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ON STRONGLY PRIME SPECTRUM OF Γ-NEAR RINGS

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Abstract

In this paper, we prove that if N is a subcommutative Γ-near ring with a right unity and a strong left unity, then

(i) there is a one-to-one order preserving correspondence between pretopologies on N and pretopologies on the left operator near-ring L of N and

(ii) there is a one-to-one order preserving correspondence between topologies (Gabriel) on N and topologies (Gabriel) on L.

Finally, we show that if N is a distributive Γ-near ring with right unity and a strong left unity, there is a one-to-one order preserving correspondence between bases for strongly prime spectrum of N and bases for strongly prime spectrum of L.

1. Introduction

The concept of Γ-near ring, a generalization of both the concepts near-ring and Γ-ring was introduced by Satyanarayana [6]. Later, several authors such as Booth [1, 2] and Selvaraj et al. [7] studied the ideal theory of Γ-near ring. We use $SSpec(N)$ and $SSpec(L)$ to denote the strongly prime spectrum of N and the strongly prime spectrum of the left operator near-ring L of N, respectively. In this paper, we introduce pretopology and Gabriel topology on N. It is shown that if N is a distributive Γ-near ring with a right unity and strong left unity, there is a one-to-one order preserving correspondence

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between bases for strongly prime spectrum of N and bases for strongly prime spectrum of L.

2. Preliminaries

Throughout this paper N stands for a zero symmetric Γ-near ring. For basic terminology in near-rings we refer to Pilz [5] and in Γ-near rings we refer to Satyanarayana [6]. In this section we recall certain definitions needed for our purpose.

Definition 2.1. A Γ - near ring is a triple $(N, +, \Gamma)$, where

(i) $(N, +)$ is a (not necessarily abelian) group;

(ii) Γ is a non-empty set of binary operations on N such that for each $\gamma \in \Gamma$, $(N, +, \gamma)$ is a right near -ring and;

(iii) $(x\gamma y)\,\mu z = x\gamma\,(y\mu z)$ for all $x, y, z \in N$ and $\gamma, \mu \in \Gamma$.

Γ-near rings generalize near-rings in the sense that every near-ring N is a Γ-near ring with $\Gamma = \{\cdot\}$, where $\cdot$ is the multiplication defined on N.

Example 2.2. Let $(G, +)$ be a group and X a non-empty set. Let $M = \{f | f : X \to G\}$. Then M is a group under pointwise addition. If G is non-abelian, then $(M, +)$ is also non-abelian. To see this, let $a, b \in G$ such that $a + b \neq b + a$. Now define $f_a, f_b : X \to G$ by $f_a(x) = b$, $f_b(x) = a$ for every $x \in X$. Then $f_a, f_b \in M$ and $f_a + f_b \neq f_b + f_a$. Thus if G is non-abelian then N is also non-abelian.

Let Γ be the set of all mappings from G into X. If $f_1, f_2 \in M$ and $g \in \Gamma$ then obviously $f_1 g f_2 \in M$. For all $f_1, f_2, f_3 \in M$ and $g_1, g_2 \in \Gamma$, it is clear that

(i) $(f_1 g_1 f_2) g_2 f_3 = f_1 g_1 (f_2 g_2 f_3)$ and

(ii) $(f_1 + f_2) g_1 f_3 = f_1 g_1 f_3 + f_2 g_1 f_3$.

But $f_1 g_1 (f_2 + f_3)$ need not be equal to $f_1 g_1 f_2 + f_1 g_1 f_3$.
To verify this, fix $0 \neq z \in G$ and $u \in X$. Define $g_u : G \to X$ by $g_u(x) = u$ for all $x \in G$, and $f_z : X \to G$ by $f_z(x) = z$ for all $x \in X$. Now for any

two elements $f_2, f_3 \in M$, consider $f_z g_u(f_2+f_3)$ and $f_z g_u f_2 + f_z g_u f_3$. For all $x \in X$,

$$[f_z g_u(f_2 + f_3)](x) = f_z[g_u(f_2(x) + f_3(x))] = f_z(u) = z.$$

and

$$[f_z g_u f_2 + f_z g_u f_3](x) = f_z g_u f_2(x) + f_z g_u f_3(x) = f_z(u) + f_z(u) = z + z.$$

Since $z \neq 0$, we have $z \neq z + z$ and hence $f_z g_u(f_2 + f_3) \neq f_z g_u f_2 + f_z g_u f_3$. Therefore M is a $\Gamma-$near ring.

Definition 2.3. Let N be a Γ-near ring, then a normal subgroup I of $(N, +)$ is said to be

(i) left ideal if $a\alpha\,(b + i) - a\alpha b \in I \quad \forall a, b \in N, i \in I$ and $\alpha \in \Gamma$,

(ii) right ideal if $i\alpha a \in I \quad \forall i \in I, a \in N$ and $\alpha \in \Gamma$,

(iii) ideal if it is both left and right ideal of N.

If I is an ideal of N, then it is denoted by $I \lhd N$.

Definition 2.4. Let N be a Γ-near ring. Let $\mathcal{L}$ be the set of all mappings of N into itself which act on the left. Then $\mathcal{L}$ is a right near-ring with operations pointwise addition and composition of mappings. Let $x \in N$and $\alpha \in \Gamma$. We define the mapping $[x, \alpha] : N \to N$ by $[x, \alpha]\, y = x\alpha y \quad \forall y \in N$. The sub near-ring L of $\mathcal{L}$ generated by the set $\{[x, \alpha] \mid x \in N, \alpha \in \Gamma\}$ is called the left operator near-ring of N.

A right operator near-ring R of N is defined analogously to the definition of L. Let $\mathcal{R}$ be the left near-ring of all mappings of N in to itself which act on the right. If $\gamma \in \Gamma$, $y \in N$, we define $[\gamma, y] : N \to N$ by $x\,[\gamma, y] = x\gamma y$ for all $x \in N$. R is the sub near-ring of $\mathcal{R}$ generated by the set $\{[\gamma, y] \mid \gamma \in \Gamma,\, y \in N\}$.

