Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **6** (2011), No. 3, pp. 321-328

CERTAIN PROPERTIES OF A SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS

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Abstract

The main object of the present paper is to derive several further interesting properties of the subclass $C_\lambda(p,\alpha)$ which was recently introduced and studied by Aouf et al.[Comput. Math. Applic. 39(2000)39-48].

1. Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad (p \in N = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z\colon z \in C \text{ and } |z| < 1\}$.

For functions $f \in A(p)$, given by (1.1), and $g \in A(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k},$$

Received MArch 29, 2010 and in revised form July 20, 2010.

AMS Subject Classification: 30C45, 26A33.

Key words and phrases: Analytic function, Hadamard product (or convolution), neighborhood, partial sums.

we define the Hadamard product (or convolution) of f and g by

$$(f*g)(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k}. \tag{1.2}$$

A function $f \in A(p)$ is said to be in the class $K(p,\alpha)$ of p -valently convex functions of order α if it satisfies the condition

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in U; 0 \leq \alpha < p). \tag{1.3}$$

Next we denote by $T(p)$ the subclass of the class $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \quad (a_{p+k} \geq 0) \tag{1.4}$$

and define further class $C(p,\alpha)$ by

$$C(p,\alpha) = K(p,\alpha) \bigcap T(p). \tag{1.5}$$

For the class $C(p,\alpha)$, the following characterization was given by Owa [8].

Lemma 1 (see [8]). Let the function f be defined by (1.4). Then f is in the class $C(p,\alpha)$ if and only if

$$\sum_{k=1}^{\infty} (p+k)(p+k-\alpha) a_{p+k} \leq p(p-\alpha). \tag{1.6}$$

The result is sharp.

For a function f defined by (1.4) and in the class $C(p,\alpha)$, Lemma 1 immediately yields

$$a_{p+1} \leq \frac{p(p-\alpha)}{(p+1)(p+1-\alpha)}. \tag{1.7}$$

In view of the coefficient inequalities (1.7), Aouf et al. [3] recently introduced and studied a subclass $C_\lambda(p,\alpha)$ of $C(p,\alpha)$ consisting of functions of the form

$$f(z) = z^p - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z^{p+1} - \sum_{k=2}^{\infty} a_{p+k} z^{p+k}, \qquad (1.8)$$
$$(a_{p+k} \geq 0; 0 \leq \alpha < p; 0 \leq \lambda < 1).$$

In particular, the class $C_\lambda(\alpha) = C_\lambda(1, \alpha)$ was considered earlier by Silverman and Silvia [10].

Many interesting properties such as integral inequalities, coefficient estimates, closure theorems, radius of convexity for the class $C_\lambda(p, \alpha)$ were given by Aouf et al. [3]. In the present sequel to these earlier works, we shall derive several interesting properties and characteristics of the δ-neighborhood and partial sums associated with the class $C_\lambda(p, \alpha)$.

2. Main Results

We begin by recalling the following lemma which will be required in our present investigation.

Lemma 2 (see [3]). Let the function f be defined by (1.8). Then f is in the class $C_\lambda(p, \alpha)$ if and only if

$$\sum_{k=2}^{\infty} (p+k)(p+k-\alpha) a_{p+k} \leq p(p-\alpha)(1-\lambda) \quad (0 \leq \alpha < p; 0 \leq \lambda < 1). \quad (2.1)$$

The result is sharp for the function f given by

$$f(z) = z^p - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z^{p+1} - \frac{p(p-\alpha)(1-\lambda)}{(p+k)(p+k-\alpha)} z^{p+k}, \quad (k \geq 2). \qquad (2.2)$$

Following the earlier works ([4, 9]; see also [1, 2, 5, 6, 7, 11]), we now define the δ-neighborhood of a function $f \in A(p)$ of the form (1.8) by

$$N_\delta(f) = \Bigg\{ g \in A(p) \colon g(z) = z^p - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z^{p+1} - \sum_{k=2}^{\infty} b_{p+k} z^{p+k}$$
$$(b_{p+k} \geq 0)$$

and

$$\left.\sum_{k=2}^{\infty}\frac{(p+k)(p+k-\alpha)}{p(p-\alpha)(1-\lambda)}|a_{p+k}-b_{p+k}|\leq\delta,(0\leq\alpha<p;0\leq\lambda<1;\delta>0)\right\}. \tag{2.3}$$

