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SOLVING BOLTZMANN EQUATION, PART I: GREEN'S FUNCTION

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Abstract

We present an approach for solving the Boltzmann equation based on the explicit construction of the Green's functions. The Green's function approach has been useful for the study of nonlinear interior waves and the boundary waves. In this Part I we study the Green's functions for the initial and initial-boundary value problems. In the forthcoming Part II, we will study the general solutions of the Boltzmann equation. Our presentation is self-contained, and, besides the synthesis of the existing ideas, we also established the detailed analysis of the leading terms of the Green's function.

1. Introduction

The *kinetic theory* starts with the *density distribution function* $\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})$ where $\boldsymbol{x}=(x^1,x^2,x^3)\in\mathbb{R}^3$ is the space variables, $t\geq 0$ is the time variable, and $\boldsymbol{\xi}=(\xi^1,\xi^2,\xi^3)\in\mathbb{R}^3$ is the *microscopic velocity.* Thus the kinetic theory differs from the *fluid dynamic* description of the gases in the inclusion of the microscopic velocity $\boldsymbol{\xi}$ as an independent variables. With the knowledge of the density distribution function $\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})$ one can derive the *macroscopic*

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variables, functions of the space and time variables $(\boldsymbol{x}, t)$ through computing the moments:

$$\begin{cases} \rho(\boldsymbol{x},t) \equiv \int_{\mathbb{R}^3} \mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})d\boldsymbol{\xi}, \text{ density,} \\ \rho\boldsymbol{v}(\boldsymbol{x},t) \equiv \int_{\mathbb{R}^3} \boldsymbol{\xi}\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})d\boldsymbol{\xi}, \text{ momentum, } \boldsymbol{v}=(v^1,v^2,v^3) \text{ fluid velocity,} \\ \rho e(\boldsymbol{x},t) \equiv \int_{\mathbb{R}^3} \frac{|\boldsymbol{\xi}-\boldsymbol{v}|^2}{2}\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})d\boldsymbol{\xi}, \text{ internal energy,} \\ \rho E(\boldsymbol{x},t) \equiv \int_{\mathbb{R}^3} \frac{|\boldsymbol{\xi}|^2}{2}\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})d\boldsymbol{\xi} = \rho e + \frac{1}{2}\rho|\boldsymbol{v}|^2, \text{ total energy,} \\ p^{ij}(\boldsymbol{x},t) \equiv \int_{\mathbb{R}^3}(\xi^i-v^i)(\xi^j-v^j)\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})d\boldsymbol{\xi}, \boldsymbol{P}=(p^{ij})_{1\le i,j\le 3}, \text{ stress tensor,} \\ q^i(\boldsymbol{x},t) \equiv \int_{\mathbb{R}^3}(\xi^i-v^i)\frac{1}{2}|\boldsymbol{v}-\boldsymbol{\xi}|^2\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})d\boldsymbol{\xi}, \text{ heat flux.} \end{cases} \tag{1.1}$$

The most important equation for the kinetic theory for the gases is the *Boltzmann equation*, [2],

$$\partial_t\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi}) + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi}) = \frac{1}{k}Q(\mathsf{f},\mathsf{f})(\boldsymbol{x},t,\boldsymbol{\xi}), \tag{1.2}$$

The left hand side of the equation, $\partial_t\mathsf{f}+\boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}\mathsf{f}$, is the *transport term*, measuring the time rate of change along the particle path. The Boltzmann equation says that the change is due to the collision operator $Q(\mathsf{f},\mathsf{f})$, which takes the form of the binary collision:

$$\begin{aligned} \boldsymbol{Q}(\mathsf{g},\mathsf{h}) \equiv \frac{1}{2}\int_{\substack{\mathbb{R}^3\times\boldsymbol{S}^2 \\ (\boldsymbol{\xi}-\boldsymbol{\xi}_*)\cdot\Omega\ge 0}} & (-\mathsf{g}(\boldsymbol{\xi})\mathsf{h}(\boldsymbol{\xi}_*) - \mathsf{h}(\boldsymbol{\xi})\mathsf{g}(\boldsymbol{\xi}_*) + \mathsf{g}(\boldsymbol{\xi}')\mathsf{h}(\boldsymbol{\xi}'_*) \\ & +\mathsf{h}(\boldsymbol{\xi}')\mathsf{g}(\boldsymbol{\xi}'_*))B(\boldsymbol{\xi}-\boldsymbol{\xi}_*,\Omega)d\boldsymbol{\xi}_*d\Omega; \end{aligned} \tag{1.3}$$

$$\begin{cases} \boldsymbol{\xi}' = \boldsymbol{\xi} - [(\boldsymbol{\xi}-\boldsymbol{\xi}_*)\cdot\Omega]\,\Omega, \\ \boldsymbol{\xi}'_* = \boldsymbol{\xi}_* + [(\boldsymbol{\xi}-\boldsymbol{\xi}_*)\cdot\Omega]\,\Omega. \end{cases} \tag{1.4}$$

The cross section $B(\boldsymbol{\xi}-\boldsymbol{\xi}_*,\Omega)$ depends on the inter-molecular forces between the particles. For the hard sphere models

$$B(\boldsymbol{\xi}-\boldsymbol{\xi}_*,\Omega) = (\boldsymbol{\xi}-\boldsymbol{\xi}_*)\cdot\Omega. \tag{1.5}$$

The Maxwellian, thermo-equilibrium states have the defining property that

$$\boldsymbol{Q}(\mathsf{M},\mathsf{M}) = 0$$

and are of the form

$$\mathsf{M} = \mathsf{M}_{(\rho,\boldsymbol{v},\theta)} \equiv \frac{\rho(\boldsymbol{x},t)}{(2\pi R\theta(\boldsymbol{x},t))^{3/2}} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{v}(\boldsymbol{x},t)|^2}{2R\theta(\boldsymbol{x},t)}},$$

determined by the macroscopic variables. Here θ is the temperature defined by $p/\rho = R\theta$. For simplicity, in most part of our presentation, we will set the *mean free path* $k = 1$.

The main purpose of this Part I and the forthcoming Part II of the present paper is to present the *Green's function approach* for the study of the solutions of the Boltzmann equation. The Green's function approach aims at *pointwise, more quantitative understanding of the solutions* of the Boltzmann equation. This is necessary for the qualitative understanding of the rich phenomena that the kinetic theory can model, particularly the nonlinear interior waves and the behavior of the solution near the boundary. There are studies based on physical considerations and asymptotic theory for the Boltzmann equation, see [42], [43], [18], [19], [20]. We start with the construction and analysis of the Green's function in this Part I. The construction involves explicit inversion of the Fourier transform for the fluid-like waves and Picard iterations for particle, particle-like and other singular waves. The Green's functions are for the linearized Boltzmann equation. The simplest situation is to consider the Boltzmann equation linearized around a global Maxwellian M:

$$\mathsf{f} = \mathsf{M} + \sqrt{\mathsf{M}}\mathsf{g}$$

so that the linearization yields

$$\begin{cases} \mathsf{g}_t + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} \mathsf{g} = \mathsf{L}\mathsf{g}, & \text{linearized Boltzmann equation,} \\ \mathsf{L}\mathsf{g} = \dfrac{2\boldsymbol{Q}(\sqrt{\mathsf{M}}\mathsf{g}, \mathsf{M})}{\sqrt{\mathsf{M}}}, & \text{linearized collision operator.} \end{cases} \tag{1.6}$$

The most basic Green's function is for the initial value problem for the above linearized Boltzmann equation. Because we are considering the Boltzmann equation linearized around a fixed Maxwellian, the equation is of constant coefficients in $(\boldsymbol{x}, t)$. Thus the equation is both time and space translational invariant and so the Green's function has the property:

$$\mathbb{G}(\boldsymbol{x}, \boldsymbol{x}_0, t, s, \boldsymbol{\xi}; \boldsymbol{\xi}_0) = \mathbb{G}(\boldsymbol{x} - \boldsymbol{x}_0, t - s, \boldsymbol{\xi}; \boldsymbol{\xi}_0)$$

and satisfies

$$\begin{cases} \mathbb{G}_t + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} \mathbb{G} = \mathsf{L}\mathbb{G}, \\ \mathbb{G}(\boldsymbol{x}, 0, \boldsymbol{\xi}; \boldsymbol{\xi}_0) = \delta(\boldsymbol{x})\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0). \end{cases} \tag{1.7}$$

The Green's function $\mathbb{G}(\boldsymbol{x} - \boldsymbol{x}_0, t - s, \boldsymbol{\xi}; \boldsymbol{\xi}_0)$ describes the propagation of the perturbation over the Maxwellian M when at time s the perturbation consists of particles concentrated at the location $\boldsymbol{x} = \boldsymbol{x}_0$ and with the microscopic velocity concentrated at $\boldsymbol{\xi}_0$. The basic reason for the study of the Green's function is that it can be used to study the general solutions. The simplest situation is to consider the initial value problem for the linear Boltzmann equation with a given source $\mathsf{h} = \mathsf{h}(\boldsymbol{x}, t, \boldsymbol{\xi})$:

$$\begin{cases} \mathsf{g}_t + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} \mathsf{g} = \mathsf{L}\mathsf{g} + \mathsf{h}, \\ \mathsf{g}(\boldsymbol{x}, 0, \boldsymbol{\xi}) = \mathsf{g}_0(\boldsymbol{x}, \boldsymbol{\xi}). \end{cases} \tag{1.8}$$

Multiply the equation with the Green's function and integrate to yield the solution representation:

$$\begin{aligned} \mathsf{g}(\boldsymbol{x}, t, \boldsymbol{\xi}) \; = \; & \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{G}(\boldsymbol{x} - \boldsymbol{y}, t, \boldsymbol{\xi}; \boldsymbol{\xi}_0) \mathsf{g}_0(\boldsymbol{y}, \boldsymbol{\xi}_0) \; d\boldsymbol{\xi}_0 d\boldsymbol{y} \\ & + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathbb{G}(\boldsymbol{x} - \boldsymbol{y}, t - s, \boldsymbol{\xi}; \boldsymbol{\xi}_0) \mathsf{h}(\boldsymbol{y}, s, \boldsymbol{\xi}_0) \; d\boldsymbol{\xi}_0 d\boldsymbol{y} ds. \end{aligned} \tag{1.9}$$

We will construct this Green's function first for the 1-dimensional case, $\boldsymbol{x} \in \mathbb{R}$, and then for the 3-dimensional case, $\boldsymbol{x} \in \mathbb{R}^3$. In preparation for these constructions, we make the following preliminaries. In Section 2, we will first review the basics for the Boltzmann equation and in Section 3 for the linearized Boltzmann equation. For the study of the fluid aspects of the Boltzmann equation, we consider in Section 4 the Euler and Navier-Stokes equations in gas dynamics. The construction of the Green's function is done first for the case of plane waves, i.e. the case when the space variable is 1-dimensional, $x \in \mathbb{R}$,

$$\begin{cases} \mathbb{G}_t + \xi^1 \partial_x \mathbb{G} = \mathsf{L}\mathbb{G}, \\ \mathbb{G}(x, 0, \boldsymbol{\xi}; \boldsymbol{\xi}_0) = \delta^1(x)\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0); \end{cases} \tag{1.10}$$

and the solution formula for the general initial value problem becomes

$$\begin{cases} \mathsf{g}_t + \xi^1 \partial_x \mathsf{g} = \mathsf{L}\mathsf{g}, \\ \mathsf{g}(x,0,\boldsymbol{\xi}) = \mathsf{g}_0(x,\boldsymbol{\xi}), \\ \mathsf{g}(x,t,\boldsymbol{\xi}) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \mathbb{G}(x-y,t,\boldsymbol{\xi};\boldsymbol{\xi}_0)\mathsf{g}_0(y,\boldsymbol{\xi}_0)\, d\boldsymbol{\xi}_0 dy. \end{cases} \tag{1.11}$$

We will consider the fluid-like waves in Section 5 and the particle-like and other singular waves in Section 6. The fluid-like waves are studied by the Fourier transform and the spectral analysis near the origin. The particle-like waves are constructed by Picard iterations and their regularity properties are studied by the *Mixture Lemma*. To combine these two types of waves for the complete picture of the Green's function requires intricate analysis.

In Section 7 we construct the Green's function for the initial value problem for the general waves in 3-dimensional space. There are Huygens waves and other fluid-like waves.

After the Green's function for a global Maxwellian is constructed for the initial value problem, we will use it to construct the Green's function for the initial-boundary value problem in Section 8.

These construction are pointwisely explicit and will be used in the forthcoming Part II for the study of nonlinear waves and boundary layers for the Boltzmann equation. Our presentation is essentially self-contained. The existing materials, [31], [32], for the initial value problem, [33], for initial-boundary value problem, are reorganized with new perspectives. Some issues are clarified and new materials are added. We raise open problems from time to time. For the Green's function approach aimed at the understanding of the dissipation effects of the boundary, see [34] and [47].

2. Boltzmann Equation

In this section we review two basic properties of the Boltzmann equation (1.2): the conservation laws and the H-Theorem.

2.1. Conservation laws

The collision operator $\boldsymbol{Q}(\mathsf{f},\mathsf{f})$, (1.3), redistributes the density function

$\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})$ as a function of the microscopic velocity $\boldsymbol{\xi}$. Nevertheless, on the macroscopic level, the collision operator conserves the mass, momentum and energy:

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ \boldsymbol{\xi} \\ \frac{1}{2}|\boldsymbol{\xi}|^2 \end{pmatrix} \boldsymbol{Q}(\mathsf{g},\mathsf{h})d\boldsymbol{\xi} = 0, \quad \begin{pmatrix} mass \\ momentum \\ energy \end{pmatrix}. \tag{2.1}$$

The above is proved by simple change of variables, noting that the transformation (1.4) has Jacobian one:

$$\int_{\mathbb{R}^3} \Psi(\boldsymbol{\xi})\boldsymbol{Q}(\mathsf{g},\mathsf{h})(\boldsymbol{\xi})d\boldsymbol{\xi} = \frac{1}{4}\int_{\mathbb{R}^3} (\Psi(\boldsymbol{\xi}) + \Psi(\boldsymbol{\xi}_*) - \Psi(\boldsymbol{\xi}') - \Psi(\boldsymbol{\xi}'_*))\boldsymbol{Q}(\mathsf{g},\mathsf{h})(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{2.2}$$

The key property is thus that the functions

$$\Psi_0(\boldsymbol{\xi}) \equiv 1,\ \Psi_i(\boldsymbol{\xi}) \equiv \xi^i,\ i = 1,2,3,\ \Psi_4(\boldsymbol{\xi}) \equiv |\boldsymbol{\xi}|^2/2,\ \Psi = \Psi_j,\ j = 0,\dots,4, \tag{2.3}$$

are *collision invariants* in the following sense.

Definition 2.1. A function $\Psi(\boldsymbol{\xi})$ of the microscopic velocity $\boldsymbol{\xi}$ is a collision invariant if

$$\Psi(\boldsymbol{\xi}) + \Psi(\boldsymbol{\xi}_*) = \Psi(\boldsymbol{\xi}') + \Psi(\boldsymbol{\xi}'_*) \tag{2.4}$$

under the transformation (1.4).

In fact, the relation (1.4) is equivalent to the fact that the functions $1,\ \boldsymbol{\xi},\ |\boldsymbol{\xi}|^2/2$ are collision invariant. Thus any function in the span of these functions are also collision invariant and has the property that the integration of them times the collision operator vanishes. There is the basic theorem of Boltzmann saying that these are all the collision invariants; its proof is omitted, [8].

Theorem 2.2 (Boltzmann)**.** *Any collision invariant is a linear combination of* $1,\ \boldsymbol{\xi},\ |\boldsymbol{\xi}|^2/2$.

Multiply $1,\ \boldsymbol{\xi},\ |\boldsymbol{\xi}|^2/2$ with the Boltzmann equation (1.2) and integrate to yield the *conservation laws* for the macroscopic variables of (1.1):

$$\begin{cases} \partial_t\rho + \partial_{\boldsymbol{x}} \cdot (\rho\boldsymbol{v}) = 0, & \text{conservation of mass,} \\ \partial_t(\rho\boldsymbol{v}) + \partial_{\boldsymbol{x}} \cdot (\rho\boldsymbol{v} \otimes \boldsymbol{v} + \boldsymbol{P}) = 0, & \text{conservation of momentum,} \\ \partial_t(\rho E) + \partial_{\boldsymbol{x}} \cdot (\rho\boldsymbol{v}E + \boldsymbol{P}\boldsymbol{v} + \boldsymbol{q}) = 0, & \text{conservation of energy.} \end{cases} \tag{2.5}$$

Remark 2.3. There are 5 conservation laws: 1 for the mass, 3 for the momentum, and 1 for the energy. On the other hand, there are 14 unknowns: 1 for the density ρ, 3 for the fluid velocity $\boldsymbol{v}$, 6 for the symmetric stress tensor $\boldsymbol{P}$, 3 for the heat flux $\boldsymbol{q}$, and 1 for the internal energy e. Thus the conservation laws are not a system of self-contained partial differential equations. The macroscopic quantities are computed as moments of the density function $\mathsf{f}(\boldsymbol{x}, t, \boldsymbol{\xi})$. The stress tensor consists of second moments and the heat flux consists of third moments. To derive equations for these would require the fourth moments and so on. For instance, one may multiply the Boltzmann equation by $(\xi^i - v^i)(\xi^j - v^j)$ and integrate to derive the evolutionary equation for the stress tensor p^{ij}, a second moment:

$$\partial_t(p^{ij}) + \partial_{\boldsymbol{x}} \cdot \int_{\mathbb{R}^3} (\xi^i - v^i)(\xi^j - v^j)\boldsymbol{\xi}\mathsf{f}d\boldsymbol{\xi} = \int_{\mathbb{R}^3} (\xi^i - v^i)(\xi^j - v^j)Q(\mathsf{f}, \mathsf{f})d\boldsymbol{\xi}.$$

This would involve the third moment, the integration of $(\xi^i - v^i)(\xi^j - v^j)\boldsymbol{\xi}\mathsf{f}$. The evolution of the third moments then involves the fourth moments and so on. Also the right hand side is not zero anymore, as $(\xi^i - v^i)(\xi^j - v^j)$ is not a collision invariant. Thus the Boltzmann equation is equivalent to a system of *infinite* partial differential equations. Another way to view the kinetic theory is to consider the distribution function $\mathsf{f}(\boldsymbol{x}, t, \boldsymbol{\xi})$, for each fixed space and time variables $(\boldsymbol{x}, t)$, as a function of the microscopic variables $\boldsymbol{\xi}$. The space of functions in $\boldsymbol{\xi}$ is infinite dimensional.

For the fluid dynamics, the conservation laws are closed by making assumptions on the dependence of the stress tensor and heat flux on the conserved quantities and their differentials. For the Euler equations in gas dynamics,

$$p_E^{ij} = p\delta_{ij}, \ \boldsymbol{q}_E = 0, \ \text{Euler stress and heat flux;} \tag{2.6}$$

and for the Navier-Stokes equations,

$$\begin{cases} p^{ij}_{NS} = p\delta_{ij} - \mu[\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i} - \frac{2}{3}\sum_{k=1}^3 \frac{\partial v^k}{\partial x^k}\delta_{ij}] - \mu_B \sum_{k=1}^3 \frac{\partial v^k}{\partial x^k}\delta_{ij}, \\ \text{Navier-Stokes stress;} \\ \boldsymbol{q}_{NS} = \kappa \nabla_{\boldsymbol{x}}\theta, \text{ Navier-Stokes heat flux.} \end{cases} \tag{2.7}$$

We will derive the viscosity μ and the heat conductivity κ later. They are functions of the temperature

$$\mu = \mu(\theta), \ \kappa = \kappa(\theta).$$

For the in-depth study of the fluid dynamics from the point of view of the kinetic theory, the readers are referred to [42], [43].