Definition 2.5. An element x of a Γ-near ring N is called distributive if $x\alpha\,(a + b) = x\alpha a + x\alpha b$ for all $a, b \in N$ and $\alpha \in \Gamma$. If all the elements of a Γ-near ring N are distributive, then N is said to be a distributive Γ-near ring.

Definition 2.6. A Γ-near ring N is said to be zero symmetric if $a\gamma 0 = 0$ $\forall a \in N, \gamma \in \Gamma$.

Definition 2.7. Let N be a Γ-near ring with left operator near-ring L. If $\sum_i [d_i, \delta_i] \in L$ has the property that $\sum_i d_i \delta_i x = x \ \forall x \in N$, then $\sum_i [d_i, \delta_i]$ is called a left unity for N. A strong left unity for N is an element $[d, \delta]$ of L such that $d\delta x = x \ \forall x \in N$.

N is said to have a right unity if there exist $d_1, d_2, \ldots, d_n \in N$ and $\delta_1, \delta_2, \cdots, \delta_n \in \Gamma$, for all $x \in N$, $\sum_i x\delta_i d_i = x$.

Definition 2.8. An ideal I of a Γ-near ring N is called a completely prime ideal of N if for $a, b \in N$ and $\alpha \in \Gamma, a\alpha b \in I$ implies $a \in I$ or $b \in I$.

Definition 2.9. A Γ-near ring N is said to be subcommutative if $a\gamma N = N\gamma a$ for all $a \in N$ and for all $\gamma \in \Gamma$.

3. Gabriel topology for Γ-near rings

Throughout this section by a Γ-near ring N we mean a zero-symmetric Γ-near ring with left unity.

In this section, we introduce a Gabriel topology for Γ-near ring and we prove that if N is a subcommutative Γ-near ring with a right unity and a strong left unity, and if L is the left operator near ring of N, then there is a one-to-one order preserving correspondence between topologies (Gabriel) on N and topologies (Gabriel) on L.

Definition 3.1. Let I be a left ideal in a Γ-near ring N and P a left ideal in L. Then for each $x \in N$ and $\alpha \in \Gamma$, we define.

$$\begin{aligned} (I : x)_\alpha &= \{y \in N \mid y\alpha x \in I\} \\ I^{(x)} &= \{\ell \in L \mid \ell x \in I\} \\ P^{(\alpha)} &= \{y \in N \mid [y, \alpha] \in P\} \end{aligned}$$

Lemma 3.2. *Let I be a left ideal of a Γ-near ring N. Then*

(a) $(I : x)_\alpha$ *is a left ideal of* N.

(b) $I^{(x)}$ *is a left ideal in* L.

(c) $P^{(\alpha)}$ *is a left ideal in* N.

Proof.

(a) For $\ell, m \in (I : x)_\alpha$, $(\ell - m)\alpha x = \ell\alpha x - m\alpha x \in I$ since I is a left ideal of N. Therefore $\ell - m \in (I : x)_\alpha$. For $m \in (I : x)_\alpha$, $n \in N$, $(n + m - n)\alpha x{=}n\alpha x + m\alpha x - n\alpha x \in I$ since I is a left ideal of N. Therefore, $n + m - n \in (I : x)_\alpha$. Thus $(I : x)_\alpha$ is a normal subgroup of $(N, +)$. For all $a, b \in N$, $i \in (I : x)_\alpha$ and $\beta \in \Gamma$,

$$\begin{aligned}(a\beta(b+i) - a\beta b)\alpha x &= a\beta(b+i)\alpha x - a\beta b\alpha x\\ &= a\beta[(b+i)\alpha x] - a\beta b\alpha x\\ &= a\beta(b\alpha x + i\alpha x) - a\beta(b\alpha x) \in I.\end{aligned}$$

Thus $a\beta(b+i) - a\beta b \in (I : x)_\alpha$. This implies that $(I : x)_\alpha$ is a left ideal of N.

(b) For $\ell, m \in I^{(x)}, (\ell - m)x = \ell x - mx \in I$. This implies that $\ell - m \in I^{(x)}$. For $\ell \in L$ and $i \in I^{(x)}$, $(\ell + i - \ell)x = \ell x + ix - \ell x \in I$ since I is a left ideal of N. Therefore $I^{(x)}$ is a normal subgroup of $(L, +)$. For $x \in N$, $(b + i)x = bx + ix \in bx + I$ since $ix \in I$. By [1, Lemma 4], $a(b + i)x + I \subseteq abx + I$, i.e., $a(b + i)x - abx \in I$. This implies that $(a(b + i) - ab)x \in I$. Thus $a(b + i) - ab \in I^{(x)}$. Therefore $I^{(x)}$ is a left ideal of L.

(c) Let $x, y \in P^{(\alpha)}$. Then $[x - y, \alpha] = [x, \alpha] - [y, \alpha] \in P$. Therefore $x - y \in P^{(\alpha)}$. For $n \in N$ and $x \in P^{(\alpha)}$, $[n + x - n, \alpha] = [n, \alpha] + [x, \alpha] - [n, \alpha] \in P$ since P is a left ideal in L. Therefore $n + x - n \in P^{(\alpha)}$. Thus $P^{(\alpha)}$ is a normal subgroup of N.
For $a, b \in N$ and $x \in P^{(\alpha)}$,

$$\begin{aligned}[a\beta(b+x) - a\beta b, \alpha]y &= [a\beta(b+x), \alpha]y - [a\beta b, \alpha]y\\ &= a\beta(b+x)\alpha y - a\beta b\alpha y\\ &= a\beta(b\alpha y + x\alpha y) - a\beta(b\alpha y)\\ &= a\beta([b,\alpha]y + [x,\alpha]y) - a\beta[b,\alpha]y\\ &= [a,\beta]([b,\alpha] + [x,\alpha])y - [a,\beta][b,\alpha]y\\ &= ([a,\beta]([b,\alpha] + [x,\alpha]) - [a,\beta][b,\alpha])y.\end{aligned}$$

Since P is a left ideal, $[a,\beta]([b,\alpha]+[x,\alpha])-[a,\beta][b,\alpha]\in P$. This implies that $a\beta(b+x)-a\beta b\in P^{(\alpha)}$. Therefore $P^{(\alpha)}$ is a left ideal in N. □

Lemma 3.3. *Let I, J be left ideals in N and Q a left ideal in L. Then for all $x, y\in N$ and $\alpha,\beta\in\Gamma$,*

(a) $(I:x)_\alpha = N$ *for all* $\alpha\in\Gamma$ *if and only if* $x\in I$.

