Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **6** (2011), No. 3, pp. 347-360

TRAPEZOIDAL RULE REVISITED

ARKADIUSZ LISAK[1,a] AND MACIEJ SABLIK[2,b]

[1] Rzeszów School of Engineering and Economics, Miłocińska 40, 35-232 Rzeszów, Poland.

[a]E-mail: lisakarek@gmail.com

[2]Institute of Mathematics, Silesian University, Bankowa 14, 40-007 Katowice, Poland.

[b]E-mail: maciej.sablik@us.edu.pl

Abstract

P. K. Sahoo in [7] has arrived at the functional equation stemming from trapezoidal rule

$$g(y)-g(x)=\frac{y-x}{6}\left[f(x)+2f\left(\frac{2x+y}{3}\right)+2f\left(\frac{x+2y}{3}\right)+f(y)\right],$$

for $x,y\in\mathbb{R}$, where f and g are unknown functions. In fact, Sahoo considered more general equations

$$g(y)-h(x)=(y-x)[f(x)+2k(sx+ty)+2k(tx+sy)+f(y)] \tag{1}$$

with four unknown functions (cf. [7]) and

$$f_1(y)-g_1(x)=(y-x)[f_2(x)+f_3(sx+ty)+f_4(tx+sy)+f_5(y)] \tag{2}$$

with six unknown functions (cf. [8]), where s and t are two fixed real parameters. The equations have been solved in [7] and [8] for $s^2=t^2$ or $s=0$ or $t=0$ without any regularity assumptions, and in the case $s^2\neq t^2$ (with $st\neq 0$) the solutions have been determined under high regularity assumptions on unknown functions (differentiability of second or fourth order).

In this paper we solve equations (1) and (2) in the case of $s^2\neq t^2$ (with $st\neq 0$) with no regularity assumptions on unknown functions for rational parameters s and t, and under very weak assumptions in other cases.

Received November 24, 2009 and in revised form March 23, 2010.

Key words and phrases: Functional equations, trapezoidal rule, generalized polynomials, no regularity assumptions.

1. Introduction

In references [7] and [8] P. K. Sahoo considered the functional equations (1) and (2), stemming from the trapezoidal rule for estimation of Riemann integral, with assumptions of high regularity of the unknown functions (two times differentiability, or even four times differentiability), thus the results were obtained. In the present paper we use a slightly different method of solve (1) and (2), based on Lemma 1 below. In addition, we show that with *no regularity at all* in case of rational parameters or with *very weak regularity* in general case we are able to obtain similar results. Obviously, in places where in Sahoo's results appear to be linear functions, we get general additive mappings. Sometimes even the general solutions of the Cauchy equation have to be linear as well. This is due to the special shape of equations (1) and (2).

In this paper, functional equations (1) and (2) in the case $s^2 \neq t^2$ with $s, t \in \mathbb{Q}$, $s, t \neq 0$ are going to be solved without any regularity assumptions. First let us recall notations used in reference [4].

Let G and H be commutative groups. Then $SA^i(G;H)$ denotes the group of all i-additive, symmetric mappings from G^i into H for $i \geq 2$, while $SA^0(G;H)$ denotes the family of constant functions from G to H and $SA^1(G;H) = \mathrm{Hom}(G;H)$. We also denote by $\mathcal{I}$ the subset of $\mathrm{Hom}(G;G) \times \mathrm{Hom}(G;G)$ containing all pairs (α, β) for which $\mathrm{Ran}(\alpha) \subset \mathrm{Ran}(\beta)$. Furthermore, a convention that a sum over empty set of indices equals 0 is adopted.

Now we present the following useful lemma which is generalization of lemma presented in reference [4] (Lemma 2.3) with a modification from [2].

