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GENERALIZED SKEW DERIVATIONS WITH ENGEL CONDITIONS ON LIE IDEALS

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Abstract

Let R be a prime ring and L a noncommutative Lie ideal of R. Suppose that f is a nonzero right generalized β-derivation of R associated with a β-derivation δ such that $[f(x), x]_k = 0$ for all $x \in L$, where k is a fixed positive integer. Then either there exists $s \in C$ scuh that $f(x) = sx$ for all $x \in R$ or $R \subseteq M_2(F)$ for some field F. Moreover, if the latter case holds, then either char$R = 2$ or char$R \neq 2$ and $f(x) = bx - xc$ for all $x \in R$, where $b, c \in \mathscr{F}R$ and $b + c \in C$.

Recently, M. C. Chou and C. K. Liu [5] proved that if δ is a nonzero σ-derivation of R and L is a noncommutative Lie ideal of R such that $[\delta(x), x]_k = 0$ for all $x \in L$, where k is a fixed positive integer, then char$R = 2$ and $R \subseteq M_2(F)$ for some field F. This result generalizes some known results, see for instances, [15] and [20]. In this paper we extend [5] further to the so-called right generalized skew derivations. Notice that our result also generalizes the case of generalized derivations by N. Argac, L. Carini and V. De Fillipis [1].

Throughout this paper, R is always a prime ring with center Z. For $x, y \in R$, set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_k = [[x, y]_{k-1}, y]$ for $k > 1$. Notice that an Engel condition is a polynomial $[x, y]_k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}y^i x y^{k-i}$ in noncommutative indeterminantes x and y. For two subsets A and B of R, $[A, B]$ is defined to be the additive subgroup of R generated by all elements

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$[a,b]$ with $a \in A$ and $b \in B$. An additive subgroup L of R is said to be a Lie ideal if $[l,r] \in L$ for all $l \in L$ and $r \in R$. A Lie ideal L is said to be noncommutative if $[L,L] \neq 0$.

Let β be an automorphism of R. A β-derivation of R is an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + \beta(x)\delta(y)$ for all $x, y \in R$. β-derivations are also called skew derivations. When $\beta = 1$, the identity map of R, β-derivations are merely ordinary derivations. If $\beta \neq 1$, then $1 - \beta$ is a β-derivation. An additive mapping $f : R \to R$ is a right generalized β-derivation if there exists a β-derivation $\delta : R \to R$ such that $f(xy) = f(x)y + \beta(x)\delta(y)$ for all $x, y \in R$. The right generalized β-derivations generalize both β-derivations and generalized derivations. If $a, b \in R$ and $\beta \neq 1$ is an automorphism of R, then $f(x) = ax - \beta(x)b$ is a right generalized β-derivation. Moreover, if δ is a β-derivation of R, then $f(x) = ax + \delta(x)$ is a right generalized β-derivation.

We let $_{\mathscr{F}}R$ denote the right Martindale quotient ring of R and Q the two sided Martindale quotient ring of R. Let C be the center of Q and $_{\mathscr{F}}R$, which is called the extended centroid of R. Note that Q and $_{\mathscr{F}}R$ are also prime rings and C is a field (see [2]). It is known that automorphisms, derivations and β-derivations of R can be uniquely extended to Q and $_{\mathscr{F}}R$. In [4], we know that right generalized β-derivations of R can also be uniquely extended to $_{\mathscr{F}}R$. Indeed, if f is a right generalized β-derivation of R, then there exists $s \in {_{\mathscr{F}}R}$ such that $f(x) = sx + \delta(x)$ for all $x \in R$, where δ is a β-derivation of R (Lemma 2 in [4]).

A β-derivation δ of R is called X-inner if $\delta(x) = bx - \beta(x)b$ for some $b \in Q$. δ is called X-outer if it is not X-inner. An automorphism β is called X-inner if $\beta(x) = uxu^{-1}$ for some invertible $u \in Q$. β is called X-outer if it is not X-inner.

We are now ready to state the main result:

Main Theorem. *Let R be a prime ring and L a noncommutative Lie ideal of R. Suppose that f is a nonzero right generalized β-derivation of R associated with a β-derivation δ such that $[f(x), x]_k = 0$ for all $x \in L$, where k is a fixed positive integer. Then either there exists $s \in C$ such that $f(x) = sx$ for all $x \in R$ or $R \subseteq M_2(F)$ for some field F. Moreover, if the latter case holds, then either* $\mathrm{char}R = 2$ *or* $\mathrm{char}R \neq 2$ *and $f(x) = bx - xc$ for all $x \in R$, where $b, c \in {_{\mathscr{F}}R}$ and $b + c \in C$.*

As a corollary, we have

Corollary 1. *Let R be a prime ring and L a noncommutative Lie ideal of R. Suppose that $\beta \neq 1_R$ and f is a nonzero right generalized β-derivation of R associated with a β-derivation δ such that $[f(x), x]_k = 0$ for all $x \in L$, where k is a fixed positive integer. Then there exists $s \in C$ such that $f(x) = sx$ for all $x \in R$ unless* $\mathrm{char} R = 2$ *and $R \subseteq M_2(F)$ for some field F.*

Corollary 2. *Let R be a prime ring and L a noncommutative Lie ideal of R. Suppose that f is a nonzero generalized derivation of R associated with derivation d such that $[f(x), x]_k = 0$ for all $x \in L$, where k is a fixed positive integer. Then either there exists $s \in C$ such that $f(x) = sx$ for all $x \in R$ or $R \subseteq M_2(F)$ for some field F. Moreover, if the latter case holds, then either* $\mathrm{char} R = 2$ *or* $\mathrm{char} R \neq 2$ *and $f(x) = bx - xc$ for all $x \in R$, where $b, c \in \mathscr{F}R$ and $b + c \in C$.*

We begin with a lemma which is a consequence of [1].