(b) $(I\cap J:x)_\alpha=(I:x)_\alpha\cap(J:x)_\alpha$.

(c) $((I:x)_\alpha:y)_\beta=(I:y\alpha x)_\beta$

Proof.

(a) Suppose that $(I:x)_\alpha=N$ for all $\alpha\in\Gamma$ and $x\in N$. Let $n\in N$. Then $n\in(I:x)_\alpha$. This implies that $n\alpha x\in I$. Then $n\alpha x=n\alpha(0+x)-n\alpha 0\in I$. This implies that $x\in I$ since I is a left ideal of N. Conversely, suppose that $x\in I$. $(I:x)_\alpha\subseteq N$ is obvious. Suppose $N\nsubseteq(I:x)_\alpha$. Then there exists some $n\in N$ such that $n\notin(I:x)_\alpha$. This implies that $n\alpha x\notin I$. But $n\alpha x=n\alpha(0+x)-n\alpha 0\in I$ since I is a left ideal of N and $x\in I$, a contradiction. Therefore $N=(I:x)_\alpha$.

(b) Let $n\in(I\cap J:x)_\alpha$. This implies that

$$\begin{aligned} n\alpha x\in I\cap J &\Leftrightarrow n\alpha x\in I \ \ \textit{and}\ \ n\alpha x\in J\\ &\Leftrightarrow n\in(I:x)_\alpha \ \ \textit{and}\ \ n\in(J:x)_\alpha\\ &\Leftrightarrow n\in(I:x)_\alpha\cap(J:x)_\alpha. \end{aligned}$$

(c) Let $n\in((I:x)_\alpha:y)_\beta$. This implies that

$$\begin{aligned} n\beta y\in(I:x)_\alpha &\Leftrightarrow n\beta y\alpha x\in I\\ &\Leftrightarrow n\in(I:y\alpha x)_\beta. \end{aligned}$$ □

Definition 3.4. *Let I be a left ideal in N and Q a left ideal in L. Then we define*

$$\begin{aligned} I^+ &= \{\ell\in L\,|\,\ell N\subseteq I\}\\ Q^{+'} &= \{x\in N\,|\,[x,\Gamma]\subseteq Q\}. \end{aligned}$$

It is clear that I^+ and $Q^{+'}$ are left ideals in L and N, respectively.

Lemma 3.5. *Let N be a subcommutative Γ-near ring. Let I and Q be left ideals in N and L, respectively. Then, for all $x, y, \ell \in N$ and $\alpha \in \Gamma$,*

(a) $(I^+ : [x,\alpha]) = ((I : x)_\alpha)^+$.

(b) $(Q : [x,\alpha])^{+'} = (Q^{+'} : x)_\alpha$.

(c) $(I^{(x)} : [y : \alpha]) = ((I : x)_\alpha)^{(y)}$.

(d) $(I \cap J)^{(x)} = I^{(x)} \cap J^{(x)}$.

(e) $I^{(\ell x)} = (I^{(x)} : \ell)$.

Proof.

(a)

$$\begin{aligned} \ell \in ((I : x)_\alpha)^+ &\Leftrightarrow \ell N \subseteq (I : x)_\alpha \\ &\Leftrightarrow \ell N \alpha x \subseteq I \\ &\Leftrightarrow \ell x \alpha N \subseteq I \quad [\because N \text{ is subcommutative}] \\ &\Leftrightarrow \ell[x,\alpha] \in I^+ \\ &\Leftrightarrow \ell \in (I^+ : [x,\alpha]). \end{aligned}$$

(b)

$$\begin{aligned} y \in (Q^{+'} : x)_\alpha &\Leftrightarrow y\alpha x \in Q^{+'} \\ &\Leftrightarrow [y\alpha x, \Gamma] \subseteq Q \\ &\Leftrightarrow [y,\alpha][x,\Gamma] \subseteq Q \\ &\Leftrightarrow [y,\alpha][x,\alpha] \in Q \\ &\Leftrightarrow [y,\alpha] \in (Q : [x,\alpha]) \text{ for any } \alpha \in \Gamma \\ &\Leftrightarrow [y,\Gamma] \subseteq (Q : [x,\alpha]) \\ &\Leftrightarrow y \in (Q : [x,\alpha])^{+'}. \end{aligned}$$

(c)

$$\begin{aligned} \ell \in (I^{(x)} : [y,\alpha]) &\Leftrightarrow \ell[y,\alpha] \in I^{(x)} \\ &\Leftrightarrow \ell[y,\alpha]x \in I \\ &\Leftrightarrow \ell y \alpha x \in I \\ &\Leftrightarrow \ell y \in (I : x)_\alpha \end{aligned}$$

$$\Leftrightarrow \ \ell \in ((I:x)_\alpha)^{(y)}.$$

(d)

$$\begin{aligned} \ell \in (I \cap J)^{(x)} &\Leftrightarrow \ \ell x \in I \cap J \\ &\Leftrightarrow \ \ell x \in I \text{ and } \ell x \in J \\ &\Leftrightarrow \ \ell \in I^{(x)} \text{ and } \ell \in J^{(x)} \\ &\Leftrightarrow \ \ell \in I^{(x)} \cap J^{(x)}. \end{aligned}$$

(e)

$$\begin{aligned} m \in I^{(\ell x)} &\Leftrightarrow \ m\ell x \in I \\ &\Leftrightarrow \ m\ell \in I^{(x)} \\ &\Leftrightarrow \ m \in (I^{(x)} : \ell). \end{aligned}$$

□

Lemma 3.6. *Let I be a left ideal in N and P, Q left ideals in L. Then for all $x \in N$ and $\alpha, \beta \in \Gamma$,*

(a) $(P \cap Q)^{(\alpha)} = P^{(\alpha)} \cap Q^{(\alpha)}$.