Lemma 1. *Fix $N, M, K \in \mathbb{N} \cup \{0\}$ and let $I_0, \ldots, I_{M+K}$ be finite subsets of $\mathcal{I}$. Suppose further that H is uniquely divisible by $N!$ and let functions $\varphi_i : G \to SA^i(G;H)$, $i \in \{0, \ldots, N\}$ and functions $\psi_{i,(\alpha,\beta)} : G \to SA^i(G;H)$, $(\alpha, \beta) \in I_i$, $i \in \{0, \ldots, M+K\}$ satisfy*

$$\begin{aligned}\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \varphi_i(x)(y^i) \; &= \; \sum_{i=0}^{M} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x) + \beta(y))(y^i) \\ &\quad + \sum_{i=M+1}^{M+K} \sum_{(\alpha,\beta)\in I_i} \psi_{i,(\alpha,\beta)}(\alpha(x) + \beta(y))(x^i)\end{aligned}$$

for every $x, y \in G$. Then φ_N is a generalized polynomial of degree at most equal to

$$\sum_{i=0}^{M+K} \mathrm{card}(\bigcup_{s=i}^{M+K} I_s) - 1.$$

Proof. The Lemma is proved by using induction with respect to $M+K$. If $M+K=0$ then $M=K=0$ and we can use the famous theorem of Székelyhidi (Theorem 9.5 in reference [9]). Suppose that $M \geq 1$ or $K \geq 1$ and the assertion holds true for some $M+K-1$. Let us consider the case where on the right hand side there are $M+K+1$ summands. It follows that at least one of the sets I_i, $1 \leq i \leq M+K$ is nonempty. If I_i is a set appearing in the first sum on the right-hand side then, as in the proof of Lemma 2.3 in reference [4], we obtain for some $u_1, \ldots, u_k$ ($k = \mathrm{card} I_i$)

$$\begin{aligned}
&\Delta_{u_1,\ldots,u_k}\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \hat{\varphi}_i(x)(y^i) \\
&= \sum_{j \neq i} \sum_{(\alpha,\beta)\in L_j} \hat{\psi}_{j,(\alpha,\beta)}(\alpha(x)+\beta(y))(y^j) \\
&\quad + \sum_{j=M+1}^{M+K} \sum_{(\alpha,\beta)\in L_j} \hat{\psi}_{j,(\alpha,\beta)}(\alpha(x)+\beta(y))(x^j),
\end{aligned}$$

where $\hat{\varphi}_i$ and $\hat{\psi}_{j,(\alpha,\beta)}$ are some functions mapping G into $SA^i(G;H)$ and $SA^j(G;H)$, respectively. We admit $L_j = \bigcup_{s=j}^{M+K} I_s$, $j \in \{0, \ldots, M+K\} \setminus \{i\}$. By induction hypothesis, we obtain that $\Delta_{u_1,\ldots,u_k}\varphi_N$ is a polynomial function of order at most equal to

$$\sum_{j \neq i} \mathrm{card} L_j - 1 = \sum_{j \neq i} \mathrm{card}(\bigcup_{s=j}^{M+K} I_s) - 1.$$

In particular, the function φ_N is polynomial of order

$$\sum_{j=0}^{M+K} \mathrm{card} L_j - 1 = \sum_{j=0}^{M+K} \mathrm{card}(\bigcup_{s=j}^{M+K} I_s) - 1.$$

Analogously, if the set I_i is appearing in the second sum on the right-hand side, we get for $u_1, \ldots, u_k$ ($k = \mathrm{card} I_i$)

$$\begin{aligned}
&\Delta_{u_1,\ldots,u_k}\varphi_N(x)(y^N) + \sum_{i=0}^{N-1} \hat{\varphi}_i(x)(y^i) \\
&\quad = \sum_{j=0}^{M} \sum_{(\alpha,\beta)\in L_j} \hat{\psi}_{j,(\alpha,\beta)}(\alpha(x)+\beta(y))(y^j) \\
&\qquad + \sum_{j\neq i} \sum_{(\alpha,\beta)\in L_j} \hat{\psi}_{j,(\alpha,\beta)}(\alpha(x)+\beta(y))(x^j),
\end{aligned}$$

where L_j, $\hat{\varphi}_i$ and $\hat{\psi}_j$ have the same meaning as above. Also for this case by using induction hypothesis it is obtained that φ_N is a polynomial function of order bounded by

$$\sum_{j=0}^{M+K} \mathrm{card}\Big(\bigcup_{s=j}^{M+K} I_s\Big) - 1. \qquad \square$$

The following lemma is also required.

Lemma 2.([3]) *For every $k \in \mathbb{N}$, if $B \in SA^k(\mathbb{R};\mathbb{R})$ satisfies*

$$B(x^{k-1}, y) = yB(x^{k-1}, 1)$$

for every $x, y \in \mathbb{R}$, then B is k-linear, i.e.

$$B(x_1, \ldots, x_k) = B(1^k)x_1 \cdot \cdots \cdot x_k$$

for every $x_1, \ldots, x_k \in \mathbb{R}$, where 1^k is the k-tuple $(\underbrace{1, \ldots, 1}_{k})$.