2.2. H-Theorem and Maxwellians

The Boltzmann equation has the so-called *molecular chaos* hypothesis built in the collision operator in that the *loss term* $-\mathsf{f}(\boldsymbol{\xi})\mathsf{f}(\boldsymbol{\xi}_*)$ and the *gain term* $\mathsf{f}(\boldsymbol{\xi}')\mathsf{f}(\boldsymbol{\xi}'_*)$ in the collision operator $\boldsymbol{Q}(\mathsf{f},\mathsf{f})$ are supposed to be the two particles distributions *before* the collision, taken here as the products of the single particle distributions f. This is the hypothesis of *no correlation before collision.* An important consequence is the *H-Theorem* on the *irreversibility* of the Boltzmann process:

$$\int_{\mathbb{R}^3} \log \mathsf{f}\boldsymbol{Q}(\mathsf{f},\mathsf{f})d\boldsymbol{\xi} = \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2_+} \log\frac{\mathsf{f}\mathsf{f}_*}{\mathsf{f}'\mathsf{f}'_*}[\mathsf{f}'\mathsf{f}'_* - \mathsf{f}\mathsf{f}_*]Bd\Omega d\boldsymbol{\xi}_* d\boldsymbol{\xi} \le 0. \tag{2.8}$$

The above is proved by simple change of variables, again using fact that the transformation (1.4) has Jacobian one, (2.2). From (2.8), the Boltzmann Theorem yields that

$$\int_{\mathbb{R}^3} \log \mathsf{f}\boldsymbol{Q}(\mathsf{f},\mathsf{f})d\boldsymbol{\xi} = 0, \quad \text{if and only if } \log \mathsf{f} \text{ is a collision invariant,} \\ \text{in the span of } 1,\ \boldsymbol{\xi},\ |\boldsymbol{\xi}|^2/2. \tag{2.9}$$

In other words, $\log \mathsf{f}$, as a function of the microscopic velocity $\boldsymbol{\xi}$, is quadratic, or, equivalently, f is gaussian in $\boldsymbol{\xi}$. Direct calculations show that the distribution function is explicitly given in terms of the macroscopic variables:

$$\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})=\frac{\rho(\boldsymbol{x},t)}{(2\pi R\theta(\boldsymbol{x},t))^{3/2}}e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{v}(\boldsymbol{x},t)|^2}{2R\theta(\boldsymbol{x},t)}}\equiv \mathsf{M}_{[}(\rho,\boldsymbol{v},\theta)]. \tag{2.10}$$

Here we have introduced the macroscopic quantity θ, the *temperature*, through the *ideal gas relation* between the temperature θ, the pressure p and the density ρ:

$$p=R\rho\theta. \tag{2.11}$$

The states $\mathsf{M}_{(\rho,\boldsymbol{v},\theta)}$ are called the *Maxwellian distributions.* They are the *thermo-equilibrium states* in that

$$\boldsymbol{Q}(\mathsf{M},\mathsf{M})=0. \tag{2.12}$$

The *H-Theorem* is obtained by integrating the Boltzmann equation (1.2) times $\log \mathsf{f}$

$$\partial_t H+\partial_{\boldsymbol{x}}\cdot\boldsymbol{H}\le 0,\ H\equiv\int_{\mathbb{R}^3}\mathsf{f}\log\mathsf{f}d\boldsymbol{\xi},\ \boldsymbol{H}\equiv\int_{\mathbb{R}^3}\boldsymbol{\xi}\mathsf{f}\log\mathsf{f}d\boldsymbol{\xi}. \tag{2.13}$$

Thus a Boltzmann solution has the tendency to reach the thermo-equilibrium and the Boltzmann process is *irreversible.*

Remark 2.4. From the kinetic theory point of view, the fluid dynamics and thermodynamics are the study around the Maxwellians. The fact that the Maxwellian distribution (2.10) is determined by only two state variables ρ and θ, and is consistent with the basic axiom in thermodynamics that there are exactly two independent state variables. One can choose any two state variables and all the other state variables are functions, the *constitutive relations*, of the two chosen state variables. With the given distribution (2.10), this thermodynamics axiom clearly holds. For the non-equilibrium situation one can define, symbolically, the pressure

$$p\equiv\frac{p^{11}+p^{22}+p^{33}}{3}=\frac{1}{3}\int_{\mathbb{R}^3}|\boldsymbol{v}-\boldsymbol{\xi}|^2\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})d\boldsymbol{\xi},\ \text{pressure,}$$

and we have from (1.1) that the internal energy and the pressure are related as:

$$p=\frac{2}{3}\rho e. \tag{2.14}$$

This is the relation for the *monatomic gases* and, when combined with the ideal gas relation (2.11), we have

$$e = \frac{3}{2}R\theta. \tag{2.15}$$

In fact, one often takes (2.15) as the defining property of the temperature θ in terms of the internal energy e. The monatomic gases have the 3 degrees of freedom of translational energy. For the non-monatomic gases, there are other degree of freedom of rotational and vibrational energy, for instance. For a gas with $\alpha,\ \alpha \geq 3$, degrees of freedom, we would have $e = \frac{\alpha}{2}R\theta$.

2.3. One-dimensional flows

For plane waves, the density function $\mathsf{f}(\boldsymbol{x}, t, \boldsymbol{\xi})$ depends only on 1-dimensional space variable $x^1,\ \boldsymbol{x} = (x^1, x^2, x^3)$. We will write $x^1 = x$. The Boltzmann equation becomes

$$\partial_t \mathsf{f}(x, t, \boldsymbol{\xi}) + \xi^1 \partial_x \mathsf{f}(x, t, \boldsymbol{\xi}) = Q(\mathsf{f}, \mathsf{f})(x, t, \boldsymbol{\xi}), \tag{2.16}$$

Note that the microscopic velocity $\boldsymbol{\xi} = (\xi^1, \xi^2, \xi^3)$ must remain 3-dimensional. The Green's function $\mathbb{G}(x, t, \boldsymbol{\xi}; \boldsymbol{\xi}_0),\ x \in \mathbb{R}$, for the initial value problem, c.f. (1.7), becomes

$$\begin{cases} \mathbb{G}_t + \xi^1 \partial_x \mathbb{G} = \mathsf{L}\mathbb{G} \text{ for } x \in \mathbb{R}, \\ \mathbb{G}(x, 0, \boldsymbol{\xi}; \boldsymbol{\xi}_0) = \delta^1(x)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0). \end{cases} \tag{2.17}$$

Here we have highlighted the dimension of the delta functions. For convenience, we assume that the density distribution

$$\mathsf{f}(x, t, \boldsymbol{\xi}) \text{ is even in } \xi^2 \text{ and } \xi^3. \tag{2.18}$$

It is easy to see that if this assumption is made for the initial data, it remains so for later time. With this, the fluid velocity $\boldsymbol{v} = (v^1, v^2, v^3)$ becomes

$$\boldsymbol{v} = (v^1, 0, 0) \equiv (v, 0, 0), \tag{2.19}$$

and we will write the Maxwellians as

$$\mathsf{M}_{[\rho, \boldsymbol{v}, \theta]} \equiv \mathsf{M}_{[\rho, v, \theta]}. \tag{2.20}$$

The thermo-equilibrium manifold, the collection of all Maxwellians, is then 3-dimensional. There are 3 effective collision invariants:

$$1,\ \xi^1,\ \frac{|\boldsymbol{\xi}|^2}{2},\ 1-\text{dimensional collision invariants.} \tag{2.21}$$

3. Linearized Boltzmann Equation

Consider the solution f of the Boltzmann equation (1.2) as a perturbation of the Maxwellian $\mathsf{M} = \mathsf{M}_{[\rho,\boldsymbol{v},\theta]}$:

$$\begin{cases} \mathsf{f} = \mathsf{M} + \sqrt{\mathsf{M}}\mathsf{g}, \\ \mathsf{g}_t + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}\mathsf{g} = \mathsf{L}\mathsf{g} + \Gamma(\mathsf{g}), \\ \mathsf{L}\mathsf{g} = \dfrac{2\boldsymbol{Q}(\sqrt{\mathsf{M}}\mathsf{g},\mathsf{M})}{\sqrt{\mathsf{M}}},\ \text{linearized collision operator,} \\ \Gamma(\mathsf{g}) = \dfrac{\boldsymbol{Q}(\sqrt{\mathsf{M}}\mathsf{g},\sqrt{\mathsf{M}}\mathsf{g})}{\sqrt{\mathsf{M}}},\ \text{nonlinear term.} \end{cases} \tag{3.1}$$

The Boltzmann equation linearized around M with the weight function $\sqrt{\mathsf{M}}$ is

$$\mathsf{g}_t + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}\mathsf{g} = \mathsf{L}\mathsf{g}. \tag{3.2}$$

3.1. Linear collision operator

Any Maxwellian $\bar{\mathsf{M}} = \mathsf{M}_{[\rho,v,\theta]}$ is a thermo-equilibrium state, $\boldsymbol{Q}(\bar{\mathsf{M}},\bar{\mathsf{M}}) = 0$. The equilibrium manifold is a 5-dimensional manifold of the Maxwellians, parametrized by $(\rho,\boldsymbol{v},\theta)$. As a consequence, the kernel of linearized collision operator $\mathsf{L}\mathsf{g} = 2\boldsymbol{Q}(\sqrt{\mathsf{M}}\mathsf{g},\mathsf{M})/\sqrt{\mathsf{M}}$ is the tangent space of the equilibrium manifold at the base Maxwellian M of the linearization. This is found by the differentiation of the explicit form (2.10) of the Maxwellian state with respect to the 5 parameters $(\rho,\boldsymbol{v},\theta)$. For instance the differentiation with respect to $\boldsymbol{v}$ yields

$$\frac{\partial\mathsf{M}}{\partial\boldsymbol{v}} = \frac{\boldsymbol{\xi}-\boldsymbol{v}}{R\theta}\mathsf{M},$$

which is, as a function of the microscopic velocity $\boldsymbol{\xi}$, in the span of $\{\mathsf{M},\ \boldsymbol{\xi}\mathsf{M}\}$. In the form of the linearized variable g with the weight $\sqrt{\mathsf{M}}$, (3.1), it is in the span of $\{\sqrt{\mathsf{M}},\ \boldsymbol{\xi}\sqrt{\mathsf{M}}\}$. The other differentiations yield altogether the span $\{\sqrt{\mathsf{M}},\ \boldsymbol{\xi}\sqrt{\mathsf{M}},\ |\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\}$ of the tangent space, and we have

$$\begin{cases} \mathsf{L}\psi_j = 0,\ j = 0, \dots, 4, \\ \psi_0 \equiv \sqrt{\mathsf{M}},\ \psi_i = \xi^i\sqrt{\mathsf{M}},\ i = 1,2,3,\ \psi_4 \equiv \frac{|\boldsymbol{\xi}|^2}{2}\sqrt{\mathsf{M}},\ \text{linear collision kernel.} \end{cases} \tag{3.3}$$

There is also the algebraic way of finding the linear collision kernel, cf. (2.2), (3.13).

Remark 3.1. The Boltzmann Theorem says that the linear collision kernel is exactly the above 5-dimensional space, the linear equilibrium states. The study of the Boltzmann solutions contains the following two important considerations. The first is the tendency of the Boltzmann solutions converging to the local Maxwellians, as predicted by the H-Theorem. There is the Cercignani conjecture, [7] related to this efforts, see [12], [6], [46], [21]. Then there is the study of the Boltzmann solutions flowing around the 5-dimensional equilibrium manifold. The study of the later is essential for the understanding of the relation of the kinetic theory with the *fluid dynamics.* On the linear level, we call the projection, in the function space of the microscopic variable, onto the linear collision kernel the linear macro projection. Its orthogonal projection is called the micro projection. These projections have been used for a long while, see for instance, in the content of energy method, the recent papers, [22], [23], [24], and the hypocoercivity theory, [39]; its introduction for the above specific purpose was done in [30]. A good part of the present effort is to illustrate the Boltzmann flow from the perspective of these two basic considerations.

For the hard sphere models, (1.5), and taking the base Maxwellian M for linearization to be

$$\mathsf{M}_{[1,0,1]} = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|\boldsymbol{\xi}|^2}{2}} \tag{3.4}$$

the linearized collision operator has the explicit form, e.g. [9],

$$\begin{cases} \mathsf{L}\mathsf{g}(\boldsymbol{\xi}) = (-\nu + \mathsf{K})\mathsf{g}(\boldsymbol{\xi}) \equiv -\nu(\boldsymbol{\xi})\mathsf{g}(\boldsymbol{\xi}) + \int_{\mathbb{R}^3} K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)\mathsf{g}(\boldsymbol{\xi}_*)d\boldsymbol{\xi}_*, \\ \nu(\boldsymbol{\xi}) \equiv \frac{1}{\sqrt{2\pi}}\left(2e^{-\frac{|\boldsymbol{\xi}|^2}{2}} + 2(|\boldsymbol{\xi}| + \frac{1}{|\boldsymbol{\xi}|})\int_0^{|\boldsymbol{\xi}|} e^{-\frac{\eta^2}{2}}d\eta\right), \\ C_1(1+|\boldsymbol{\xi}|) \le \nu(\boldsymbol{\xi}) \le C_2(1+|\boldsymbol{\xi}|) \text{ for some } C_1,\ C_2 > 0, \\ K(\boldsymbol{\xi}, \boldsymbol{\xi}_*) \equiv \frac{2}{\sqrt{2\pi}|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}\exp\left(-\frac{(|\boldsymbol{\xi}|^2 - |\boldsymbol{\xi}_*|^2)^2}{8|\boldsymbol{\xi}-\boldsymbol{\xi}_*|^2} - \frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|^2}{8}\right) \\ \qquad\qquad - \frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}{2}\exp\left(-\frac{(|\boldsymbol{\xi}|^2 + |\boldsymbol{\xi}_*|^2)}{4}\right). \end{cases} \tag{3.5}$$

The linearized collision operator around a Maxwellian $\mathsf{M}_{[\rho,\boldsymbol{v},\theta]}$ is a simple translation and dilation of the above. Here, as in what follows, we have taken, for simplicity, the gas constant $R = 1$.

The Hilbert space $L^2_{\boldsymbol{\xi}}$ for functions of the microscopic variables will be used. We will also use the weighted spaces:

$$\|\mathsf{g}\|_{L^\infty_{\boldsymbol{\xi},\beta}} \equiv \sup_{\boldsymbol{\xi}\in\mathbb{R}^3} |\mathsf{g}(\boldsymbol{\xi})|(1+|\boldsymbol{\xi}|)^\beta,\ \beta \ge 0. \tag{3.6}$$

The integral operator K has some boundedness and smoothing properties. The following lemma follows from direct computations.

Lemma 3.2. *The operator* K *is compact and the integral operators* $\mathsf{K}_{\boldsymbol{\xi}},\ \mathsf{K}_{\boldsymbol{\xi}_*}$*, defined by the kernel* $K_{\boldsymbol{\xi}},\ K_{\boldsymbol{\xi}_*}$ *respectively, are bounded in* $L^2_{\boldsymbol{\xi}}$*. For any* $\beta > 0$ *there exist positive constants* $C(\beta)$ *and* C_1 *such that*

$$\begin{cases} \|\mathsf{K}\,\mathsf{j}\|_{L^\infty_{\boldsymbol{\xi},\beta+1}} \le C(\beta)\|\mathsf{j}\|_{L^\infty_{\boldsymbol{\xi},\beta}}, \\ \|\mathsf{K}\mathsf{j}\|_{L^\infty_{\boldsymbol{\xi},0}} \le C_1\|\mathsf{j}\|_{L^2_{\boldsymbol{\xi}}}. \end{cases} \tag{3.7}$$

The integral operator K has limited smoothing property in $\boldsymbol{\xi}$, (3.7). This is due to the singularity of the kernel $K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ at $\boldsymbol{\xi} = \boldsymbol{\xi}_*$, (3.5). We decompose it into

$$\begin{cases} \mathsf{K} = \mathsf{K}_0 + \mathsf{K}_1 = \mathsf{K}_{0,D} + \mathsf{K}_{1,D}, \\ \mathsf{K}_i\mathsf{z}(\boldsymbol{\xi}) \equiv \int_{\mathbb{R}^3} K_i(\boldsymbol{\xi}, \boldsymbol{\xi}_*)\mathsf{z}(\boldsymbol{\xi}_*)\, d\boldsymbol{\xi}_* \text{ for } i = 0, 1, \\ K_0(\boldsymbol{\xi}, \boldsymbol{\xi}_*) = \chi_0\left(\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}{D\nu_0}\right) K(\boldsymbol{\xi}, \boldsymbol{\xi}_*), \\ K_1(\boldsymbol{\xi}, \boldsymbol{\xi}_*) = \left(1 - \chi_0\left(\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}{D\nu_0}\right)\right) K(\boldsymbol{\xi}, \boldsymbol{\xi}_*), \\ \chi_0(r) \equiv 1 \text{ for } r \in [-1, 1], \\ supp(\chi_0) \subset [-2, 2],\ \chi_0 \in C_c^\infty(\mathbb{R}),\ \chi_0 \geq 0. \end{cases} \tag{3.8}$$

Here the cutoff parameter D will be chosen to be small. We may write the linearized Boltzmann equation (3.2) as:

$$\partial_t \mathsf{g} + \xi^1 \partial_x \mathsf{g} + \nu(\boldsymbol{\xi})\mathsf{g} = (\mathsf{K}_0 + \mathsf{K}_1)\mathsf{g}. \tag{3.9}$$

The operator K_0 shares the same smoothing property as K and has strength of the order of the cut-off parameter D:

$$\|\mathsf{K}_0\mathsf{h}\|_{L^\infty_{\xi,\beta+1}} \leq C_\beta D \|\mathsf{h}\|_{L^\infty_{\xi,\beta}} \text{ for } \beta \geq 0. \tag{3.10}$$

The operator K_1 is a smoothing operator because it does not inherit the singular nature of the kernel $K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ at $\boldsymbol{\xi} = \boldsymbol{\xi}_*$.

Lemma 3.3. *The operator* K_1 *given in* (3.8) *is a smoothing operator in* $\boldsymbol{\xi}$*: for any* $\mathsf{h} \in L^2_{\boldsymbol{\xi}}$, $i \geq 0$,

$$\|\mathsf{K}_1\mathsf{h}\|_{H^i_{\boldsymbol{\xi}}} = O(1)\|\mathsf{h}\|_{L^2_{\boldsymbol{\xi}}}.$$

Here $H^i_{\boldsymbol{\xi}}$ *are the standard Sobolev spaces on* $L^2_{\boldsymbol{\xi}}$*.*

Proof. From the definition of K_1 in (3.8),

$$\partial^\alpha_{\boldsymbol{\xi}} \mathsf{K}_1\mathsf{h}(\boldsymbol{\xi}) \equiv \int_{\mathbb{R}^3} \mathsf{h}(\boldsymbol{\xi}_*)\partial^\alpha_{\boldsymbol{\xi}} \left(\left(1 - \chi_0\left(\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}{D\nu_0}\right)\right) K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)\right) d\boldsymbol{\xi}_*.$$

The function $K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)$ is smooth for $|\boldsymbol{\xi}-\boldsymbol{\xi}_*| > D\nu_0$. Thus, $\left(\left(1-\chi_0\left(\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}{D\nu_0}\right)\right) K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)\right)$ is a globally smooth function. It is easy to see that for any $i \equiv |\alpha| \geq 0$ the function $\partial^\alpha_{\boldsymbol{\xi}}((1 - \chi_0(\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}{D\nu_0}))K(\boldsymbol{\xi}, \boldsymbol{\xi}_*)) \in L^1_{\boldsymbol{\xi}}$ and therefore defines a bounded operator from $L^2_{\boldsymbol{\xi}}$ to $L^2_{\boldsymbol{\xi}}$, and the lemma is proved. □

We have the following basic theorems, [4], [5], [9].