(b) $(P : [x, \alpha])^{(\beta)} = (P^{(\alpha)} : x)_\beta$.

(c) $(P^{(\alpha)})^{(x)} = (P : [x, \alpha])$.

(d) $(I^{(x)})^{(\alpha)} = (I : x)_\alpha$.

Proof.

(a)

$$\begin{aligned} x \in (P \cap Q)^{(\alpha)} &\Leftrightarrow \ [x, \alpha] \in P \cap Q \\ &\Leftrightarrow \ [x, \alpha] \in P \text{ and } [x, \alpha] \in Q \\ &\Leftrightarrow \ x \in P^{(\alpha)} \text{ and } x \in Q^{(\alpha)} \\ &\Leftrightarrow \ x \in P^{(\alpha)} \cap Q^{(\alpha)}. \end{aligned}$$

(b)

$$\begin{aligned} y \in (P : [x, \alpha])^{(\beta)} &\Leftrightarrow \ [y, \beta] \in (P : [x, \alpha]) \\ &\Leftrightarrow \ [y, \beta][x, \alpha] \in P \\ &\Leftrightarrow \ [y\beta x, \alpha] \in P \end{aligned}$$

$$\Leftrightarrow\ y \in (P^{(\alpha)} : x)_\beta.$$

(c)

$$\begin{aligned} \ell \in (P^{(\alpha)})^{(x)} &\Leftrightarrow \ell x \in P^{(\alpha)} \\ &\Leftrightarrow [\ell x, \alpha] \in P \\ &\Leftrightarrow \ell [x, \alpha] \in P \\ &\Leftrightarrow l \in (P : [x, \alpha]). \end{aligned}$$

(d)

$$\begin{aligned} y \in (I^{(x)})^{(\alpha)} &\Leftrightarrow [y, \alpha] \in I^{(x)} \\ &\Leftrightarrow [y, \alpha] x \in I \\ &\Leftrightarrow y \alpha x \in I \\ &\Leftrightarrow y \in (I : x)_\alpha. \end{aligned}$$ □

Definition 3.7. A nonempty family $\mathcal{F}(N)$ of left ideals of N is said to be a pretopology on N if

(T1) $I \in \mathcal{F}(N)$ implies $(I : x)_\alpha \in \mathcal{F}(N)$ for all $x \in N$ and $\alpha \in \Gamma$.

(T2) $I \in \mathcal{F}(N)$, $I \subseteq J$ implies $J \in \mathcal{F}(N)$ for all left ideals J of N.

(T3) $I, J \in \mathcal{F}(N)$ implies $I \cap J \in \mathcal{F}(N)$.

A pretopology on N is said to be a (Gabriel) topology on N if

(T4) $(I : x)_\alpha \in \mathcal{F}(N)$ for all $\alpha \in \Gamma$ and $x \in J$ for some $J \in \mathcal{F}(N)$ implies $I \in \mathcal{F}(N)$.

Proposition 3.8. *If $\mathcal{F}(N)$ is a topology on N and $I, J \in \mathcal{F}(N)$ then $I\Gamma J \in \mathcal{F}(N)$.*

Proof. For all $x \in J$ and $\alpha \in \Gamma$, $I\alpha x \subseteq I\Gamma J$ implies $I \subseteq (I\Gamma J : x)_\alpha$. By $(T2)$, $(I\Gamma J : x)_\alpha \in \mathcal{F}(N)$ for all $x \in J$ and $\alpha \in \Gamma$. By$(T4)$, $I\Gamma J \in \mathcal{F}(N)$. □

Lemma 3.9. *Let $\mathcal{F}(N)$ be a pretopology on N. Then $\mathcal{F}(L)$={ left ideals P of $L |\, P^{(\alpha)} \in \mathcal{F}(N)$ for all $\alpha \in \Gamma$} is a pretopology on L.*

Proof. Let $P \in \mathcal{F}(L)$. Then $P^{(\beta)} \in \mathcal{F}(N)$ for all $\beta \in \Gamma$ implies $(P^{(\beta)} : y)_\alpha = (P : [y, \beta])^{(\alpha)} \in \mathcal{F}(N)$ for all $\alpha \in \Gamma$ and $y \in N$ by Lemma 3.6(b), and so $(P : [y, \beta]) \in \mathcal{F}(L)$. If $P \in \mathcal{F}(L)$ and Q is any left ideal of L such that $P \subseteq Q$, then for all $\alpha \in \Gamma, P^{(\alpha)} \subseteq Q^{(\alpha)}$ implies $Q^{(\alpha)} \in \mathcal{F}(N)$ and so $Q \in \mathcal{F}(L)$. If $P, Q \in \mathcal{F}(L)$ then $P \cap Q \in \mathcal{F}(L)$ by Lemma 3.6(a). □

Lemma 3.10. *Let L be the left operator near-ring of N and $\mathcal{F}(L)$ a pretopology on L. Then $\mathcal{F}(N)$=$\{$left ideals I of $N | I^{(x)} \in \mathcal{F}(L)$ for all $x \in N\}$ is a pretopology on N.*

Proof. Let $I \in \mathcal{F}(N)$. Then for all $x, y \in N$ and $\alpha \in \Gamma$, $I^{(x)} \in \mathcal{F}(L)$ implies $(I^{(x)} : [y, \alpha]) = ((I : x)_\alpha)^{(y)} \in \mathcal{F}(L)$ by Lemma 3.5(c) and so $(I : x)_\alpha \in \mathcal{F}(N)$. If $I \in \mathcal{F}(N)$ and J is any left ideal of N such that $I \subseteq J$, then $I^{(x)} \subseteq J^{(x)}$ for all $x \in N$ implies $J^{(x)} \in \mathcal{F}(L)$ and so $J \in \mathcal{F}(N)$. If $I, J \in \mathcal{F}(N)$ then $I \cap J \in \mathcal{F}(N)$ by Lemma 3.5(d). Thus $\mathcal{F}(N)$ is a pretopology on N. □