2. Main Results

Now we are able to solve functional equations (1) and (2) in the case $s^2 \neq t^2$ for rational s, t without any regularity assumptions. We start with the equation (1).

Let $s,t\in\mathbb{Q}$ with $s^2\neq t^2$ and $s,t\neq 0$. Taking in (1) $y=x$ we get

$$h(x)=g(x). \tag{3}$$

We may treat the multiplication $y\cdot\varphi(x)$ as action of the homomorphism $\varphi(x)$ on the argument y, i.e. $\varphi(x)(y)=y\cdot\varphi(x)$. If we put $y=y-x$ in (1) after some simplifications we can write (1) in the form

$$\begin{aligned}&f(x)(y)+g(x)\\&\quad = [-f(x+y)-2k((s+t)x+ty)-2k((s+t)x+sy)](y)+g(x+y).\end{aligned}$$

Using Lemma 1 for $N=M=1$, $K=0$, $I_1=\{(\mathrm{id},\mathrm{id}),((s+t)\mathrm{id},t\,\mathrm{id}),((s+t)\mathrm{id},s\,\mathrm{id})\}$, $I_0=\{(\mathrm{id},\mathrm{id})\}$, $\varphi_1=f$, $\varphi_0=h$, $\psi_{0,(\mathrm{id},\mathrm{id})}=g$, $\psi_{1,(\mathrm{id},\mathrm{id})}=-f$, $\psi_{1,((s+t)\mathrm{id},t\,\mathrm{id})}=\psi_{1,((s+t)\mathrm{id},s\,\mathrm{id})}=-2k$ we get (with assumptions $s^2\neq t^2$ and $s,t\neq 0$) that f is a generalized polynomial of degree at most $\sum_{i=0}^{1}\mathrm{card}(\bigcup_{s=i}^{1} I_s)-1=5$, that is

$$f(x)=\sum_{i=0}^{5}B_i^d(x), \tag{4}$$

where B_i^d are diagonalizations of i-additive symmetric functions B_i for $i=0,1,\dots,5$.

Putting $u=sx+ty$, $v=y-x$ we have $x=\frac{1}{s+t}u-\frac{t}{s+t}v$ and $y=\frac{1}{s+t}u+\frac{s}{s+t}v$ ($s+t\neq 0$ since $s^2\neq t^2$). Followed with obvious transformations and simplifications we get from (1)

$$\begin{aligned}k(u)(v) \;=\; &-\frac{1}{2}f\left(\frac{1}{s+t}u-\frac{t}{s+t}v\right)(v)-\frac{1}{2}f\left(\frac{1}{s+t}u+\frac{s}{s+t}v\right)(v)\\&-k(u+(s-t)v)(v)+\frac{1}{2}g\left(\frac{1}{s+t}u+\frac{s}{s+t}v\right)\\&-\frac{1}{2}g\left(\frac{1}{s+t}u-\frac{t}{s+t}v\right).\end{aligned}$$

Again using Lemma 1 for $N=M=1$, $K=0$, $I_0=\{(\frac{1}{s+t}\mathrm{id},\frac{s}{s+t}\mathrm{id}),(\frac{1}{s+t}\mathrm{id},-\frac{t}{s+t}\mathrm{id})\}$, $I_1=\{(\frac{1}{s+t}\mathrm{id},-\frac{t}{s+t}\mathrm{id}),(\frac{1}{s+t}\mathrm{id},\frac{s}{s+t}\mathrm{id}),(\mathrm{id},(s-t)\mathrm{id})\}$, $\varphi_1=k$, $\varphi_0=0$, $\psi_{0,(\frac{1}{s+t}\mathrm{id},\frac{s}{s+t}\mathrm{id})}=\psi_{0,(\frac{1}{s+t}\mathrm{id},-\frac{t}{s+t}\mathrm{id})}=\frac{1}{2}g$, $\psi_{1,(\frac{1}{s+t}\mathrm{id},\frac{s}{s+t}\mathrm{id})}=\psi_{1,(\frac{1}{s+t}\mathrm{id},-\frac{t}{s+t}\mathrm{id})}=-\frac{1}{2}f$, $\psi_{1,(\mathrm{id},(s-t)\mathrm{id})}=-k$ we obtain (with assumptions $s^2\neq t^2$, $s,t\neq 0$) that k is a generalized polynomial of degree at most $\sum_{i=0}^{1}\mathrm{card}(\bigcup_{s=i}^{1} I_s)-1=5$, that is

$$k(x)=\sum_{i=0}^{5} A_i^d(x), \tag{5}$$

where A_i^d are diagonalizations of i-additive symmetric functions A_i for $i=0,1,\ldots,5$.