Theorem 3.4. *The linearized collision operator* L *is non-negative self-adjoint in* $L^2_{\boldsymbol{\xi}}$ *and the Green's function* $\mathbb{G}$ *is bounded in* $L^2_{\boldsymbol{x}}(L^2_{\boldsymbol{\xi}})$*:*

$$\|\mathbb{G}\|_{L^2_{\boldsymbol{x}}(L^2_{\boldsymbol{\xi}})} \le 1. \tag{3.11}$$

Proof. The form of the linear collision operator with the weight of $\sqrt{\mathsf{M}}$ serves the convenient purpose that the collision operator is self-adjoint in the Hilbert space $L^2_{\boldsymbol{\xi}}$:

$$(\mathsf{Lg},\mathsf{h}) = (\mathsf{g},\mathsf{Lh}),\ \ (\mathsf{g},\mathsf{h}) = (\mathsf{g},\mathsf{h})_{L^2_{\boldsymbol{\xi}}} \equiv \int_{\mathbb{R}^3} \mathsf{g}(\boldsymbol{\xi})\mathsf{h}(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{3.12}$$

This is shown by simple changes of variables in the integrations such as (2.2),

$$\begin{cases} (\mathsf{Lg},\mathsf{h}) = (\mathsf{g},\mathsf{Lh}) \\ = -\frac{1}{16}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\boldsymbol{S}^2} \mathsf{M}\mathsf{M}_*[\frac{\mathsf{g}'_*}{\sqrt{\mathsf{M}'_*}} + \frac{\mathsf{g}'}{\sqrt{\mathsf{M}'}} - \frac{\mathsf{g}}{\sqrt{\mathsf{M}}} - \frac{\mathsf{g}_*}{\sqrt{\mathsf{M}_*}}] \\ \qquad\qquad \times[\frac{\mathsf{h}'_*}{\sqrt{\mathsf{M}'_*}} + \frac{\mathsf{h}'}{\sqrt{\mathsf{M}'}} - \frac{\mathsf{h}}{\sqrt{\mathsf{M}}} - \frac{\mathsf{h}_*}{\sqrt{\mathsf{M}_*}}]Bd\Omega d\boldsymbol{\xi}_* d\boldsymbol{\xi}, \end{cases} \tag{3.13}$$

making use of the fact that the Maxwellians have the defining property that $\log \mathsf{M}$ is collision invariant, or, $\mathsf{M}'\mathsf{M}'_* = \mathsf{M}\mathsf{M}_*$. To prove the boundedness of the Green's function, we use the energy method by integrating the linear Boltzmann equation (3.2) times g to yield, using the non-negativeness of L:

$$\frac{d}{dt}\int_{\mathbb{R}^3} (\mathsf{g},\mathsf{g})_{L^2_{\boldsymbol{\xi}}} d\boldsymbol{x} = \int_{\mathbb{R}^3} (\mathsf{Lg},\mathsf{g})_{L^2_{\boldsymbol{\xi}}} d\boldsymbol{x} \le 0. \tag{3.14}$$

Thus the Green's function as an operator in propagating the solution of the initial value problem, (1.8), (1.9), is contractive in $L^2_{\boldsymbol{x}}(L^2_{\boldsymbol{\xi}})$ and so (3.11) holds. □

3.2. Macro-micro projections

One forms the orthogonal basis for the kernel of the collision operator

(3.3):

$$\begin{cases} \chi_0 \equiv \mathsf{M}^{1/2}, \\ \chi_i \equiv (\xi^i - v^i)\mathsf{M}^{1/2},\ i = 1, 2, 3, \\ \chi_4 \equiv \frac{1}{\sqrt{6}}(|\boldsymbol{\xi} - \boldsymbol{v}|^2 - 3)\mathsf{M}^{1/2}. \end{cases} \tag{3.15}$$

The *macro projection* P_0 and the *micro projection* P_1 are

$$\mathsf{P}_0\mathsf{g} \equiv \sum_{j=0}^{4} (\mathsf{g}, \chi_j)\chi_j,\ \mathsf{P}_1 \equiv I - \mathsf{P}_0. \tag{3.16}$$

We will write

$$\mathsf{g}_0 \equiv \mathsf{P}_0\mathsf{g},\ \mathsf{g}_1 \equiv \mathsf{P}_1\mathsf{g},\ \mathsf{g} = \mathsf{g}_0 + \mathsf{g}_1,\ (\mathsf{g}_0, \mathsf{g}_1) = 0. \tag{3.17}$$

The macro projection P_0 consists of the *isotropic part* P_0^{iso} and the *momentum part* P_0^m:

$$\mathsf{P}_0 = \mathsf{P}_0^{iso} + \mathsf{P}_0^m,\ \mathsf{P}_0^{iso}\mathsf{g} = (\mathsf{g}, \chi_0)\chi_0 + (\mathsf{g}, \chi_4)\chi_4,\ \mathsf{P}_0^m \equiv \sum_{i=1}^{3} (\mathsf{g}, \chi_i)\chi_i. \tag{3.18}$$

Clearly, the linear collision operator L satisfies

$$\mathsf{P}_0\mathsf{L} = \mathsf{L}\mathsf{P}_0 = 0,\ \mathsf{P}_1\mathsf{L} = \mathsf{L}\mathsf{P}_1 = \mathsf{L}; \tag{3.19}$$

and, from the Boltzmann Theorem, the nonlinear collision term $\Gamma(\mathsf{g})$, (3.1), also satisfies

$$\mathsf{P}_0\Gamma = 0,\ \mathsf{P}_1\Gamma = \Gamma. \tag{3.20}$$

Theorem 3.5. *The spectrum $\sigma(\mathsf{L})$ of the linearized collision operator L has the following property:*

(1) *There is the eigenvalue 0 with corresponding eigenfunctions the 5-dimensional collision invariants,* (3.3)*.*

(2) *Besides the 0 eigenvalues, the maximum value of the spectrum is a negative number $-\nu_0$. In other words, there is a spectrum gap between 0 and the rest of the spectrum.*

Consequently,

$$(\mathsf{Lg}, \mathsf{g}) \le -\nu_1(\mathsf{P}_1\mathsf{g}, \mathsf{P}_1\mathsf{g}), \tag{3.21}$$

for any function $\mathsf{g} \in L^2_{\boldsymbol{\xi}}$.

Proof. From (3.5), the collision operator can be viewed as the compact perturbation K of the multiplicative operator $-\nu(\boldsymbol{\xi})$. From the Weyl's Theorem, the essential spectrum $\sigma_{ess} = \{y : \ y \le -\nu_0\}$ of L is the values taken by $\nu(\boldsymbol{\xi})$. As $\nu(\boldsymbol{\xi})$ is an increasing function of $|\boldsymbol{\xi}|$,

$$\nu_0 \equiv \nu(0) = \min_{\boldsymbol{\xi}} \nu(\boldsymbol{\xi}), \tag{3.22}$$

As L is self-adjoint and nonpositive, the discrete eigenvalues $\sigma_{discrete}(\mathsf{L}) = \sigma(\mathsf{L})\backslash\sigma_{essential}(\mathsf{L}) \subset (-\nu(0), 0]$. Therefore, there are finite eigenvalues in interval $[-\nu_0 + \varepsilon, 0]$ for any small positive ε. In particular, the zero eigenvalue is isolated. Thus the maximum of the rest of the spectrum $-\nu_1$ is negative, $\nu_0 > \nu_1 > 0$. Finally, (3.21) follows from the fact that the micro projection P_1 projects to the orthogonal complement of the kernel of L. This establishes (3.21) and the theorem is proved. □

The above theorem is due to the efforts of Hilbert, Carleman, and Grad. For constructive estimates of the spectrum gap, see [1], [37], [38].

Corollary 3.6. *There exist positive constants* C_1, C_2 *such that*

$$(\mathsf{Lg}, \mathsf{g}) \ \le \ -C_1((1 + |\boldsymbol{\xi}|)\mathsf{P}_1\mathsf{g}, \mathsf{P}_1\mathsf{g}), \tag{3.23}$$

$$|(|\boldsymbol{\xi}|\mathsf{g}, \mathsf{g})| \ \le \ C_2[(\mathsf{g}, \mathsf{g}) + (-\mathsf{Lg}, \mathsf{g})]. \tag{3.24}$$

Proof. From the expression $\mathsf{L} = \mathsf{K} - \nu(\boldsymbol{\xi})$ and the estimates, (3.5),

$$\nu(\boldsymbol{\xi}) \ge D_1(1 + |\boldsymbol{\xi}|), \ \|\mathsf{K}\|_{L^2_{\boldsymbol{\xi}}} = D_2 < \infty,$$

for some positive constants D_1, D_2, and so we have

$$(\mathsf{Lg}, \mathsf{g}) = (\mathsf{LP}_1\mathsf{g}, \mathsf{P}_1\mathsf{g}) \le -D_1((1 + |\boldsymbol{\xi}|)\mathsf{P}_1\mathsf{g}, \mathsf{P}_1\mathsf{g}) + D_2(\mathsf{P}_1\mathsf{g}, \mathsf{P}_1\mathsf{g}). \tag{3.25}$$

The estimate (3.23) is shown with $C_1 = D_1\nu_1/(D_2 + \nu_1)$ by adding (3.21) times D_2/ν_1 and (3.25). We have from the Cauchy-Schwarz inequality that

$$|(|\boldsymbol{\xi}|\mathsf{g}, \mathsf{g})| \ = \ |(|\boldsymbol{\xi}|(\mathsf{g}_0 + \mathsf{g}_1), \mathsf{g}_0 + \mathsf{g}_1)|$$

$$= O(1)[((1+|\boldsymbol{\xi}|)\mathsf{P}_0\mathsf{g},\mathsf{P}_0\mathsf{g})+((1+|\boldsymbol{\xi}|)\mathsf{P}_1\mathsf{g},\mathsf{P}_1\mathsf{g})].$$

Thus (3.24) follows from (3.23) if

$$((1+|\boldsymbol{\xi}|)\mathsf{P}_0\mathsf{g},\mathsf{P}_0\mathsf{g}) = O(1)(\mathsf{P}_0\mathsf{g},\mathsf{P}_0\mathsf{g}),$$

which holds trivially because, after the macro projection P_0, it is a finite dimensional situation. This completes the proof of the corollary. □

3.3. Macroscopic variables

The fluid variables for the conservation laws

$$\begin{cases} \rho \equiv (\mathsf{g},\mathsf{M}^{1/2}) = (\mathsf{g}_0,\mathsf{M}^{1/2}), \\ m_i \equiv (\mathsf{g},\xi^i\mathsf{M}^{1/2}) = (\mathsf{g}_0,\xi^i\mathsf{M}^{1/2}),\ \boldsymbol{m} \equiv (m_1,m_2,m_3), \\ E \equiv (\mathsf{g},\frac{1}{2}|\boldsymbol{\xi}|^2\mathsf{M}^{1/2}) = (\mathsf{g}_0,\frac{1}{2}|\boldsymbol{\xi}|^2\mathsf{M}^{1/2}). \end{cases} \tag{3.26}$$

There is some ambiguity with earlier notations, as, for instance, the ρ here is not the density, but the perturbation of the density. There is no confusion here and we use these for the sake of notational simplicity. We will also use the projection to the orthogonal basis $\chi_i,\ i=0,1,2,3,4$:

$$\begin{cases} \bar\rho \equiv (\mathsf{g},\chi_0) = (\mathsf{g}_0,\chi_0), \\ \bar m_i \equiv (\mathsf{g},\chi_i) = (\mathsf{g}_0,\chi_i), \\ \bar E \equiv (\mathsf{g},\chi_4) = (\mathsf{g}_0,\chi_4), \end{cases} \tag{3.27}$$

$$\mathsf{g}_0(\boldsymbol{x},t,\boldsymbol{\xi}) = \bar\rho(\boldsymbol{x},t)\chi_0(\boldsymbol{\xi}) + \sum_{i=1}^{3}\bar m_i(\boldsymbol{x},t)\chi_i(\boldsymbol{\xi}) + \bar E(\boldsymbol{x},t)\chi_4(\boldsymbol{\xi}). \tag{3.28}$$

4. Euler and Navier-Stokes

For the discussion of this section, the mean free path k is retrieved to highlight the dependence of the hydrodynamics dissipation parameters, the viscosities and heat conductivity on k.

Boltzmann equation has its fluid dynamics aspects. The most basic equations for gas dynamics are the *Euler equations*. The Euler equations in gas dynamics can be directly derived from the Boltzmann equation and provide the basic information on the propagation of fluid-like waves in the kinetic theory. The Navier-Stokes equations and the Boltzmann equation share the basic property that they are both dissipative. Boltzmann equation is dissipative partly as a consequence of the H-Theorem. The heat conductivity and viscosity coefficients for the Navier-Stokes equations are also the basic dissipation parameters for the Boltzmann solutions. We will derive the compressible Navier-Stokes equations in the spirit of Chapman-Enskog expansion.

4.1. Euler equations

Each Maxwellian is a constant solution of the Boltzmann equation, (2.12). In general, local Maxwellians do not form a solution of the Boltzmann equation. As the mean free path k tends to zero, for smooth solutions, one expects $\boldsymbol{Q}(\mathsf{f},\mathsf{f})$ to go to zero:

$$Q(\mathsf{f},\mathsf{f})(\boldsymbol{x},t,\boldsymbol{\xi}) = k[\partial_t \mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi}) + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}\mathsf{f}(\boldsymbol{x},t,\boldsymbol{\xi})] \to 0.$$

And so, by the H-Theorem, (2.12), $\mathsf{f} \to \mathsf{M}_\mathsf{f}$, where M_f are the Maxwellians determined by the macroscopic variables of the solution f. We call

$$(\mathsf{M}_\mathsf{f})_t + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}(\mathsf{M}_\mathsf{f}) = 0 \tag{4.1}$$

the *Euler equations in the kinetic form*. Assuming that the distribution function is locally Maxwellian

$$\mathsf{f} = \mathsf{M}_\mathsf{f},$$

then, by direct calculations, it is easy to see that

$$\text{stress tensor } \ \boldsymbol{P} = pI, \text{ the pressure } p; \text{ heat flux } \boldsymbol{q} = 0.$$

Here I is the 3 by 3 identity matrix. Thus in this *zero mean free path limit*, $k \to 0+$, the conservation laws (2.5) are simplified to the *Euler equations in* *gas dynamics*

$$\begin{cases} \partial_t \rho + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v}) = 0, \\ \partial_t (\rho \boldsymbol{v}) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} \otimes \boldsymbol{v} + pI) = 0, \\ \partial_t (\rho E) + \partial_{\boldsymbol{x}} \cdot (\rho \boldsymbol{v} E + p \boldsymbol{v}) = 0. \end{cases} \tag{4.2}$$

The state variables are related by the constitutive relations for the monatomic gases:

$$p = R\rho\theta = \frac{2}{3}\rho e = p_0 e^{As} \rho^{\frac{5}{3}}. \tag{4.3}$$

Here the first relation is the ideal gas law, (2.11). We will take the gas constant $R = 1$. The second equality comes from the monatomic gases law (2.14). From these we have the last equality by the thermal dynamics relation $de = -pd(1/\rho) + \theta ds$, with s the *entropy*, for any choice of positive constants A and p_0.

The Euler equations are self-contained, as the stress tensor is now a given function of the conserved quantity and the heat flux is zero. So the number of the dependent variables is now $14 - 6 - 3 = 5$, the same as the number of equations.

Similarly, the linear Euler equations are the projection of the linearized Boltzmann equation (3.2) to the tangent space of the equilibrium Maxwellian states manifold at the base Maxwellian $\mathsf{M} = \mathsf{M}_0 = \mathsf{M}_{[\rho_0, \boldsymbol{v}_0, \theta_0]}$. Thus the *linear Euler equations in the kinetic form* is the macro projection of the linearized Boltzmann equation, (3.19):

$$(\mathsf{P}_0 \mathsf{g})_t + \nabla_{\boldsymbol{x}} \cdot \mathsf{P}_0 \boldsymbol{\xi} \mathsf{P}_0 \mathsf{g} = 0. \tag{4.4}$$

The resulting conservation laws, in terms of the macroscopic variables are the *linear Euler equations in gas dynamics*. There are two versions. The first is using the conservative macroscopic variables (3.26). This one is the same as the linearized version of the full Euler equations (4.2):

$$\begin{cases} \rho_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{m} = 0, \\ \boldsymbol{m}_t + \frac{2}{3} \nabla_{\boldsymbol{x}} E = 0, \\ E_t + \frac{5}{2} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{m} = 0. \end{cases} \tag{4.5}$$

Here we have, for simplicity, assumed that the base state for the linearization is taken to be $\rho_0 = 1,\ \boldsymbol{v}_0 = 0,\ \theta_0 = 1$. The second version is using the macroscopic variables through the orthogonal projections (3.27), (3.28):

$$\begin{cases} \bar{\rho}_t + \boldsymbol{v}_0 \cdot \nabla_{\boldsymbol{x}} \bar{\rho} + \nabla_{\boldsymbol{x}} \cdot \bar{\boldsymbol{m}} = 0, \\ \bar{\boldsymbol{m}}_t + \boldsymbol{v}_0 \cdot \nabla_{\boldsymbol{x}} \bar{\boldsymbol{m}} + \nabla_{\boldsymbol{x}} \bar{\rho} + \sqrt{\frac{2}{3}} \nabla_{\boldsymbol{x}} \bar{E} = 0, \\ \bar{E}_t + \boldsymbol{v}_0 \cdot \nabla_{\boldsymbol{x}} \bar{E} + \sqrt{\frac{2}{3}} \nabla_{\boldsymbol{x}} \cdot \bar{\boldsymbol{m}} = 0. \end{cases} \tag{4.6}$$

4.2. Euler, Euler flux projections

In the linear Euler equations (4.4), the flux operators $\mathsf{P}_0 \xi^i \mathsf{P}_0, i = 1, 2, 3,$ is on the 5-dimensional space $\{\chi_j,\ j = 0, \ldots, 4\}$. For definiteness, we consider $\mathsf{P}_0 \xi^1 \mathsf{P}_0$. By straightforward calculations, we have the following *Euler characteristics* λ_i^1 and *Euler characteristic directions* E_i^1:

$$\begin{cases} \mathsf{P}_0 \xi^1 \mathsf{E}_j^1 = \lambda_j^1 \mathsf{E}_j^1,\ j = 1, \ldots, 5, \\ \lambda_1^1 = v^1 - \boldsymbol{c},\ \lambda_2^1 = v^1,\ \lambda_3^1 = v^1 + \boldsymbol{c},\ \lambda_4^1 = \lambda_5^1 = v^1, \\ \mathsf{E}_1^1 = \sqrt{\frac{3}{10}} \chi_0 - \sqrt{\frac{1}{2}} \chi_1 + \sqrt{\frac{1}{5}} \chi_4, \\ \mathsf{E}_2^1 = -\sqrt{\frac{2}{5}} \chi_0 + \sqrt{\frac{3}{5}} \chi_4, \\ \mathsf{E}_3^1 = \sqrt{\frac{3}{10}} \chi_0 + \sqrt{\frac{1}{2}} \chi_1 + \sqrt{\frac{1}{5}} \chi_4, \\ \mathsf{E}_4^1 = \chi_2, \\ \mathsf{E}_5^1 = \chi_3. \end{cases} \tag{4.7}$$

Here we have taken the base state to be $(\rho, \boldsymbol{v}, \theta)$ and so the *sound speed* $\boldsymbol{c}$ is

$$\boldsymbol{c} = \sqrt{\frac{5\theta}{3}}. \tag{4.8}$$

The Euler characteristic directions are orthogonal:

$(\mathsf{E}_j^1, \mathsf{E}_k^1) = 0$ for $\lambda_j^1 \neq \lambda_k^1$. This is seen easily as follows:

$$\begin{aligned} \lambda_j^1 (\mathsf{E}_j^1, \mathsf{E}_k^1) &= (\mathsf{P}_0 \xi^1 \mathsf{E}_j^1, \mathsf{E}_k^1) = (\xi^1 \mathsf{E}_j^1, \mathsf{E}_k^1) = (\mathsf{E}_j^1, \xi^1 \mathsf{E}_k^1) \\ &= (\mathsf{E}_j^1, \mathsf{P}_0 \xi^1 \mathsf{E}_k^1) = \lambda_k^1 (\mathsf{E}_j^1, \mathsf{E}_k^1). \end{aligned}$$

The above Euler characteristic directions have been normalized:

$$(\mathsf{E}_i^1, \mathsf{E}_j^1) = \delta_{ij}, \ i, j = 1, \ldots, 5, \tag{4.9}$$

so that the macro projection can also be written as

$$\mathsf{P}_0 \mathsf{g} = \sum_{j=0}^{4} (\mathsf{g}, \mathsf{E}_j^1) \mathsf{E}_j^1.$$

We define the *Euler projections* B_j:

$$\mathsf{B}_j \mathsf{g} = (\mathsf{g}, \mathsf{E}_j^1) \mathsf{E}_j^1, \ j = 1, \ldots, 5. \tag{4.10}$$

For the 1-dimensional Boltzmann equation, (2.16), the linearized Euler equations are

$$\mathsf{P}_0 \mathsf{g}_t + (\mathsf{P}_0 \xi^1 \mathsf{P}_0 \mathsf{g})_x = 0. \tag{4.11}$$

The kernel of the linearized collision operator is the span of

$$\begin{aligned} &\mathsf{L}\psi_j = 0, \ j = 0, 1, 4, \ \psi_0 \equiv \sqrt{\mathsf{M}}, \ \psi_1 = \xi^1 \sqrt{\mathsf{M}}, \ \psi_4 \equiv \frac{|\boldsymbol{\xi}|^2}{2} \sqrt{\mathsf{M}}, \\ &\text{1-dimensional linear collision kernel,} \end{aligned} \tag{4.12}$$

and the macro and micro projections become, with $\boldsymbol{v} = (v, 0, 0)$,

$$\begin{cases} \mathsf{P}_0 \mathsf{g} \equiv (\mathsf{g}, \chi_0)\chi_0 + (\mathsf{g}, \chi_1)\chi_1 + (\mathsf{g}, \chi_4)\chi_4; \ \mathsf{P}_1 \equiv I - \mathsf{P}_0, \\ \chi_0 \equiv \mathsf{M}^{1/2}, \\ \chi_1 \equiv (\xi^1 - v)\mathsf{M}^{1/2}, \\ \chi_4 \equiv \frac{1}{\sqrt{6}}(|\boldsymbol{\xi} - \boldsymbol{v}|^2 - 3)\mathsf{M}^{1/2}. \end{cases} \tag{4.13}$$

The Euler characteristics are

$$\begin{cases} \mathsf{P}_0 \xi^1 \mathsf{E}_j = \lambda_j^1 \mathsf{E}_j, \ j = 1, 2, 3, \\ \lambda_1 = v - \boldsymbol{c}, \ \lambda_2 = v, \ \lambda_3 = v + \boldsymbol{c}, \\ \mathsf{E}_1 = \sqrt{\frac{3}{10}}\chi_0 - \sqrt{\frac{1}{2}}\chi_1 + \sqrt{\frac{1}{5}}\chi_4, \\ \mathsf{E}_2 = -\sqrt{\frac{2}{5}}\chi_0 + \sqrt{\frac{3}{5}}\chi_4, \\ \mathsf{E}_3 = \sqrt{\frac{3}{10}}\chi_0 + \sqrt{\frac{1}{2}}\chi_1 + \sqrt{\frac{1}{5}}\chi_4. \end{cases} \tag{4.14}$$

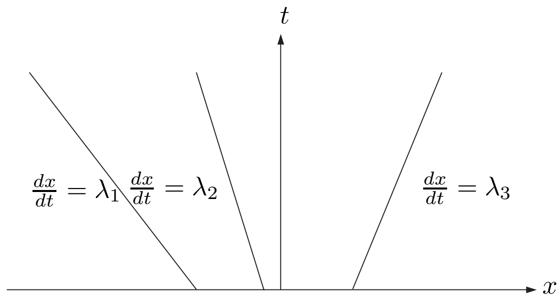


Figure 1: Euler waves.