Lemma 1. *Let R be a prime ring with center Z and $b \in R$. Let L be a noncommutative Lie ideal of R. If $[bx, x]_k = 0$ for all $x \in L$, where k is a fixed positive integer, then $b \in Z$ unless* $\mathrm{char} R = 2$ *and $R \subseteq M_2(F)$ for some field F.*

Lemma 2. *Let R be a dense subring of* $\mathrm{End}(V_D)$, *containing nonzero linear transformations of finite rank, where D is division ring and* $\dim V_D \geq 3$. *Let $f(x) = bx - \phi(x)c$ where $b, c \in R$ and ϕ is an automorphism of R. If $[f([x, y]), [x, y]]_k = 0$ for all $x, y \in R$, where k is fixed positive integer, then $b - c \in Z$ and $f(x) = (b - c)x$ for all $x \in R$.*

Proof. We will adopt the proof of Lemma 2 in [5] with some necessary modification. Since R is a primitive ring with nonzero socle, by a result in [12, p.79], there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\phi(x) = TxT^{-1}$ for all $x \in R$. Moreover, $T(vs) = T(v)\tau(s)$ for all $v \in V$ and $s \in D$, where τ is an automorphism of D.

If v and $T^{-1}cv$ are D-dependent for all $v \in V$, then as before, there exists $\lambda \in D$ such that $T^{-1}cv = v\lambda$ for all $v \in V$. This imply

$$\begin{aligned} f(x)v &= (bxv - \phi(x)c)v = (bx - TxT^{-1}c)v \\ &= bxv - Txv\lambda = bxv - T(T^{-1}cxv) \end{aligned}$$

$$= bxv - cxv = ((b-c)x)v$$

for all $x \in R$ and for all $v \in V$. Hence $(f(x) - (b-c)x)V = 0$ for all $x \in R$. Since V is faithful, we have $f(x) = (b-c)x$ for all $x \in R$ and hence, by the assumption, we have

$$[(b-c)[x,y],[x,y]]_k = 0 \tag{2}$$

for all $x, y \in R$. By (2) and Lemma 1, it follows that $b - c \in Z$ and we are down.

So we may assume that v_0 and $T^{-1}cv_0$ are D-independent for some $v_0 \in V$. If $\dim V_D \geq 4$, then we can choose $u, w \in V$ such that v_0, $T^{-1}cv_0$, u and w are D-independent. By the density of R, there exists $x, y \in R$ such that

$$xv_0 = 0,\ xT^{-1}cv_0 = 0,\ xu = T^{-1}w,\ xw = u$$

and

$$yv_0 = 0,\ yT^{-1}cv_0 = u,\ yu = -w,\ yw = 0.$$

Hence $[x,y]v_0 = 0$, $[x,y]T^{-1}cv_0 = T^{-1}w$, $[x,y]w = w$ and $(b[x,y] - \phi([x,y])c)v_0 = (b[x,y] - T[x,y]T^{-1}c)v_0 = w$. With all these, we obtain from the assumption that

$$\begin{aligned}
0 &= [f([x,y]),[x,y]]_k v_0 \\
&= [b[x,y] - \phi([x,y])c,[x,y]]_k v_0 \\
&= \sum_{i=0}^{k} (-1)^i \binom{k}{i} [x,y]^i (b[x,y] - \phi([x,y])c)[x,y]^{k-i} v_0 \\
&= (-1)^k [x,y]^k (b[x,y] - \phi([x,y])c) v_0 \\
&= (-1)^k w,
\end{aligned}$$

a contradiction.

Therefore, we may assume $\dim V_D = 3$. In this case, we can choose $w \in V$ such that v_0, $T^{-1}cv_0$ and w are D-independent and $\{v_0, T^{-1}cv_0, w\}$ forms a basis for V. If V. If $T(v_0 + T^{-1}cv_0 + w)$, $T(T^{-1}cv_0 + w) \in v_0 D$, then $T(v_0) \in v_0 D$ and hence v_0, $T^{-1}cv_0 + w \in T^{-1}cv_0 + w \in T^{-1}(v_0 D) = (T^{-1}v_0)D$ contrary to the fact that v_0 and $T^{-1}cv_0 + w$ are D-independent. Therefore if $u = v_0\lambda + T^{-1}cv_0 + w$, where $\lambda \in \{0,1\}$, then $T(u) \notin v_0 D$.

Write $T(u) = v_0\alpha + T^{-1}cv_0\beta + w\gamma$, where $\alpha, \beta, \gamma \in D$ with $\beta \neq 0$ or $\gamma \neq 0$. By the density of R, there exists $x, y \in R$ such that

$$xv_0 = 0,\ xT^{-1}cv_0 = w,\ xw = 0$$

and

$$yv_0 = 0,\ yT^{-1}cv_0 = 0,\ yw = -u.$$

In particular, $xu = w$, $yu = -u$, $xT(u) = w\beta$ and $yT(u) = -u\gamma$. Therefore, $[x, y]v_0 = 0$, $[x, y]T^{-1}cv_0 = u$, $[x, y]w = -w$ and $(b[x, y] - \phi([x, y])c)v_0 = (b[x, y] - T[x, y]T^{-1}c)v_0 = T(u)$. Also $[x, y]u = u - w$, $[x, y]T(u) = u\beta - w\gamma$, $[x, y]^{2i-1}T(u) = u\beta - w\gamma$ and $[x, y]^{2i}T(u) = (u - w)\beta + w\gamma$ for $i \geq 1$. Since β, γ are not all zero and u, w are D-independent, it is easy to see that $[x, y]^iT(u) \neq 0$ for $i \geq 1$. With all these and the assumption, we have