Lemma 3.11. *Let N be a subcommutative Γ-near ring. If $\mathcal{F}(N)$ is a pretopology on N, then $\mathcal{F}(L) = \{I^+ | I \in \mathcal{F}(N)\}$ is a pretopology on L.*

Proof. Let $I^+ \in \mathcal{F}(L)$. We have to prove that $(I^+ : [x, \alpha]) \in \mathcal{F}(L)$ for all $x \in N$ and $\alpha \in \Gamma$. But $(I^+ : [x, \alpha]) = ((I : x)_\alpha)^+ \in \mathcal{F}(L)$ by Lemma 3.5(a), since $(I : x)_\alpha \in \mathcal{F}(N)$. Let I and J be left ideals of N and $I \subseteq J$. Since $\mathcal{F}(N)$ is a pretopology on N, $J \in \mathcal{F}(N)$. Since $I \subseteq J \Rightarrow I^+ \subseteq J^+$ and $J \in \mathcal{F}(N)$, $J^+ \in \mathcal{F}(L)$. Since $I \cap J \in \mathcal{F}(N)$, $(I \cap J)^+ \in \mathcal{F}(L)$. Also $(I \cap J)^+$=$I^+ \cap J^+$. Thus $I^+ \cap J^+ \in \mathcal{F}(L)$. Therefore $\mathcal{F}(L)$ is a pretopology on L. □

Lemma 3.12. *Let $\mathcal{F}(L)$ be a pretopology on L. Then $\mathcal{F}(N) = \{P^{+'} | P \in \mathcal{F}(L)\}$ is a pretopology on N.*

Proof. Let $I^{+'} \in \mathcal{F}(N)$. We have to prove that $(I^{+'} : x)_\alpha \in \mathcal{F}(N)$. By Lemma 3.5(b), $(I^{+'} : x)_\alpha = (I : [x, \alpha])^{+'}$. Since $I \in \mathcal{F}(L)$, $(I : [x, \alpha])^{+'} \in \mathcal{F}(N)$. Therefore $(I^{+'} : x)_\alpha \in \mathcal{F}(N)$. Let $I^{+'}$ and $J^{+'}$ be the left ideals in N and $I^{+'} \subseteq J^{+'}$. Then $I^{+'} \subseteq J^{+'} \Rightarrow I \subseteq J$. Since $I \in \mathcal{F}(L)$ and $I \subseteq J$ and $\mathcal{F}(L)$ is a pretopology on L, $J \in \mathcal{F}(L)$. This implies that $J^{+'} \in \mathcal{F}(N)$. Let $I^{+'}$ and $J^{+'} \in \mathcal{F}(N)$. This implies that $I \in \mathcal{F}(L)$ and $J \in \mathcal{F}(L)$. Since

$\mathcal{F}(L)$ is a pretopology on L, $I \cap J \in \mathcal{F}(L)$. Thus $(I \cap J)^{+'} \in \mathcal{F}(N)$. It is clear that $(I \cap J)^{+'} = I^{+'} \cap J^{+'}$. Therefore $I^{+'} \cap J^{+'} \in \mathcal{F}(N)$. Hence $\mathcal{F}(N)$ is a pretopology on N. □

Proposition 3.13. *Let N be a subcommutative $\Gamma-$near ring with a right unity and a strong left unity. Then there is a one-to-one order preserving correspondence between pretopologies on N and pretopologies on L.*

Proof. Starting with a pretopology $\mathcal{F}(N)$ on N, we get a pretopology $\mathcal{F}(L)$ on L given by $\mathcal{F}(L) = \{I^+ | I \in \mathcal{F}(N)\}$. This in turn induces a pretopology $\mathcal{F}_1(N)$ on N given by $\mathcal{F}_1(N) = \{(I^+)^{+'} | I^+ \in \mathcal{F}(L)\} = \mathcal{F}(N)$ since $(I^+)^{+'}{=}I$ by [1, Proposition 5].

On the other hand, if we start with a pretopology $\mathcal{F}(L)$ on L we get a pretopology $\mathcal{F}(N)$ on N given by $\mathcal{F}(N) = \{P^{+'} | P \in \mathcal{F}(L)\}$. This in turn induces a pretopology $\mathcal{F}_1(L)$ on L given by $\mathcal{F}_1(L){=}\{(P^{+'})^+ | P^{+'} \in \mathcal{F}(N)\}{=}\mathcal{F}(L)$, since $(P^{+'})^+ = P$ by [1, Proposition 5]. Thus this correspondence is order preserving and the proof is complete. □

Definition 3.14. *A left ideal I of N is said to be essential in N if $I \cap J \neq 0$ for all non zero left ideals J of N.*

Lemma 3.15.

(a) *If P is an essential left ideal in L then $P^{(\alpha)}$ is an essential left ideal in N for all $\alpha \in \Gamma$.*

(b) *If I is an essential left ideal in N then $I^{(x)}$ is an essential left ideal in L and $(I : x)_\alpha$ is an essential left ideal in N for all $x \in N$ and $\alpha \in \Gamma$*

Proof.

(a) Let J be a nonzero left ideal in N. Then $[J, \alpha]$ is a left ideal in L. If $[J, \alpha] = 0$, then $J \subseteq P^{(\alpha)}$. If $[J, \alpha] \neq 0$ then since P is essential, $[J, \alpha] \cap P \neq 0$. Therefore there exists $x \in J$ such that $0 \neq [x, \alpha] \in P$, i.e., $P^{(\alpha)} \cap J \neq 0$ and so $P^{(\alpha)}$ is essential.

(b) Let P be any nonzero left ideal in L. If $Px = 0$ then $P \subseteq I^{(x)}$. If $Px \neq 0$ then $Px \cap I \neq 0$ implies that there exists $0 \neq r \in P$ such that $rx \in I$, i.e., $r \in P \cap I^{(x)}$. Thus $I^{(x)}$ is essential in L. Moreover, since $(I : x)_\alpha = (I^{(x)})^{(\alpha)}$ by(a), $(I : x)_\alpha$ is essential in N. □

Lemma 3.16.