Putting now $y=0$ in (1) we have

$$g(x)=[f(x)+2k(sx)+2k(tx)+f(0)](x)+g(0)$$

and using previous calculations we get

$$g(x)=x\left[\sum_{i=0}^{5} B_i^d(x)+2\sum_{i=0}^{5} A_i^d(sx)+2\sum_{i=0}^{5} A_i^d(tx)+B_0\right]+c_0, \tag{6}$$

where $c_0=g(0)$.

Remark 1. M. Sablik during The Forty-fourth ISFE held in May 2006 in Louisville, Kentucky, USA, has made a remark (refer to reference [5]) that using Lemma 1 we are able to show that if functions f,g,h,k satisfy equation (1) then f and k are generalized polynomial functions of order at most 5, and $g=h$ is a polynomial of order at most 6. If we assume that every A_i and every B_i for $i=0,1,\ldots,5$ are continuous (to get homogeneity of monomial functions forming solutions) or satisfy any condition implying that A_i and B_i are homogeneous (that is measurability or boundedness on a set of positive Lebesgue measure) we get that functions f,g,h,k are polynomials and then they satisfy the assertion of Sahoo's theorems. So inserting the general forms of functions f,g,h,k into equation (1) and assuming continuity (or measurability or boundedness on a set of positive Lebesgue measure) of functions f and k, we obtain the solution mentioned by Sahoo in his talk at the meeting (reference [6]) and the differentiability (up to four times) of unknown functions (for real parameters s and t) is not required. In addition, using the same condition, further result is obtained under no regularity assumption in the case where s and t are rational numbers.

Substituting formulas (3), (4), (5), (6) to the equation (1) we have

$$\begin{aligned}
&2y\sum_{i=0}^{5}A_i^d(sy)+2y\sum_{i=0}^{5}A_i^d(ty)-2x\sum_{i=0}^{5}A_i^d(sx)-2x\sum_{i=0}^{5}A_i^d(tx)\\
&= y\sum_{i=1}^{5}B_i^d(x)-x\sum_{i=1}^{5}B_i^d(y)+2y\sum_{i=0}^{5}A_i^d(sx+ty)-2x\sum_{i=0}^{5}A_i^d(sx+ty)\\
&\quad+2y\sum_{i=0}^{5}A_i^d(tx+sy)-2x\sum_{i=0}^{5}A_i^d(tx+sy).
\end{aligned}$$

Using i-additivity of functions A_i and B_i for $i=0,1,\ldots,5$ and their homogeneity with respect to $s,t\in\mathbb{Q}$ we can write the previous equation as

$$\begin{aligned}
&2y\sum_{i=0}^{5}(s^i+t^i)A_i^d(y)-2x\sum_{i=0}^{5}(s^i+t^i)A_i^d(x)\\
&= y\sum_{i=1}^{5}B_i^d(x)+2y\sum_{i=0}^{5}\sum_{j=0}^{i}\binom{i}{j}(s^{i-j}t^j+t^{i-j}s^j)A_i(x^{i-j},y^j)\\
&- x\sum_{i=1}^{5}B_i^d(y)-2x\sum_{i=0}^{5}\sum_{j=0}^{i}\binom{i}{j}(s^{i-j}t^j+t^{i-j}s^j)A_i(x^{i-j},y^j),
\end{aligned}$$

that is

$$\begin{aligned}
&x\sum_{i=1}^{5}B_i^d(y)-y\sum_{i=1}^{5}B_i^d(x)\\
&= 2y\sum_{i=1}^{5}\sum_{j=0}^{i-1}\binom{i}{j}(s^{i-j}t^j+t^{i-j}s^j)A_i(x^{i-j},y^j)\\
&\quad-2x\sum_{i=1}^{5}\sum_{j=1}^{i}\binom{i}{j}(s^{i-j}t^j+t^{i-j}s^j)A_i(x^{i-j},y^j).
\end{aligned}$$