$$\begin{aligned}
0 &= [f([x, y]), [x, y]]_k v_0 \\
&= [b[x, y] - \phi([x, y])c, [x, y]]_k v_0 \\
&= \left(\sum_{i=0}^{k}(-1)^i\binom{k}{i}[x, y]^i(b[x, y] - \phi([x, y])c)[x, y]^{k-i}\right) v_0 \\
&= (-1)^k[x, y]^k(b[x, y] - \phi([x, y])c)v_0 \\
&= (-1)^k[x, y]^kT(u),
\end{aligned}$$

a contradiction. So the proof of the lemma is complete. $\square$

Lemma 3. *Let F be field with $\mathrm{char}F \neq 2$, V_F a vector space over F with $\dim V_F = 2$, and $R = \mathrm{End}(V_F)$. Let $f(x) = bx - \phi(x)c$ for all $x \in R$, where $b, c \in R$ and ϕ is an automorphism of R. If $[f([x, y]), [x, y]]_k = 0$ for all $x, y \in R$, where k is a fixed positive integer, then either $b - c \in Z$ and $f(x) = (b - c)x$ for all $x \in R$ or $\phi = 1_R$ and $b + c \in Z$.*

Proof. Again, by [12, p.79], there exists a semi-linear automorphism $T \in \mathrm{End}(V)$ such that $\phi(x) = TxT^{-1}$ for all $x \in R$. Moreover, $T(vs) = T(v)\tau(s)$ for all $v \in V$, $s \in F$, where τ is an automorphism of F. If v and $T^{-1}cv$ are F-dependent for all $v \in V$, as the second paragraph in the proof of Lemma 2, then $b - c \in Z$ and $f(x) = (b - c)x$ for all $x \in R$. So we may assume that

v_0 and $T^{-1}cv_0$ are F-independent for some $v_0 \in V$. Clearly, $\{v_0, T^{-1}cv_0\}$ is a basis for V_F. It is easy to see that there exists $x, y \in R$ such that

$$xv_0 = T^{-1}cv_0, \quad xT^{-1}cv_0 = 0, \quad yv_0 = 0, \quad yT^{-1}cv_0 = v_0.$$

Therefore, $[x, y]v_0 = -v_0$, $[x, y]T^{-1}cv_0 = T^{-1}cv_0$ and $(b[x, y]-\phi([x, y])c)v_0 = (b[x, y] - T[x, y]T^{-1}c)v_0 = (b + c)v_0$. Consequently, we have

$$\begin{aligned} 0 &= [b[x,y] - \phi([x,y])c, [x,y]]_k v_0 \\ &= \sum_{i=0}^{k}(-1)^i \binom{k}{i}[x,y]^i(b[x,y] - \phi([x,y])c)[x,y]^{k-i}v_0 \\ &= (-1)^k \sum_{i=0}^{k} \binom{k}{i}[x,y]^i(b[x,y] - \phi([x,y])c)v_0 \\ &= (-1)^{k+1} \sum_{i=0}^{k} \binom{k}{i}[x,y]^i(b + c)v_0. \end{aligned}$$

Clearly, $(b + c)v_0 = v_0 r + T^{-1}cv_0 s$ for some $r, s \in F$. If $s \neq 0$, by the last equation, we have $(-1)^{k+1}2^k T^{-1}cv_0 s = 0$, a contradiction. Therefore $(b + c)v_0 = v_0 r$.

We also can choose $x, y \in R$ such that

$$xv_0 = -v_0 - T^{-1}cv_0, \quad xT^{-1}cv_0 = 0, \quad yv_0 = 0, \quad yT^{-1}cv_0 = -v_0.$$

Then $[x, y]v_0 = -v_0$ and $[x, y]T^{-1}cv_0 = v_0 + T^{-1}cv_0$. Moreover, $[x, y]^i v_0 = (-1)^i v_0$, $v_0 = (-1)^i v_0$, $[x, y]^{2i-1}T^{-1}cv_0 = v_0 + T^{-1}cv_0$ and $[x, y]^{2i}T^{-1}cv_0 = T^{-1}cv_0$ for $i \geq 1$. If $T(v_0) = v_0 q + T^{-1}cv_0 p$, then $(b[x, y] - \phi([x, y])c)v_0 = (b[x, y] - T[x, y]T^{-1}c)v_0 = -(b + c)v_0 - T(v_0) = -v_0 r - v_0 q - T^{-1}cv_0 p = -v_0(r + q) - T^{-1}cv_0 p$ and

$$\begin{aligned} 0 &= [b[x,y] - \phi([x,y])c, [x,y]]_k v_0 \\ &= \sum_{i=0}^{k}(-1)^i \binom{k}{i}[x,y]^i(b[x,y] - \phi([x,y])c)[x,y]^{k-i}v_0 \\ &= (-1)^k \sum_{i=0}^{k} \binom{k}{i}[x,y]^i(b[x,y] - \phi([x,y])c)v_0 \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{k+1}\sum_{i=0}^{k}\binom{k}{i}[x,y]^i(v_0(r+q)+T^{-1}cv_0p) \\
&= (-1)^{k+1}\left\{\sum_{i=0}^{k}(-1)^i\binom{k}{i}v_0(r+q)+\left(\sum_{i=0}^{k}\binom{k}{i}[x,y]^iT^{-1}cv_0\right)p\right\} \\
&= (-1)^{k+1}(jv_0p+2^kT^{-1}cv_0p)
\end{aligned}
$$

where $j=\binom{k}{1}+\binom{k}{3}+\cdots+\binom{k}{2\lfloor\frac{k+1}{2}\rfloor-1}$. If $p\neq 0$, then the last equation leads a contradiction since $\mathrm{char}R\neq 2$. Therefore, $T(v_0)=v_0q$, $q\neq 0$.

