2 \dots s'_f$ be a reduced expression of w_0. We can find a sequence $s''_1, s''_2, \dots, s''_h$ in S such that $z = \hat{s}''_1 \hat{s}''_2 \dots \hat{s}''_h$. We have

$$(\hat{s}_1 \hat{s}_2 \dots \hat{s}_k)(\hat{s}_1^{\bullet} \hat{s}_2^{\bullet} \dots \hat{s}_k^{\bullet}) \dots (\hat{s}_1^{\bullet^{e-1}} \hat{s}_2^{\bullet^{e-1}} \dots \hat{s}_k^{\bullet^{e-1}}) = \hat{s}'_1 \hat{s}'_2 \dots \hat{s}'_f \hat{s}''_1 \hat{s}''_2 \dots \hat{s}''_h.$$

(The left (resp. right) hand side contains ke (resp. $f+h$) elements of S.) We must have $ke = f+h$. Moreover by the definition of β^+ there exist $\mathbf{s}^1, \mathbf{s}^2, \dots, \mathbf{s}^m$ ($m \geq 2$) such that each $\mathbf{s}^r$ is a sequence $\mathbf{s}^r_1, \mathbf{s}^r_2, \dots, \mathbf{s}^r_{ke}$ in S, $\mathbf{s}^1$ is the sequence

$$s_1, s_2, \dots, s_k, s_1^{\bullet}, s_2^{\bullet}, \dots, s_k^{\bullet}, \dots, s_1^{\bullet^{e-1}}, s_2^{\bullet^{e-1}}, \dots, s_k^{\bullet^{e-1}},$$

(ke terms), $\mathbf{s}^m$ is the sequence

$$s'_1, s'_2, \dots, s'_f, s''_1, s''_2, s''_h$$

and for any $r \in [1, m-1]$ the sequence $\mathbf{s}^{r+1}$ is obtained from the sequence $\mathbf{s}^r$ by replacing a string $\mathbf{s}^r_{e+1}, \mathbf{s}^r_{e+2}, \dots, \mathbf{s}^r_{e+u}$ of the form $s, t, s, t, \dots$ (u terms, $s \neq t$ in S, st of order u in $\mathbf{W}$) by the string $t, s, t, s, \dots$ (u terms).

Now let $(g, g'U^*_w) \in \tilde{\mathfrak{B}}_w$ (if $q = 1$) and $g'U^*_w \in \tilde{X}_w$ (if $q > 1$). Let $\mathfrak{Z} = \{c \in G; cgc^{-1} = g, cg'U^*_w = g'U^*_w\}$ (if $q = 1$), $\mathfrak{Z} = \{c \in G^F; cg'U^*_w = g'U^*_w\}$ (if $q > 1$). If $c \in \mathfrak{Z}$ then $g'^{-1}cg' \in U^*_w$ hence c is unipotent. Thus $\mathfrak{Z}$ is a unipotent group contained in $B := g'B^*g'^{-1}$. We define a sequence $B_0, B_1, \dots, B_{ke}$ in $\mathcal{B}$ by the following requirements:

$B_{ik} = g^i B g^{-i}$ (if $q = 1$) and $B_{ik} = F^i(B)$ (if $q > 1$) for $i \in [0, e]$,

$(B_{ik+j-1}, B_{ik+j}) \in \mathcal{O}_{s_j^{\bullet i}}$ for $i \in [0, e-1], j \in [1, k]$.

This sequence is uniquely determined. Now conjugation by any $c \in \mathfrak{Z}$ preserves each of $B, gBg^{-1}, g^2Bg^{-2}, \dots, g^eBg^{-e}$ (if $q = 1$) and each of $B, F(B), F^2(B), \dots, F^e(B)$ (if $q > 1$) hence (by uniqueness) it automatically preserves each B_v, $v \in [0, ke]$. Thus $\mathfrak{Z} \subset B_v$ for any $v \in [0, ke]$. We define a sequence $B_*^1, B_*^2, \dots, B_*^m$ such that each B_*^r is a sequence $(B_0^r, B_1^r, \dots, B_{ke}^r)$ in $\mathcal{B}$ satisfying $(B_{j-1}^r, B_j^r) \in \mathcal{O}_{\mathbf{s}_j^r}$ for $j \in [1, ke]$, as follows: $B_*^1 = (B_0, B_1, \dots, B_{ke})$ and for $r \in [1, m-1]$, B_*^{r+1} is obtained from B_*^r by replacing the string $B_a^r, B_{a+1}^r, \dots, B_{a+u}^r$ (where

$$(\mathbf{s}_{a+1}^r, \mathbf{s}_{a+2}^r, \dots, \mathbf{s}_{a+u}^r) = (s, t, s, t, \dots)$$

as above) by the string $B_a^{r+1}, B_{a+1}^{r+1}, \dots, B_{a+u}^{r+1}$ defined by

$$\begin{aligned} B_a^{r+1} &= B_a^r, B_{a+u}^{r+1} = B_{a+u}^r, (B_a^{r+1}, B_{a+1}^{r+1}) \in \mathcal{O}_t, (B_{a+1}^{r+1}, B_{a+2}^{r+1}) \in \mathcal{O}_s, \\ &\quad (B_{a+2}^{r+1}, B_{a+3}^{r+1}) \in \mathcal{O}_t, \dots. \end{aligned}$$

(Note that $B_a^{r+1}, B_{a+1}^{r+1}, \dots, B_{a+u}^{r+1}$ are uniquely determined since $(B_a^r, B_{a+u}^r) \in \mathcal{O}_{stst\dots} = \mathcal{O}_{tsts\dots}$ and $stst\dots, tsts\dots$ are reduced expressions in $\mathbf{W}$.) We note that for any $r \in [1, m]$ any Borel subgroup in the sequence B_*^r is stable under conjugation by any $c \in \mathfrak{Z}$. (For $r = 1$ this has been already observed. The general case follows by induction on r using the uniqueness in the previous sentence.) In particular any Borel subgroup in the sequence B_*^m is stable under conjugation by any $c \in \mathfrak{Z}$. From the definitions we see that $(B_0^m, B_f^m) \in \mathcal{O}_{w_0}$ that is, B_0^m, B_f^m are opposed Borel subgroups. Since both are stable under conjugation by any $c \in \mathfrak{Z}$ we see that $\mathfrak{Z} \subset B_0^m \cap B_f^m$, a torus. Since $\mathfrak{Z}$ is a unipotent group we see that $\mathfrak{Z} = \{1\}$. This proves 0.3(a).

We prove 0.3(b). Let $(g, B) \in \mathfrak{B}_w$ (if $q = 1$) and $B \in X_w$ (if $q > 1$). If $q = 1$ we can find $(g, g'U_w^*) \in \tilde{\mathfrak{B}}_w$ such that $\pi_w(g, g'U_w^*) = (g, B)$. If $q > 1$ we can find $g'U_w^* \in \tilde{X}_w$ such that $\pi_w(g'U_w^*) = B$. Let $\mathfrak{Z}_0 = \{c \in G; cgc^{-1} = g, cBc^{-1} = B\}$ (if $q = 1$), $\mathfrak{Z}_0 = \{c \in G^F; cBc^{-1} = B\}$ (if $q > 1$). If $c \in \mathfrak{Z}_0$ then $\pi_w(g, cg'U_w^*) = \pi_w(g, g'U_w^*)$ (for $q = 1$) and $\pi_w(cg'U_w^*) = \pi_w(g'U_w^*)$ (for $q > 1$); hence we have $cg'U_w^* = g't^{-1}U_w^*$ for a unique $t \in T_w^*$. Note that $c \mapsto t$ is a group homomorphism $\mathfrak{Z}_0 \to T_w^*$. If c is in the kernel of this homomorphism then c is in the isotropy group of $(g, g'U_w^*)$ (for $q = 1$) and of $g'U_w^*$ (for $q > 1$); hence by 0.3(a) we have $c = 1$. Thus $\mathfrak{Z}_0 \to T_w^*$ is injective. This proves (b). More precisely, we see that $\mathfrak{Z}_0 g'U_w^* \subset g'T_w^*U_w^*$

hence $g'^{-1}\mathfrak{z}_0 g' \subset T_w^* U_w^*$. Since $g'^{-1}\mathfrak{z}_0 g'$ is a finite diagonalizable subgroup of $T_w^* U_w^*$, it is conjugate under some element of U_w^* to a subgroup of T_w^*.