Theorem 1. *Let* $f\in C_\lambda(p,\alpha)$ *be given by* (1.8). *If* f *satisfies the inclusion condition*

$$(f(z)+\varepsilon z^p)(1+\varepsilon)^{-1}\in C_\lambda(p,\alpha),\quad (\varepsilon\in C;|\varepsilon|<\delta;\delta>0), \tag{2.4}$$

then

$$N_\delta(f)\subset C_\lambda(p,\alpha). \tag{2.5}$$

Proof. It is not difficult to see that a function f belongs to $C_\lambda(p,\alpha)$ if and only if

$$\frac{zf''(z)-(p-1)f'(z)}{zf''(z)+(p+1-2\alpha)f'(z)}\neq\sigma,\quad (z\in U;\sigma\in C,|\sigma|=1), \tag{2.6}$$

which is equivalent to

$$(f*h)(z)/z^p\neq 0,\quad (z\in U), \tag{2.7}$$

where, for convenience,

$$\begin{aligned} h(z) &:= z^p+\sum_{k=1}^{\infty}c_{p+k}z^{p+k} \\ &= z^p+\sum_{k=1}^{\infty}\frac{(p+k)[(2p+k-2\alpha)\sigma-k]}{2p\sigma(p-\alpha)}z^{p+k}. \end{aligned} \tag{2.8}$$

We easily find from (2.8) that

$$\begin{aligned} |c_{p+k}| &= \left|\frac{(p+k)[(2p+k-2\alpha)\sigma-k]}{2p\sigma(p-\alpha)}\right|\leq\frac{(p+k)(p+k-\alpha)}{p(p-\alpha)} \\ &\leq \frac{(p+k)(p+k-\alpha)}{p(p-\alpha)(1-\lambda)},\quad (k=1,2,\ldots;0\leq\lambda<1). \end{aligned} \tag{2.9}$$

Furthermore, under the hypotheses of the theorem, (2.7) yields the following inequality:

$$\left|\frac{(f*h)(z)}{z^p}\right| \geq \delta, \quad (z \in U; \delta > 0). \tag{2.10}$$

Now, if we let

$$\varphi(z) = z^p - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z^{p+1} - \sum_{k=2}^{\infty} b_{p+k} z^{p+k} \in N_\delta(f),$$

then

$$\begin{aligned}\left|\frac{(f(z)-\varphi(z))*h(z)}{z^p}\right| &= \left|\sum_{k=2}^{\infty}(a_{p+k}-b_{p+k})c_{p+k}z^k\right| \\ &\leq \sum_{k=2}^{\infty} \frac{(p+k)(p+k-\alpha)}{p(p-\alpha)(1-\lambda)}|a_{p+k}-b_{p+k}|\cdot|z|^k \\ &< \delta, \quad (z \in U; 0 \leq \lambda < 1; \delta > 0).\end{aligned}$$

Thus, for any complex number σ such that $|\sigma| = 1$,we have

$$(\varphi * h)(z)/z^p \neq 0, \quad (z \in U),$$

which implies that $\varphi \in C_\lambda(p,\alpha)$. The proof is complete. □

Next, we prove

Theorem 2. *Let $f \in A(p)$ be given by (1.8) and define the partial sums $s_n(z)$ by*

$$s_n(z) = \begin{cases} z^p - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z^{p+1}, & n = 1; \\ z^p - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z^{p+1} - \sum_{k=2}^{n} a_{p+k} z^{p+k}, & n = 2, 3, \ldots. \end{cases} \tag{2.11}$$

Suppose also that

$$\sum_{k=2}^{\infty} l_{p+k} a_{p+k} \leq 1 - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)}$$

$$(\textit{where } l_{p+k} = \tfrac{(p+k)(p+k-\alpha)}{p(p-\alpha)(1-\lambda)}; 0 \leq \alpha < p; \quad 0 \leq \lambda < 1). \tag{2.12}$$

Then for $n \geq 2$, we have

$$Re\left(\frac{f(z)}{s_n(z)}\right) > 1 - \frac{1}{l_{p+n+1}} \tag{2.13}$$

and

$$Re\left(\frac{s_n(z)}{f(z)}\right) > \frac{l_{p+n+1}}{1 + l_{p+n+1}}. \tag{2.14}$$

Each of the bounds in (2.13) *and* (2.14) *is the best possible.*

Proof. From (2.12) and Lemma 2, we have immediately that $f \in C_\lambda(p, \alpha)$.Under the hypothesis of the theorem, we can see from (2.12) that

$$l_{p+k+1} > l_{p+k} > 1, \quad (k = 2, 3, \ldots).$$

Therefore, we have

$$\sum_{k=2}^{n} a_{p+k} + l_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} \leq \sum_{k=2}^{\infty} l_{p+k} a_{p+k} \leq 1 - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} \tag{2.15}$$

by using (2.12) again.

Set

$$\begin{aligned} h_1(z) &= l_{p+n+1}\left[\frac{f(z)}{s_n(z)} - (1 - \frac{1}{l_{p+n+1}})\right] \\ &= 1 - \frac{l_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} z^k}{1 - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z - \sum_{k=2}^{n} a_{p+k} z^k}. \end{aligned} \tag{2.16}$$

By applying (2.15) and (2.16), we find that

$$\begin{aligned} \left|\frac{h_1(z) - 1}{h_1(z) + 1}\right| &= \left|\frac{-l_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 - \frac{2p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z - 2\sum_{k=2}^{n} a_{p+k} z^k - l_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k} z^k}\right| \\ &\leq \frac{l_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k}}{2 - \frac{2p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} - 2\sum_{k=2}^{n} a_{p+k} - l_{p+n+1} \sum_{k=n+1}^{\infty} a_{p+k}} \\ &\leq 1, \quad (z \in U; n \geq 2), \end{aligned} \tag{2.17}$$

which shows that $Reh_1(z) > 0$ $(z \in U)$. From (2.16), we immediately obtain the inequality (2.13).