Assume further that $cv_0=v_0\alpha+T^{-1}cv_0\beta$, $cT^{-1}cv_0=v_0m+T^{-1}cv_0n$, $bv_0=v_0\lambda+T^{-1}cv_0\gamma$ and $bT^{-1}cv_0=v_0l+T^{-1}cv_0h$, where $\alpha,\beta,m,n,\lambda,\gamma,l,h\in F$. Since $(b+c)v_0=v_0r$, then $\alpha+\lambda=r$ and $\beta+\gamma=0$. Now for each $s\in F\backslash\{0\}$, we can choose $x,y\in R$ such that

$$xv_0=-T^{-1}cv_0s,\quad xT^{-1}cv_0=0,\quad yv_0=0,\quad yT^{-1}cv_0=-v_0.$$

Then we have $[x,y]v_0=v_0s$, $[x,y]T^{-1}cv_0=-T^{-1}cv_0s$, and $(b[x,y]-\phi([x,y])c)v_0=(b[x,y]-T[x,y]T^{-1}c)v_0=bv_0s+T(T^{-1}cv_0s)=bv_0s+cv_0\tau(s)=(v_0\lambda+T^{-1}cv_0\gamma)s+(v_0\alpha+T^{-1}cv_0\beta)\tau(s)=v_0(s\lambda+\tau(s)\alpha)+T^{-1}cv_0\beta(\tau(s)-s)$. Let $\eta=s\lambda+\tau(s)\alpha$ and $\mu=\beta(\tau(s)-s)$. Then

$$
\begin{aligned}
0 &= [b[x,y]-\phi([x,y])c,[x,y]]_kv_0 \\
&= \sum_{i=0}^{k}(-1)^i\binom{k}{i}[x,y]^i(b[x,y]-\phi([x,y])c)[x,y]^{k-i}v_0 \\
&= \sum_{i=0}^{k}(-1)^i\binom{k}{i}[x,y]^i(b[x,y]-\phi([x,y])c)v_0s^{k-i} \\
&= \sum_{i=0}^{k}(-1)^i\binom{k}{i}[x,y]^i(v_0\eta+T^{-1}cv_0\mu)s^{k-i} \\
&= \sum_{i=0}^{k}(-1)^i\binom{k}{i}v_0s^k\eta+\sum_{i=0}^{k}(-1)^k\binom{k}{i}T^{-1}cv_0s^k\mu \\
&= (-1)^k2^kT^{-1}cv_0s^k\mu.
\end{aligned}
$$

This implies $\mu=0$ since $\mathrm{char}R\neq 2$. Hence $\beta=0$ or $\tau(s)=s$ for all $s\in F\backslash\{0\}$, that is, $\beta=0$ or $\tau=1_F$. If $\beta=0$, $T(T^{-1}cv_0)=cv_0=v_0\alpha$, which is absurd since we also have $T(v_0) = v_0 q$. Hence $\tau = 1_F$ and T is F-linear.

Finally, we want to show that T is indeed a scalar linear transformation and hence $\phi = 1_R$, the identity automorphism of R. Since all the objects involved in the equation $[b[x,y]-T([x,y])T^{-1}c,[x,y]]_k v_0 = 0$ are all F-linear transformations, in the rest of the proof, we will use matrices to represent all the elements of R relative to the basis $\{v_0, T^{-1}cv_0\}$. Indeed, we have

$$b = \begin{bmatrix} \lambda & l \\ \gamma & h \end{bmatrix}, \quad c = \begin{bmatrix} \alpha & m \\ \beta & n \end{bmatrix}, \quad T = \begin{bmatrix} q & \alpha \\ 0 & \beta \end{bmatrix},$$

where $\alpha + \lambda = r$, $\gamma + \beta = 0$, $q \neq 0$ and $\beta \neq 0$. For any $s, t, u \in F$ we can also choose $x, y \in R$ such that $[x,y] = \begin{bmatrix} s & t \\ u & -s \end{bmatrix}$. Hence

$$\begin{aligned} & b[x,y] - T[x,y]T^{-1}c \\ = & \begin{bmatrix} (\lambda+\alpha)s + lu - qt & \left(\frac{2\alpha n}{\beta} - l - m\right)s + \left(\lambda - \frac{qn}{\beta}\right)t + \left(\frac{\alpha^2 n}{q\beta} - \frac{\alpha m}{q}\right)u \\ hu & (n-h)s + t\gamma + \left(\frac{n\alpha - \beta m}{q}\right)u \end{bmatrix}. \end{aligned}$$