We prove 0.3(c) by a method inspired by the Bonnafé-Rouquier [1] proof of 0.3(d). We can again assume that w satisfies $(*)$. Let Y be the set of all sequences $(B_0, B_1, \dots, B_{e-1}) \in \mathcal{B}^e$ such that $(B_i, B_{i+1}) \in \mathcal{O}_{w^{\bullet i}}$ for $i \in [0, e-2]$. By [1, Proposition 3], Y is an affine variety. Hence $D \times Y$ is an affine subvariety of $D \times \mathcal{B}^e$. Let Y' be the set of all $(g, B_0, B_1, \dots, B_{e-1}) \in D \times \mathcal{B}^e$ such that $B_i = g^i B_0 g^{-i}$ for $i \in [1, e-1]$; this is a closed subvariety of $D \times \mathcal{B}^e$. Hence $(D \times Y) \cap Y'$ is a closed subvariety of $D \times Y$ so that it is affine. The map $\mathfrak{B}_w \to Y'$ given by $(g, B) \mapsto (g, B, gBg^{-1}, g^2Bg^{-2}, \dots, g^{e-1}Bg^{-e+1})$ is an isomorphism of $\mathfrak{B}_w$ onto $(G \times Y) \cap Y'$. Hence $\mathfrak{B}_w$ is affine. Since $\tilde{\mathfrak{B}}_w$ is a principal bundle over $\mathfrak{B}_w$ with (finite) group T_w^* and $\mathfrak{B}_w$ is affine, we see that $\tilde{\mathfrak{B}}_w$ is affine. This proves 0.3(c).

Corollary 3.2. *We preserve the setup of* 0.3.

(a) *If $q = 1$, any isotropy group of the U_w^* action $u_1 : u \mapsto \dot{w}^{-1} u_1 \dot{w} u d u_1^{-1} d^{-1}$ on U^* is $\{1\}$.*

(b) *If $q > 1$, any isotropy group of the U_w^* action $u_1 : u \mapsto \dot{w}^{-1} u_1 \dot{w} u F(u_1)^{-1}$ on U^* is $\{1\}$.*

We prove (a). Let $u_1 \in U_w^*, u \in U^*$ be such that $\dot{w}^{-1} u_1 \dot{w} u d u_1^{-1} d^{-1} = u$. We must show that $u_1 = 1$. Note that $(\dot{w}ud, U_w^*) \in \tilde{\mathfrak{B}}_{\dot{w}}$ and $(u_1 \dot{w} u d u_1^{-1}, u_1 U_w^*) = (\dot{w}ud, U_w^*)$. Thus u_1 is in the isotropy group at $(\dot{w}u, U_w^*)$ for the G-action on $\tilde{\mathfrak{B}}_w$. Using 0.3(a) we deduce that $u_1 = 1$, as required.

We prove (b). Let $u_1 \in U_w^*, u \in U^*$ be such that $\dot{w}^{-1} u_1 \dot{w} u F(u_1^{-1}) = u$. We must show that $u_1 = 1$. By Lang's theorem we can find $z \in G$ such that $z^{-1}F(z) = \dot{w}u$. We have $u_1 z^{-1} F(z) F(u_1^{-1}) = z^{-1}F(z)$ that is $z u_1 z^{-1} = F(z u_1 z^{-1})$. We set $u_1' = z u_1 z^{-1}$ so that $u_1' \in G^F$. In the G^F-action on $\tilde{X}_w$, $u_1' \in G^F$ sends $zU_w^* \in \tilde{X}_w$ to $u_1' z U_w^* = z u_1 U_w^* = z U_w^*$. Thus u_1' is in the isotropy group at zU_w^* for the G^F-action. Using 0.3(a) we deduce that $u_1' = 1$ hence $u_1 = 1$, as required.

3.3. We preserve the setup of 0.3. Let $U_w^* \backslash\backslash U^*$ be the set of orbits of the U_w^* action on U^* given in 3.2(a) (if $q = 1$) or 3.2(b) (if $q > 1$). The statements (a), (b) below are immediate.

(a) If $q>1$ we have a bijection $G^F\backslash\tilde{X}_w\xrightarrow{\sim}U_w^*\backslash\backslash U^*$, $g'U_w^*\mapsto\dot{w}^{-1}g'^{-1}F(g')$ with inverse induced by $u\mapsto g'U_w^*$ where $g'\in G$, $g'^{-1}F(g')=\dot{w}u$. (See [4, 1.12].)

(b) If $q=1$ we have a bijection $G\backslash\tilde{\mathfrak{B}}_w\xrightarrow{\sim}U_w^*\backslash\backslash U^*$, $(g,g'U_w^*)\mapsto\dot{w}^{-1}g'^{-1}gg'd^{-1}$ with inverse induced by $u\mapsto(\dot{w}ud,U^*)$.

4. Proof of Theorem 0.4

4.1. In this section we prove the assertions about $G^F\backslash\tilde{\mathbf{X}}_w$ in Theorem 0.4. (The assertions about $G^F\backslash\mathbf{X}_w$ are then an immediate consequence.) Using 3.3 we see that it is enough to consider one group in each isogeny class. Using 0.3(a) and 1.1(b) we see that it is enough to consider a single w (of minimal length) in each elliptic conjugacy class of $\mathbf{W}$.

Let V be a $\mathbf{k}$-vector space of finite dimension $n\geq 2$. In this subsection we assume that (if $q=1$) we have $\hat{G}=G=D=SL(V)$; if $q>1$ (so that $\mathbf{k}$ is an algebraic closure of F_q) we assume that V has a fixed F_q-rational structure with Frobenius map $F:V\to V$ (thus V^F is an n-dimensional F_q-vector space) and that $G=SL(V)$ with the F_q-rational structure and Frobenius map induced by those of V.

Let ω be a basis element of Λ^nV such that $F(\omega)=\omega$ for the map $F:\Lambda^nV\to\Lambda^nV$ given by $v_1\wedge v_2\wedge\ldots\wedge v_n\mapsto F(v_1)\wedge F(v_2)\wedge\ldots\wedge F(v_n)$. (Recall that if $q=1$ we have $F=1$.) If $q>1$ we denote by $\mathbf{s}(V)$ the set of all bijective group homomorphisms $F':V\to V$ such that $F'(\lambda v)=\lambda^qF'(v)$ for all $v\in V,\lambda\in\mathbf{k}$.

If $q>1$ let $\mathbf{s}_\omega(V)$ be the set of all $F'\in\mathbf{s}(V)$ such that $F'(\omega)=\omega$. We have $F\in\mathbf{s}_\omega(V)$. Note that G acts on $\mathbf{s}_\omega(V)$ by $x:F'\mapsto xF'x^{-1}$ and that this action is transitive; the stabilizer of F is G^F.

Let $\mathcal{F}$ be the set of all sequences $V_*=(0=V_0\subset V_1\subset V_2\subset\ldots\subset V_n=V)$ of subspaces of V such that $\dim V_i=i$ for $i\in[0,n]$. Now G acts naturally (transitively) on $\mathcal{F}$. For any $V_*\in\mathcal{F}$ we set $B_{V_*}=\{g\in G;gV_*=V_*\}$, a Borel subgroup of G.

If $q=1$ let Z be the set of all pairs $(g,V_*)\in G\times\mathcal{F}$ such that $V_1\neq gV_1\subset V_2$, $V_2\neq gV_2\subset V_3$, $\ldots$, $V_{n-1}\neq gV_{n-1}\subset V_n$. If $q>1$ let Z be the set of all $V_*\in\mathcal{F}$ such that $V_1\neq F(V_1)\subset V_2$, $V_2\neq F(V_2)\subset V_3$, $\ldots$, $V_{n-1}\neq F(V_{n-1})\subset V_n$. Now $(g,V_*)\mapsto(g,B_{V_*})$ (if $q=1$) and $V_*\mapsto B_{V_*}$ (if

$q > 1$) defines an isomorphism $Z \xrightarrow{\sim} \mathfrak{B}_w$ (if $q = 1$) or $Z \xrightarrow{\sim} X_w$ (if $q > 1$) for a well defined Coxeter element w of length $n-1$ in $\mathbf{W}$ (an elliptic element of minimal length in its conjugacy class).

If $q = 1$ let Z' be the set of pairs (g, L) where $g \in G$ and L is a line in V such that $V = \oplus_{i \in [0,n-1]} g^i(L)$; if $q > 1$ let Z' be the set of lines L in V such that $V = \oplus_{i \in [0,n-1]} F^i(L)$. We have an isomorphism $Z \xrightarrow{\sim} Z'$ given by $(g, V_*) \mapsto (g, V_1)$ if $q = 1$ and by $V_* \mapsto V_1$ if $q > 1$. Combining with the earlier isomorphism we obtain an isomorphism $Z' \xrightarrow{\sim} \mathfrak{B}_w$ if $q = 1$ and $Z' \xrightarrow{\sim} X_w$ if $q > 1$. (For $q > 1$ the last isomorphism appears in [4, Sec.2].)