Now, comparing the terms with respect to degree of x and y, we arrive at the following conditions

$$xB_1(y)-yB_1(x)=2(s+t)yA_1(x)-2(s+t)xA_1(y), \tag{7}$$

$$y\sum_{i=2}^{5}B_i^d(x)=2x\sum_{i=2}^{5}i(st^{i-1}+ts^{i-1})A_i(x^{i-1},y)-2y\sum_{i=2}^{5}(s^i+t^i)A_i^d(x), \tag{8}$$

$$
\begin{aligned}
2y\sum_{i=3}^{5}\sum_{j=1}^{i-2}\binom{i}{j}&(s^{i-j}t^{j}+t^{i-j}s^{j})A_i(x^{i-j},y^{j})\\
&=2x\sum_{i=3}^{5}\sum_{j=2}^{i-1}\binom{i}{j}(s^{i-j}t^{j}+t^{i-j}s^{j})A_i(x^{i-j},y^{j}) \qquad (9)
\end{aligned}
$$

holding for all $x, y \in \mathbb{R}$ (expressions with degree of x equal to 1 for $i \geq 2$ generate the same condition as expressions with degree of y equal to 1 therefore they are omitted).

If we put $y = 1$ in (7) we get

$$B_1(x) = (b_1 + 2(s+t)a_1)x - 2(s+t)A_1(x), \qquad (10)$$

where $a_1 = A_1(1)$, $b_1 = B_1(1)$.

From (8) we directly get

$$yB_i^d(x) = 2i(s^{i-1}t + t^{i-1}s)xA_i(x^{i-1}, y) - 2(s^i + t^i)yA_i^d(x) \qquad (11)$$

for every $2 \leq i \leq 5$. Letting $y = x$ we have

$$B_i^d(x) = 2[i(s^{i-1}t + t^{i-1}s) - s^i - t^i]A_i^d(x) \qquad (12)$$

for every $2 \leq i \leq 5$.

On the other side putting $y = 1$ in (8) and multiplying both sides by y we get

$$yB_i^d(x) = 2i(s^{i-1}t + t^{i-1}s)xyA_i(x^{i-1}, 1) - 2(s^i + t^i)yA_i^d(x) \qquad (13)$$

for every $2 \leq i \leq 5$. Now, if we compare (11) and (13) we obtain

$$2i(s^{i-1}t + t^{i-1}s)xA_i(x^{i-1}, y) = 2i(s^{i-1}t + t^{i-1}s)xyA_i(x^{i-1}, 1).$$

Since $s^2 \neq t^2$ and $s, t \neq 0$ then $s^{i-1}t + t^{i-1}s = st(s^{i-2} + t^{i-2}) \neq 0$ and thus

$$A_i(x^{i-1}, y) = yA_i(x^{i-1}, 1).$$

Using Lemma 2 we get that A_i are linear, that is

$$A_i^d(x) = a_i x^i \qquad (14)$$

where a_i are arbitrary real constants for every $2 \le i \le 5$. Using (12) we also get

$$B_i^d(x) = 2(is^{i-1}t + it^{i-1}s - s^i - t^i)a_i x^i \tag{15}$$

for every $2 \le i \le 5$. Now, putting (14) in (9) and comparing the terms of the same degree we obtain

$$\sum_{j=1}^{i-2} \binom{i}{j} (s^{i-j}t^j + t^{i-j}s^j)a_i x^{i-j}y^{j+1} = \sum_{j=2}^{i-1} \binom{i}{j} (s^{i-j}t^j + t^{i-j}s^j)a_i x^{i-j+1}y^j$$

for $i = 3, 4, 5$. For $i = 3$ we have identity. For $i = 4$ we get

$$4(s^3t + t^3s)a_4x^3y^2 + 12s^2t^2a_4x^2y^3 = 12s^2t^2a_4x^3y^2 + 4(st^3 + ts^3)a_4x^2y^3,$$

thus

$$st(s^2 + t^2 - 3st)a_4x^2y^2(x - y) = 0.$$

Since x and y are arbitrary, $s, t \neq 0$ and the equality $s^2 + t^2 - 3st = 0$ is fulfilled for $s = \frac{3\pm\sqrt{5}}{2}t$, which is impossible as we assumed that $s, t \in \mathbb{Q}$, we have