If we take

$$f(z) = z^p - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z^{p+1} - \frac{1 - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)}}{l_{p+n+1}} z^{p+n+1}, \quad (n \geq 2), \tag{2.18}$$

then $f(z)$ satisfies the condition (2.12) and $f \in C_\lambda(p, \alpha)$. Thus

$$\begin{aligned} \frac{f(z)}{s_n(z)} &= 1 - \left(\frac{1 - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)}}{l_{p+n+1}} \right) z^{n+1} \Big/ \left(1 - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z \right) \\ &\to 1 - \frac{1}{l_{p+n+1}}, (\text{ as } z \to 1^-), \end{aligned} \tag{2.19}$$

which shows that the bound in (2.13) is the best possible.

Similarly, if we put

$$\begin{aligned} h_2(z) &= (1 + l_{p+n+1}) \left(\frac{s_n(z)}{f(z)} - \frac{l_{p+n+1}}{1 + l_{p+n+1}} \right) \\ &= 1 + \frac{(1 + l_{p+n+1}) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{1 - \frac{p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z - \sum_{k=2}^{\infty} a_{p+k} z^k} \end{aligned} \tag{2.20}$$

and make use of (2.15), we can deduce that

$$\begin{aligned} &\left| \frac{h_2(z) - 1}{h_2(z) + 1} \right| \\ &= \left| \frac{(1 + l_{p+n+1}) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 - \frac{2p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z - 2 \sum_{k=2}^{n} a_{p+k} z^k + (1 + l_{p+n+1}) \sum_{k=n+1}^{\infty} a_{p+k} z^k} \right| \\ &= \left| \frac{(1 + l_{p+n+1}) \sum_{k=n+1}^{\infty} a_{p+k} z^k}{2 - \frac{2p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} z - 2 \sum_{k=2}^{n} a_{p+k} z^k + (l_{p+n+1} - 1) \sum_{k=n+1}^{\infty} a_{p+k} z^k} \right| \\ &\leq \frac{(1 + l_{p+n+1}) \sum_{k=n+1}^{\infty} a_{p+k}}{2 - \frac{2p(p-\alpha)\lambda}{(p+1)(p+1-\alpha)} - 2 \sum_{k=2}^{n} a_{p+k} - (l_{p+n+1} - 1) \sum_{k=n+1}^{\infty} a_{p+k}} \\ &\leq 1, \quad (z \in U), \end{aligned} \tag{2.21}$$

which leads us immediately to assertion (2.14) of the theorem.

The bound in (2.14) is sharp with the extremal function given by (2.18). The proof of the theorem is thus completed. □

Acknowledgment

The author would like to express sincere thanks to the referee for careful reading and suggestions which helped me to improve the paper.

References

1. O. Altintas and S. Owa, Neighborhoods of certain analytic functions with negative coefficients, *Internat. J. Math. Math. Sci.*, **19**(1996), 797-800.
2. O. Altintas, O. Ozkan and H.M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, *Appl. Math. Lett.*, **13**(2000), No.3, 63-67.
3. M. K. Aouf, H.M. Hossen and H.M. Srivastava, Some families of multivalent functions, *Comput. Math. Applic.*, **39**(2000), 39-48.
4. A. W. Goodman, Univalent functions and nonanalytic curves, *Proc. Amer. Math. Soc.*, **8**(1957), 598-601.
5. J.-L. Liu, Further properties of a certain subclass of analytic and multivalent functions, *Appl. Math. Comput.*, **187**(2007), 290-294.
6. J.-L. Liu and H.M. Srivastava, A linear operator and associated families of meromorphically multivalent functions, *J. Math. Anal. Appl.*, **259**(2001), 566-581.
7. J.-L. Liu and H.M. Srivastava, Classes of meromorphically multivalent functions associated with the generalized hypergeometric function, *Math. Comput. Modelling*, **39**(2004), 21-34.
8. S. Owa, On certain classes of p-valent functions with negative coefficients, *Simon Stevin* **59**(1985), 385-402.
9. S. Ruscheweyh, Neighborhoods of univalent functions, *Proc. Amer. Math. Soc.*, **81**(1981), 521-527.
10. H. Silverman and E.M. Silvia, Fixed coefficients for subclasses of starlike functions, *Houston J. Math.* **7**(1981), 129-136.
11. H. M. Srivastava and S. Owa (Editors), *Current Topics in Analytic Function Theory*, World Scientific, Singapore, 1992.