If $q = 1$, let $\tilde{Z}'$ be the set of pairs $(g, v) \in G \times V$ such that $v \wedge g(v) \wedge \ldots \wedge g^{n-1}(v) = \omega$; if $q > 1$, let $\tilde{Z}'$ be the set of all $v \in V$ such that $v \wedge F(v) \wedge \ldots \wedge F^{n-1}(v) = \omega$. Note that G^F acts on $\tilde{Z}'$ by $x : (g, v) \mapsto (xgx^{-1}, x(v))$ (if $q = 1$) and by $x : v \mapsto x(v)$ (if $q > 1$). Define $\pi : \tilde{Z}' \to Z$ by $(g, v) \mapsto (g, L)$ (if $q = 1$) and by $v \mapsto L$ (if $q > 1$) where L is the line spanned by v. We can identify $\tilde{Z}'$ with $\tilde{\mathfrak{B}}_w$ (if $q = 1$) or with $\tilde{X}_w$ (if $q > 1$) in a way compatible with the G^F-actions and so that, if $q = 1$, the diagram

$$\begin{array}{ccc} \tilde{Z}' & \xrightarrow{\sim} & \tilde{\mathfrak{B}}_w \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_w}\downarrow \\ Z' & \xrightarrow{\sim} & \mathfrak{B}_w \end{array}$$

(and the analogous diagram with $\mathfrak{B}_w, \tilde{\mathfrak{B}}_w$ replaced by $X_w, \tilde{X}_w$ if $q > 1$) is commutative. (For $q > 1$ see [4, Sec.2].) If $q = 1$, let $\tilde{Z}''$ be the set of all $(g, v_0, v_1, \ldots, v_{n-1}) \in G \times V^n$ such that $v_i = g^i(v_0)$ for $i \in [0, n-1]$, $v_0 \wedge v_1 \wedge \ldots \wedge v_{n-1} = \omega$. If $q > 1$, let $\tilde{Z}''_0$ be the set of all $(v_0, v_1, \ldots, v_{n-1}) \in V^n$ such that $v_i = F^i(v_0)$ for $i \in [0, n-1]$, $v_0 \wedge v_1 \wedge \ldots \wedge v_{n-1} = \omega$; let $\tilde{Z}''$ be the set of all $(F', v_0, v_1, \ldots, v_{n-1}) \in \mathbf{s}_\omega(V) \times V^n$ such that $v_i = F'^i(v_0)$ for $i \in [0, n-1]$, $v_0 \wedge v_1 \wedge \ldots \wedge v_{n-1} = \omega$.

If $q = 1$ we have an isomorphism $\tilde{Z}'' \xrightarrow{\sim} \tilde{Z}'$ given by $(g, v_0, v_1, \ldots, v_{n-1}) \mapsto (g, v_0)$; if $q > 1$ we have an isomorphism $\tilde{Z}''_0 \xrightarrow{\sim} \tilde{Z}'$ given by $(v_0, v_1, \ldots, v_{n-1}) \mapsto v_0$. Combining with the earlier isomorphism we obtain an isomorphism $\tilde{Z}'' \xrightarrow{\sim} \tilde{\mathfrak{B}}_w$ if $q = 1$ and $\tilde{Z}''_0 \xrightarrow{\sim} \tilde{X}_w$ if $q > 1$.

If $q=1$ the G-action on $\tilde{\mathfrak{B}}_w$ becomes the G-action on $\tilde{Z}''$ given by

$$x:(g,v_0,v_1,\dots,v_{n-1})\mapsto(xgx^{-1},x(v_0),x(v_1),\dots,x(v_{n-1})).$$

If $q>1$ the G^F-action on $\tilde{X}_w$ becomes the G^F-action on $\tilde{Z}''_0$ given by $x:(v_0,v_1,\dots,v_{n-1})\mapsto(x(v_0),x(v_1),\dots,x(v_{n-1}))$. If $q>1$, G acts (freely) on $\tilde{Z}''$ by $x:(F',v_0,v_1,\dots,v_{n-1})\mapsto(xF'x^{-1},x(v_0),x(v_1),\dots,x(v_{n-1}))$. Since G acts transitively on $\mathbf{s}_\omega(V)$ and the stabilizer of F is G^F we see that the space of G^F-orbits on $\tilde{Z}''_0$ may be identified with the space of G-orbits on $\tilde{Z}''$. We must show that the space of G-orbits on $\tilde{Z}''$ is an affine space for any q. We define $\tilde{Z}''\to\mathbf{k}^{n-1}$ by $(g,v_0,v_1,\dots,v_{n-1})\mapsto(a_1,a_2,\dots a_{n-1})$ if $q=1$ and by $(F',v_0,v_1,\dots,v_{n-1})\mapsto(a_1,a_2,\dots a_{n-1})$ if $q>1$ where $a_i\in\mathbf{k}$ are given by $g^n(v_0)=a_0v_0+a_1v_1+\dots+a_{n-1}v_{n-1}$ (if $q=1$) and by $F'^n(v_0)=a_0v_0+a_1v_1+\dots+a_{n-1}v_{n-1}$ (if $q>1$); the coefficient a_0 is equal to $(-1)^{n-1}$. This map is constant on the orbits of G hence it induces a map $\mu:G\backslash\tilde{Z}''\to\mathbf{k}^{n-1}$. Next we define a map in the opposite direction $\tau:\mathbf{k}^{n-1}\to G\backslash\tilde{Z}''$.

Let $(a_1,a_2,\dots a_{n-1})\in\mathbf{k}^n$. Let $v_0,v_1,\dots,v_{n-1}$ be any basis of V such that $v_0\wedge v_1\wedge\dots\wedge v_{n-1}=\omega$. If $q=1$ define $g\in GL(V)$ by $g(v_0)=v_1$, $g(v_1)=v_2,\dots,g(v_{n-2})=v_{n-1}$, $g(v_{n-1})=(-1)^{n-1}v_0+a_1v_1+\dots+a_{n-1}v_{n-1}$. We have $(g,v_0,v_1,\dots,v_{n-1})\in\tilde{Z}''$ and the G-orbit of this element of $\tilde{Z}''$ is independent of the choices and is by definition $\tau(a_1,a_2,\dots,a_{n-1})$. If $q>1$ we define $F'\in\mathbf{s}_\omega(V)$ by the requirement that $F'(v_0)=v_1$, $F'(v_1)=v_2$, $\dots$, $F'(v_{n-2})=v_{n-1}$, $F'(v_{n-1})=(-1)^{n-1}v_0+a_1v_1+\dots+a_{n-1}v_{n-1}$. We have $(F',v_0,v_1,\dots,v_{n-1})\in\tilde{Z}''$ and the G-orbit of this element of $\tilde{Z}''$ is independent of the choices and is by definition $\tau(a_1,a_2,\dots a_{n-1})$.

It is clear that τ is an inverse of μ. This completes the proof of Theorem 0.4 in our case.

4.2. Let V be a $\mathbf{k}$-vector space of finite dimension $\mathbf{n}\ge 3$. We set $\kappa=0$ if $\mathbf{n}$ is even, $\kappa=1$ if $\mathbf{n}$ is odd and $n=(\mathbf{n}-\kappa)/2$. Assume that V has a fixed bilinear form $(,):V\times V\to\mathbf{k}$ and a fixed quadratic form $Q:V\to\mathbf{k}$ such that either

$$Q=0,\ (x,x)=0 \text{ for all } x\in V,\ V^\perp=0;$$

or

$$Q\ne 0,\ (x,y)=Q(x+y)-Q(x)-Q(y) \text{ for } x,y\in V,$$

$Q : V^\perp \to \mathbf{k}$ is injective.
Here, for any subspace V' of V we set $V'^\perp = \{x \in V; (x, V') = 0\}$. If $Q \neq 0$ it follows that $V^\perp = 0$ unless $\kappa = 1$ and $p = 2$ in which case $\dim V^\perp = 1$. If $Q = 0$ we set $\epsilon = -1$; if $Q \neq 0$ we set $\epsilon = 1$. We have $(x, y) = \epsilon(y, x)$ for any $x, y \in V$. A subspace V' of V is said to be isotropic if $(,)$ and Q are zero on V'. In the case where $\kappa = 0, Q \neq 0$, we fix a connected component $\mathcal{I}$ of the space of isotropic subspaces of dimension n of V.