(a) *P is an essential left ideal in L if and only if $P^{(\alpha)}$ is essential in N for all $\alpha \in \Gamma$.*

(b) *I is an essential left ideal in N if and only if $I^{(x)}$ is an essential left ideal in L.*

Proof.

(a) One implication was proved in Lemma 3.15. Conversely, let $P^{(\alpha)}$ be essential in N for all $\alpha \in \Gamma$. Let Q be a nonzero left ideal in L. Since $P^{(\alpha)}$ is an essential in N, $P^{(\alpha)} \cap Q^{(\alpha)} \neq 0$. But $(P \cap Q)^{(\alpha)} = P^{(\alpha)} \cap Q^{(\alpha)} \neq 0$ by Lemma 3.6(a). This implies that there exists $y \in N$ such that $[y, \alpha] \in P \cap Q$. This implies that $P \cap Q \neq 0$. Therefore P is an essential left ideal in L.

(b) One implication was proved in Lemma 3.15. Conversely, let $I^{(x)}$ be an essential left ideal in L. Let J be a nonzero left ideal in N. Since $I^{(x)}$ is essential, $I^{(x)} \cap J^{(x)} \neq 0$. But $I^{(x)} \cap J^{(x)} = (I \cap J)^{(x)} \neq 0$, by Lemma 3.5(d). This implies that there exists $l \in L$ such that $\ell x \in I \cap J$. Therefore $I \cap J \neq 0$. Thus I is essential in N. □

Lemma 3.17.

(a) *I is an essential left ideal in N if and only if I^{+} is an essential left ideal in L.*

(b) *Q is an essential left ideal in L if and only if $Q^{+'}$ is an essential left ideal in N.*

Proof.

(a) Let I be an essential left ideal in N. Let P be a non zero left ideal in L. If $PN = 0$, then $P \subseteq I^{+}$. If $PN \neq 0$, then $PN \cap I \neq 0$ implies that there exists $0 \neq r \in P$ such that $rN \subseteq I$. That is $r \in P \cap I^{+}$. Thus I^{+} is essential in L.
Conversely, let I^{+} be an essential left ideal in L. Let J be a non zero left ideal in N. Then J^{+} is a non zero left ideal in L. Since I^{+} is essential, $I^{+} \cap J^{+} \neq 0$. Then there exists $l \in L$ such that $lx \in I \cap J$. This implies that $I \cap J \neq 0$. Thus I is an essential in N.

(b) Let Q be an essential left ideal in L. Since $Q = (Q^{+'})^{+}$ by [1, Proposition 5], $(Q^{+'})^{+}$ is an essential left ideal in L. Thus, by(a), $(Q^{+'})$ is an essential left ideal in N. The converse is similar. □

Proposition 3.18. *The family of all essential left ideals in N is a pretopology on N.*

Proof. Let $I \in \mathcal{F}(N)$, the family of all essential left ideal in N. Then by Lemma 3.15(b), $(I : x)_\alpha$ is an essential left ideal in N. Since $(T2)$ and $(T3)$ are obvious, $\mathcal{F}(N)$ is a pretopology on N. □

Theorem 3.19. *Let N be a subcommutative Γ-near ring with a right unity and a strong left unity. Let the left operator near-ring L of N be commutative. Then there is a one-to-one order preserving correspondence between topologies on N and topologies on L.*

Proof. Let $\mathcal{F}(N)$ be a topology. Suppose I is a left ideal in N such that $(I^+ : \ell) \in \mathcal{F}(L)$ for all $\ell \in J^+$ for some $J \in \mathcal{F}(N)$. Then by Lemma 3.5(a), $((I : x)_\alpha)^+ = (I^+ : [x, \alpha]) \in \mathcal{F}(L)$ for all $x \in J$ and $\alpha \in \Gamma$. This shows that $(I : x)_\alpha \in \mathcal{F}(N)$ and so $I \in \mathcal{F}(N)$ by $(T4)$ of Definition 3.7. Therefore $I^+ \in \mathcal{F}(L)$ and hence $\mathcal{F}(L)$ is a topology on L.

Conversely, let $\mathcal{F}(L)$ be a topology on L and $\mathcal{F}(N)$ the corresponding pretopology on N. We have to prove that $\mathcal{F}(N)$ satisfies $(T4)$. Let Q be a left ideal in L such that $(Q^{+'} : x)_\alpha \in \mathcal{F}(N)$ for all $\alpha \in \Gamma$ and $x \in P^{+'}$ for some $P \in \mathcal{F}(L)$. Then for all $\ell \in P$, $y \in L$ and $\alpha \in \Gamma$,

$$\begin{aligned} (Q : \ell[y, \alpha])^{+'} &= (Q : [\ell y, \alpha])^{+'} \\ &= (Q^{+'} : \ell y)_\alpha \in \mathcal{F}(N) \qquad \text{by Lemma 3.5(b)} \end{aligned}$$

since $\ell y \in P^{+'}$.

Since L is commutative, $(Q : \ell[y, \alpha]) = (Q : [y, \alpha]\ell)$ and $[y, \alpha]\ell{=}[y, \alpha][0+\ell] - [y, \alpha]0 \in P$ since P is left ideal. Therefore $(Q : \ell[y, \alpha]){=}(Q : \ell_1) \in \mathcal{F}(L)$ for all $\ell_1 \in P$. Hence $Q \in \mathcal{F}(L)$ by $(T4)$ of Definition 3.7. Therefore $Q^{+'} \in \mathcal{F}(N)$ by Lemma 3.17(b). Thus $\mathcal{F}(N)$ is a topology on N. □

Proposition 3.20. *Let N be a subcommutative Γ-near ring. Let $\sum_{j=1}^{n}[y_j,\beta_j]$ be left unity in N. Then a non empty family $\mathcal{F}(N)$ of left ideals in N is a topology on N if and only if*

(i) *$I \in \mathcal{F}(N)$ implies $(I:x)_{\beta_j} \in \mathcal{F}(N)$ for all $x \in N$ and $j = 1,2,\ldots,n$.*