The *Euler projections* B_j are defined as:

$$\mathsf{B}_j\mathsf{g} = (\mathsf{g}, \mathsf{E}_j)\mathsf{E}_j, \;\; j = 1, 2, 3. \tag{4.15}$$

The initial value problem for the Euler equations

$$\begin{cases} \mathsf{P}_0\mathsf{g}_t + (\mathsf{P}_0\xi^1\mathsf{P}_0\mathsf{g})_x = 0, \\ \mathsf{P}_0\mathsf{g}(x, 0, \boldsymbol{\xi}) = \bar{\mathsf{g}}(x, \boldsymbol{\xi}), \end{cases} \tag{4.16}$$

is decoupled by the Euler projections:

$$\begin{cases} g_j \equiv (\mathsf{g}, \mathsf{E}_j), \;\; \bar{g}_j \equiv (\bar{\mathsf{g}}, \mathsf{E}_j), \\ (g_j)_t + \lambda_j(g_j)_x = 0, \\ g_j(x, 0) = \bar{g}_j(x). \end{cases} \tag{4.17}$$

This can be solve by the characteristic method to yield the *Euler waves*, Figure 1:

$$\begin{cases} \mathsf{g}(x, t, \boldsymbol{\xi}) = \sum_{j=1}^{3} g_j(x, t)\mathsf{E}_j(\boldsymbol{\xi}), \\ g_j(x, t) = \bar{g}_j(x - \lambda_j t). \end{cases} \tag{4.18}$$

The *Euler flux projections* $\tilde{\mathsf{B}}_j$ are defined for nonzero Euler characteristic speeds λ_j as follows:

$$\tilde{\mathsf{B}}_j\mathsf{g} = \frac{1}{\lambda_j}(\xi^1\mathsf{g}, \mathsf{E}_j)\mathsf{E}_j, \;\; j = 1, 2, 3, \;\; \tilde{\mathsf{P}}_0 \equiv \sum_{j=1}^{3} \tilde{\mathsf{B}}_j. \tag{4.19}$$

It is easy to see that

$$\begin{aligned} \mathsf{P}_0 &= \sum_{j=1}^{3} \mathsf{B}_j,\ \mathsf{P}_1\mathsf{B}_i = \mathsf{B}_i\mathsf{P}_1 = 0, \\ \tilde{\mathsf{B}}_i\mathsf{E}_j &= \delta_{ij}\mathsf{E}_i,\ \mathsf{B}_i\xi^1\tilde{\mathsf{B}}_j = \delta_{ij}\lambda_i\tilde{\mathsf{B}}_i,\ i,j=1,2,3,\ \mathsf{P}_1\tilde{\mathsf{B}}_i = 0. \end{aligned} \tag{4.20}$$

Although $\mathsf{B}_i\mathsf{P}_1 = 0$, it is not true in general that $\tilde{\mathsf{B}}_i\mathsf{P}_1 = 0$.

For the study of initial-boundary and the boundary value problems, it is essential to differentiate the direction of the Euler waves. For this, we define the *upwind Euler projection* B_+ and the *downwind Euler projection* B_-:

$$\mathsf{B}_+ \equiv \sum_{\lambda_i>0} \mathsf{B}_i,\ \mathsf{B}_- \equiv \sum_{\lambda_i<0} \mathsf{B}_i. \tag{4.21}$$

Similarly, the *upwind-downwind Euler flux projections* $\tilde{\mathsf{B}}_\pm$ are defined as

$$\tilde{\mathsf{B}}_+ \equiv \sum_{\lambda_i>0} \tilde{\mathsf{B}}_i,\ \tilde{\mathsf{B}}_- \equiv \sum_{\lambda_i<0} \tilde{\mathsf{B}}_i. \tag{4.22}$$

4.3. Navier-Stokes dissipation parameters

The full Euler equations (4.2) are derived by assuming that the distribution function is locally Maxwellian, $\mathsf{f} = \mathsf{M}_{\mathsf{f}}$. For the derivation of the Navier-Stokes equations, the distribution function is assumed to deviates slightly from the local Maxwellian. For our purpose of the construction of the Green's function, we will consider the case of linear Navier-Stokes equations only. In the linear case, the Euler equations (4.6) are derived from the linear Boltzmann equation (3.2) by considering only the macro projection of the macro part $\mathsf{g}_0 = \mathsf{P}_0\mathsf{g}$ of the Boltzmann solution g, (4.6). The macro projection is the same as considering the conservation laws (4.6) induced by the macro part. We now derive the linear Navier-Stokes equations by the same procedure. First we write the linear Boltzmann equation (3.2) as:

$$(\mathsf{g}_0 + \mathsf{g}_1)_t + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}(\mathsf{g}_0 + \mathsf{g}_1) = \frac{1}{k}\mathsf{L}\mathsf{g}_1,\ \mathsf{g}_0 \equiv \mathsf{P}_0\mathsf{g},\ \mathsf{g}_1 \equiv \mathsf{P}_1\mathsf{g}.$$

The macro and micro projections of this equation are

$$(\mathsf{g}_0)_t + \partial_{\boldsymbol{x}} \cdot \mathsf{P}_0 \boldsymbol{\xi}(\mathsf{g}_0 + \mathsf{g}_1) = 0, \text{ macro part of linear Boltzmann equation;} \tag{4.23}$$

$$(\mathsf{g}_1)_t + \partial_{\boldsymbol{x}} \cdot \mathsf{P}_1 \boldsymbol{\xi}(\mathsf{g}_0 + \mathsf{g}_1) = \frac{1}{k}\mathsf{L}\mathsf{g}_1, \text{ micro part of linear Boltzmann equation.} \tag{4.24}$$

In the spirit of *Chapman-Enskog expansion*, we assume that the macro part g_0 dominates the micro part g_1 and that the differential $\nabla_{\boldsymbol{x},t}\mathsf{h}$ is dominated by the quantity h itself. Another way of thinking of this is to consider the *time-asymptotic dissipation* of the Boltzmann solution toward a given Maxwellian. As we will see, the macro part decays slower than the micro part. Thus we will make the simplification of the micro part (4.24) of the linear Boltzmann equation into

$$\partial_{\boldsymbol{x}} \cdot \mathsf{P}_1(\boldsymbol{\xi}\mathsf{g}_0) \equiv \frac{1}{k}\mathsf{L}\mathsf{g}_1, \text{ or } \mathsf{g}_1 \equiv k\mathsf{L}^{-1}[\partial_{\boldsymbol{x}} \cdot \mathsf{P}_1(\boldsymbol{\xi}\mathsf{g}_0)] \text{ Chapman-Enskog relation.} \tag{4.25}$$

Plug the *Chapman-Enskog relation* (4.25) into the macro part (4.23) of the Boltzmann equation:

$$\begin{aligned}
(\mathsf{g}_0)_t + \partial_{\boldsymbol{x}} \cdot \mathsf{P}_0(\boldsymbol{\xi}\mathsf{g}_0) &= k\partial_{\boldsymbol{x}} \cdot \mathsf{P}_0\boldsymbol{\xi}(-\mathsf{L})^{-1}[\partial_{\boldsymbol{x}} \cdot \mathsf{P}_1(\boldsymbol{\xi}\mathsf{g}_0)]; \text{ or,}\\
(\mathsf{g}_0)_t + \partial_{\boldsymbol{x}} \cdot \mathsf{P}_0(\boldsymbol{\xi}\mathsf{g}_0) &= k\sum_{i,j=1}^{3} \partial_{x^i}\partial_{x^j}\mathsf{P}_0\xi^i(-\mathsf{L})^{-1}[\mathsf{P}_1(\xi^j\mathsf{g}_0)],\\
&\qquad \text{Navier-Stokes equations in kinetic form.}
\end{aligned} \tag{4.26}$$

The linear Navier-Stokes equations in gas dynamics form are the conservation laws, obtained by integrating the kinetic equation (4.26) by the linear collision invariants, the kernel of the linear collision operator, (3.15), (3.26). The integration of (4.26) times χ_0 yields the usual conservation of mass:

$$\rho_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{m} + v_0 \cdot \nabla\rho = 0.$$

The integration of (4.26) times χ_4 yields the conservation of energy:

$$E_t + v_0 \cdot \nabla E + \sqrt{\frac{2}{3}}\nabla_{\boldsymbol{x}} \cdot \boldsymbol{m} = \kappa\triangle_{\boldsymbol{x}}E,$$

with the *heat conductivity coefficient* κ:

$$\kappa = -k\left(\mathsf{P}_1\xi^1\chi_4, \mathsf{L}^{-1}\mathsf{P}_1\xi^1\chi_4\right). \tag{4.27}$$

We now compute the integration of (4.26) times $\chi_l,\ l = 1,2,3$, to yield the conservation of momentum and the viscosity coefficients:

$$\begin{aligned}
&\frac{\partial \boldsymbol{m}_l}{\partial t} + \sum_{i=1}^{3} v_0^i \frac{\partial \boldsymbol{m}_l}{x^i} + \frac{\partial \rho}{\partial x^l} + \sqrt{\frac{2}{3}}\frac{\partial E}{\partial x^l} \\
&\quad = -k \sum_{i,j,n=1}^{3} \frac{\partial^2 \boldsymbol{m}_n(\boldsymbol{x},t)}{\partial x^i \partial x^j}\left(\chi_l, \mathsf{P}_0\left(\xi^i \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^j\chi_n)\right]\right)\right).
\end{aligned} \tag{4.28}$$

Observe that

$$\begin{aligned}
&\sum_{i,j,n=1}^{3} \frac{\partial^2 \boldsymbol{m}_n(\boldsymbol{x},t)}{\partial x^i \partial x^j}\left(\chi_l, \mathsf{P}_0\left(\xi^i \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^j\chi_n)\right]\right)\right) \\
&= \sum_{i,j,n=1}^{3} \frac{\partial^2 \boldsymbol{m}_n(\boldsymbol{x},t)}{\partial x^i \partial x^j}\left(\mathsf{P}_1\left(\xi^i\chi_l\right), \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^j\chi_n)\right]\right) \\
&= \sum_{i,j,n=1}^{3} \frac{\partial^2 \boldsymbol{m}_n(\boldsymbol{x},t)}{\partial x^i \partial x^j}\left(\mathsf{P}_1\left(\xi^i\left(\xi^l - v_0^l\right)\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^j\left(\xi^n - v_0^n\right)\sqrt{M})\right]\right) \\
&= \sum_{i,j,n=1}^{3} \frac{\partial^2 \boldsymbol{m}_n(\boldsymbol{x},t)}{\partial x^i \partial x^j}\left(\mathsf{P}_1\left(\xi^i\xi^l\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^j\xi^n\sqrt{M})\right]\right) \\
&= \sum_{j=1}^{3} \frac{\partial^2 \boldsymbol{m}_j(\boldsymbol{x},t)}{\partial x^l \partial x^j}\left(\mathsf{P}_1\left((\xi^l)^2\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1((\xi^j)^2\sqrt{M})\right]\right)(i=l,\ n=j) \\
&\quad + \sum_{n\neq l} \frac{\partial^2 \boldsymbol{m}_n(\boldsymbol{x},t)}{\partial x^n \partial x^l}\left(\mathsf{P}_1\left(\xi^n\xi^l\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^l\xi^n\sqrt{M})\right]\right)(j=l,\ i=n) \\
&\quad + \sum_{j\neq l} \frac{\partial^2 \boldsymbol{m}_l(\boldsymbol{x},t)}{\partial x^j \partial x^j}\left(\mathsf{P}_1\left(\xi^j\xi^l\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^j\xi^l\sqrt{M})\right]\right)(n=l,\ i=j).
\end{aligned}$$

Due to the rotational invariance of the L^{-1} and integration, for $i \neq j$,

$$\begin{aligned}
&\left(\mathsf{P}_1\left(\xi^i\xi^j\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left(\xi^i\xi^j\sqrt{M}\right)\right]\right) \\
&\quad = \left(\mathsf{P}_1\left((\xi^i)^2\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left((\xi^j)^2\sqrt{M}\right)\right]\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\left(\mathsf{P}_1\left((\xi^i)^2\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left((\xi^i)^2\sqrt{M}\right)\right]\right)\\
&= \left(\mathsf{P}_1\left(\left(\frac{\xi^i+\xi^j}{\sqrt{2}}\right)^2\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left(\left(\frac{\xi^i+\xi^j}{\sqrt{2}}\right)^2\sqrt{M}\right)\right]\right)\\
&= \left(\mathsf{P}_1\left(\left(\frac{(\xi^i)^2+(\xi^j)^2}{2}+\xi^i\xi^j\right)\sqrt{M}\right),\right.\\
&\qquad \left.\mathsf{L}^{-1}\left[\mathsf{P}_1\left(\left(\frac{(\xi^i)^2+(\xi^j)^2}{2}+\xi^i\xi^j\right)\sqrt{M}\right)\right]\right)\\
&= \frac{1}{2}\left(\mathsf{P}_1\left((\xi^i)^2\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left((\xi^i)^2\sqrt{M}\right)\right]\right)\\
&\qquad +\frac{3}{2}\left(\mathsf{P}_1\left(\xi^i\xi^j\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left(\xi^i\xi^j\sqrt{M}\right)\right]\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\left(\mathsf{P}_1\left((\xi^i)^2\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left((\xi^i)^2\sqrt{M}\right)\right]\right)\\
&\quad = 3\left(\mathsf{P}_1\left(\xi^i\xi^j\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left(\xi^i\xi^j\sqrt{M}\right)\right]\right).
\end{aligned}
\tag{4.29}
$$

With these, (4.28) becomes

$$
\begin{aligned}
&\frac{\partial \boldsymbol{m}_l}{\partial t}+\sum_{i=1}^{3} v_0^i\frac{\partial \boldsymbol{m}_l}{x^i}+\frac{\partial \rho}{\partial x^l}+\sqrt{\frac{2}{3}}\frac{\partial E}{\partial x^l}\\
&= 3\mu\frac{\partial^2 \boldsymbol{m}_l(\boldsymbol{x},t)}{\partial x^l\partial x^l}+\mu\left\{\sum_{j\neq l}\frac{\partial^2 \boldsymbol{m}_j(\boldsymbol{x},t)}{\partial x^l\partial x^j}+\sum_{n\neq l}\frac{\partial^2 \boldsymbol{m}_n(\boldsymbol{x},t)}{\partial x^n\partial x^l}+\sum_{j\neq l}\frac{\partial^2 \boldsymbol{m}_l(\boldsymbol{x},t)}{\partial x^j\partial x^j}\right\}\\
&= \mu\sum_{j=1}^{3}\left\{\frac{\partial^2 \boldsymbol{m}_l(\boldsymbol{x},t)}{\partial (x^j)^2}+2\frac{\partial^2 \boldsymbol{m}_j(\boldsymbol{x},t)}{\partial x^l\partial x^j}\right\}\\
&= \sum_{j=1}^{3}\frac{\partial}{\partial x^j}\left\{\mu\left(\frac{\partial \boldsymbol{m}_l}{\partial x^j}+\frac{\partial \boldsymbol{m}_j}{\partial x^l}-\frac{2}{3}\delta_{jl}\sum_{n=1}^{3}\frac{\partial \boldsymbol{m}_n}{\partial x^n}\right)+\frac{5}{3}\mu\delta_{jl}\sum_{n=1}^{3}\frac{\partial \boldsymbol{m}_n}{\partial x^n}\right\}
\end{aligned}
$$

Here the *viscosity* μ and the *bulk viscosity* μ_B are given as

$$
\mu \equiv -k\left(\mathsf{P}_1\left(\xi^1\xi^2\sqrt{M}\right), \mathsf{L}^{-1}\left[\mathsf{P}_1\left(\xi^1\xi^2\sqrt{M}\right)\right]\right),\ \mu_B \equiv \frac{5}{3}\mu. \tag{4.30}
$$

Using (4.27) and (4.30) the *linear Navier-Stokes equations in gas dynamics* *form* are then:

$$\begin{cases} \rho_t + \nabla_{\boldsymbol{x}} \cdot \boldsymbol{m} + v_0 \cdot \nabla_{\boldsymbol{x}} \rho = 0, \\ \frac{\partial \boldsymbol{m}_l}{\partial t} + \sum_{i=1}^{3} v_0^i \frac{\partial \boldsymbol{m}_l}{x^i} + \frac{\partial \rho}{\partial x^l} + \sqrt{\frac{2}{3}} \frac{\partial E}{\partial x^l} \\ = \sum_{j=1}^{3} \frac{\partial}{\partial x^j} \left\{ \mu \left(\frac{\partial \boldsymbol{m}_l}{\partial x^j} + \frac{\partial \boldsymbol{m}_j}{\partial x^l} - \frac{2}{3} \delta_{jl} \sum_{n=1}^{3} \frac{\partial \boldsymbol{m}_n}{\partial x^n} \right) - \frac{5}{3} \mu \delta_{jl} \sum_{n=1}^{3} \frac{\partial \boldsymbol{m}_n}{\partial x^n} \right\}, \ l = 1,2,3, \\ E_t + v_0 \cdot \nabla_{\boldsymbol{x}} E + \sqrt{\frac{2}{3}} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{m} = \kappa \triangle_{\boldsymbol{x}} E. \end{cases} \tag{4.31}$$

Another form of the Navier-Stokes equations that are convenient for the study of fluid waves are obtained by integrating the kinetic equation (4.26) times the Euler characteristic directions. We do this for 1-D case so that the Navier-Stokes equation in the kinetic form is

$$\frac{\partial g_0}{\partial t} + \frac{\partial}{\partial x} \mathsf{P}_0 \left(\xi^1 g_0 \right) = -k \frac{\partial^2}{\partial x^2} \mathsf{P}_0 \left(\xi^1 \mathsf{L}^{-1} \left[\mathsf{P}_1 (\xi^1 g_0) \right] \right). \tag{4.32}$$

Express

$$g_0 = \sum_{j=1}^{3} g_j \mathsf{E}_j$$

in terms of the Euler characteristic directions $\mathsf{E}_j,\ j = 1, 2, 3$, (4.14). This yields the Navier-Stokes equations in gas dynamics form:

$$\begin{cases} (g_i)_t + \lambda_i (g_i)_x = \sum_{j=1}^{3} A_{ij} (g_j)_{xx}, \ i = 1, 2, 3, \\ A_{ij} = -k \left(\mathsf{P}_1 \xi^1 \mathsf{E}_i, \mathsf{L}^{-1} \left[\mathsf{P}_1 (\xi^1 \mathsf{E}_j) \right] \right). \end{cases} \tag{4.33}$$

The dissipation parameters matrix A_{ij} are related to the viscosity μ and heat conductivity κ. For instance, from (4.14),

$$\begin{aligned} -\frac{1}{k} A_{11} &= \Big(\mathsf{P}_1 \xi^1 \mathsf{E}_1, \mathsf{L}^{-1} \Big[\mathsf{P}_1 (\xi^1 \mathsf{E}_1) \Big] \Big) \\ &= \frac{1}{5} \Big(\mathsf{P}_1 \xi^1 \Big(\sqrt{\frac{3}{2}} \chi_0 - \sqrt{\frac{5}{2}} \chi_1 + \chi_4 \Big), \mathsf{L}^{-1} \Big[\mathsf{P}_1 \xi^1 \Big(\sqrt{\frac{3}{2}} \chi_0 - \sqrt{\frac{5}{2}} \chi_1 + \chi_4 \Big) \Big] \Big) \end{aligned}$$