Let $Is(V)$ be the group of all $g \in GL(V)$ such that $(gx, gy) = (x, y)$ for all $x, y \in V$ and $Q(gx) = Q(x)$ for all $x \in V$ (a closed subgroup of $GL(V)$). In this section we assume that (if $q = 1$) $G = D$ is the identity component of $Is(V)$; if $q > 1$ (so that $\mathbf{k}$ is an algebraic closure of F_q) we assume that V has a fixed F_q-rational structure with Frobenius map $F : V \to V$ (so that V^F is an $\mathbf{n}$-dimensional F_q-vector space), that $(F(x), F(y)) = (x, y)^q$ for all $x, y \in V$, that $Q(F(x)) = Q(x)^q$ for al $x \in V$ and that G is the identity component of $Is(V)$ with the F_q-rational structure and Frobenius map induced by those of V; in addition we assume that G is F_q-split.

Let $\mathcal{F}'$ be the set of all sequences $V_* = (0 = V_0 \subset V_1 \subset V_2 \subset \dots \subset V_{\mathbf{n}} = V)$ of subspaces of V such that $\dim V_i = i$ for $i \in [0, \mathbf{n}]$, $Q|_{V_i} = 0$, $V_i^\perp = V_{\mathbf{n}-i}$ for all $i \in [0, n]$ and (in the case where $\kappa = 0, Q \neq 0$), $V_n \in \mathcal{I}$. Now G acts naturally (transitively) on $\mathcal{F}'$.

As in 1.5, let W be the group of permutations of $[1, \mathbf{n}]$ which commute with the involution $i \mapsto \mathbf{n} - i + 1$ of $[1, \mathbf{n}]$. Let V_*, V'_* be two sequences in $\mathcal{F}'$. Let $a_{V_*, V'_*} : i \mapsto a_i$ be the permutation of $[1, \mathbf{n}]$ defined in [15, 1.4]. When $\kappa = 0, Q \neq 0$ let W' be the group of even permutations in W (a subgroup of index 2 of W), see 1.6. Let $s_i \in W (i \in [1, n])$ be as in 1.5. Then $(W, \{s_1, s_2, \dots, s_{n-1}, s_n\})$ is a Weyl group of type B_n. If $\kappa = 0, Q \neq 0$, we have $s_i \in W'$ for $i \in [1, n-1]$; as in 1.6 we set $s_{(n-1)'} = s_n s_{n-1} s_n \in W'$. Then $(W', \{s_1, s_2, \dots, s_{n-1}, s_{(n-1)'}\})$ is a Weyl group of type D_n. We identify $\mathbf{W}$ with W (if $(1 - \kappa)Q = 0$) and with W' (if $(1 - \kappa)Q \neq 0$) as Coxeter groups as in [15, 1.5]. For any $V_* \in \mathcal{F}'$ we set $B_{V_*} = \{g \in G; gV_* = V_*\}$, a Borel subgroup of G. We identify $\mathcal{F}' = \mathcal{B}$ via $V_* \mapsto B_{V_*}$.

4.3. In the remainder of this paper we preserve the setup of 4.2.

Let $p_* = (p_1 \geq p_2 \geq \dots \geq p_\sigma)$ be a sequence in $\mathbf{Z}_{>0}$ such that $p_1 + \dots + p_\sigma = n$. In the case where $\kappa = 0, Q \neq 0$ we assume in addition that σ is even. For any $r \in [1, \sigma]$ we set $p_{<r} = \sum_{r' \in [1, r-1]} p_{r'}$. Let $w \in W$ be the

permutation of $[1, \mathbf{n}]$ defined in 1.5. If $(1-\kappa)Q = 0$, then w is elliptic in $\mathbf{W}$ and it has minimal length in its conjugacy class C in $\mathbf{W}$. If $\kappa = 0, Q \neq 0$, then $w \in W' = \mathbf{W}$ is elliptic and it has minimal length in its conjugacy class C' in $\mathbf{W}$.

If $q = 1$ let $Z = \{(g, V_*, V'_*) \in G \times \mathcal{F}' \times \mathcal{F}'; V'_* = g(V_*), a_{V_*,V'_*} = w\}$.

If $q > 1$ let $Z = \{(V_*, V'_*) \in \mathcal{F}' \times \mathcal{F}'; V'_* = F(V_*), a_{V_*,V'_*} = w\}$.
Note that $Z = \mathfrak{B}_w$ (if $q = 1$) and $Z = X_w$ (if $q > 1$),

If $q = 1$ let $\tilde{Z}'$ be the set of all sequences $(g, v_1, v_2, \dots, v_\sigma) \in G \times V^\sigma$ such that

$$(g^i v_t, v_r) = 0 \text{ for any } 1 \le t < r \le \sigma,\ i \in [-p_t, p_t - 1]$$
$$(v_r, g^i v_r) = 0 \text{ for } i \in [-p_r+1, p_r-1],\ Q(v_r) = 0 \text{ and } (v_r, g^{p_r} v_r) = 1,\ r \in [1, \sigma];$$

if $\kappa = 0, Q \neq 0$, the span of $g^j v_k$ $(k \in [1, \sigma], j \in [0, p_k - 1])$ belongs to $\mathcal{I}$. (The span in the last condition is automatically an n-dimensional isotropic subspace.)

If $q > 1$ let $\tilde{Z}'_0$ be the set of all sequences $(v_1, v_2, \dots, v_\sigma) \in V^\sigma$ such that

$$(F^i(v_t), v_r) = 0 \text{ for any } 1 \le t < r \le \sigma,\ i \in [-p_t, p_t - 1];$$
$$(v_r, F^i(v_r)) = 0 \text{ for } i \in [-p_r+1, p_r-1],\ Q(v_r) = 0 \text{ and } (v_r, F^{p_r}(v_r)) = 1,\ r \in [1, \sigma];$$

if $\kappa = 0, Q \neq 0$, the span of $F^j(v_k)$ $(k \in [1, \sigma], j \in [0, p_k - 1])$ belongs to $\mathcal{I}$. (The span in the last condition is automatically an n-dimensional isotropic subspace.) Let

$$\mathcal{T} = \{(\lambda_1, \lambda_2, \dots, \lambda_\sigma) \in (\mathbf{k}^*)^\sigma; \lambda_r^{q^{p_r}+1} = 1 \text{ for } r \in [1, \sigma]\},$$

a finite group isomorphic to T^*_w). Then if $q = 1$, $\mathcal{T}$ acts (freely) on $\tilde{Z}'$ by

$$(\lambda_1, \lambda_2, \dots, \lambda_\sigma) : (g, v_1, v_2, \dots, v_\sigma) \mapsto (g, \lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_\sigma v_\sigma)$$

and if $q > 1$, $\mathcal{T}$ acts (freely) on $\tilde{Z}'_0$ by

$$(\lambda_1, \lambda_2, \dots, \lambda_\sigma) : (v_1, v_2, \dots, v_\sigma) \mapsto (\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_\sigma v_\sigma).$$

Let Z' (if $q=1$) and Z'_0 (if $q>1$) be the space of orbits of this $\mathcal{T}$-action. The following result is equivalent to [15, 3.3].

If $q=1$ we have an isomorphism $Z'\xrightarrow{\sim}Z$ induced by $(g,v_1,v_2,\dots,v_\sigma)\mapsto(g,V_*,g(V_*))$ where for any $r\in[1,\sigma]$, $i\in[0,p_r]$, $V_{p_{<r}+i}$ is the subspace of V spanned by g^jv_k $(k\in[0,r-1],j\in[0,p_k-1])$ and by g^jv_r $(j\in[0,i-1])$; moreover, $V_i^\perp=V_{\mathbf{n}-i}$ for all $i\in[0,n]$.

Exactly the same proof as in [15, 3.3] (with the action of g replaced by the action of F) gives the following result.

If $q>1$ we have an isomorphism $Z'_0\xrightarrow{\sim}Z$ induced by $(v_1,v_2,\dots,v_\sigma)\mapsto(V_*,g(V_*))$ where for any $r\in[1,\sigma]$, $i\in[0,p_r]$, $V_{p_{<r}+i}$ is the subspace of V spanned by $F^j(v_k)$ $(k\in[0,r-1],j\in[0,p_k-1])$ and by $F^j(v_r)$ $(j\in[0,i-1])$; moreover, $V_i^\perp=V_{\mathbf{n}-i}$ for all $i\in[0,n]$.