$$a_4 = 0. \tag{16}$$

For $i = 5$ we get

$$\begin{aligned} &5(s^4t + t^4s)a_5x^4y^2 + 10(s^3t^2 + t^3s^2)a_5x^3y^3 + 10(s^2t^3 + t^2s^3)a_5x^2y^4 \\ &= 10(s^3t^2 + t^3s^2)a_5x^4y^2 + 10(s^2t^3 + t^2s^3)a_5x^3y^3 + 5(st^4 + s^4t)a_5x^2y^4, \end{aligned}$$

thus

$$st(s^3 + t^3 - 2s^2t - 2st^2)a_5x^2y^2(x^2 - y^2) = 0.$$

Again, since x and y are arbitrary, $s, t \neq 0$ and the equality $s^3 + t^3 - 2st(s + t) = 0$ is fulfilled for $s = -t$ or $s = \frac{3\pm\sqrt{5}}{2}t$, which is impossible as we assumed that $s^2 \neq t^2$ and $s, t \in \mathbb{Q}$, then we also have

$$a_5 = 0. \tag{17}$$

Finally from (5), (14), (16) and (17) we obtain

$$k(x) = a_3x^3 + a_2x^2 + A_1(x) + a_0,$$

from (4), (10), (15), (16) and (17) we get

$$f(x) = 2(3st(s+t) - s^3 - t^3)a_3x^3 + 2(4st - s^2 - t^2)a_2x^2 + B_1(x) + b_0$$

and from (3), (6), (10), (15), (16) and (17) we have $h(x) = g(x)$ and

$$\begin{aligned} g(x) &= 2(3st(s+t) - s^3 - t^3)a_3x^4 + 2(4st - s^2 - t^2)a_2x^3 + B_1(x)x + b_0x \\ &\quad +2(s^3 + t^3)a_3x^4 + 2(s^2 + t^2)a_2x^3 + 2(s+t)A_1(x)x + 4a_0x + b_0x + c_0 \\ &= 6st(s+t)a_3x^4 + 8sta_2x^3 + (b_1 + 2(s+t)a_1)x^2 + 2(b_0 + 2a_0)x + c_0. \end{aligned}$$

Now let us state the first of our main results.

Theorem 1. *Let $s, t \in \mathbb{Q}$ be any two nonzero parameters with $s^2 \neq t^2$. The functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the equation (1) for all $x, y \in \mathbb{R}$ if and only if $h(x) = g(x)$ and*

$$g(x) = 6st(s+t)a_3x^4 + 8sta_2x^3 + (b_1 + 2(s+t)a_1)x^2 + 2(b_0 + 2a_0)x + c_0$$

$$f(x) = 2(3st(s+t) - s^3 - t^3)a_3x^3 + 2(4st - s^2 - t^2)a_2x^2 + B_1(x) + b_0$$

$$k(x) = a_3x^3 + a_2x^2 + A_1(x) + a_0,$$

where $A_1 : \mathbb{R} \to \mathbb{R}$ is an arbitrary additive function, $B_1 : \mathbb{R} \to \mathbb{R}$ is given by (10)*, $a_1 = A_1(1)$, $b_1 = B_1(1)$ and a_3, a_2, a_0, b_0, c_0 are arbitrary real constants.*

Now we will solve functional equation (2) in the case $s^2 \neq t^2$ for rational s, t without any regularity assumptions. We will use Theorem 1 and the following lemma (cf.[8; Lemma 2]).

Lemma 3. *Let s and t be any two nonzero real parameters with $s^2 \neq t^2$. Functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ satisfy the equation*

$$(y - x)[\psi(x) + \phi(sx + ty) - \phi(tx + sy) - \psi(y)] = 0 \tag{18}$$

for all $x, y \in \mathbb{R}$ if and only if

$$\begin{aligned} \psi(x) &= (t^2 - s^2)E_2^d(x) + (t - s)E_1(x) + F_0, && (19) \\ \phi(x) &= E_2^d(x) + E_1(x) + E_0, && (20) \end{aligned}$$

where E_2^d is a diagonalization of a biadditive symmetric function $E_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $E_1 : \mathbb{R} \to \mathbb{R}$ is an additive function and E_0, F_0 are arbitrary real constants.