Note that $\xi^1 \chi_0$ is a linear collision invariant and so $\mathsf{P}_1 \xi^1 \chi_0 = 0$, and that $\xi^1 \chi_4$ is odd in ξ^1 and $\xi^1 \chi_1$ is even in ξ^1 and so $(\mathsf{P}_1 \xi^1 \chi_1, \mathsf{L}^{-1} \xi^1 \chi_4) = 0$. Thus it follows from (4.27), (4.30), and (4.29) that

$$\begin{aligned} A_{11} &= -k\frac{1}{5}\left\{\frac{5}{2}\left(\mathsf{P}_1\xi^1\chi_1, \mathsf{L}^{-1}\left[\mathsf{P}_1\xi^1\chi_1\right]\right) + \left(\mathsf{P}_1\xi_1\chi_4, \mathsf{L}^{-1}\mathsf{P}_1\xi_1\chi_4\right)\right\} \\ &= \frac{3}{2}\mu + \frac{1}{5}\kappa. \end{aligned}$$

Similarly, we have

$$\begin{aligned} A_{12} &= -k(\mathsf{P}_1\xi^1\mathsf{E}_1, \mathsf{L}^{-1}[\mathsf{P}_1(\xi^1\mathsf{E}_2)]) \\ &= \frac{\sqrt{3}}{5}\Big(\mathsf{P}_1\xi^1\Big(\sqrt{\frac{3}{2}}\chi_0 - \sqrt{\frac{5}{2}}\chi_1 + \chi_4\Big), \mathsf{L}^{-1}\Big[\mathsf{P}_1\xi^1\Big(-\sqrt{\frac{2}{3}}\chi_0 + \chi_4\Big)\Big]\Big) \\ &= \frac{\sqrt{3}}{5}\left(\mathsf{P}_1\xi_1\chi_4, \mathsf{L}^{-1}\mathsf{P}_1\xi_1\chi_4\right) = \frac{\sqrt{3}}{5}\kappa, \\ A_{13} &= -k\left(\mathsf{P}_1\xi^1\mathsf{E}_1, \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^1\mathsf{E}_3)\right]\right) \\ &= \frac{1}{5}\Big(\mathsf{P}_1\xi^1\Big(\sqrt{\frac{3}{2}}\chi_0 - \sqrt{\frac{5}{2}}\chi_1 + \chi_4\Big), \mathsf{L}^{-1}\Big[\mathsf{P}_1\xi^1\Big(\sqrt{\frac{3}{2}}\chi_0 + \sqrt{\frac{5}{2}}\chi_1 + \chi_4\Big)\Big]\Big) \\ &= \frac{1}{5}\left\{-\frac{5}{2}\left(\mathsf{P}_1\xi^1\chi_1, \mathsf{L}^{-1}\left[\mathsf{P}_1\xi^1\chi_1\right]\right) + \left(\mathsf{P}_1\xi_1\chi_4, \mathsf{L}^{-1}\mathsf{P}_1\xi_1\chi_4\right)\right\} \\ &= -\frac{3}{2}\mu + \frac{1}{5}\kappa, \\ A_{22} &= -k\left(\mathsf{P}_1\xi^1\mathsf{E}_2, \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^1\mathsf{E}_2)\right]\right) \\ &= -k\frac{3}{5}\Big(\mathsf{P}_1\xi^1\Big(-\sqrt{\frac{2}{3}}\chi_0 + \chi_4\Big), \mathsf{L}^{-1}\Big[\mathsf{P}_1\xi^1\Big(-\sqrt{\frac{2}{3}}\chi_0 + \chi_4\Big)\Big]\Big) \\ &= -k\frac{3}{5}\left(\mathsf{P}_1\xi_1\chi_4, \mathsf{L}^{-1}\mathsf{P}_1\xi_1\chi_4\right) = \frac{3}{5}\kappa, \\ A_{23} &= -k\left(\mathsf{P}_1\xi^1\mathsf{E}_2, \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^1\mathsf{E}_3)\right]\right) \\ &= -k\frac{\sqrt{3}}{5}\Big(\mathsf{P}_1\xi^1\Big(-\sqrt{\frac{2}{3}}\chi_0 + \chi_4\Big), \mathsf{L}^{-1}\Big[\mathsf{P}_1\xi^1\Big(\sqrt{\frac{3}{2}}\chi_0 + \sqrt{\frac{5}{2}}\chi_1 + \chi_4\Big)\Big]\Big) \\ &= -k\frac{\sqrt{3}}{5}\left(\mathsf{P}_1\xi_1\chi_4, \mathsf{L}^{-1}\mathsf{P}_1\xi_1\chi_4\right) = \frac{\sqrt{3}}{5}\kappa, \\ A_{33} &= -k\left(\mathsf{P}_1\xi^1\mathsf{E}_3, \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^1\mathsf{E}_3)\right]\right) \\ &= -k\frac{1}{5}\Big(\mathsf{P}_1\xi^1\Big(\sqrt{\frac{3}{2}}\chi_0 + \sqrt{\frac{5}{2}}\chi_1 + \chi_4\Big), \mathsf{L}^{-1}\Big[\mathsf{P}_1\xi^1\Big(\sqrt{\frac{3}{2}}\chi_0 + \sqrt{\frac{5}{2}}\chi_1 + \chi_4\Big)\Big]\Big) \\ &= -k\frac{1}{5}\left\{\frac{5}{2}\left(\mathsf{P}_1\xi^1\chi_1, \mathsf{L}^{-1}\left[\mathsf{P}_1\xi^1\chi_1\right]\right) + \left(\mathsf{P}_1\xi_1\chi_4, \mathsf{L}^{-1}\mathsf{P}_1\xi_1\chi_4\right)\right\} \\ &= \frac{3}{2}\mu + \frac{1}{5}\kappa. \end{aligned}$$

Due to the symmetry of A_{ij} we have

$$[A_{ij}] = \begin{bmatrix} \frac{3}{2}\mu + \frac{1}{5}\kappa & \frac{\sqrt{3}}{5}\kappa & -\frac{3}{2}\mu + \frac{1}{5}\kappa \\ \frac{\sqrt{3}}{5}\kappa & \frac{3}{5}\kappa & \frac{\sqrt{3}}{5}\kappa \\ -\frac{3}{2}\mu + \frac{1}{5}\kappa & \frac{\sqrt{3}}{5}\kappa & \frac{3}{2}\mu + \frac{1}{5}\kappa \end{bmatrix} \tag{4.34}$$

where the viscosity μ and heat conductivity κ are given by (4.30) and (4.27).

The dissipation parameters matrix $[A_{ij}]$ is non-negative, and but not positive definite. This is typical for systems of viscous conservation laws with physical viscosity matrix, [25]. It follows from the general theory for the system of hyperbolic-parabolic equations that the dominant dissipation waves are given by the diagonal system, [35]:

$$(\bar{g}_j)_t + \lambda_j(\bar{g}_j)_x = A_j(\bar{g}_j)_{xx},\ j = 1, 2, 3, \tag{4.35}$$

with the dissipation parameters

$$\begin{cases} A_1 \equiv A_{11} = -k\left(\mathsf{P}_1\xi^1\mathsf{E}_1, \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^1\mathsf{E}_1)\right]\right) = \frac{3}{2}\mu + \frac{1}{5}\kappa, \\ A_2 \equiv A_{22} = -k\left(\mathsf{P}_1\xi^1\mathsf{E}_2, \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^1\mathsf{E}_2)\right]\right) = \frac{3}{5}\kappa, \\ A_3 \equiv A_{33} = -k\left(\mathsf{P}_1\xi^1\mathsf{E}_3, \mathsf{L}^{-1}\left[\mathsf{P}_1(\xi^1\mathsf{E}_3)\right]\right) = A_1 = \frac{3}{2}\mu + \frac{1}{5}\kappa. \end{cases} \tag{4.36}$$

The Green's function

$$\begin{cases} (\bar{G}_j)_t + \lambda_j(\bar{G}_j)_x = A_j(\bar{G}_j)_{xx},\ j = 1, 2, 3, \\ \bar{G}(x, 0) = \delta(x), \end{cases} \tag{4.37}$$

consists of dissipation waves for this simplified system are the collection of heat kernels $H(x, t; A)$, Figure 2, c,f, Figure 1:

$$\begin{cases} \bar{G}_j(x, t) = H(x - \lambda_j t, t; A_j), \\ H(x, t; A) \equiv \frac{1}{\sqrt{4\pi A t}} e^{-\frac{x^2}{4At}}. \end{cases} \tag{4.38}$$

In the kinetic form, (4.32), (4.33), this is translated into

$$\bar{\mathbb{G}}(x, t) = \sum_{j=1}^{3} H(x - \lambda_j t, t; A_j)\mathsf{E}_j \otimes \langle \mathsf{E}_j|, \tag{4.39}$$

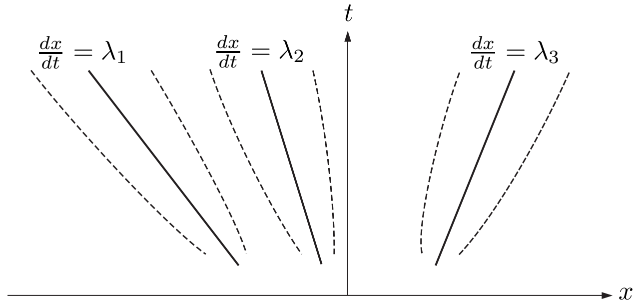


Figure 2: Navier-Stokes leading waves.

containing the operator in the function of the microscopic velocity:

$$\mathsf{E}_j \otimes \langle \mathsf{E}_j | \mathsf{g} \equiv (\mathsf{E}_j, \mathsf{g})\mathsf{E}_j. \tag{4.40}$$

5. 1-D Green's Function, Fluid-Like Waves

In this and the following two sections, we will construct the Green's function for the initial value problem for the linear Boltzmann equation. This and the next sections will concentrate on the 1-dimensional linearized Boltzmann equation, cf. (3.2),

$$\mathsf{g}_t + \xi^1 \partial_x \mathsf{g} = \mathsf{L}\mathsf{g}. \tag{5.1}$$

For simplicity, we will take the base Maxwellian for linearization, (3.1) to be $\mathsf{M} = \mathsf{M}_0 \equiv \mathsf{M}_{[1,0,1]}$ so that the explicit expression of the linearized collision operator is given in (3.5) and the Euler characteristic values are

$$\lambda_1 = -\sqrt{\frac{5}{3}},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{\frac{5}{3}}.$$

The *1-D Green's function* $\mathbb{G}(x, x_0, t, s, \boldsymbol{\xi}; \boldsymbol{\xi}_0) = \mathbb{G}(x - x_0, t - s, \boldsymbol{\xi}; \boldsymbol{\xi}_0)$ satisfies (2.17):

$$\begin{cases} \mathbb{G}_t + \xi^1 \partial_x \mathbb{G} = \mathsf{L}\mathbb{G} \text{ for } -\infty < x < \infty,\ t > 0, \\ \mathbb{G}(x, 0, \xi; \xi_0) = \delta(x)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0). \end{cases} \tag{5.2}$$

The Green's function $\mathbb{G}(x - y, t - s, \boldsymbol{\xi}; \boldsymbol{\xi}_0)$ describes the propagation of the perturbation over the Maxwellian M_0 when at time σ the perturbation consists of particles concentrated at space $x = 0$ and with microscopic velocity $\boldsymbol{\xi}_0$.

Consider the initial value problem

$$\begin{cases} \mathsf{g}_t + \xi^1 \partial_x \mathsf{g} = \mathsf{L}\mathsf{g}, \\ \mathsf{g}(x, 0, \boldsymbol{\xi}) = \mathsf{g}_0(x, \boldsymbol{\xi}). \end{cases} \tag{5.3}$$

Multiply the equation with the Green's function and integrate to yield the solution representation for (5.3):

$$\mathsf{g}(x,t,\boldsymbol{\xi}) = \int_{\mathbb{R}} \int_{\mathbb{R}^3} \mathbb{G}(x-y,t,\boldsymbol{\xi};\boldsymbol{\xi}_0)\mathsf{g}_0(y,\boldsymbol{\xi}_0)\, d\boldsymbol{\xi}_0 dy \text{ (locally in } (x,t,\boldsymbol{\xi})), \tag{5.4}$$

$$\mathsf{g}(x,t) \equiv \mathbb{G}^t \mathsf{g}_0(x) \text{ (locally in } (x,t) \text{ and in a semi-group format).} \tag{5.5}$$

Here the formula (5.4) is for pointwise expression; (5.5) is to view the Green's function as an operator. Consider the Fourier transform in the space variable x:

$$\hat{\mathsf{g}}(\eta,t) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\eta x} \mathsf{g}(x,t) dx, \ \mathsf{g}(x,t) \equiv \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\eta x} \hat{\mathsf{g}}(\eta) d\eta.$$

Take the Fourier transform of (2.17) to obtain

$$\begin{cases} \hat{\mathbb{G}}_t + i\xi^1 \eta \hat{\mathbb{G}} = \mathsf{L}\hat{\mathbb{G}}, \ -\infty < \eta < \infty, \ t > 0, \\ \hat{\mathbb{G}}(x,0,\boldsymbol{\xi};\boldsymbol{\xi}_0) = \frac{1}{\sqrt{2\pi}} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0). \end{cases} \tag{5.6}$$

With this, one can use the isometry property of the Fourier transform in the L^2_x to study the dissipation property of the Boltzmann solutions. Instead, we will *invert the Fourier transform for the fluid-like part* of $\hat{\mathbb{G}}$, to gain quantitative information on the Green's function in the physical space variable x. The formal expression of the Green's function is

$$\mathbb{G}(x,t,\boldsymbol{\xi};\boldsymbol{\xi}_0) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta x + (-i\xi^1\eta + \mathsf{L})t} \delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0) d\eta. \tag{5.7}$$

Plug this into the solution formula (5.4) to yield

$$\mathsf{g}(x,t,\boldsymbol{\xi}) = \int_{\mathbb{R}} \left[\frac{1}{2\pi} \int_{\mathbb{R}} e^{i\eta(x-y) + (-i\xi^1\eta + \mathsf{L})t} d\eta \mathsf{g}_0(y,\cdot) \right] (\boldsymbol{\xi}) dy. \tag{5.8}$$

Thus in the operator expression, c.f. (5.5), we can write

$$\mathbb{G}(x,t) \equiv \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta x+(-i\xi^1\eta+\mathsf{L})t}\,d\eta. \tag{5.9}$$

Remark 5.1. The operator $-i\xi^1\eta+\mathsf{L}$ on functions of the microscopic velocity $\boldsymbol{\xi}$ is the focus of our study for this section. As the operator has the one-dimensional Fourier parameter η, we do not have complete information of its spectrum. Nevertheless, the point spectrum for η near origin has explicit expression and correspond to the *fluid-like waves*, similar to that for the compressible Navier-Stokes equations. These waves dominate the Green's function time-asymptotically. The construction and description of these waves is the main concern of this section.

To obtain the complete description of the Green's function, there are two further steps to take. The first is to construct the *singular waves* through a series of *Picard iterations*, done in Section 6. These two constructions of waves offers two distinct decompositions of the Green's function, and allows us finally to apply the soft analysis of weighted energy method and Sobolev calculus toward the end of Section 6 for the complete pointwise description of the Green's function.

5.1. Spectral near origin

From the expression (5.9) for the Green's function, it is natural to consider the spectrum of the operator $-i\xi^1\eta+\mathsf{L}$. We consider first the situation near the origin, $|\eta|$ small. Let $(\sigma,\mathsf{e})=(\sigma(\eta),\mathsf{e}(\eta))$ be the eigenvalue-eigenfunctions for $-i\xi^1\eta+\mathsf{L}$:

$$(-i\xi^1\eta+\mathsf{L})\mathsf{e}=\sigma\mathsf{e}. \tag{5.10}$$

We know from Theorem 3.5 that the zero eigenvalue of the linear collision operator L is isolated and with multiplicity 3, (4.12). Thus for $|\eta|\ll 1$ the spectrum of $-i\xi^1\eta+\mathsf{L}$ should be also be 3 eigenvalues near zero.

The following Lemma 5.4 provides the spectrum information we need. Statement (I)–(III) of Lemma 5.4 are true for hard potentials with Grad cutoff. However, the analytic property, (IV), is only true for the hard sphere model, see Remark 5.5. Since analyticity is required for the following analysis, *we will assume the hard sphere model throughout*. Still, for the sake of completeness, we point out the fact that (I)–(III) are generally true for hard potentials with Grad cutoff.

Definition 5.2. For any complex $\phi, \psi \in L^2_{\boldsymbol{\xi}}$, define the pseudo inner product as

$$[\phi, \psi] = \int_{\mathbb{R}^3} \phi(\boldsymbol{\xi})\psi(\boldsymbol{\xi})d\boldsymbol{\xi}. \tag{5.11}$$

Remark 5.3. Note that we do not take complex conjugate in (5.11), so $[\cdot, \cdot]$ is not an inner product.

Since $(-i\eta\xi^1 + \mathsf{L})$ is the sum of two multiplicative operators, $-i\eta\xi^1, -\nu(\boldsymbol{\xi})$, and an integral operator K, $(-i\eta\xi^1 + \mathsf{L})$, is symmetric with respect to $[\cdot, \cdot]$. Namely,

$$\Big[\phi, (-i\eta\xi^1 + \mathsf{L})\psi\Big] = \Big[(-i\eta\xi^1 + \mathsf{L})\phi, \psi\Big]. \tag{5.12}$$

Consequently, if $\sigma_i \neq \sigma_k$, $[\mathsf{e}_j, \mathsf{e}_k] = 0$.

Lemma 5.4. *Consider the spectrum* $\mathrm{Spec}\,(\eta)$ *of the operator* $-i\xi^1\eta + \mathsf{L}$, $\eta \in \mathbb{R}$. *For hard potentials with Grad cutoff, the following statement* (I)–(III) *are true.*

(I) *For any* $0 < \delta \ll 1$, *there corresponds* $\tau = \tau(\delta) > 0$ *such that*

  (i) *For* $|\eta| > \delta$,

$$\mathrm{Spec}\,(\eta) \subset \{z \in \mathbb{C} : \mathrm{Re}\,(z) < -\tau\}.$$

  (ii) *For* $|\eta| \leq \delta$, *the spectrum within the region* $\{z \in \mathbb{C} : -\tau \leq \mathrm{Re}\,(z)\}$ *consisting of exactly three eigenvalues* $\sigma_1(\eta), \sigma_2(\eta), \sigma_3(\eta)$:

$$\mathrm{Spec}\,(\eta) \cap \{z \in \mathbb{C} : -\tau \leq \mathrm{Re}\,(z)\} = \{\sigma_1(\eta), \sigma_2(\eta), \sigma_3(\eta)\}.$$

(II) *For* $|\eta| \ll 1$, *the eigenvalues* $\sigma_1(\eta), \sigma_2(\eta), \sigma_3(\eta)$ *satisfy*

$$\begin{cases} \sigma_1(\eta) = -i\eta\lambda_1 - A_1\eta^2 + O(|\eta|^3), \\ \sigma_2(\eta) = -i\eta\lambda_2 - A_2\eta^2 + O(|\eta|^3), \\ \sigma_3(\eta) = -i\eta\lambda_3 - A_3\eta^2 + O(|\eta|^3), \end{cases} \tag{5.13}$$

cf. Figure 3*, where* $A_j = -(\mathsf{P}_1\xi^1\mathsf{E}_j, \mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_j)$ *is the Navier-Stokes dissipation coefficient, cf.* (4.36)*.*

In view of (5.13)*, for* $0 < |\eta| \ll 1$*, the eigenvalues* $\sigma_1(\eta), \sigma_2(\eta), \sigma_3(\eta)$ *are distinct. After being normalized according to* $[\mathsf{e}_j, \mathsf{e}_k] = \delta_{jk}$*, cf. Remark* 5.3*, the eigenvectors* $\mathsf{e}_1(\eta), \mathsf{e}_2(\eta), \mathsf{e}_3(\eta)$ *take the following form:*

$$\begin{aligned} \mathsf{e}_j(\eta) =& \mathsf{E}_j + i\eta\Big(\sum_{k\neq j} \frac{A_{jk}}{\lambda_k - \lambda_j}\mathsf{E}_k + \mathsf{e}_j^{\perp}\Big) + O(|\eta|^2), \\ \mathsf{e}_j^{\perp} =& \mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_j, \end{aligned} \tag{5.14}$$

where $A_{jk} = -(\mathsf{P}_1\xi^1\mathsf{E}_j, \mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_k)$ *is the Navier-Stokes dissipation coefficient, cf.* (4.34)*.*

(III) *For all* $0 < \delta \ll 1$*, the semigroup* $e^{(-i\xi^1\eta+\mathsf{L})t}$ *can be decomposed as*

$$e^{(-i\eta\xi^1+\mathsf{L})t} = \Pi_\delta + \chi_{\{|\eta|<\delta\}}\frac{1}{2\pi i}\oint_\Gamma e^{zt}\Big(z - (-i\eta\xi^1 + \mathsf{L})\Big)^{-1}dz, \tag{5.15}$$

where $\|\Pi_\delta\|_{L^2_{\boldsymbol{\xi}}} = O(1)e^{-a(\tau)t}$*,* $a(\tau) > 0$ *depends on* τ *(and therefore on* δ*),* $\chi_{\{\cdot\}}$ *is the indicator function, and* Γ *can be any close curve that lies entirely on* $\{\mathrm{Re}\, z > -\tau\}$ *and that encloses the three eigenvalues* $\sigma_1(\eta), \sigma_2(\eta), \sigma_3(\eta)$*, Figure* 3*.*

(IV) *Particularly, for the hard sphere model,* $\sigma_j(\eta), \mathsf{e}_j(\eta)$ *are holomorphic in* η *for all* $|\eta| \ll 1$*. Consequently,* (5.13) *and* (5.14) *hold even for complex* η*.*

Proof. The spectrum of $-i\xi^1\eta + \mathsf{L}$ has been studied and this lemma has been proved, [41], [36], [14], [45], by making use of the spectral gap at origin for L, Theorem 3.5. Here we emphasize the computation of eigenvalues and eigenvectors, (5.13) and (5.14), and prove (II) only.