Combining with an earlier identification we get an isomorphism $Z'\xrightarrow{\sim}\mathfrak{B}_w$ (if $q=1$) and $Z'_0\xrightarrow{\sim}X_w$ (if $q>1$). Similarly we get an isomorphism $\tilde{Z}'\xrightarrow{\sim}\tilde{\mathfrak{B}}_w$ (if $q=1$) and $\tilde{Z}'_0\xrightarrow{\sim}\tilde{X}_w$ (if $q>1$) compatible with the isomorphism in the previous sentence and such that the T_w^*-action and $\mathcal{T}$-action are compatible.

If $q>1$ let $\mathbf{s}_1(V)$ be the set of all $F'\in\mathbf{s}(V)$ (see 3.1) such that $(F'(x),F'(y))=(x,y)^q$ for all $x,y\in V$, $Q(F'(x))=Q(x)^q$ for all $x\in V$ and such that (in the case where $\kappa=0,Q\neq0$) F' maps $\mathcal{I}$ onto itself and (in the case where $\kappa=1$) F' induces the same map as F on $\Lambda^{\mathbf{n}}(V)$. Note that G acts on $\mathbf{s}_1(V)$ transitively by $x:F'\mapsto xF'x^{-1}$ and the stabilizer of $F\in\mathbf{s}_1(V)$ is G^F.

If $q>1$ let $\tilde{Z}'$ be the set of all sequences $(F',v_1,v_2,\dots,v_\sigma)$ where $F'\in\mathbf{s}_1(V)$ and $v_1,v_2,\dots,v_\sigma$ are vectors in V such that

$$(F'^i(v_t),v_r)=0 \text{ for any } 1\le t<r\le\sigma,\ i\in[-p_t,p_t-1];$$
$$(v_r,F'^i(v_r))=0 \text{ for } i\in[-p_r+1,p_r-1],\ Q(v_r)=0 \text{ and } (v_r,F'^{p_r}(v_r))=1,\ r\in[1,\sigma];$$

if $\kappa=0,Q\neq0$, the span of $F'^j(v_k)$ $(k\in[1,\sigma],j\in[0,p_k-1])$ belongs to $\mathcal{I}$. Note that G acts naturally on $\tilde{Z}'$; since G acts on $\mathbf{s}_1(V)$ transitively and the stabilizer of $F\in\mathbf{s}_1(V)$ is G^F we see that the space of G^F-orbits on $\tilde{Z}'_0$ can be identified with the space of G-orbits on $\tilde{Z}'$.

Let $\tilde{Z}'_1$ be the set of all collections

$$(g \in G; w_i^r \in V(r \in [1,\sigma], i \in [0, p_r - 1]); z_j^r \in V(r \in [1,\sigma], j \in [1, p_r]) \quad (\text{if } q = 1),$$

$$(F' \in \mathbf{s}_1(V); w_i^r \in V(r \in [1,\sigma], i \in [0, p_r - 1]); z_j^r \in V(r \in [1,\sigma], j \in [1, p_r]) \quad (\text{if } q > 1)$$

such that

(a) $(w_i^t, w_{i'}^r) = 0$ for all t, r, i, i';

(b) $Q(w_i^t) = 0$ for all t, i;

(c) $(z_j^t, z_{j'}^r) = 0$ for all $t, r, j > 0, j' > 0$;

(d) $Q(z_j^t) = 0$ for all $t, j > 0$;

(e) $(w_i^r, z_j^r) = (w_0^r, z_{i+j}^r)^{q^i}$ if $j > 0, i + j < p_r$;

(f) $(w_i^r, z_j^r) = 1$ if $j > 0, i + j = p_r$;

(g) $(w_i^r, z_j^r) = 0$ if $j > 0, i + j > p_r$;

(h) $(w_i^t, z_j^r) = (w_0^t, z_{i+j}^r)^{q^i}$ if $j > 0, i + j < p_r, t < r$;

(i) $(w_i^t, z_j^r) = 0$ if $j > 0, i + j \geq p_r, t < r$;

(j) $(w_i^t, z_j^r) = (w_0^t, z_{i+j}^r)^{q^i}$ if $j > 0, i + j \leq p_t, t > r$;

(k) $(w_i^t, z_j^r) = 0$ if $j > 0, i + j > p_t, t > r$;

(l) if $\kappa = 0, Q \neq 0$, the span of z_j^r $(r \in [1,\sigma], j \in [1, p_r])$ belongs to $\mathcal{I}$;

(m) $gw_i^r = w_{i+1}^r$ for $r \in [1,\sigma], i \in [0, p_r - 2]$, $gw_{p_r-1}^r = z_{p_r}^r$ for $r \in [1,\sigma]$, $gz_j^r = z_{j-1}^r$ for $r \in [1,\sigma], j \in [2, p_r]$ (if $q = 1$).

(n) $F'(w_i^r) = w_{i+1}^r$ for $r \in [1,\sigma], i \in [0, p_r - 2]$, $F'(w_{p_r-1}^r) = z_{p_r}^r$ for $r \in [1,\sigma]$, $F'(z_j^r) = z_{j-1}^r$ for $r \in [1,\sigma], j \in [2, p_r]$ (if $q > 1$).

If $q = 1$ we have an isomorphism $\tilde{Z}' \xrightarrow{\sim} \tilde{Z}'_1$ given by

$$(g, v_1, v_2, \dots, v_\sigma) \mapsto (g, w_i^r, z_j^r)$$

where $w_i^r = g^{-p_r+i} v_r$ for $r \in [1,\sigma], i \in [0, p_r - 1]$, $z_j^r = g^{p_r - j} v_r$ for $r \in [1,\sigma], j \in [1, p_r]$.

If $q>1$ we have an isomorphism $\tilde{Z}'\xrightarrow{\sim}\tilde{Z}'_1$ given by

$$(F', v_1, v_2, \dots, v_\sigma) \mapsto (F', w_i^r, z_j^r)$$

where $w_i^r = F'^{-p_r+i}v_r$ for $r\in[1,\sigma], i\in[0,p_r-1]$, $z_j^r = F'^{p_r-j}v_r$ for $r\in[1,\sigma], j\in[1,p_r]$.

Let $\tilde{Z}'_2$ be the set of all collections

$$(w_i^r \in V(r\in[1,\sigma], i\in[0,p_r-1]); z_j^r\in V(r\in[1,\sigma], j\in[0,p_r])$$

such that that equations (a)-(l) hold and in addition the following equations hold:

(I) $(z_0^t, w^s_{p_s-h}) = (z_1^t, w^s_{p_s-h-1})^q$ for $t,s\in[1,\sigma]$, $h\in[1,p_s-1]$;

(II) $(z_0^t, z^s_{p_s-h}) = 0$ for $t,s\in[1,\sigma]$, $h\in[1,p_s-1]$; $(z_0^t, z^s_{p_s}) = (z_1^t, w^s_{p_s-1})^q$ for $t,s\in[1,\sigma]$;

(III) $(z_0^t, z_0^{t'})=0$ for $t<t'$ in $[1,\sigma]$;

(IV) $Q(z_0^t)=0$ for $t\in[1,\sigma]$.

It is easy to verify that the elements w_i^r, z_j^r associated with a collection in $\tilde{Z}'_2$ form a basis of V except if $\kappa=1$ when they form a basis of a hyperplane in V on which $(,)$ is nondegenerate.

We have an isomorphism $\tilde{Z}'_1\xrightarrow{\sim}\tilde{Z}'_2$ given by

$$(g, (w_i^r)_{r\in[1,\sigma], i\in[0,p_r-1]}; (z_j^r)_{r\in[1,\sigma], j\in[1,p_r]}) \mapsto$$
$$((w_i^r)_{r\in[1,\sigma], i\in[0,p_r-1]}; (z_j^r)_{r\in[1,\sigma], j\in[0,p_r]}) \quad (\text{if } q=1),$$

$$(F', (w_i^r)_{r\in[1,\sigma], i\in[0,p_r-1]}; (z_j^r)_{r\in[1,\sigma], j\in[1,p_r]}) \mapsto$$
$$((w_i^r)_{r\in[1,\sigma], i\in[0,p_r-1]}; (z_j^r)_{r\in[1,\sigma], j\in[0,p_r]}) \quad (\text{if } q>1)$$

where

(o) $z_0^r = gz_1^r$ for $r\in[1,\sigma]$ (if $q=1$) and $z_0^r = F'(z_1^r)$ for $r\in[1,\sigma]$ (if $q>1$).