Without loss of the generality, we may assume that k is odd in the rest of the proof. Hence

$$\begin{aligned} & [b[x,y] - T([x,y])T^{-1}c, [x,y]]_k v_0 \\ = & \sum_{i=0}^{k} (-1)^i \binom{k}{i} [x,y]^i (b[x,y] - \phi([x,y])c)[x,y]^{k-i} v_0 \\ = & \sum_{i=\text{odd}}^{k} (-1)^i \binom{k}{i} [x,y]^i (b[x,y] - \phi([x,y])c)[x,y]^{k-i} v_0 \\ & + \sum_{i=\text{even}}^{k} (-1)^i \binom{k}{i} [x,y]^i (b[x,y] - \phi([x,y])c)[x,y]^{k-i} v_0 \\ = & \sum_{i=\text{odd}}^{k} (-1)^i \binom{k}{i} \begin{bmatrix} (s^2+tu)^{\frac{k-1}{2}} & 0 \\ 0 & (s^2+tu)^{\frac{k-1}{2}} \end{bmatrix} \begin{bmatrix} (\lambda+\alpha)s^2 + lus - qts + hut \\ (\lambda+\alpha)su + lu^2 - qtu - hsu \end{bmatrix} \\ & + \sum_{i=\text{even}}^{k} (-1)^i \binom{k}{i} \begin{bmatrix} (s^2+tu)^{\frac{k-1}{2}} & 0 \\ 0 & (s^2+tu)^{\frac{k-1}{2}} \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
&\begin{bmatrix} (\lambda+\alpha)s^2 - qts - \left(m-\frac{2\alpha n}{\beta}\right)su + \left(\lambda-\frac{qn}{\beta}\right)tu + \left(\frac{\alpha^2 n}{q\beta}-\frac{\alpha m}{q}\right)u^2 \\ nsu+\gamma tu + \left(\frac{n\alpha - m\beta}{q}\right)u^2 \end{bmatrix}\\
=\ & 2^{k-1}\begin{bmatrix} (s^2+tu)^{\frac{k-1}{2}} & 0 \\ 0 & (s^2+tu)^{\frac{k-1}{2}} \end{bmatrix}\\
&\begin{bmatrix} -\left(m+l-\frac{2\alpha n}{\beta}\right)su + \left(\lambda-h-\frac{qn}{\beta}\right)tu + \left(\frac{\alpha^2 n}{q\beta}-\frac{\alpha m}{q}\right)u^2 \\ (-\lambda-\alpha+n+h)su+(\gamma+q)tu+\left(-l+\frac{n\alpha-m\beta}{q}\right)u^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

So we have

$$(s^2+tu)\left(\left(m+l-\frac{2\alpha n}{\beta}\right)su-\left(\lambda-h-\frac{qn}{\beta}\right)tu-\left(\frac{\alpha^2 n}{q\beta}-\frac{\alpha m}{q}\right)u^2\right)=0 \tag{3}$$

and

$$(s^2+tu)\left((-\lambda-\alpha+n+h)su+(\gamma+q)tu+\left(-l+\frac{n\alpha-m\beta}{q}\right)u^2\right)=0. \tag{4}$$

Substituting $s=t=u=1$ into (3), we obtain

$$m+l-\frac{2\alpha n}{\beta}-\left(\lambda-h-\frac{qn}{\beta}\right)-\left(\frac{\alpha^2}{q\beta}-\frac{\alpha m}{q}\right)=0. \tag{5}$$

Substituting $s=0$ and $t=u=1$ into (3), we obtain

$$\left(\lambda-h-\frac{qn}{\beta}\right)-\left(\frac{\alpha^2}{q\beta}-\frac{\alpha m}{q}\right)=0. \tag{6}$$

Combining (5) and (6), we obtain

$$m+l-\frac{2\alpha n}{\beta}=0. \tag{7}$$

Again, substituting $s=u=1$ and $t=0$ into (3), we obtain

$$m+l-\frac{2\alpha n}{\beta}-\left(\frac{\alpha^2}{q\beta}-\frac{\alpha m}{q}\right)=0. \tag{8}$$

Combining (5) and (8), we obtain

$$\lambda-h-\frac{qn}{\beta}=0. \tag{9}$$

Similarly, we can deduce from (4) to get

$$\lambda+\alpha=n+h \quad \text{and} \quad \gamma+q=0. \tag{10}$$

But it is known that $\gamma+\beta=0$, hence $\beta=q$. Now from (9) we have $\lambda=h+n$. Comparing this to (10), we have $\alpha=0$. Hence T is a scalar linear transformation. From (5) we also have $m+l=0$ and hence $b+c\in Z$. The proof is complete. □

Lemma 4. *Let R be a noncommutative prime ring and let $f(x)=bx-\beta(x)c$, where β is an X-inner automorphism of R. If $[f([x,y]),[x,y]]_k=0$ for all $x,y\in R$, where k is a fixed positive integer, then either $b-c\in Z$ and $f(x)=(b-c)x$ for all $x\in R$ or $R\subseteq M_2(F)$ for some field F. Moreover, if the latter case holds, then either* $\mathrm{char}R=2$ *or* $\mathrm{char}R\neq 2$, *$\beta=1_R$, and $b+c\in Z$.*

Proof. By the assumption, there exists an invertible element $g\in Q$ such that $\beta(x)=gxg^{-1}$ for all $x\in R$. If $g^{-1}c\in C$, then $f(x)=bx-gxg^{-1}c=bx-cx=(b-c)x$ for all $x\in R$ and we are down by Lemma 1. So we may assume $g^{-1}c\notin C$. Let

$$\varphi(x,y)=[f([x,y]),[x,y]]_k=\sum_{i=0}^{k}(-1)^i\binom{k}{i}[x,y]^i(b[x,y]-g[x,y]g^{-1}c)[x,y]^{k-i}.$$