Apply Macro-Micro projection to the equation $(-i\xi^1\eta + \mathsf{L})\mathsf{e}_j = \sigma_j\mathsf{e}_j$:

$$-i\eta\mathsf{P}_0\xi^1\Big((\mathsf{P}_0\mathsf{e}_j) + (\mathsf{P}_1\mathsf{e}_j)\Big) = \sigma_j(\mathsf{P}_0\mathsf{e}_j), \tag{5.16a}$$

$$-i\eta\mathsf{P}_1\xi^1(\mathsf{P}_0\mathsf{e}_j) - i\eta\mathsf{P}_1\xi^1(\mathsf{P}_1\mathsf{e}_j) + \mathsf{L}(\mathsf{P}_1\mathsf{e}_j) = \sigma_j(\mathsf{P}_1\mathsf{e}_j). \tag{5.16b}$$

In view of (5.16a), it is convenient to set $\sigma_j = i\eta\gamma_j$. From (5.16b) we can solve $\mathsf{P}_1\mathsf{e}_j$ in terms of $\mathsf{P}_0\mathsf{e}_j$ as $\mathsf{P}_1\mathsf{e}_j = i\eta[\mathsf{L} - i\eta\mathsf{P}_1\xi^1 - i\eta\gamma_j]^{-1}\mathsf{P}_1\xi^1(\mathsf{P}_0\mathsf{e}_j)$.

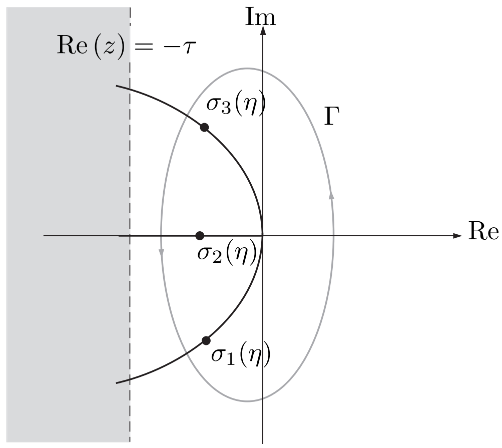


Figure 3: Spectrum near origin.

Substituting this back to (5.16a), we obtain the following equation for $\mathsf{P}_0\mathsf{e}_j$:

$$\left(\mathsf{P}_0\xi^1 + i\eta\mathsf{P}_0\xi^1\Big(\mathsf{L} - i\eta\mathsf{P}_1\xi^1 - i\eta\gamma_j\Big)^{-1}\mathsf{P}_1\xi^1\right)(\mathsf{P}_0\mathsf{e}_j) = -\gamma_j(\mathsf{P}_0\mathsf{e}_j). \quad (5.17)$$

Through Macro-Micro decomposition, we have transformed the original infinite dimensional eigen-equation into the 3 dimensional equation (5.17). Our next step is to solve (5.17) by the implicit function theorem. We will do this only for e_1, since $\mathsf{e}_2, \mathsf{e}_3$ can be treated similarly.

Instead of normalizing e_1 according to $[\mathsf{e}_1, \mathsf{e}_1] = 1$, we temporarily adopt the normalization e_1 to e'_1 by $(\mathsf{E}_1, \mathsf{e}'_1) = 1$. Set

$$\mathsf{P}_0\mathsf{e}'_1 = \mathsf{E}_1 + \beta_2\mathsf{E}_2 + \beta_3\mathsf{E}_3.$$

Under this setup, (5.17) becomes

$$\begin{aligned}
&\mathscr{G}_1(\gamma_1, \beta_2, \beta_3; \eta) \\
&\quad \equiv \lambda_1 + i\eta\Big(\mathsf{P}_0\xi^1(\mathsf{L} - i\eta\mathsf{P}_1\xi^1 - i\eta\gamma_j)^{-1}\mathsf{P}_1\xi^1(\mathsf{E}_1 + \beta_2\mathsf{E}_2 + \beta_3\mathsf{E}_3), \mathsf{E}_1\Big) \\
&\qquad +\gamma_1 = 0, \\
&\mathscr{G}_2(\gamma_1, \beta_2, \beta_3; \eta) \\
&\quad \equiv \lambda_2\beta_2 + i\eta\Big(\mathsf{P}_0\xi^1(\mathsf{L} - i\eta\mathsf{P}_1\xi^1 - i\eta\gamma_j)^{-1}\mathsf{P}_1\xi^1(\mathsf{E}_1 + \beta_2\mathsf{E}_2 + \beta_3\mathsf{E}_3), \mathsf{E}_2\Big) \\
&\qquad +\gamma_1\beta_2 = 0,
\end{aligned}$$

$$\begin{aligned}&\mathscr{G}_3(\gamma_1,\beta_2,\beta_3;\eta)\\ &\quad\equiv\ \lambda_3\beta_3+i\eta\Big(\mathsf{P}_0\xi^1(\mathsf{L}-i\eta\mathsf{P}_1\xi^1-i\eta\gamma_j)^{-1}\mathsf{P}_1\xi^1(\mathsf{E}_1+\beta_2\mathsf{E}_2+\beta_3\mathsf{E}_3),\mathsf{E}_3\Big)\\ &\qquad+\gamma_1\beta_3=0. \qquad (5.18)\end{aligned}$$

This equation has a solution $(\gamma_1,\beta_2,\beta_3;\eta)=(-\lambda_1,0,0;0)$. Moreover,

$$\left|\frac{\partial(\mathscr{G}_1,\mathscr{G}_2,\mathscr{G}_3)}{\partial(\gamma_1,\beta_2,\beta_3)}\right|_{(-\lambda_1,0,0,0)}=\begin{vmatrix}1&0&0\\0&(\lambda_2-\lambda_1)&0\\0&0&(\lambda_3-\lambda_1)\end{vmatrix}\neq 0.$$

Consequently, by the implicit function theorem, for $|\eta|\ll 1$, (5.18) has the solution $(\gamma_1(\eta),\beta_2(\eta),\beta_3(\eta))$, smooth in η, with $(\gamma_1(0),\beta_2(0),\beta_3(0))=(-\lambda_1,0,0)$.

Differentiating (5.18) with respect to η yields

$$\begin{aligned}\frac{\partial\gamma_1}{\partial\eta}(0)&=-i\Big(\mathsf{P}_0\xi^1\mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_1,\mathsf{E}_1\Big)=iA_1,\\ \frac{\partial\beta_2}{\partial\eta}(0)&=-i\Big(\mathsf{P}_0\xi^1\mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_1,\mathsf{E}_2\Big)\frac{1}{\lambda_2-\lambda_1}=\frac{iA_{12}}{\lambda_2-\lambda_1},\\ \frac{\partial\beta_3}{\partial\eta}(0)&=-i\Big(\mathsf{P}_0\xi^1\mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_1,\mathsf{E}_3\Big)\frac{1}{\lambda_3-\lambda_1}=\frac{iA_{13}}{\lambda_3-\lambda_1}.\end{aligned}$$

Hence $\sigma_1(\eta)=-i\eta\lambda_1-A_1\eta^2+O(\eta^3)$. Moreover, since $\mathsf{P}_1\mathsf{e}_1'=i\eta[\mathsf{L}-i\eta\mathsf{P}_1\xi^1-i\eta\gamma_1]^{-1}\mathsf{P}_1\xi^1(\mathsf{P}_0\mathsf{e}_1')$,

$$\mathsf{e}_1'(\eta)=\mathsf{E}_1+i\eta\left(\frac{A_{12}}{\lambda_2-\lambda_1}\mathsf{E}_2+\frac{A_{13}}{\lambda_3-\lambda_1}\mathsf{E}_3+\mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_1\right)+O(\eta^2).$$

We now normalize e_1': put $\mathsf{e}_1=\frac{\mathsf{e}_1'}{\sqrt{[\mathsf{e}_1',\mathsf{e}_1']}}$. Note that $[\mathsf{e}_1',\mathsf{e}_1']$ is generally a complex number. By some direct computation, $[\mathsf{e}_1',\mathsf{e}_1']=1+O(\eta^2)\approx 1$. We unambiguously refer to $\sqrt{[\mathsf{e}_1',\mathsf{e}_1']}$ as the complex square root closer to 1. Under this choice, $\sqrt{[\mathsf{e}_1',\mathsf{e}_1']}$ is smooth in η, and, in the case of (IV), holomorphic in η. Also we have (5.14):

$$\begin{aligned}\mathsf{e}_1(\eta)=\big(1+O(\eta^2)\big)\left(\mathsf{E}_1+i\eta\left(\frac{A_{12}}{\lambda_2-\lambda_1}\mathsf{E}_2+\frac{A_{13}}{\lambda_3-\lambda_1}\mathsf{E}_3+\mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_1\right)\right)\\ +O(\eta^2)\end{aligned}$$

$$=\mathsf{E}_1 + i\eta\left(\frac{A_{12}}{\lambda_2-\lambda_1}\mathsf{E}_2 + \frac{A_{13}}{\lambda_3-\lambda_1}\mathsf{E}_3 + \mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_1\right) + O(\eta^2). \qquad \square$$

Remark 5.5. The linear collision operator $\mathsf{L} = -\nu + \mathsf{K}$ has the property that

$$\nu(\boldsymbol{\xi}) \sim 1 + |\boldsymbol{\xi}|^\alpha,$$

where $\alpha = 1$ for the hard sphere models and $0 < \alpha < 1$ for Grad cut-off hard potentials. The analyticity of the eigenvalues $\sigma_i(\eta)$ holds only for the hard sphere models because of this. To see that, note that

$$\frac{\gamma_1+\lambda}{-i\eta} = \big(\mathsf{E}_1, \mathsf{P}_0\xi^1[\mathsf{L} - i\eta\mathsf{P}_1\xi^1 + i\eta(\lambda_1+\rho)]^{-1}\mathsf{P}_1\xi^1(\mathsf{E}_1+\mathsf{b}_0)\big)$$

The key is the behavior at $|\boldsymbol{\xi}| = \infty$:

$$\frac{\partial^n}{\partial\eta}[\mathsf{L} - i\eta\mathsf{P}_1\xi^1 + i\eta(\lambda_1+\rho)]^{-1} \sim n!\frac{|\xi|^n}{\nu^n(\boldsymbol{\xi})}e^{-\frac{|\xi|^2}{2}},$$

for hard sphere model. Hence,

$$\frac{\gamma_1+\lambda}{-i\eta} \sim \sum_{n=1}^{\infty}\frac{n(n+1)(n-1)!}{(n+1)!}\eta^{n+1} \sim -i\lambda_1\eta + \sum_{n=1}^{\infty}\eta^{n+1},$$

which converges as $|\eta| \ll 1$. On the other hand, for hard potentials

$$\frac{\gamma_1+\lambda}{-i\eta} \sim \sum_{n=1}^{\infty} n!\int |\boldsymbol{\xi}|^{n(1-\alpha)}e^{-\frac{|\xi|^2}{2}}d\boldsymbol{\xi} \sim \sum_{n=1}^{\infty}\Gamma(n(1-\alpha))\eta^{n+1},$$

which diverges for any $0 < \alpha < 1$ and $\eta \neq 0$. Thus the complex analytic method that we will use to obtain the explicit expression of the inverse Fourier transform for the hard sphere models cannot be applied to the hard potential models. There is the study for the hard potential models, [28], which yields a version of the pointwise estimates for the Green's function weaker than that of Theorem 5.9 for hard sphere models. The real analytic method used in [28] is motivated by [29] for viscous conservation laws and [10] for the boundary layers. It would be interesting to obtain, for the hard potential models, similar estimates as for the hard sphere models in Theorem 5.9.

We now proceed to compute explicitly the second term of (7.21).

Definition 5.6. For $\phi \in L^2_{\boldsymbol{\xi}}$,

$$\Big(\mathsf{e}_j \otimes [\mathsf{e}_j|\Big)\phi \equiv [\phi, \mathsf{e}_j]\mathsf{e}_j.$$

Lemma 5.7. *For* $|\eta| \ll 1$,

$$\frac{1}{2\pi i}\oint_\Gamma e^{zt}\Big(z - (-i\eta\xi^1 + \mathsf{L})\Big)^{-1} dz = \sum_{j=1}^{3} e^{\sigma_j t}\mathsf{e}_j \otimes [\mathsf{e}_j|. \tag{5.19}$$

Proof. Define a subspace $\mathcal{H}_\eta$ of $L^2_{\boldsymbol{\xi}}$ as

$$\mathcal{H}_\eta \equiv \{\phi : [\phi, \mathsf{e}_j(\eta)] = 0, \text{ for } j = 1,2,3\}.$$

Let $\langle\cdot\rangle$ be the linear span. Clearly, $L^2_{\boldsymbol{\xi}} = \langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle \oplus H_\eta$, $\sum_1^3 \mathsf{e}_j \otimes [\mathsf{e}_j|$ is the projection onto $\langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle$ along H_η, and $1 - \sum_1^3 \mathsf{e}_j \otimes [\mathsf{e}_j|$ is the projection onto H_η along $\langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle$. Moreover, since e_j is an eigenvector of $(-i\eta\xi^1 + \mathsf{L})$ and $(-i\eta\xi^1 + \mathsf{L})$ is symmetric with respect to $[\cdot,\cdot]$, (5.12), both $\langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle$ and H_η are $(z - (-i\eta\xi^1 + \mathsf{L}))$–invariant:

$$\Big(z - (-i\eta\xi^1 + \mathsf{L})\Big)\langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle \subset \langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle, \quad \Big(z - (-i\eta\xi^1 + \mathsf{L})\Big)\mathcal{H}_\eta \subset \mathcal{H}_\eta. \tag{5.20}$$

Set

$$S_\eta = \Big(-i\eta\xi^1 + \mathsf{L}\Big)\sum_{j=1}^{3} \mathsf{e}_j \otimes [\mathsf{e}_j|, \quad T_\eta = \Big(-i\eta\xi^1 + \mathsf{L}\Big)\Big(1 - \sum_{j=1}^{3} \mathsf{e}_j \otimes [\mathsf{e}_j|\Big).$$

As a consequent of (5.20), $(z - S_\eta) : \langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle \longrightarrow \langle \mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\rangle$, $(z - T_\eta) : \mathcal{H}_\eta \longrightarrow \mathcal{H}_\eta$, and $(z - (-i\eta\xi^1 + \mathsf{L}))^{-1} = (z - S_\eta)^{-1} + (z - T_\eta)^{-1}$. S_η becomes a 3 by 3 diagonal matrix under the basis $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$, so by some direct computations:

$$\frac{1}{2\pi i}\oint_\Gamma e^{zt}(z - S_\eta)^{-1} dz = \sum_{j=1}^{3} e^{\sigma_j t}\mathsf{e}_j \otimes [\mathsf{e}_j|.$$

It remains only to show $\oint_\Gamma e^{zt}(z - T_\eta)^{-1} dz = 0$. First, since $(-i\eta\xi^1 + \mathsf{L}) \sim$

$[-(1+|\boldsymbol{\xi}|+O(\eta))\nu_0+\mathsf{K}]$, for $|\eta|\ll 1$,

$$(-i\eta\xi^1+\mathsf{L})^{-1}=L^{-1}+\mathcal{O}(\eta), \tag{5.21}$$

where $\mathcal{O}(\eta)$ symbolizes an operator with $\|\mathcal{O}(\eta)\|_{L^2_{\boldsymbol{\xi}}}=O(\eta)$. Next, for each $\phi\in\mathcal{H}_\eta$,

$$\begin{aligned}&[\phi,\mathsf{e}_j(\eta)]=0=[\phi,\mathsf{E}_j]+[\phi,O(\eta)]=(\phi,\mathsf{E}_j)+\|\phi\|\,O(\eta)\\ &\qquad\Longrightarrow\quad(\phi,\mathsf{E}_j)=\|\phi\|\,O(\eta).\end{aligned}$$

Therefore, $\mathsf{P}_0\phi=O(\eta)\,\|\phi\|$. Combine this fact with (5.21) and Lemma 3.5 to obtain

$$\begin{aligned}\frac{\left\|T_\eta^{-1}\phi\right\|_{L^2_{\boldsymbol{\xi}}}}{\|\phi\|_{L^2_{\boldsymbol{\xi}}}}&=\frac{\left\|\left(\mathsf{L}^{-1}+\mathcal{O}(\eta)\right)\left(\mathsf{P}_1\phi+\|\phi\|\,O(\eta)\right)\right\|_{L^2_{\boldsymbol{\xi}}}}{\|\phi\|_{L^2_{\boldsymbol{\xi}}}}\\ &\le\frac{\nu_1\,\|\mathsf{P}_1\phi\|_{L^2_{\boldsymbol{\xi}}}+O(\eta)\,\|\phi\|_{L^2_{\boldsymbol{\xi}}}}{\|\phi\|_{L^2_{\boldsymbol{\xi}}}}=\nu_1+O(\eta).\end{aligned}$$

Consequently, T_η^{-1} is uniformly bounded for $|\eta|\ll 1$. Since Γ encloses $\sigma_1,\sigma_2,\sigma_3$ and $\sigma_j=O(\eta)$, in the region enclosed by Γ we may assume $|z|\sim|\eta|\ll 1$. Under this assumption, $(z-T_\eta)^{-1}=-T_\eta^{-1}(1+z(T_\eta)^{-1}+(zT_\eta^{-1})^2+\ldots)$ is holomorphic in z and therefore $\oint_\Gamma e^{zt}(z-T_\eta)^{-1}dz=0$. $\square$

5.2. Fluid-like Waves

Plugging (7.21) and (5.19) to (5.7), we have

$$\mathbb{G}(x,t,\boldsymbol{\xi};\boldsymbol{\xi}_0)=\frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\circ\Pi_\delta+\sum_{j=1}^{3}\frac{1}{2\pi}\int_{-\delta}^{\delta}e^{ix\eta+\sigma_j(\eta)t}\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|\,d\eta,$$

where $\mathscr{F}^{-1}$ symbolizes the inverse Fourier transform. Since $\mathscr{F}^{-1}:L^2_\eta\to L^2_x$ is an isometry,

$$\left\|\mathscr{F}^{-1}\circ\Pi_\delta\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}=O(1)e^{-\frac{t}{C(\delta)}}, \tag{5.22}$$

for some $C(\delta)>0$ depending on δ.

Definition 5.8. Define the fluid-like waves as

$$\mathbb{G}_L(x,t,\boldsymbol{\xi};\delta) \equiv \sum_{j=1}^{3} \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{ix\eta+\sigma_j(\eta)t} \mathsf{e}_j(\eta) \otimes [\mathsf{e}_j(\eta)|\, d\eta. \tag{5.23}$$

To highlight the dependence on t, we will frequently abbreviate $\mathbb{G}_L(x,t,\boldsymbol{\xi};\delta)$ as $\mathbb{G}_L^t$.

Our goal of this subsection is to the extract leading terms, the main fluid-like waves, from $\mathbb{G}_L$.

Theorem 5.9. *For any fixed $C_j > A_j,\ j=1,2,3,$ there exists $C>0$ and $\delta > 0$ such that*

$$\begin{aligned} &\left\| \mathbb{G}_L(x,t,\boldsymbol{\xi};\delta) - \sum_{j=1}^{3} \frac{e^{-\frac{(x-\lambda_j t)^2}{4A_j t}}}{\sqrt{4\pi A_j(t+1)}} \mathsf{E}_j \otimes \langle \mathsf{E}_j| \right\|_{L^2_{\boldsymbol{\xi}}} \\ &= O(1)\left(\sum_{j=1}^{3} \frac{e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}}{t+1} + e^{-\frac{t}{C}} \right). \end{aligned} \tag{5.24}$$

Note that here our estimate is pointwise in x, unlike (), which is L^2 in x.