The inverse map is given by

$$((w_i^r)_{r\in[1,\sigma], i\in[0,p_r-1]}; (z_j^r)_{r\in[1,\sigma], j\in[0,p_r]}) \mapsto$$
$$(g, (w_i^r)_{r\in[1,\sigma], i\in[0,p_r-1]}; (z_j^r)_{r\in[1,\sigma], j\in[1,p_r]}) \quad (\text{if } q=1),$$

$$((w_i^r)_{r\in[1,\sigma],i\in[0,p_r-1]};(z_j^r)_{r\in[1,\sigma],j\in[0,p_r]})\mapsto$$
$$(F',(w_i^r)_{r\in[1,\sigma],i\in[0,p_r-1]};(z_j^r)_{r\in[1,\sigma],j\in[1,p_r]})\quad (\text{if } q>1),$$

where $g\in G$ (if $q=1$) and $F'\in \mathbf{s}_1(V)$ (if $q>1$) is defined on

$$w_i^r(r\in[1,\sigma],i\in[0,p_r-1],\quad z_j^r(r\in[1,\sigma],j\in[1,p_r])$$

by (p), (q), (o); if $\kappa=1$, we denote by ξ the unique vector in V such that $(w_i^r,\xi)=0$, $(z_j^r,\xi)=0$ for all $r,i,j>0$ and such that

$$w_0^1\wedge w_{p_1-1}^1\wedge\ldots\wedge w_0^\sigma\wedge w_{p_\sigma-1}^\sigma\wedge z_1^1\wedge z_{p_1}^1\wedge\ldots\wedge z_1^\sigma\wedge z_{p_\sigma}^\sigma\wedge\xi=\omega$$

(with ω being a fixed basis element of $\Lambda^{\mathbf{n}}(V)$) and the value $g(\xi)\in V$ (resp. $F'(\xi)\in V$) is uniquely determined by the requirement that $g\in G$ (resp. $F'\in\mathbf{s}_1(V)$. For future reference we note that $\zeta:=(\xi,\xi)$ and $\zeta_0=Q(\xi)$ depend only on ω and not on w_i^r,z_j^r.

Let $\tilde{Z}_3'$ be the set of all collections

$$(c_h^r\in\mathbf{k}(r\in[1,\sigma],h\in[1,p_r-1];d_h^{t,r}\in\mathbf{k}(1\le t<r\le\sigma;h\in[1,p_r-1]);e_h^{t,r}\in\mathbf{k}$$
$$(1\le r<t\le\sigma;h\in[1,p_t]);x_i^{t,r}\in\mathbf{k}(t,r\in[1,\sigma],i\in[0,p_r-1]);y_j^{t,r}\in\mathbf{k}$$
$$(t,r\in[1,\sigma],j\in[1,p_r]);u^t\in\mathbf{k}(t\in[1,\sigma]);u^t=0\text{ unless }\kappa=1)$$

such that the equations (i), (ii), (iii), (iv) below are satisfied.

$$\text{(i)}\quad y_h^{t,s}+\sum_{j\in[1,h-1]}(c_{p_s+j-h}^s)^{q^{p_s-h}}y_j^{t,s}+\sum_{r,j;r<s;j\in[1,h]}(e_{p_s+j-h}^{s,r})^{q^{p_s-h}}y_j^{t,r}$$
$$+\sum_{r,j;r>s;j\in[1,p_r-p_s+h-1]}(d_{p_s+j-h}^{s,r})^{q^{p_s-h}}y_j^{t,r}=M$$

for any $t,s\in[1,\sigma],h\in[1,p_s-1]$, where $M=(d_{p_s-h}^{s,t})^{q^{p_s-h}}\epsilon$ if $s<t$, $M=(e_{p_s-h}^{s,t})^{q^{p_s-h}}\epsilon$ if $s>t$, $M=(c_{p_s-h}^t)^{q^{p_s-h}}\epsilon$ if $s=t$;

$$\text{(ii)}\quad x_h^{t,s}+\sum_{i\in[0,h-1]}(c_{p_s+i-h}^s)^{q^i}x_i^{t,s}+\sum_{r,i;r<s;i\in[0,h-1]}(d_{p_s+i-h}^{r,s})^{q^i}x_i^{t,r}$$
$$+\sum_{r,i;r>s;i\in[0,p_r-p_s+h]}(e_{p_s+i-h}^{r,s})^{q^i}x_i^{t,r}=M'$$

for any $t, s \in [1,\sigma], h \in [0, p_s - 1]$, where $M' = (e^{s,t}_{p_s})^{q^{p_s}}\epsilon$ if $s > t$, $h = 0$, $M' = \epsilon$ if $s = t$, $h = 0$, $M' = 0$ if $s < t$ or if $h > 0$;

(iii)
$$\begin{aligned}&\sum_{r<r';i\in[0,p_r-1],j\in[1,p_{r'}];i+j<p_{r'}}(x_i^{t,r}y_j^{t',r'}+x_i^{t',r}y_j^{t,r'}\epsilon)(d^{r,r'}_{i+j})^{q^i}\\ &+\sum_{r>r';i\in[0,p_r-1],j\in[1,p_{r'}];i+j\le p_r}(x_i^{t,r}y_j^{t',r'}+x_i^{t',r}y_j^{t,r'}\epsilon)(e^{r,r'}_{i+j})^{q^i}\\ &+\sum_{r;i\in[0,p_r-1],j\in[1,p_r];i+j<p_r}(x_i^{t,r}y_j^{t',r}+x_i^{t',r}y_j^{t,r}\epsilon)(c^{r}_{i+j})^{q^i}\\ &+\sum_{r;i\in[0,p_r-1],j\in[1,p_r];i+j=p_r}(x_i^{t,r}y_j^{t',r}+x_i^{t',r}y_j^{t,r}\epsilon)+u^tu^{t'}\zeta=0;\end{aligned}$$

for any $t < t'$ in $[1,\sigma]$;

(iv)
$$\begin{aligned}&\sum_{r<r';i\in[0,p_r-1],j\in[1,p_{r'}];i+j<p_{r'}}x_i^{t,r}y_j^{t,r'}(d^{r,r'}_{i+j})^{q^i}\\ &+\sum_{r>r';i\in[0,p_r-1],j\in[1,p_{r'}];i+j\le p_r}x_i^{t,r}y_j^{t,r'}(e^{r,r'}_{i+j})^{q^i}\\ &+\sum_{r;i\in[0,p_r-1],j\in[1,p_r];i+j<p_r}x_i^{t,r}y_j^{t',r}(c^{r}_{i+j})^{q^i}\\ &+\sum_{r;i\in[0,p_r-1],j\in[1,p_r];i+j=p_r}x_i^{t,r}y_j^{t,r}+(u^t)^2\zeta_0=0\end{aligned}$$

for any $t \in [1,\sigma]$ (if $Q \neq 0$).

We define $\tilde{Z}'_2 \to \tilde{Z}'_3$ by setting

$$\begin{aligned}c^r_h &= (w^r_0, z^r_h)\ (h \in [1, p_r-1]),\ d^{t,r}_h = (w^t_0, z^r_h)\ t<r, h\in[1,p_r-1]),\\ e^{t,r}_h &= (w^t_0, z^r_h)\ (t>r, h\in[1,p_t]),\end{aligned}$$

and defining $x_i^{t,r}, y_j^{t,r}$ and u^t (if $\kappa = 1$) by:

$$\begin{aligned}z^t_0 &= \sum_{r;i\in[0,p_r-1]}x_i^{t,r}w_i^r+\sum_{r;j\in[1,p_r]}y_j^{t,r}z_j^r \quad (\text{if } \kappa=0)\\ z^t_0 &= \sum_{r;i\in[0,p_r-1]}x_i^{t,r}w_i^r+\sum_{r;j\in[1,p_r]}y_j^{t,r}z_j^r+u^t\xi \quad (\text{if } \kappa=1)\end{aligned}$$

(ξ as in the definition of the inverse of $\tilde{Z}'_1 \to \tilde{Z}'_2$.) This map is well defined (the equations (i), (ii), (iii), (iv) come from I, II, III, IV). Consider the fibre $\mathfrak{F}$ of this map at a point of $\tilde{Z}'_3$. Then $\mathfrak{F}$ consists of all bases of V (if $\kappa = 0$) or "bases" (=bases with one missing element) of V spanning a nondegenerate hyperplane (if $\kappa = 1$), with a fixed index set, such that the value of $(,)$ at any two basis (or "basis") elements is prescribed, the value of Q at any basis (or "basis") element is prescribed, and such that (in the case $\kappa = 0, Q \neq 0$) the elements of type z in this basis span a subspace in $\mathcal{I}$. These bases (or "bases") clearly form a single G-orbit; note that the elements in such a basis (or "basis") will automatically satisfy the equations I, II, III, IV. We see that $\tilde{Z}'_3$ may be identified with the space of G-orbits on $\tilde{Z}'_2$ for the obvious (free) G-action.