Then it is easy to see that $\varphi(x,y)=0$ is a nontrivial GPI of R. By [6] or [2, Theorem 6.4.4], $\varphi(x,y)=0$ is also a nontrivial GPI of Q. Let F be the algebraic closure of C if C is infinite and $F=C$ otherwise. By [2], $\varphi(x,y)=0$ is also a nontrivial GPI of $Q\otimes{}_CF$. Moreover, since $Q\otimes{}_CF$ is centrally closed prime algebra [11, Theorem 3.5], by replacing R, C with $Q\otimes{}_CF$, F respectively, we may assume that R is centrally closed and the field C is either algebraically closed or finite. By [12, p.75], R is isomorphic to a dense subring of the ring of linear transformations on a vector space over C, containing nonzero linear transformations of finite rank. Since R is not commutative, we have $\dim V_C\geq 2$. If $\dim V_C\geq 3$, then by Lemma 2, $b-c\in Z$ and $f(x)=(b-c)x$ for all $x\in R$. Hence we may assume that $\dim V_C=2$ and $\mathrm{char}R\neq 2$. Then by Lemma 3, we have either $b-c\in Z$ and $f(x)=(b-c)x$ for all $x\in R$ or $\beta=1_R$ and $b+c\in Z$. If $f(x)=(b-c)x$ for all $x\in R$, then $bx-\beta(x)c=bx-gxg^{-1}c=(b-c)x$ for all $x\in R$. Hence

$gxg^{-1}c = cx$ and $g^{-1}cx = xg^{-1}c$ for all $x \in R$. That is, $g^{-1}c \in C$, which is a contradiction. Therefore, $\beta = 1_R$, $b + c \in Z$ and the proof is complete. □

Proof of Main Theorem. By [4, Lemma 2], we can write $f(x) = sx + \delta(x)$ for all $x \in R$, where $s \in {}_{\mathscr{F}}R$ and δ is a β-derivation of R. Since L is a noncommutative Lie ideal, by [16, Remark 2], there is a nonzero ideal I of R such that $[I, I] \subseteq L$. By the hypothesis, we have

$$h(x, y) = [f([x, y]), [x, y]]_k = [s[x, y] + \delta([x, y]), [x, y]]_k = 0 \text{ for all } x, y \in I. \tag{11}$$

We divide the proof into three cases.

Case 1. Suppose that $\delta = 0$. Then $f(x) = sx$ for all $x \in R$. By (11), we have $[s[x, y], [x, y]]_k = 0$ for all $x, y \in I$ and by [6], we have $[s[x, y], [x, y]]_k = 0$ for all $x, y \in {}_{\mathscr{F}}R$. Now we done by Lemma 1.

For the rest of proof we need the following fact which is a consequence of [5].

Fact. *If $\delta \neq 0$, $s \in Z$ and $[s[x, y] + \delta([x, y]), [x, y]]_k = 0$ for all $x, y \in R$, then* $\mathrm{char} R = 2$ *and $R \subseteq M_2(F)$ for some field F.*

Case 2. Suppose that δ is X-outer. By [9], $[s[x, y] + \delta([x, y]), [x, y]]_k = 0$ for all $x, y \in {}_{\mathscr{F}}R$. Notice that $s[x, y] + \delta([x, y]) = s[x, y] + \delta(xy) - \delta(yx) = s[x, y] + \delta(x)y + \beta(x)\delta(y) - \delta(y)x - \beta(y)\delta(x)$. So

$$[s[x, y] + \delta(x)y + \beta(x)\delta(y) - \delta(y)x - \beta(y)\delta(x), [x, y]]_k = 0 \text{ for all } x, y \in {}_{\mathscr{F}}R. \tag{12}$$

Assume first that β is X-inner, that is, $\beta(x) = gxg^{-1}$ for some invertible element $g \in Q$. From (12), we have

$$[s[x, y] + \delta(x)y + gxg^{-1}\delta(y) - \delta(y)x - gyg^{-1}\delta(x), [x, y]]_k = 0 \text{ for all } x, y \in {}_{\mathscr{F}}R. \tag{13}$$

By [9], we get from (13) that

$$[s[x, y] + zy + gxg^{-1}u - ux - gyg^{-1}z, [x, y]]_k = 0 \text{ for all } x, y, z, u \in {}_{\mathscr{F}}R.$$

Setting $z = u = 0$, we have $[s[x, y], [x, y]]_k = 0$ for all $x, y \in {}_{\mathscr{F}}R$. By Lemma 1 and the fact mentioned above, we have $\mathrm{char} R = 2$ and $R \subseteq M_2(F)$ for some field F. So we are done.

Assume next that β is X-outer. By (12) and [9, Theorem 1], we have

$$[s[x,y]+zy+wu-ux-tz,[x,y]]_k=0 \text{ for all } x,y,z,u,w,t\in {}_{\mathscr{F}}R. \quad (14)$$

Again, setting $z=u=0$, (14) implies $[s[x,y],[x,y]]_k=0$ for all $x,y\in {}_{\mathscr{F}}R$. We are done as before.