Proof. Pick $C_j > C_j'' > C_j' > A_j$. We then fix some $\delta > 0$, so small such that whenever $|\eta| < 2\delta$:

(1) σ_j, e_j are holomorphic in η.

(2) (5.13), (5.14), and Lemma 5.7 hold. Moreover,

$$\left|\sigma_j(\eta) - \Big(-i\lambda_j\eta - A_j\eta^2\Big)\right| < \min\left\{ \left(1-\frac{A_j}{C_j'}\right), \frac{1}{3} \right\} \times A_j|\eta|^2. \tag{5.25}$$

Because of the analyticity property of the eigenvalues and eigenfunctions, we can apply the *complex analysis method to compute explicitly the inverse Fourier transform* by the contour integral. This will yield the fluid-like behavior of the Green's function. By Cauchy integral theorem, we can substitute the path $[-\delta,\delta]$ by any other path, lying entirely in the analytic region $\{|\eta| < 2\delta\}$, with the same endpoints. We choose the contour

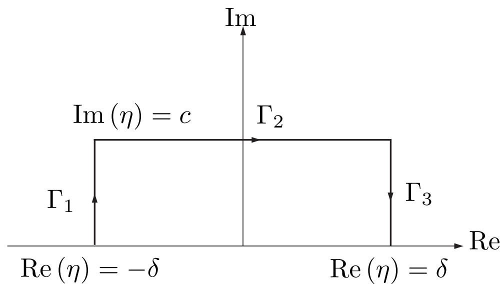


Figure 4: Path of integration.

$\Gamma_1 + \Gamma_2 + \Gamma_3$, Figure 4,

$$\begin{aligned}
\Gamma_1 = \Gamma_1(c) &\equiv \{\eta : \ \mathrm{Re}\,(\eta) = -\delta, \ \mathrm{Im}\,(\eta) \text{ lies between } 0 \text{ and } c\}, \\
\Gamma_2 = \Gamma_2(c) &\equiv \{\eta : \ -\delta \le \mathrm{Re}\,(\eta) \le \delta, \ \mathrm{Im}\,(\eta) = c\}, \\
\Gamma_3 = \Gamma_3(c) &\equiv \{\eta : \ \mathrm{Re}\,(\eta) = \delta, \ \mathrm{Im}\,(\eta) \text{ lies between } 0 \text{ and } c\},
\end{aligned} \tag{5.26}$$

where c is a constant, specified later. On Γ_2, put $\mathrm{Re}\,(\eta) = u$ so that $\eta = u + ic$.

Since

$$e^{ix\eta+\sigma_j(\eta)t} = \exp\left[-A_j t\left(\eta - i\frac{x-\lambda_j t}{2A_j t}\right)^2 - \frac{(x-\lambda_j t)^2}{4A_j t} + O(\eta^3)t\right],$$

it is desirable to set $c = \frac{x-\lambda_j t}{2A_j t}$. However, in order that $\Gamma_1 + \Gamma_2 + \Gamma_3$ lies entirely in $\{|\eta| < 2\delta\}$, $|c|$ cannot be arbitrarily large. For this reason, we will process the integral on Γ_2 in two separated situations.

Case 1: $\left|\frac{x-\lambda_j t}{2A_j t}\right| < \frac{\delta}{2}$ In this case, set $c = \frac{x-\lambda_j t}{2A_j t}$.

On Γ_2,

$$e^{ix\eta+\sigma_j(\eta)t} = e^{-\frac{(x-\lambda_j t)^2}{4A_j t}} e^{-A_j t u^2} + e^{-\frac{(x-\lambda_j t)^2}{4A_j t}} e^{-A_j t u^2}\left(e^{O(|\eta|^3)t} - 1\right).$$

By (5.25),

$$\begin{aligned}
& e^{-\frac{(x-\lambda_j t)^2}{4A_j t}} e^{-A_j t u^2} \left\{ \begin{array}{l} O(1)|\eta|^3 t, \text{ for } |\eta|^3 t < 1 \\ e^{O(|\eta|^3)t} \quad, \text{ for } |\eta|^3 t > 1 \end{array} \right\} \\
= \ & O(1) e^{-\frac{(x-\lambda_j t)^2}{4A_j t}} e^{-A_j t u^2} e^{O(|\eta|^3)t} |\eta|^3 t
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\exp\left[-\frac{(x-\lambda_j t)^2}{4A_j t} - A_j t u^2 + \min\left\{\left(1-\frac{A_j}{C_j'}\right),\frac{1}{3}\right\}\right.\\
&\quad \left.\times tA_j\left(u^2+\left(\frac{x-\lambda_j t}{2A_j t}\right)^2\right)\right]|\eta|^3 t\\
&= O(1)\exp\left[-\frac{(x-\lambda_j t)^2}{4A_j t} - A_j t u^2 + \left(1-\frac{A_j}{C_j'}\right)\frac{(x-\lambda_j t)^2}{4A_j t} + \frac{A_j t u^2}{3}\right]\\
&\quad \times\left(|u|^3 t + \frac{|x-\lambda_j t|^3}{t^2}\right)\\
&= O(1)e^{-\frac{(x-\lambda_j t)^2}{4C_j'' t}} e^{-\frac{A_j t u^2}{3}}\left(|u| + \left|\frac{x-\lambda_j t}{t}\right|\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{\Gamma_2} e^{ix\eta+\sigma_j(\eta)}\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|d\eta\\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-\frac{(x-\lambda_j t)^2}{4A_j t}} e^{-u^2 tA_j}\Big(\mathsf{E}_j\otimes\langle\mathsf{E}_j| + |\eta|\mathcal{O}(1)\Big)du\\
&\quad + \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{-\frac{(x-\lambda_j t)^2}{4C_j'' t}} e^{-u^2 t\frac{A_j}{3}}\left(|u| + \left|\frac{x-\lambda_j t}{t}\right|\right)\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|du\\
&= \frac{e^{-\frac{(x-\lambda_j t)^2}{4A_j t}}}{\sqrt{4\pi}}\left(\frac{1}{\sqrt{\pi}}\int_{-\delta}^{\delta} e^{-A_j t u^2}du\right)\mathsf{E}_j\otimes\langle\mathsf{E}_j|\\
&\quad + e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}\int_{-\delta}^{\delta} e^{-u^2 t\frac{A_j}{4}}\frac{1}{\sqrt{t+1}}\mathcal{O}(1)du\\
&= \left(\frac{e^{-\frac{(x-\lambda_j t)^2}{4A_j t}}}{\sqrt{4\pi A_j t}}\mathsf{E}_j\otimes\langle\mathsf{E}_j| + \frac{e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}}{\sqrt{(t+1)t}}\mathcal{O}(1)\right)\left(\frac{1}{\sqrt{\pi}}\int_{-\delta\sqrt{A_j t}}^{\delta\sqrt{A_j t}} e^{-u^2}du\right)\\
&= \frac{e^{-\frac{(x-\lambda_j t)^2}{4A_j t}}}{\sqrt{4\pi A_j(t+1)}}\mathsf{E}_j\otimes\langle\mathsf{E}_j|\\
&\quad + e^{-\frac{(x-\lambda_j t)^2}{4A_j t}}\left[\frac{1}{\sqrt{t}}\left(\frac{1}{\sqrt{\pi}}\int_{-\delta\sqrt{A_j t}}^{\delta\sqrt{A_j t}} e^{-u^2}du\right) - \frac{1}{\sqrt{t+1}}\right]\mathcal{O}(1)\\
&\quad + \frac{e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}}{\sqrt{(t+1)}}\left(\frac{1}{\sqrt{t}}\int_{-\delta\sqrt{A_j t}}^{\delta\sqrt{A_j t}} e^{-u^2}du\right)\mathcal{O}(1)
\end{aligned}
$$

$$= \frac{e^{-\frac{(x-\lambda_j t)^2}{4A_j t}}}{\sqrt{4\pi A_j(t+1)}}\mathsf{E}_j \otimes \langle \mathsf{E}_j| + \frac{e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}}{t+1}\mathcal{O}(1).$$

Case 2: $\left|\frac{x-\lambda_j t}{2A_j t}\right| > \frac{\delta}{2}$ In this case, set $c = \frac{\delta}{2}\text{sign}\,(x-\lambda_j t)$.

Note that in this case we have $|x-\lambda_j t| > \delta A_j t$. On Γ_2, by (5.25),

$$\begin{aligned}\left|e^{ix\eta+\sigma_j(\eta)t}\right| &\leq \left|\exp\left[i(x-\lambda_j t)\eta - A_j t\eta^2 + \frac{A_j}{3}|\eta|^2 t\right]\right| \\ &= \exp\left[-|x-\lambda_j t|\frac{\delta}{2} + A_j t\left(\frac{\delta^2}{4} - u^2\right) + \frac{A_j t}{3}\left(\frac{\delta^2}{4} + u^2\right)\right] \\ &\leq \exp\left[-A_j t\left(\frac{\delta^2}{2} - \frac{\delta^2}{3}\right)\right] = O(1)e^{-\frac{t}{C}}.\end{aligned}$$

Consequently,

$$\frac{1}{2\pi}\int_{\Gamma_2} e^{ix\eta+\sigma_j(\eta)}\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|d\eta = \mathcal{O}(e^{-\frac{t}{C}}).$$

For both cases, on Γ_1, Γ_3, $\text{Im}\,(\eta)$ takes the same sign as $x-\lambda_j t$. Therefore, $|e^{i(x-\lambda_j t)\eta}| \leq 1$. Moreover, for both cases, $|\text{Im}\,(\eta)| \leq |c| \leq \frac{\delta}{2}$. Combining this fact with (5.25), we have

$$\begin{aligned}\left|e^{i(x-\lambda_j)t\eta - \eta^2 A_j t + O(\eta^3)t}\right| &\leq \exp\left[-A_j t\left(\delta^2 - \frac{\delta^2}{4}\right) + \frac{A_j t}{3}\left(\delta^2 + \frac{\delta^2}{4}\right)\right] \\ &= O(1)e^{-\frac{t}{C}}.\end{aligned}$$

This implies

$$\frac{1}{2\pi}\int_{\Gamma_1 \text{ or } \Gamma_3} e^{ix\eta+\sigma_j(\eta)}\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|d\eta = \mathcal{O}(e^{-\frac{t}{C}}). \qquad \square$$

Remark 5.10. The computation of the inverse Fourier transform is an essential part of the study of the Green's function here. The approach goes back to Zeng 1994, [48], Liu-Zeng 1997, [35] for the viscous conservation laws. This is generalized to the Boltzmann equation, Liu-Yu 2004 [31]. For the generalization to the 3-D case, Liu-Yu 2006 [32], to be presented later, the inversion of the Fourier transform using the complex analytic method requires additional thinking. There is an essential difference between Green's function for the viscous conservation laws and that for the Boltzmann equation in that the former contains heat kernel with singularity, c.f. (4.39), while the later's singularity resides not in the heat kernels, (5.24). Instead, the singularity for the Boltzmann equation is contained in the essential kinetic waves that will be constructed in the next section.

5.3. Scale separations

Boltzmann equation has much richer wave phenomena than the equations for the fluid dynamics. We illustrate here a basic separation of scales property, that the micro part decays at a faster rate than the macro part.

Theorem 5.11. *For any fixed $C_j > A_j$, $j = 1, 2, 3$, there exists C and δ such that*

$$\|\mathbb{G}_L(x,t,\boldsymbol{\xi};\delta)\mathsf{P}_1\|_{L^2_{\boldsymbol{\xi}}}, \|\mathsf{P}_1\mathbb{G}_L(x,t,\boldsymbol{\xi};\delta)\|_{L^2_{\boldsymbol{\xi}}} = O(1)\left(\sum_{j=1}^{3}\frac{e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}}{t+1} + e^{-\frac{t}{C}}\right), \tag{5.27a}$$

$$\|\mathsf{P}_1\mathbb{G}_L(x,t,\boldsymbol{\xi};\delta)\mathsf{P}_1\|_{L^2_{\boldsymbol{\xi}}} = O(1)\left(\sum_{j=1}^{3}\frac{e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}}{(t+1)^{\frac{3}{2}}} + e^{-\frac{t}{C}}\right). \tag{5.27b}$$

Proof. Since $\mathsf{P}_1(\mathsf{E}_j \otimes \langle \mathsf{E}_j|) = (\mathsf{E}_j \otimes \langle \mathsf{E}_j|)\mathsf{P}_1 = 0$, Theorem 5.9 immediately implies (5.27a).

For (5.27b), after referring to the proof of Theorem 5.9, one finds that:

$$\text{In Case 1, } \frac{1}{2\pi}\int_{\Gamma_1 \text{ or } \Gamma_3} e^{ix\eta+\sigma_j(\eta)}\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|d\eta = \mathcal{O}(e^{-\frac{t}{C}}),$$

$$\text{In Case 2, } \frac{1}{2\pi}\int_{\Gamma_1+\Gamma_2+\Gamma_3} e^{ix\eta+\sigma_j(\eta)}\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|d\eta = \mathcal{O}(e^{-\frac{t}{C}}).$$

Now that $\mathsf{P}_1\mathcal{O}(e^{-t/C})\mathsf{P}_1 = \mathcal{O}(e^{-t/C})$, it remains only to consider

$$\text{Case 1:} \quad \frac{1}{2\pi}\int_{\Gamma_2} e^{ix\eta+\sigma_j(\eta)}\mathsf{P}_1\Big(\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|\Big)\mathsf{P}_1 d\eta.$$

$\mathsf{P}_1\mathsf{e}_j = O(\eta)$ implies $\mathsf{P}_1\Big(\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|\Big)\mathsf{P}_1 = \mathcal{O}(\eta^2)$. Therefore,

$$\begin{aligned}
\frac{1}{2\pi}\int_{\Gamma_2} e^{ix\eta+\sigma_j(\eta)}\mathsf{P}_1\Big(\mathsf{e}_j(\eta)\otimes[\mathsf{e}_j(\eta)|\Big)\mathsf{P}_1 d\eta &= \int_{\Gamma_2} e^{ix\eta+\sigma_j(\eta)}|\eta|^2 d\eta\mathcal{O}(1)\\
&= \int_{-\delta}^{\delta} e^{-\frac{(x-\lambda_j t)^2}{4C_j'' t}} e^{-u^2 t\frac{A_j}{3}}\left(u^2+\left(\frac{x-\lambda_j t}{t}\right)^2\right)du\mathcal{O}(1)\\
&= e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}\int_{-\delta}^{\delta}\frac{1}{t+1}e^{-u^2 t\frac{A_j}{4}}du\mathcal{O}(1)\\
&= \frac{e^{-\frac{(x-\lambda_j t)^2}{4C_j t}}}{(t+1)^{\frac{3}{2}}}\mathcal{O}(1). \qquad \square
\end{aligned}$$

6. 1-D Green's Function, Particle-Like Waves

The Boltzmann equation is the *meso-scopic equation*, being between the microscopic interacting particle systems and the macroscopic fluid dynamics equations. In the last section we have shown that the fluid-like waves for the Boltzmann solution have as their leading terms closely related to the Euler and Navier-Stokes waves. In this section we finish the construction of the Green's function by considering the short waves in the Green's function. However, because the spectrum the short waves represent is not explicit, the construction is indirect and elaborate, starting with the construction of the *singular waves*. These waves contains particle wave, particle-like waves, essential kinetic waves and a series of increasingly smooth waves. We now outline the definition of these waves. For this, we need the following solution operators, c.f. (3.8):

Definition 6.1. Denote by $\mathbb{S}^t$ and $\mathbb{O}^t_D$ the solution operators of the equations

$$\begin{cases}\mathsf{h}_t+\xi^1\partial_x\mathsf{h}+\nu(\boldsymbol{\xi})\mathsf{h}=0,\\ \mathsf{h}(x,0)=\mathsf{h}_0(x),\ x\in\mathbb{R},\end{cases} \tag{6.1}$$

$$\mathsf{h}(x,t) \equiv \mathbb{S}^t \mathsf{h}_0(x),$$

and

$$\begin{cases} \mathsf{j}_t + \xi^1 \partial_x \mathsf{j} + \nu(\boldsymbol{\xi})\mathsf{j} = \mathsf{K}_0 \mathsf{j}, \\ \mathsf{j}(x,0) = \mathsf{j}_0(x), \ x \in \mathbb{R}, \end{cases} \tag{6.2}$$

$$\mathsf{j}(x,t) \equiv \mathbb{O}_D^t \mathsf{j}_0(x).$$

These operators are studied in the first subsection. For the complete construction of the singular waves, the first step is to use the damped transport operator (6.1) and the essential kinetic operator (6.2) to extract the *particle-like waves* from the Green's function. Recall the Green's function, (2.17),

$$\mathbb{G}_t + \xi^1 \partial_x \mathbb{G} + \nu(\boldsymbol{\xi})\mathbb{G} = \mathsf{K}\mathbb{G}, \ x \in \mathbb{R}, \ \boldsymbol{\xi} \in \mathbb{R}^3,$$
$$\mathbb{G}(x,0,\boldsymbol{\xi};\boldsymbol{\xi}_0) = \delta(x)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0), \ x \in \mathbb{R}.$$

The first term is the *particle waves* defined by

$$\begin{cases} \mathsf{h}^0_t + \xi^1 \partial_x \mathsf{h}^0 + \nu(\boldsymbol{\xi})\mathsf{h}^0 = 0, \\ \mathsf{h}^0(x,0) = \delta(x)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0). \end{cases} \tag{6.3}$$

We then define the *particle-like waves* $\mathsf{h}^j, \ j = 1, 2, \ldots$, as follows: We have

$$(\mathbb{G} - \mathsf{h}^0)_t + \xi^1 \partial_x (\mathbb{G} - \mathsf{h}^0) + \nu(\boldsymbol{\xi})(\mathbb{G} - \mathsf{h}^0) = \mathsf{K}(\mathbb{G} - \mathsf{h}^0) + \mathsf{K}\mathsf{h}^0,$$
$$(\mathbb{G} - \mathsf{h}^0)(x,0,\boldsymbol{\xi};\boldsymbol{\xi}_0) = 0.$$

Thus we define the second term h^1 by

$$\begin{cases} \mathsf{h}^1_t + \xi^1 \partial_x \mathsf{h}^1 + \nu(\boldsymbol{\xi})\mathsf{h}^1 = \mathsf{K}\mathsf{h}^0 = \mathsf{K}\mathbb{S}^t \delta(x)\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0), \\ \mathsf{h}^1(x,0) = 0. \end{cases} \tag{6.4}$$

In general, the particle-like waves are defined through the Picard iterations as:

$$\begin{cases} \mathsf{h}^j_t + \xi^1 \partial_x \mathsf{h}^j + \nu(\boldsymbol{\xi})\mathsf{h}^j = \mathsf{K}\mathsf{h}^{j-1}, \\ \mathsf{h}^j(x,0) = 0, \ j = 1, 2, \ldots. \end{cases} \tag{6.5}$$

The Green's function minus these particle-like waves

$$\mathsf{g}_k \equiv \mathbb{G} - \sum_{i=0}^{k} \mathsf{h}^i \tag{6.6}$$

satisfies

$$\begin{cases} \mathsf{g}_{kt} + \xi^1 \mathsf{g}_{kx} = \mathsf{L}\mathsf{g}_k + \mathsf{K}\mathsf{h}^k, \\ \mathsf{g}_k(x,0,\boldsymbol{\xi}) = 0. \end{cases} \tag{6.7}$$

The source $\mathsf{K}\mathsf{h}^k$ is bounded for $k \geq 2$, Lemma 6.9; and as it turns out, this is so also for the $3-D$ case. We next use the essential kinetic operator $\mathbb{O}_D^t$, (6.2), (3.8), to construct an *essential kinetic wave* $\bar{\mathsf{h}}$:

$$\begin{cases} \bar{\mathsf{h}}_t + \xi^1 \bar{\mathsf{h}}_x + \nu(\boldsymbol{\xi})\bar{\mathsf{h}} = \mathsf{K}_0\bar{\mathsf{h}} + \mathsf{K}\mathsf{h}^2, \\ \bar{\mathsf{h}}(\boldsymbol{x},0,\boldsymbol{\xi}) = 0. \end{cases} \tag{6.8}$$

The function $\bar{\mathsf{h}}$ is named the essential kinetic wave as it is localized. Moreover, the new source $\mathsf{K}_1\bar{\mathsf{h}}$ is now *smooth in microscopic velocity* $\boldsymbol{\xi}$, Lemma 3.3:

$$\begin{cases} (\mathbb{G} - \mathsf{h}^0 - \mathsf{h}^1 - \mathsf{h}^2 - \bar{\mathsf{h}})_t + \xi^1(\mathbb{G} - \mathsf{h}^0 - \mathsf{h}^1 - \mathsf{h}^2 - \bar{\mathsf{h}})_x \\ = \mathsf{L}(\mathbb{G} - \mathsf{h}^0 - \mathsf{h}^1 - \mathsf{h}^2 - \bar{\mathsf{h}}) + \mathsf{K}_1\bar{\mathsf{h}}, \\ (\mathbb{G} - \mathsf{h}^0 - \mathsf{h}^1 - \mathsf{h}^2 - \bar{\mathsf{h}})(x,0,\boldsymbol{\xi}) = 0. \end{cases} \tag{6.9}$$

With the source smooth in microscopic velocity, we use the Picard iterations, c.f. (6.5), again:

$$\begin{cases} \mathsf{g}^0_t + \xi^1\partial_x \mathsf{g}^0 + \nu(\boldsymbol{\xi})\mathsf{g}^0 = \mathsf{K}_1\bar{\mathsf{h}}, \\ \mathsf{g}^{j+1}_t + \xi^1\partial_x \mathsf{g}^{j+1} + \nu(\boldsymbol{\xi})\mathsf{g}^{j+1} = \mathsf{K}\mathsf{g}^j, \\ \mathsf{g}^j(x,0) = 0, \ j = 0,1,2,\dots. \end{cases} \tag{6.10}$$

This is the final step in the construction of the singular waves. It is shown by a *Mixture Lemma* that the smoothness of the source in $\boldsymbol{\xi}$ induces an increasingly more smooth functions g^j, as j increases.