We shall denote by $\mathbf{U}$ a universal polynomial with coefficients in $\mathbf{k}$ in the quantities

$$\begin{aligned} &c_h^r \quad (r \in [1, \sigma], h \in [1, p_r - 1]; d_h^{t,r}(1 \leq t < r \leq \sigma; h \in [1, p_r - 1]);\\ &e_h^{t,r} \quad (1 \leq r < t \leq \sigma; h \in [1, p_t] \end{aligned}$$

and the quantities

$$\begin{aligned} &y_{p_s}^{r,s} \quad (r \leq s \text{ in } [1, \sigma]) \text{ if } \kappa = 0, Q = 0,\\ &y_{p_s}^{r,s} \quad (r < s \text{ in } [1, \sigma]) \text{ if } \kappa = 0, Q \neq 0,\\ &y_{p_s}^{r,s} \quad (r < s \text{ in } [1, \sigma]),\\ &u^t \quad (t \in [1, \sigma]) \text{ if } \kappa = 1. \end{aligned}$$

We order the variables $y_j^{t,r}$ (with fixed t and with $j \in [1, p_r - 1]$) in the definition of $\tilde{Z}'_3$ as follows: we say that $y_j^{t,r} < y_k^{t,s}$ if $j < k$ or $j = k, r < s$. Then in the equation (i) all terms other than $y_h^{t,s}$ are $< y_h^{t,s}$ (for $r > s$ we have $j \leq p_r - p_s + h - 1 \leq h - 1$ so that $j < h$). Therefore, from (i) we see by induction on the order above that

(p) $$y_j^{t,r} = \mathbf{U} \text{ for any } j \in [1, p_r - 1].$$

We order the variables $x_i^{t,r}$ (with fixed t and with $i \in [0, p_r - 1]$) in the definition of $\tilde{Z}'_3$ as follows: we say that $x_j^{t,r} < x_k^{t,s}$ if $j < k$ or $j = k, r > s$. Then in the equation (ii) all terms other than $x_h^{t,s}$ are $< x_h^{t,s}$ (for $r > s$ we have $i \le p_r - p_s + h \le h$ so that $i \le h$). Therefore, from (ii) we see by induction on the order above that

$$x_i^{t,r} = \mathbf{U} \text{ for any } i \in [0, p_r - 1]. \tag{q}$$

For $s \le t$ and $h = 0$ we can write equation (ii) as follows:

$$\begin{cases} x_0^{t,s} + \sum_{r; r>s; p_r=p_s} e_{p_s}^{r,s} x_0^{t,r} = \epsilon & \text{if } t = s, \\ x_0^{t,s} + \sum_{r; r>s; p_r=p_s} e_{p_s}^{r,s} x_0^{t,r} = 0 & \text{if } s < t. \end{cases} \tag{r}$$

Assuming that $Q \ne 0$ we now rewrite (iv) using (p), (q) (the only quantities $y_j^{t,s}$ that are not of the form $\mathbf{U}$ are those with $j = p_s$):

$$\sum_{r>r'; p_{r'}=p_r} x_0^{t,r} y_{p_{r'}}^{t,r'} e_{p_{r'}}^{r,r'} + \sum_r x_0^{t,r} y_{p_r}^{t,r} + (u^t)^2 \zeta_0 = \mathbf{U}$$

that is,

$$\sum_r y_{p_r}^{t,r} (x_0^{t,r} + \sum_{s; s>r; p_s=p_r} x_0^{t,s} e_{p_r}^{s,r}) + (u^t)^2 \zeta_0 = \mathbf{U}.$$

Using (r) this becomes

$$y_{p_t}^{t,t} + \sum_{r; r>t} y_{p_r}^{t,r} (x_0^{t,r} + \sum_{s; s>r; p_s=p_r} x_0^{t,s} e_{p_r}^{s,r}) + (u^t)^2 \zeta_0 = \mathbf{U}$$

that is

$$y_{p_t}^{t,t} = \mathbf{U}.$$

Here we have assumed that $Q \ne 0$; but the same holds for $Q = 0$ by the definition of $\mathbf{U}$. We now rewrite (iii) for $t < t'$ using (p), (q) (again, the only quantities $y_j^{t,s}$ that are not of the form $\mathbf{U}$ are those with $j = p_s$):

$$\sum_{r>r'; p_{r'}=p_r} (x_0^{t,r} y_{p_{r'}}^{t',r'} + x_0^{t',r} y_{p_{r'}}^{t,r'} \epsilon) e_{p_{r'}}^{r,r'} + \sum_r (x_0^{t,r} y_{p_r}^{t',r} + x_0^{t',r} y_{p_r}^{t,r} \epsilon) = \mathbf{U}$$

that is

$$\sum_r y_{p_r}^{t',r} (x_0^{t,r} + \sum_{s; s>r; p_s=p_r} x_0^{t,s} e_{p_r}^{s,r})$$

$$+\sum_{r} y^{t,r}_{p_r}\epsilon(x^{t',r}_0 + \sum_{s;s>r;p_s=p_r} x^{t',s}_0 e^{s,r}_{p_r}) = \mathbf{U}.$$

Using (r) this becomes

$$\epsilon y^{t',t}_{p_t} + y^{t,t'}_{p_{t'}} + \sum_{r;r>t} y^{t',r}_{p_r}(x^{t,r}_0 + \sum_{s;s>r;p_s=p_r} x^{t,s}_0 e^{s,r}_{p_r})$$
$$+\sum_{r;r>t'} y^{t,r}_{p_r}\epsilon(x^{t',r}_0 + \sum_{s;s>r;p_s=p_r} x^{t',s}_0 e^{s,r}_{p_r}) = \mathbf{U}.$$

that is

$$\epsilon y^{t',t}_{p_t} + \sum_{r;t'>r>t} y^{t',r}_{p_r}(x^{t,r}_0 + \sum_{s;s>r;p_s=p_r} x^{t,s}_0 e^{s,r}_{p_r}) = \mathbf{U}.$$

This shows by induction on $t'-t$ that

$$y^{t',t}_{p_t} \in \mathbf{U}$$

for all $t < t'$. We now see that the equations defining $\tilde{Z}'_3$ are all of the form $b \in \mathbf{U}$ where b is any one of the variables which do not enter in the definition of $\mathbf{U}$. This shows that $\tilde{Z}'_3$ is an affine space whose dimension is equal to the number of variables which enter in the definition of $\mathbf{U}$ that is

$$\begin{cases} \sum_r (2r-1)p_r & \text{if } \kappa = 0,\ Q = 0 \text{ or if } \kappa = 1, \\ \sum_r (2r-1)p_r - \sigma & \text{if } \kappa = 0,\ Q \neq 0. \end{cases}$$

This completes the proof of Theorem 0.4.

5. Counting Rational Points

5.1. In this section we describe another example of a close relation between the varieties $\mathfrak{B}_w, X_w$.

Let $\mathcal{H}$ be the Iwahori-Hecke algebra over $\mathbf{Q}(\mathbf{q})$ ($\mathbf{q}$ is an indeterminate) with basis $t_w (w \in \mathbf{W})$ and multiplication defined by $t_w t_{w'} = t_{ww'}$ if $w, w' \in \mathbf{W}$, $l(ww') = l(w) + l(w')$ and $t^2_{s_i} = \mathbf{q} + (\mathbf{q}-1)t_{s_i}$ for $i \in I$. For any $w, w' \in \mathbf{W}$ let $n_{w,w'} \in \mathbf{Z}[\mathbf{q}]$ be the trace of the linear map $\mathcal{H} \mapsto \mathcal{H}$ given by $t_y \mapsto t_w t_{y^\bullet} t_{w'^{-1}}$ for all y.