Proof. From equation (18) we easily get

$$\psi(x) + \phi(sx + ty) - \phi(tx + sy) - \psi(y) = 0 \tag{21}$$

for $x \neq y$, and in fact for all x and y. Hence we obtain directly from Lemma 1 that ψ is a generalized polynomial of degree at most equal to 2, i.e.

$$\psi(x) = \sum_{i=0}^{2} F_i^d(x), \tag{22}$$

where F_2^d is a diagonalization of a biadditive symmetric function $F_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $F_1 : \mathbb{R} \to \mathbb{R}$ is an additive function and F_0 is an arbitrary real constant.

Taking now $u = sx + ty$, $v = x$ we get that $x = v$, $y = \frac{1}{t}u - \frac{s}{t}v$ and

$$\psi(v) + \phi(u) - \phi\left(\frac{s}{t}u + \left(t - \frac{s^2}{t}\right)v\right) - \psi\left(\frac{1}{t}u - \frac{s}{t}v\right) = 0. \tag{23}$$

Again, using Lemma 1, we get that ϕ is a generalized polynomial of degree at most equal to 2, i.e.

$$\phi(x) = \sum_{i=0}^{2} E_i^d(x), \tag{24}$$

where E_2^d is a diagonalization of a biadditive symmetric function $E_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $E_1 : \mathbb{R} \to \mathbb{R}$ is an additive function and E_0 is an arbitrary real constant.

Inserting (22) and (24) to equation (21) we obtain

$$\begin{aligned} &F_2^d(x) + F_1(x) + E_2^d(sx + ty) + E_1(sx + ty) \\ &\quad = E_2^d(tx + sy) + E_1(tx + sy) + F_2^d(y) + F_1(y). \end{aligned}$$

Putting $y = 0$ and comparing the terms of the same degree we arrive at the following conditions

$$F_2^d(x) = (t^2 - s^2)E_2^d(x),$$

$$F_1(x) = (t-s)E_1(x),$$

thus ψ and ϕ fulfill conditions (19) and (20), respectively. □

Theorem 2. *Let $s, t \in \mathbb{Q}$ be any two nonzero parameters with $s^2 \neq t^2$. Functions $g_1, f_1, f_2, f_3, f_4, f_5 : \mathbb{R} \to \mathbb{R}$ satisfy functional equation* (2) *for all $x, y \in \mathbb{R}$ if and only if $g_1(x) = f_1(x)$ and*

$$f_1(x) = 6st(s+t)a_3x^4 + 8sta_2x^3 + (b_1 + 2(s+t)a_1)x^2 + 2(b_0 + 2a_0)x + c_0$$

$$f_2(x) = 2(3st(s+t) - s^3 - t^3)a_3x^3 + 2(4st - s^2 - t^2)a_2x^2 + B_1(x) + b_0 + (t^2 - s^2)E_2^d(x) + (t-s)E_1(x) + F_0$$

$$f_3(x) = 2a_3x^3 + 2a_2x^2 + 2A_1(x) + 2a_0 + E_2^d(x) + E_1(x) + E_0$$

$$f_4(x) = 2a_3x^3 + 2a_2x^2 + 2A_1(x) + 2a_0 - E_2^d(x) - E_1(x) - E_0$$

$$f_5(x) = 2(3st(s+t) - s^3 - t^3)a_3x^3 + 2(4st - s^2 - t^2)a_2x^2 + B_1(x) + b_0 - (t^2 - s^2)E_2^d(x) - (t-s)E_1(x) - F_0,$$

where E_2^d is a diagonalization of a biadditive symmetric function $E_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $A_1 : \mathbb{R} \to \mathbb{R}$ and $E_1 : \mathbb{R} \to \mathbb{R}$ are arbitrary additive functions, $B_1 : \mathbb{R} \to \mathbb{R}$ is given by (10), *$a_1 = A_1(1)$, $b_1 = B_1(1)$ and $a_3, a_2, a_0, b_0, c_0, E_0, F_0$ are arbitrary real constants.*