(ii) *If $(I:x)_{\beta_j} \in \mathcal{F}(N)$ for all $x \in J$ for some $J \in \mathcal{F}(N)$, $j = 1,2,\ldots,n$, then $I \in \mathcal{F}(N)$.*

Proof. If $\mathcal{F}(N)$ is a topology, then clearly (i) is satisfied. Let $\mathcal{F}(N)$ be a topology on N. If I is a left ideal in N such that $(I:x)_{\beta_j} \in \mathcal{F}(N)$ for all $x \in J, J \in \mathcal{F}(N)$, $j = 1,2,\ldots,n$, then for all $\ell \in J^+$, $(I^+ : [\ell y_j,\beta_j])=((I:\ell y_j)_{\beta_j})^+ \in \mathcal{F}(L)$ by Lemma 3.5(a) and since $\ell y_j \in J$. Since $\ell = \sum_{j=1}^{n}[\ell y_j,\beta_j]$, $(I^+ : \ell) \in \mathcal{F}(L)$. This shows that $I^+ \in \mathcal{F}(L)$ since $\mathcal{F}(L)$ is a topology on L and so $I \in \mathcal{F}(N)$. This proves (ii).

Conversely, if $\mathcal{F}(N)$ satisfies (i) and (ii), then (T4) of Definition 3.7 is automatically satisfied. Also if $I \in \mathcal{F}(N)$ then for all $x,y \in N$ and $\beta \in \Gamma$,

$$((I:x)_\beta : y)_{\beta_j} = (I:y\beta x)_{\beta_j} \in \mathcal{F}(N) \text{ for } j = 1,2,\ldots,n$$

and so $(I:x)_\beta \in \mathcal{F}(N)$ by (ii). $\square$

4. Topology on the Set of All Strongly Prime Ideals in Γ-near Rings

We use $SSpec(N)$ and $SSpec(L)$ to denote the set of strongly prime ideals of N and the set of strongly prime ideals of the left operator near-ring L, respectively. In this section, we prove that there is a one-to-one order preserving correspondence between bases on $SSpec(N)$ and bases on $SSpec(L)$.

Definition 4.1. Let N be a Γ-near ring. An ideal $P \neq N$ is said to be strongly prime if for any $x \notin P$, there exist finite subsets $F \subseteq N$ and $\Delta \subseteq \Gamma$ such that for any $y \in N$, $x\alpha f\beta y \in P$ for all $\alpha,\beta \in \Delta$ and $f \in N$ implies $y \in P$.

Definition 4.2. Let N be a Γ-near ring. A basis for a topology on $SSpec(N)$ is a collection $\mathcal{B}(N)$ of subsets of $SSpec(N)$ such that

1. For each $P \in SSpec(N)$, there exists an element $B \in \mathcal{B}(N)$ containing P.
2. If $P \in B_1 \cap B_2$, where $B_1, B_2 \in \mathcal{B}(N)$, then there is a basis element B_3 containing P such that $B_3 \subseteq B_1 \cap B_2$.

Proposition 4.3. *If $P \in SSpec(N)$, then $P^+ \in SSpec(L)$, where $P^+ = \{\ell \in L | \, \ell N \subseteq P\}$.*

Proof. Let $\ell \notin P^+$. Then $\ell x \notin P$ for some $x \in N$. Since $P \in SSpec(N)$, there exists finite subsets $F = \{f_j | j = 1, 2, \ldots, m\} \subseteq N$ and $\Delta = \{\alpha_i | i = 1, 2, \ldots, n\}$ such that for any $x \in N$

$$\ell x \alpha_i f_j \alpha_k y \in P \text{ for all } \alpha_i, \alpha_k \in \Delta, f_j \in F \text{ implies } y \in P. \tag{1}$$

Let $G = \{[x\alpha_i f_j, \alpha_k] | \, 1 \leq i, k \leq n, 1 \leq j \leq m\}$ and $\ell' \in L$ such that $\ell G \ell' \subseteq P^+$, i.e., $\ell[x\alpha_i f_j, \alpha_k]\ell' \in P^+$ and so that $\ell[x\alpha_i f_j, \alpha_k]\ell' N \subseteq P$. Hence $\ell x \alpha_i f_j \alpha_k \ell' N \subseteq P$ for all $\alpha_i, \alpha_k \in \Delta, f_j \in F$. By(1), $\ell' N \subseteq P$. Therefore $\ell' \in P^+$. Thus $P^+ \in SSpec(L)$. □

Note that the elements of the right operator near-ring R are expressible in the form $\sum_i [\alpha_i, x_i]$, where $x_i \in M, \alpha_i \in \Gamma$ see [1, p.472]. But the left operator near-ring L does not, in general, consist exclusively of elements of the form $\sum_i [x_i, \alpha_i]$, where $x_i \in M, \alpha_i \in \Gamma$. If a $\Gamma-$near ring N is distributive, then the elements of L are expressible in the form $\sum_i [x_i, \alpha_i]$.

Proposition 4.4. *Let N be a distributive $\Gamma-$near ring. If $Q \in SSpec(L)$, then $Q^{+'} \in SSpec(N)$, where $Q^{+'} = \{x \in N | \, [x, \Gamma] \subseteq Q\}$.*

Proof. Let $x \in Q^{+'}$. Then there exists $\alpha \in \Gamma$ such that $[x, \alpha] \notin Q$. Since $Q \in SSpec(L)$, there exists

$$G = \left\{ \sum_{i=1}^{m} [y_{ik}, \beta_{ik}] | k = 1, 2, \ldots, n \right\} \subseteq L \text{ such that for any } \ell \in L,$$
$$[x, \alpha] G \ell \subseteq G \text{ implies } \ell \in Q. \tag{2}$$

Let $F = \{y_{i_k} | \, i = 1, 2, \ldots, m; k = 1, 2, \ldots, n\}$ and $\Delta = \{\beta_{i_k}, \alpha | i = 1, 2, \ldots, m;$ $k = 1, 2, \ldots, n\}$. Let $z \in M$ such that $x \Delta F \Delta z \subseteq Q^{+'}$. Then $x \alpha y_{i_k} \beta_{i_k} z \in$