5.2. In this subsection we assume that we are in case 1 but $\mathbf{k}$ is as in case 2 and we are given an F_q-rational structure on $\hat{G}$ with Frobenius map $\Phi : \hat{G} \to \hat{G}$ such that $\Phi(d) = d$ and $\Phi(t) = t^q$ for all $t \in T^*$. Then T^*, B^*, D are Φ-stable and Φ acts trivially on $\mathbf{W}$. We define a new F_q-rational structure on G with Frobenius map $F : G \to G$ such that $F(x) = d\Phi(x)d^{-1}$ for all $x \in G$. Note that G, F are as in case 2. Thus both $\mathfrak{B}_w$ and X_w are well defined for $w \in \mathbf{W}$. Moreover $w \mapsto w^\bullet$ defined in terms of G, D is the same as $w \mapsto w^\bullet$ defined in terms of G, F. Now let w, w' be elements of W. Let $\mathfrak{B}_w \times_D \mathfrak{B}_{w'} = \{((g_1, B), (g_1', B') \in \mathfrak{B}_w \times \mathfrak{B}_{w'}; g_1 = g_1'\}$.

Let $G^F \backslash (X_w \times X_{w'})$ be the set of orbits of the diagonal G^F-action on $X_w \times X_{w'}$. Note that for any $s \in \mathbf{Z}_{>0}$, Φ^s defines F_{q^s}-rational structures on $\mathfrak{B}_w \times_D \mathfrak{B}_{w'}$, $X_w \times X_{w'}$, $G^F \backslash (X_w \times_D X_{w'})$ with Frobenius maps denoted again by Φ^s. We have the following result.

Theorem 5.3. *Let $s \in \mathbf{Z}_{>0}$. Let $N_s = |(\mathfrak{B}_w \times_D \mathfrak{B}_{w'})(F_{q^s})|$ and $N_s' = |(G^F \backslash (X_w \times X_{w'}))(F_{q^s})|$. We have $N_s' = |G^{\Phi^s}|^{-1} N_s = n_{w,w'}|_{\mathbf{q}=q^s}$.*

The equality $N_s' = n_{w,w'}|_{\mathbf{q}=q^s}$ is proved in [12, 3.8] under the additional assumption that F^s acts trivially on $\mathbf{W}$. However exactly the same proof applies without that assumption. It remains to show that $|G^{\Phi^s}|^{-1} N_s = n_{w,w'}|_{\mathbf{q}=q^s}$. Replacing Φ by Φ^s we see that we can assume that $s = 1$. Hence it is enough to show that $N_1' = |G^\Phi|^{-1} N_1$. Let G^F_ξ be the stabilizer of $\xi \in X_w \times X_{w'}$ in G^F. We have

$$\begin{aligned}
N_1' &= |(G^F \backslash (X_w \times X_{w'}))^\Phi| = \sum_{\xi \in X_w \times X_{w'}; \Phi(\xi) = h\xi \text{ for some } h \in G^F} |G^F_\xi| / |G^F| \\
&= \sum_{\xi \in X_w \times X_{w'}; h \in G^F; \Phi(\xi) = h\xi} |G^F|^{-1} \\
&= |G^F|^{-1} |\{(h, B, B') \in G^F \times \mathcal{B} \times \mathcal{B}; (B, FB) \in \mathcal{O}_w, (B', FB') \in \mathcal{O}_{w'}, \\
&\quad \Phi(B) = hBh^{-1}, \Phi(B') = hB'h^{-1}\}| \\
&= |G^F|^{-1} |\{(h, B, B') \in G^F \times \mathcal{B} \times \mathcal{B}; (B, FB) \in \mathcal{O}_w, (B', FB') \in \mathcal{O}_{w'}, \\
&\quad d^{-1}F(B)d = hBh^{-1}, d^{-1}F(B')d = hB'h^{-1}\}|.
\end{aligned}$$

We set $h = \Phi(y)y^{-1}$ where $y \in G$ has $|G^\Phi|$ choices. The condition $F(h) = h$ becomes $F(\Phi(y))F(y)^{-1} = \Phi(y)y^{-1}$ that is $\Phi(y^{-1}F(y)) = y^{-1}F(y)$ (since

$F\Phi = \Phi F$). We get

$$\begin{aligned} N'_1 \;=\; & |G^F|^{-1}|G^\Phi|^{-1}|\{(y,B,B')\in G\times\mathcal{B}\times\mathcal{B};\Phi(y^{-1}F(y))=y^{-1}F(y),\\ & (B,F(B))\in\mathcal{O}_w,(B',F(B'))\in\mathcal{O}_{w'},y\Phi(y^{-1})d^{-1}F(B)d\Phi(y)y^{-1}=B,\\ & y\Phi(y^{-1})d^{-1}F(B')d\Phi(y)y^{-1}=B'\}|.\end{aligned}$$

We set $B_1 = y^{-1}By$, $B'_1 = y^{-1}B'y$. We get

$$\begin{aligned} N'_1 \;=\; & |G^F|^{-1}|G^\Phi|^{-1}|\{(y,B_1,B'_1)\in G\times\mathcal{B}\times\mathcal{B};\Phi(y^{-1}F(y))=y^{-1}F(y),\\ & (yB_1y^{-1},F(y)F(B_1)F(y^{-1}))\in\mathcal{O}_w,\\ & (yB'_1y^{-1},F(y)F(B'_1)F(y^{-1}))\in\mathcal{O}_{w'},\\ & d^{-1}F(B_1)d=B_1,d^{-1}F(B'_1)d=B'_1\}|,\\ N'_1 \;=\; & |G^F|^{-1}|G^\Phi|^{-1}|\{(y,B_1,B'_1)\in G\times\mathcal{B}\times\mathcal{B};\Phi(y^{-1}F(y))=y^{-1}F(y),\\ & (yB_1y^{-1},F(y)dB_1d^{-1}F(y^{-1}))\in\mathcal{O}_w,\\ & (yB'_1y^{-1},F(y)dB'_1d^{-1}F(y^{-1}))\in\mathcal{O}_{w'},\\ & d^{-1}F(B_1)d=B_1,d^{-1}F(B'_1)d=B'_1\}|.\end{aligned}$$

We set $z = y^{-1}F(y) \in G^\Phi$. Note that for any $z \in G^\Phi$ there are $|G^F|$ values of y satisfying $\Phi(y^{-1}F(y)) = y^{-1}F(y)$. We get

$$\begin{aligned} N'_1 \;=\; & |G^\Phi|^{-1}|\{(z,B_1,B'_1)\in G^\Phi\times\mathcal{B}\times\mathcal{B};(B_1,zdB_1d^{-1}z^{-1})\in\mathcal{O}_w,\\ & (B'_1,zdB'_1d^{-1}z^{-1})\in\mathcal{O}_{w'},d^{-1}F(B_1)d=B_1,d^{-1}F(B'_1)d=B'_1\}|.\end{aligned}$$

We set $z' = zd \in D^\Phi$. We get

$$\begin{aligned} N'_1 \;=\; & |G^\Phi|^{-1}|\{(z',B_1,B'_1)\in D\times\mathcal{B}\times\mathcal{B};(B_1,z'B_1z'^{-1})\in\mathcal{O}_w,\\ & (B'_1,z'B'_1z'^{-1})\in\mathcal{O}_{w'},(\Phi(z'),\Phi(B_1),\Phi(B'_1))=(z',B_1,B'_1)\}|.\end{aligned}$$

Thus $N'_1 = |G^\Phi|^{-1}N_1$. The theorem is proved.

5.4. Assume in addition that G is semisimple and that w, w' are $\bullet$-elliptic of minimal length in their $\bullet$-conjugacy class. This guarantees that $\mathfrak{B}_w \times_D \mathfrak{B}_{w'}$ is affine and the (diagonal) G action on $\mathfrak{B}_w \times_D \mathfrak{B}_{w'}\}$ has finite isotropy groups (see 0.3); thus all its orbits have the same dimensions so they are all closed and the set $G\backslash(\mathfrak{B}_w \times_D \mathfrak{B}_{w'})$ of orbits of this action is naturally an affine variety. Note that Φ defines an F_q-rational structure on $G\backslash\mathfrak{B}_w \times_D \mathfrak{B}_{w'}$. We show:

(a) *For any $s \in \mathbf{Z}_{>0}$, the affine varieties $G\backslash(\mathfrak{B}_w \times_D \mathfrak{B}_{w'})$, $G^F\backslash(X_w \times X_{w'})$ have the same number of F_{q^s}-rational points.*
In view of 5.3 it is enough to show that any Φ^s-stable G-orbit on $\mathfrak{B}_w \times_D \mathfrak{B}_{w'}$ contains exactly $|G^{\Phi^s}|$ rational points. This follows from the fact that the isotropy group in G at a point of that orbit is finite.

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