Case 3. Suppose that δ is a nonzero X-inner β-derivation defined by some element $b\in Q$, that is, $\delta(x)=bx-\beta(x)b$ for all $x\in R$. In this case, $f(x)=(s+b)x-\beta(x)b$ for all $x\in R$ and $[f(x),x]_k=[(s+b)x-\beta(x)b,x]_k=0$ for all $x\in L$. Since $[I,I]\subseteq L$, by [8, Theorem 1], it is easy to see that

$$[(s+b)x-\beta(x)b,x]_k=0 \text{ for all } x\in [{}_{\mathscr{F}}R,{}_{\mathscr{F}}R]. \quad (15)$$

In particular, we have $[(s+b)[x,y]-\beta([x,y])b,[x,y]]_k=0$ for all $x,y\in {}_{\mathscr{F}}R$. If β is X-inner then by Lemma 4 and the fact mentioned above, we have either $\mathrm{char}R=2$, $R\subseteq M_2(F)$ or $\mathrm{char}R\neq 2$, $R\subseteq M_2(F)$ and $f(x)=(s+b)x-xb$ with $s+2b\in C$. So we may assume that β is X-outer. Then by Chuang's theorem [8], ${}_{\mathscr{F}}R$ is a GPI ring and by [7], it is a primitive ring having nonzero socle and its associated division ring D is finite dimensional over C. Hence ${}_{\mathscr{F}}R$ is isomorphic to a densed subring of ring of linear transformations on a vector space V over D, containing nonzero linear transformations of finite rank. By Lemma 2, we have $\dim V_D\leq 2$. If $\dim V_D=1$ then ${}_{\mathscr{F}}R=D$. If $\dim V_D=2$, then ${}_{\mathscr{F}}R\simeq M_2(D)$. If C is finite, then $\dim D_C<\infty$ implies that D is also finite. Therefore $D\simeq C$ is a field by Wedderburn's theorem [13, p.183]. This implies either ${}_{\mathscr{F}}R=C$ or ${}_{\mathscr{F}}R\simeq M_2(C)$. But since R is noncommutative, we must have ${}_{\mathscr{F}}R\simeq M_2(C)$. By Lemma 3 and the fact mentioned above, we must have $\mathrm{char}\,{}_{\mathscr{F}}R=2$. Thus $\mathrm{char}R=2$ and $R\subseteq M_2(C)$. So for the rest of the proof, we may assume that C is infinite. By Lemma 3, we can also assume that ${}_{\mathscr{F}}R$ is not a subring of $M_2(F)$ for any field F.

Subcase 1. β is not Frobenius. Since $[(s+b)[x,y]-\beta([x,y])b,[x,y]]_k=[(s+b)[x,y]-[\beta(x),\beta(y)]b,[x,y]]_k=0$ for all $x,y\in {}_{\mathscr{F}}R$, then by [8, Theorem 2], we have $[(s+b)[x,y]-[z,u]b,[x,y]]_k=0$ for all $x,y,z,u\in {}_{\mathscr{F}}R$. Setting $z=x$ and $u=y$ we have $[(s+b)[x,y]-[x,y]b,[x,y]]_k=0$ for all $x,y\in {}_{\mathscr{F}}R$. By Lemma 2, $s\in C$ and then the fact mentioned above implies that $\mathrm{char}\,{}_{\mathscr{F}}R=2$ and ${}_{\mathscr{F}}R\subseteq M_2(F)$, a contradiction.

Subcase 2. β is Frobenius. We may assume $\mathrm{char}R = p > 0$. Otherwise, if $\mathrm{char}R = 0$, then the Frobenius automorphism β fixes C and hence must be X-inner by [13, p.140], a contradiction. So for all $\lambda \in C$, $\beta(\lambda) = \lambda^{p^n}$ for some nonzero integer n. Choose an integer m such that $p^m > k$. Then from $[[f(x), x]_k, x]_{p^m-k} = [f(x), x]_{p^m} = [f(x), x^{p^m}]$, we can reduce (15) to

$$[(s+b)x - \beta(x)b, x^{p^m}] = 0 \text{ for all } x \in [\mathscr{F}R, \mathscr{F}R]. \tag{16}$$

Assume first that $n \geq 1$. Clearly, $[\mathscr{F}R, \mathscr{F}R]$ is a C-space. For $\lambda \in C$ and $x \in [\mathscr{F}R, \mathscr{F}R]$, replacing x in (16) by λx, we have

$$0 = [\lambda(s+b)x - \lambda^{p^n}\beta(x)b, \lambda^{p^m}x^{p^m}] = \lambda^{p^m+1}[(s+b)x, x^{p^m}] - \lambda^{p^{m+n}}[\beta(x)b, x^{p^m}].$$

As C is infinite, it follows from the Vandermonde determinant argument that

$$[\beta(x)b, x^{p^m}] = 0 \text{ for all } x \in [\mathscr{F}R, \mathscr{F}R]. \tag{17}$$

For $\lambda \in C$, replacing x in (17) by $x + \lambda y$, we have

$$\begin{aligned} 0 &= [\beta(x+\lambda y)b, (x+\lambda y)^{p^m}] = \left[\beta(x)b + \lambda^{p^n}\beta(y)b, \sum_{i=0}^{p^m}\psi_i(x,y)\lambda^i\right] \\ &= \sum_{i=0}^{p^m}\lambda^i[\beta(x)b, \psi_i(x,y)] + \sum_{i=0}^{p^m-1}\lambda^{p^n+i}[\beta(y)b, \psi_i(x,y)] \end{aligned}$$

where $\psi_i(x,y)$ denotes the sum of all monic monomials with x-degree $p^m - i$ and y-degree i for $0 \leq i \leq p^m$. In particular, $\psi_1(x,y) = x^{p^m-1}y + x^{p^m-2}yx + \cdots + yx^{p^m-1} = \sum_{i=0}^{p^m-1} x^{p^m-1-i}yx^i$. As C is infinite, if follows again from the Vandermonde determinant argument that $[\beta(x)b, \psi_1(x,y)] = 0$ for all $x, y \in [\mathscr{F}R, \mathscr{F}R]$. Setting $y = [x,z]$, where $z \in \mathscr{F}R$, then $\psi_1(x,[x,z]) = x^{p^m}z - zx^{p^m}$ and $[\beta(x)b, \psi_1(x,[x,z])] = 0$ for all $x \in [\mathscr{F}R, \mathscr{F}R]$ and $z \in \mathscr{F}R$. From these it follows that

$$\beta(x)bx^{p^m}z - \beta(x)bzx^{p^m} - x^{p^m}z\beta(x)b + zx^{p^m}\beta(x)b = 0 \tag{18}$$

for all $x \in [\mathscr{F}R, \mathscr{F}R]$ and $z \in \mathscr{F}R$.