$Q^{+'}$ for all $i = 1, 2, \ldots, m$; $k = 1, 2, \ldots, n$. Hence $[x\alpha y_{i_k}\beta_{i_k}z, \beta] \in Q$ for all $\beta \in \Gamma$, i.e., $[x, \alpha][y_{i_k}, \beta_{i_k}][z, \beta] \in Q$ for all $\beta \in \Gamma$. Thus $[x, \alpha]\sum_{i=1}^{m}[y_{i_k}, \beta_{i_k}][z, \beta] \in Q$ for all $k = 1, 2, \ldots, n$ and for all $\beta \in \Gamma$. By (2), $[z, \beta] \in Q$ for all $\beta \in \Gamma$. Therefore $z \in Q^{+'}$ and consequently $Q^{+'} \in SSpec(N)$. □

Definition 4.5. For any subset $A \subseteq N$. We define $\mathcal{B}_A = \{P \in SSpec(N) | A \nsubseteq P\}$. In case $A = \{x\}$, we write $\mathcal{B}_x = \{P \in SSpec(N) | x \notin P\}$. For any subset $U \subseteq L$, we define $\mathcal{B}_U = \{I \in SSpec(L) | U \nsubseteq I\}$.

Lemma 4.6. *For any $\Gamma-$near ring N, $\mathcal{B}(N) = \{\mathcal{B}_x | x \in N\}$ forms a basis for a topology on $SSpec(N)$.*

Proof. For any $P \in SSpec(N)$, there exists $x \in N$ such that $x \notin P$, because $P \neq N$. From the definition of $\mathcal{B}_x$, $P \in \mathcal{B}_x$. If $P \in \mathcal{B}_y \cap \mathcal{B}_z$ for some $y, z \in N$, then $y \notin P$ and $z \notin P$. Since P is strongly prime, there exist finite subsets $F \subseteq N$ and $\Delta \subseteq \Gamma$ such that $y\alpha f\beta z \notin P$ for some $\alpha, \beta \in \Delta$ and $f \in F$. Hence $P \in \mathcal{B}_{y\alpha f\beta z}$. We claim that $\mathcal{B}_{y\alpha f\beta z} \subseteq \mathcal{B}_y \cap \mathcal{B}_z$. Let $Q \in \mathcal{B}_{y\alpha f\beta z}$. Then $y\alpha f\beta z \notin Q$, suppose $y \in Q$ or $z \in Q$, we have $y\alpha f\beta z \in Q$, a contradiction. Therefore $y \notin Q$ and $z \notin Q$ and consequently, $Q \in \mathcal{B}_y \cap \mathcal{B}_z$. □

Lemma 4.7. *Let N be a distributive $\Gamma-$near ring. Then $\mathcal{B}(L) = \{\mathcal{B}_{[x,\Gamma]} | x \in N\}$ forms a basis for a topology on $SSpec(L)$.*

Proof. Let $P \in SSpec(L)$. Then $P^{+'} \in SSpec(N)$ by Proposition 4.4. Since $\mathcal{B}(N)$ is a basis on $SSpec(N)$, there exists $\mathcal{B}_x \in \mathcal{B}(N)$ such that $P^{+'} \in \mathcal{B}_x$. Hence $x \notin P^{+'}$, that is $[x, \Gamma] \nsubseteq P$ and so that $P \in \mathcal{B}_{[x,\Gamma]}$. Let $Q \in \mathcal{B}_{[y,\Gamma]} \cap \mathcal{B}_{[z,\Gamma]}$ for some $y, z \in N$. Then $[y, \Gamma] \nsubseteq Q$ and $[z, \Gamma] \nsubseteq Q$. It means that $y \notin Q^{+'}$ and $z \notin Q^{+'}$. Hence $Q^{+'} \in \mathcal{B}_y \cap \mathcal{B}_z$. Since $\mathcal{B}(N)$ is a basis, there is an element $\mathcal{B}_{z'} \in \mathcal{B}(N)$ such that $Q^{+'} \in \mathcal{B}_{z'} \subseteq \mathcal{B}_y \cap \mathcal{B}_z$. It can be easily verified that $Q \in \mathcal{B}_{[z',\Gamma]} \subseteq \mathcal{B}_{[y,\Gamma]} \cap \mathcal{B}_{[z,\Gamma]}$. Thus $\mathcal{B}(L)$ forms a basis for a topology on $SSpec(L)$. □

Note that a distributive $\Gamma-$near ring N with a right unity and a strong left unity is not a Γ-ring because in a Γ-near ring, Γ is a non-empty set of binary operations on N.

Theorem 4.8. *Let N be a distributive $\Gamma-$near ring with a right unity and a strong left unity. Then there is a one-to-one order preserving correspondence between the following:*

(i) *base for $SSpec(N)$;*

(ii) *base for $SSpec(L)$.*

Proof. Since $(P^{+'})^{+} = P$ by [1, Proposition 5], the mapping $\mathcal{B}_x \mapsto \mathcal{B}_{[x,\Gamma]}$ defines a one-to-one correspondence order preserving between $\mathcal{B}(N)$ and $\mathcal{B}(L)$. □

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References

1. G. L. Booth, A note on Gamma near-rings, *Stud. Sci. Math. Hungarica*, **23** (1988), 471-475.

2. G. L. Booth, Radicals of Γ− near-rings, *Publ. Math. Debrecen*, **39** (1990), 223-230.

3. N. J. Groenewald, Strongly prime near-rings, *Proc. Edinburgh. Math. Soc.*, **31** (1988), 337-343.

4. N. J. Groenewald, Strongly prime near-rings 2, *Comm. in Algebra*, **17**(1989), no.3, 735-749.

5. G. Pilz, *Near-rings* , North Holland Publicaion Company, Amsterdam, 1983.

6. Satyanarayna Bhavanari, *Contributions to near-rings*, Doctoral Thesis, Nagarjuna University, 1984.

7. C. Selvaraj, R. George and G. L. Booth, On strongly equiprime Γ-near rings, *Bull. Institute of Math. Academia Sinica*, **4**(2009), no. 1, 35-46.