Proof. To prove Theorem 2 we repeat the method used in the proof of [8], Theorem 1.
Letting $y = x$ in (2) we see that $g_1(x) = f_1(x)$ for all $x \in \mathbb{R}$. Thus

$$f_1(y) - f_1(x) = (y-x)[f_2(x) + f_3(sx+ty) + f_4(tx+sy) + f_5(y)]. \tag{25}$$

Interchanging x and y in (25) we obtain

$$f_1(y) - f_1(x) = (y-x)[f_2(y) + f_3(sy+tx) + f_4(ty+sx) + f_5(x)]. \tag{26}$$

Adding and subtracting (25) and (26) respectively the followings are obtained

$$g(y) - g(x) = (y-x)[f(x) + 2k(sx+ty) + 2k(tx+sy) + f(y)], \tag{27}$$

$$(y-x)[\psi(x)+\phi(sx+ty)-\phi(tx+sy)-\psi(y)]=0, \tag{28}$$

where

$$\begin{cases} f(x)=f_2(x)+f_5(x) \\ k(x)=\frac{1}{2}\left[f_3(x)+f_4(x)\right], \\ g(x)=2f_1(x) \end{cases} \tag{29}$$

$$\begin{cases} \psi(x)=f_2(x)-f_5(x) \\ \phi(x)=f_3(x)-f_4(x) \end{cases} \tag{30}$$

for all $x, y \in \mathbb{R}$. Hence

$$\begin{cases} f_1(x)=\frac{1}{2}g(x) \\ f_2(x)=\frac{1}{2}\left[f(x)+\psi(x)\right] \\ f_3(x)=\frac{1}{2}\left[2k(x)+\phi(x)\right] \\ f_4(x)=\frac{1}{2}\left[2k(x)-\phi(x)\right] \\ f_5(x)=\frac{1}{2}\left[f(x)-\psi(x)\right] \end{cases} \tag{31}$$

and without loss of generality (since (2) is linear with respect to unknown functions)

$$\begin{cases} f_1(x)=g(x) \\ f_2(x)=f(x)+\psi(x) \\ f_3(x)=2k(x)+\phi(x) \\ f_4(x)=2k(x)-\phi(x) \\ f_5(x)=f(x)-\psi(x) \end{cases} \tag{32}$$

Now, using Theorem 1, Lemma 3 and (32) we obtain thesis of Theorem 2. □

Remark 2. The above results also hold true (*mutatis mutandis*) in the case where the equations are considered for mappings from a group G to a divisible group H. This is due to the fact that Lemma 2 is valid in such a general case.

Remark 3. In [8] the following problem is asked (Problem 1): *we have assumed that the functions* $f_1, f_2, f_5 : \mathbb{R} \longrightarrow \mathbb{R}$ *are twice differentiable and* $f_3, f_4 : \mathbb{R} \longrightarrow \mathbb{R}$ *are four times differentiable. The proof of* [Theorem 1 from [8]] *heavily relies on this differentiability assumption. Thus we pose the following problem: Determine the general solution of the functional* *equation* (2) *without any regularity assumptions on the unknown functions* f_1, f_2, f_3, f_4, f_5. Theorem 2 actually answers the Sahoo's question, at least for rational parameters s and t.

References

1. Kannappan, Pl., Riedel, T. and Sahoo, P. K., On a generalization of a functional equation associated with Simpson's rule. *Rocznik Nauk.-Dydakt. Prace Mat.*, **15** (1998), 85-101.

2. Pawlikowska, I., A characterization of polynomials through Flett's MVT, *Publ. Math. Debrecen* **60** (2002), 1-14.

3. Riedel, T., Sablik, M., Sklar, A., Polynomials and divided differences, *Publ. Math. Debrecen*, **66** (2005), 313-326.

4. Sablik, M., Taylor's theorem and functional equations, *Aequationes Math.*, **60** (2000), 258-267.

5. Sablik, M., Remark 10. The Fourty-fourth ISFE, May 14–20, 2006, Louisville, Kentucky, USA. *Report of Meeting, Aequationes Math.*, **73** (2007), 172-200.

6. Sahoo, P. K., On a functional equation associated with the trapezoidal rule. *Talk during the 44th ISFE, Report of Meeting, Aequationes Math.*, **73** (2007), 172-200.

7. Sahoo, P. K., On a functional equation associated with the trapezoidal rule, *Bull. Inst. Math. Acad. Sin.* (N.S.) **2** (2007), 67-82.

8. Sahoo, P. K., On a functional equation associated with the trapezoidal rule - II, *Bull. Inst. Math. Acad. Sin.* (N.S.), **4** (2009), 9-23.

9. Székelyhidi, L., *Convolution Type Functional Equations on Topological Abelian Groups*, World Scientific Publishing Co., Inc., Teaneck, NJ, 1991.