Assume that $x^{p^m} \notin C$ for some $x \in [\mathscr{F}R, \mathscr{F}R]$. Then 1 and x^{p^m} are linear independent over C. Applying [18, Lemma 1.2] to (18), $\beta(x)b$ can be expressed as a C-linear combination of 1 and x^{p^m}. In particular, $[\beta(x)b, x] = 0$. Since C is infinite, for any $y \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$, there exists infinite many $\lambda \in C$ such that $(x+\lambda y)^{p^m} \notin C$; otherwise the Vandermonde determinant argument shows that $x^{p^m} \in C$ which is a contradiction. For such $\lambda \in C$ we have

$$\begin{aligned} 0 &= [\beta(x+\lambda y)b, x+\lambda y] = [\beta(x)b + \lambda^{p^m}\beta(y)b, x+\lambda y] \\ &= \lambda[\beta(x)b, y] + \lambda^{p^m}[\beta(y)b, x] + \lambda^{p^m+1}[\beta(y)b, y]. \end{aligned}$$

Applying the Vandermonde determinant argument again, $[\beta(x)b, y] = 0$ for all $y \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$. In particular, $[\beta(x)b[z,w], [z,w]] = 0$ for all $z, w \in {}_{\mathscr{F}}R$. By Lemma 1 and the assumption made right before Subcase 1, we have $\beta(x)b \in C$. Since for any $y \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$ there exists infinite many $\lambda \in C$ such that $(x+\lambda y)^{p^m} \notin C$, we also have $\beta(x+\lambda y)b \in C$. Hence $\beta(y)b \in C$ for all $y \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$. In particular, $[x,y]a \in C$ for all $x, y \in {}_{\mathscr{F}}R$, where $a = \beta^{-1}(b)$. Since ${}_{\mathscr{F}}R \simeq M_2(D)$ is a finite dimensional central simple algebra and the assumption made right above Subcase 1, ${}_{\mathscr{F}}R \subseteq M_t(F)$, $t \geq 3$ for some field F. It is easy to see that $[x,y]a \in C$ for all $x, y \in M_t(F)$. On the other hand, let $x = e_{12}$ and $y = e_{21}$, the matrix units in $M_t(F)$, then $[x,y]a \notin C$ unless $a = 0$, which is a contradiction.

We now may assume that $x^{p^m} \in C$ for all $x \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$. In particular, $[x,y]^{p^m} \in C$ for all $x, y \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$. As before, ${}_{\mathscr{F}}R \subseteq M_t(F)$, $t \geq 3$ for some field F and $[x,y]^{p^m} \in C$ for all $x, y \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$. But setting $x = e_{12}$ and $y = e_{21}$, then we have $[x,y] = e_{11} - e_{22}$ and $[x,y]^{p^m} = e_{11} + (-1)^{p^m}e_{22} \notin C$, which is a contradiction.

Assume next that $n \leq -1$. In this case, let $n' = -n$, then $n' \geq 1$ and $\beta(\lambda^{p^{n'}}) = \lambda$ for all $\lambda \in C$. Replacing x in (16) by $\lambda^{p^{n'}}x$, we have

$$\begin{aligned} 0 &= [(b+s)\lambda^{p^{n'}}x - \beta(\lambda^{p^{n'}}x)b, (\lambda^{p^{n'}}x)^{p^m}] \\ &= [\lambda^{p^{n'}}(b+s)x - \lambda\beta(x)b, \lambda^{p^{n'+m}}x^{p^m}] \\ &= \lambda^{2n'+m}[(b+s)x, x^{p^m}] - \lambda^{p^{n'+m}+1}[\beta(x)b, x^{p^m}]. \end{aligned}$$

Again, by the Vandermonde determinant argument, $[\beta(x)b, x^{p^m}] = 0$ for all $x \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$. Replacing x by $x + \lambda^{p^{n'}}y$, we obtain as before that $[\beta(x)b, \psi_1(x,y)] = 0$ for all $x, y \in [{}_{\mathscr{F}}R, {}_{\mathscr{F}}R]$. Now we can finish the proof by using the same argument as we did for the case $n \geq 1$. The proof of the main theorem is complete. $\square$

Finally, we give an example to show that the exceptional case in the main theorem does occur.

Example. Let $R = M_2(F)$, where F is a field of characteristic 2 and let $L = [R, R] = Fe_{12} + Fe_{21} + F(e_{11} - e_{22})$. Clearly, L is a noncommutative Lie ideal of R and $x^2 \in Z(R)$ for all $x \in L$. Let f be the map defined by $f(x) = e_{11}x - xe_{12}$ for all $x \in R$. Then f is a nonzero generalized derivation of R and $[f(x), x]_2 = [f(x), x^2] = 0$ for all $x \in L$. However, e_{11}, e_{12} and $e_{11} + e_{12}$ are not in $Z(R)$.

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