The construction in the last section of the fluid-like waves and the construction of singular waves in this section offer two decompositions of the Green's function. This allows us to apply the energy method and Sobolev calculus to gain global pointwise analysis of the Green's function.

6.1. Essential kinetic operators

The operator $\mathbb{S}^t$, (6.1), is a damped transport equation and is solved by the characteristic method:

$$\mathbb{S}^t \mathsf{h}_0(x,\boldsymbol{\xi}) = e^{-\nu(\boldsymbol{\xi})t}\mathsf{h}_0(x-\xi^1 t,\boldsymbol{\xi}). \tag{6.11}$$

From this we have immediately the following lemma.

Lemma 6.2. *For any $\beta \geq 0$, the operator $\mathbb{S}^t$ satisfies*

$$\begin{cases} \|\mathbb{S}^t\|_{L^\infty_x(L^\infty_{\boldsymbol{\xi},\beta})} \leq e^{-\nu_0 t}, \\ \|\mathbb{S}^t\|_{L^2_x(L^2_{\boldsymbol{\xi}})} \leq e^{-\nu_0 t}. \end{cases} \tag{6.12}$$

The operator $\mathbb{O}^t_D$, (6.2), captures the singular part of the linear Boltzmann operator.

Lemma 6.3. *There exist positive constants D_0 and C_1 such that for any $D \in (0, D_0)$ the operator $\mathbb{O}^t_D$ satisfies*

$$\|\mathbb{O}^t_D\|_{L^2_x(L^2_{\boldsymbol{\xi}})} \leq C_1 e^{-\nu_0 t/2}.$$

Proof. First, we regard $\mathbb{O}^t_D$ as an operator on $L^2_x(L^2_{\boldsymbol{\xi}})$, and consider the initial value problem

$$\begin{cases} \mathsf{j}_t + \xi^1 \mathsf{j}_x + \nu(\xi)\mathsf{j} - \mathsf{K}_0\mathsf{j} = 0 \\ \mathsf{j}(x,0) \equiv \mathsf{g}_0(x). \end{cases}$$

Take the Fourier transform to result in

$$\mathscr{F}(\mathbb{O}^t_D \mathsf{g}_0)(\eta) = \hat{\mathsf{j}}(\eta,t) = e^{(-i\xi^1\eta - \nu(\xi) + \mathsf{K}_0)t}\hat{\mathsf{g}}_0(\eta).$$

Since the operator K_0, (3.8), is symmetry and $\|\mathsf{K}_0\|_{L^2_\xi} = O(1)D$, the real part of the spectrum of the operator $-i\xi^1\eta - \nu(\xi) + \mathsf{K}_0$ is in $\{z \in \mathbb{C} : Re(z) \leq -\nu_0 + O(1)D\}$. Thus we may choose D sufficiently small so that

$-\nu_0 + O(1)D < -\nu_0/2$. This implies that there exist positive constants $C_1,\ D_0$ such that, for $D \in (0, D_0)$

$$\|e^{(i\eta\xi^1-\nu(\xi)+\mathsf{K}_0)t}\|_{L^2_\xi} \le C_1\ e^{-\nu_0 t/2} \text{ for any } \eta \in \mathbb{R}.$$

From this

$$\begin{aligned}\|\mathbb{O}^t_D \mathsf{g}_0\|^2_{L^2_x(L^2_{\boldsymbol{\xi}})} =&\|\mathscr{F}(\mathbb{O}^t_D \mathsf{g}_0)\|^2_{L^2_\eta(L^2_{\boldsymbol{\xi}})} \le (C_1)^2 e^{-\nu_0 t}\|\hat{\mathsf{g}}_0\|^2_{L^2_\eta(L^2_{\boldsymbol{\xi}})}\\ =&(C_1)^2 e^{-\nu_0 t}\|\mathsf{g}_0\|^2_{L^2_x(L^2_{\boldsymbol{\xi}})}.\end{aligned}$$

□

This lemma results in the existence of the operator $\mathbb{O}^t_D$ in the functional space $L^2_x(L^2_{\boldsymbol{\xi}})$ and global decaying rate in time. We next use the Picard's iteration to analyze the operator $\mathbb{O}^t_D$ in the sup norm.

Lemma 6.4. *The operator $\mathbb{O}^t_D$ is a bounded operator on $L^\infty_x(L^\infty_{\boldsymbol{\xi},\beta})$ for any $\beta \ge 0$. There exist positive constants $D_0,\ C_1$ such that, for any $D \in (0, D_0)$,*

$$\|\mathbb{O}^t_D \mathsf{g}_0\|_{L^\infty_x(L^\infty_{\boldsymbol{\xi},\beta})} \le C_1 e^{-\nu_0 t/2}\|\mathsf{g}_0\|_{L^\infty_x(L^\infty_{\boldsymbol{\xi},\beta})}.$$

Proof. From Lemma 3.2

$$\|\mathsf{K}_0\mathsf{h}\|_{L^\infty_{\boldsymbol{\xi},\beta+1}} \le C_\beta D\|\mathsf{h}\|_{L^\infty_{\boldsymbol{\xi},\beta}} \text{ for } \beta \ge 0.$$

This and Lemma 6.3 yield

$$\begin{aligned}\Big\| \int_0^t \cdots \int_0^{s_k} & \mathbb{S}^{t-s_1}\mathsf{K}_0\mathbb{S}^{s_1-s_2}\mathsf{K}_0\mathbb{S}^{s_2-s_3}\mathsf{K}_0\cdots\mathbb{S}^{s_k-s_{k+1}}\mathsf{K}_0\mathbb{S}^{s_{k+1}}\\ & ds_{k+1}\cdots ds_1\Big\|_{L^\infty_x(L^\infty_{\boldsymbol{\xi},\beta})} \le (C_\beta)^{k+1}D^{k+1}e^{-\nu_0 t/2}.\end{aligned}$$

Thus, Picard's iteration gives a convergent geometric sequence in $L^\infty_x(L^\infty_{\boldsymbol{\xi},\beta})$ for sufficiently small $D > 0$:

$$\begin{aligned}\mathbb{O}^t_D \ =\ & \mathbb{S}^t + \int_0^t \mathbb{S}^{t-s_1}\mathsf{K}_0\mathbb{S}^{s_1}ds_1 + \int_0^t\int_0^{s_1}\mathbb{S}^{t-s_1}K_0\mathbb{S}^{s_1-s_2}\mathsf{K}_0\mathbb{S}^{s_2}ds_2ds_1 + \cdots\\ & + \int_0^t\cdots\int_0^{s_k}\mathbb{S}^{t-s_1}\mathsf{K}_0\mathbb{S}^{s_1-s_2}\mathsf{K}_0\mathbb{S}^{s_2-s_3}\mathsf{K}_0\cdots\mathbb{S}^{s_k-s_{k+1}}\mathsf{K}_0\mathbb{S}^{s_{k+1}}\\ & ds_{k+1}\cdots ds_1 + \cdots\ ,\end{aligned} \tag{6.13}$$

and the lemma follows. □

The following lemma yields the significant hyperbolic property of the operator $\mathbb{S}^t$ and $\mathbb{O}_D^t$. We will use the crucial property that, for the hard sphere model, (3.5), $\frac{1}{2}\nu(\boldsymbol{\xi}) \sim \nu_1|\boldsymbol{\xi}|$ as $|\boldsymbol{\xi}| \to \infty$ and so, for some positive constant $\bar{\nu}$,

$$\nu(\boldsymbol{\xi}) = O(1)(1+|\boldsymbol{\xi}|),\ \bar{\nu}|\xi^1 t| \le \nu(\boldsymbol{\xi})t \text{ for } \ \boldsymbol{\xi} \in \mathbb{R}^3. \tag{6.14}$$

Lemma 6.5. *For any given $\beta \ge 0$, there exists sufficiently small $D > 0$ such that*

$$\begin{aligned} &\|\mathbb{S}^t \mathrm{g}_0(x)\|_{L^\infty_{\boldsymbol{\xi},\beta}} \\ &\le O(1)e^{-2\nu_0 t/3}\Big[\max_{|y-x|<t}\|\mathrm{g}_0(y)\|_{L^\infty_{\boldsymbol{\xi},\beta}} + \max_{|y-x|>t} e^{-\bar{\nu}|y-x|/3}\|\mathrm{g}_0(y)\|_{L^\infty_{\boldsymbol{\xi},\beta}}\Big], \quad (6.15) \\ &\|\mathbb{O}_D^t \mathrm{g}_0(x)\|_{L^\infty_{\boldsymbol{\xi},\beta}} \\ &\le O(1)e^{-\nu_0 t/2}\Big[\max_{|y-x|<t}\|\mathrm{g}_0(y)\|_{L^\infty_{\boldsymbol{\xi},\beta}} + \max_{|y-x|>t} e^{-\bar{\nu}|y-x|/4}\|\mathrm{g}_0(y)\|_{L^\infty_{\boldsymbol{\xi},\beta}}\Big], \quad (6.16) \end{aligned}$$

where ν_0 is given in (3.22) *and $\bar{\nu}$ in* (6.14).

Proof. We use the representation (6.11) for $\mathbb{S}^t$.
For $|\xi^1| \le 1$, we have from (6.14),

$$\begin{aligned} |\mathbb{S}^t \mathrm{g}_0(x,\xi)| &\le e^{-\nu(\xi)t}(1+|\xi|)^{-\beta}\|\mathrm{g}_0(x-\xi^1 t,\cdot)\|_{L^\infty_{\xi,\beta}} \\ &\le e^{-2\nu_0 t/3}(1+|\xi|)^{-\beta}\max_{|y-x|<t}\|\mathrm{g}_0(y)\|_{L^\infty_{\xi,\beta}}. \end{aligned} \tag{6.17}$$

From (6.14), for $|\xi^1| > 1$

$$\begin{aligned} |\mathbb{S}^t \mathrm{g}_0(x,\xi)| &\le e^{-\nu(\xi)t/3-2\nu_0 t/3}(1+|\xi|)^{-\beta}\|\mathrm{g}_0(x-\xi^1 t,\cdot)\|_{L^\infty_{\xi,\beta}} \\ &\le e^{-\bar{\nu}|\xi^1 t|/3-2\nu_0 t/3}(1+|\xi|)^{-\beta}\|\mathrm{g}_0(x-\xi^1 t,\cdot)\|_{L^\infty_{\xi,\beta}} \\ &\le \max_{|y-x|>t} e^{-\bar{\nu}|x-y|/3-2\nu_0 t/3}(1+|\xi|)^{-\beta}\|\mathrm{g}_0(y)\|_{L^\infty_{\xi,\beta}}. \end{aligned} \tag{6.18}$$

The estimate (6.15) follows from (6.17) and (6.18). From the construction of $\mathbb{O}_D^t$ in Lemma 6.4, one can view $\mathbb{O}_D^t$ as a small perturbation of $\mathbb{S}^t$. From

(3.10) and (6.15), one has that

$$\begin{aligned}&\left\|\int_0^t\cdots\int_0^{s_k}\mathbb{S}^{t-s_1}\mathsf{K}_0\mathbb{S}^{s_1-s_2}\mathsf{K}_0\mathbb{S}^{s_2-s_3}\mathsf{K}_0\cdots\mathbb{S}^{s_k-s_{k+1}}\mathsf{K}_0\mathbb{S}^{s_{k+1}}\mathsf{g}_0 ds_{k+1}\cdots ds_1\right\|_{L^\infty_{\xi,\beta}}\\ &\le (C_\beta D)^{k+1}e^{-\nu_0 t/2}\left[\max_{|y-x|<t}\|\mathsf{g}_0(y)\|_{L^\infty_{\xi,\beta}}+\max_{|y-x|>t}e^{-\nu_1|y-x|/4}\|\mathsf{g}_0(y)\|_{L^\infty_{\xi,\beta}}\right].\end{aligned} \tag{6.19}$$

Thus, the Picard's iteration in (6.13) converges for sufficiently small $D>0$ and (6.16) follows. □

6.2. Particle-like waves

In this section we compute and estimate the particle-like waves $\mathsf{h}^0,\mathsf{h}^1,\mathsf{h}^2$, (6.3), (6.4). First, by (6.11), we have

$$\mathsf{h}^0(x,t,\boldsymbol{\xi};\boldsymbol{\xi}_0)=e^{-\nu(\boldsymbol{\xi})t}\delta^1(x-\xi^1 t)\delta^3(\boldsymbol{\xi}-\boldsymbol{\xi}_0). \tag{6.20}$$

From (6.20) we compute h^1 as the following:

Lemma 6.6. *Set* $t_1\equiv t-\frac{(x-\xi_0^1 t)}{\xi^1-\xi_0^1}$. *We have*

$$\mathsf{h}^1(x,t,\boldsymbol{\xi};\boldsymbol{\xi}_0)=\begin{cases}\dfrac{e^{-\nu(\boldsymbol{\xi})(t-t_1)-\nu(\boldsymbol{\xi}_0)t_1}K(\boldsymbol{\xi},\boldsymbol{\xi}_0)}{(\xi^1-\xi_0^1)}, & \textit{for } (\frac{x}{t}-\xi^1)(\frac{x}{t}-\xi_0^1)<0,\\ 0 & , \textit{otherwise.}\end{cases} \tag{6.21}$$

In particular,

$$|\mathsf{h}^1(x,t,\boldsymbol{\xi};\boldsymbol{\xi}_0)|=O(1)\begin{cases}\frac{e^{-\frac{\nu_0 t}{2}-\frac{\nu_0|x|}{2}}}{|\xi^1-\xi_0^1|}K(\boldsymbol{\xi},\boldsymbol{\xi}_0), & \textit{for } |\xi^1-\xi_0^1|\ge\frac{|x-\xi_0^1 t|}{t},\\ 0 & , \textit{otherwise.}\end{cases} \tag{6.22}$$

Proof. By the Duhamel's principle, (6.4), and (6.11), we have

$$\mathsf{h}^1(x,t;\boldsymbol{\xi},\boldsymbol{\xi}_0)=\int_0^t e^{-\nu(\boldsymbol{\xi}_0)s}e^{-\nu(\boldsymbol{\xi})(t-s)}K(\boldsymbol{\xi},\boldsymbol{\xi}_0)\delta(x-\xi^1(t-s)-\xi_0^1 s)ds. \tag{6.23}$$

The above integral is zero unless the characteristic line through (x,t) and

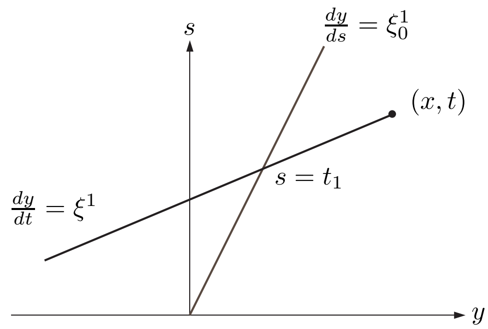


Figure 5: Characteristics.

with speed ξ^1 intersects with the source $\delta(x - \xi_0^1 s)$ at some time $s = t_1$ during the time period $(0, t)$, Figure 5,

$$x - \xi^1(t - t_1) = \xi_0^1 t_1.$$

This yields the expression (6.21). Note that we have

$$|x| = |\xi^1(t - t_1) + \xi_0^1 t_1| \leq |\xi^1|(t - t_1) + |\xi_0^1| t_1,$$

and so the estimate (6.22) follows from the linear growth $\nu_0(1 + |\boldsymbol{\xi}|)$ of the function $\nu(\boldsymbol{\xi})$, (3.5). □

Lemma 6.7.

$$\begin{aligned}&\mathsf{K}(\mathsf{h}^1)(x,t,\boldsymbol{\xi})\\&= O(1)e^{-\frac{\nu_0}{2}(t+|x|)}e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{32}}\\&\quad\times\left(1 + \left|\log|\xi^1 - \xi_0^1|\right|^2 + \left|\log\left|\frac{x - \xi_0^1 t}{t}\right|\right|^2 \chi_{\left\{\left|\frac{x-\xi_0^1 t}{t}\right|<1\right\}}\right),\end{aligned} \tag{6.24}$$

where $\chi_{\{\cdot\}}$ is the indicator function.

Proof. From the explicit expression (3.5), we obtain

$$K(\boldsymbol{\xi}, \boldsymbol{\xi}_*) = O(1)\frac{1}{|\boldsymbol{\xi}_* - \boldsymbol{\xi}|}e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2}{9}}. \tag{6.25}$$

Combining this with the estimate (6.22) for h^1, we have

$$\begin{aligned}
&\left|\mathsf{K}(\mathsf{h}^1)(x,t,\boldsymbol{\xi})\right| \\
&= O(1)e^{-\frac{\nu_0}{2}(t+|x|)} \int_{|\xi_*^1-\xi_0^1|>\left|\frac{x-\xi_0^1 t}{t}\right|} \frac{1}{|\xi_*^1-\xi_0^1|} \frac{e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}{9}}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|} \frac{e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2}{9}}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|} d\boldsymbol{\xi}_* \\
&= O(1)e^{-\frac{\nu_0}{2}(t+|x|)} e^{-\frac{2|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2+2|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}{32}} \\
&\quad \int_{|\xi_*^1-\xi_0^1|>\left|\frac{x-\xi_0^1 t}{t}\right|} \frac{1}{|\xi_*^1-\xi_0^1|} \frac{e^{-(\frac{1}{9}-\frac{1}{16})|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|} \frac{e^{-(\frac{1}{9}-\frac{1}{16})|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|} d\boldsymbol{\xi}_* \\
&= O(1)e^{-\frac{\nu_0}{2}(t+|x|)} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{32}} \\
&\quad \int_{|\xi_*^1-\xi_0^1|>\left|\frac{x-\xi_0^1 t}{t}\right|} \frac{1}{|\xi_*^1-\xi_0^1|} \frac{e^{-(\frac{1}{9}-\frac{1}{16})|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|} \frac{e^{-(\frac{1}{9}-\frac{1}{16})|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|} d\boldsymbol{\xi}_* \\
&= O(1)e^{-\frac{\nu_0}{2}(t+|x|)} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{32}} \left(\int_{\mathcal{D}_1} + \int_{\mathcal{D}_2}\right)(\dots) \\
&\equiv O(1)e^{-\frac{\nu_0}{2}(t+|x|)} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{32}} (I_1+I_2),
\end{aligned}$$

where

$$\mathcal{D}_1 \equiv \left\{1 > |\xi_*^1-\xi_0^1| > \frac{|x-\xi_0^1 t|}{t}\right\}, \quad \mathcal{D}_2 \equiv \left\{|\xi_*^1-\xi_0^1| > 1\right\}.$$

We further split I_1 into three parts:

$$\begin{aligned}
I_1 &= \left(\int_{\mathcal{A}_1} + \int_{\mathcal{A}_2} + \int_{\mathcal{A}_3}\right) \frac{1}{|\xi_*^1-\xi_0^1|} \frac{e^{-(\frac{1}{9}-\frac{1}{16})|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|} \frac{e^{-(\frac{1}{9}-\frac{1}{16})|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2}}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|} d\boldsymbol{\xi}_* \\
&\equiv I_{11} + I_{12} + I_{13},
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{A}_1 \equiv \left\{\boldsymbol{\xi}_* \in \mathcal{D}_1 : |(\xi_*^2-\xi_0^2)^2 + (\xi_*^3-\xi_0^3)^2| < 1 \,\&\, |\boldsymbol{\xi}_*-\boldsymbol{\xi}_0| < |\boldsymbol{\xi}_*-\boldsymbol{\xi}|\right\}, \\
&\mathcal{A}_2 \equiv \left\{\boldsymbol{\xi}_* \in \mathcal{D}_1 : |(\xi_*^2-\xi^2)^2 + (\xi_*^3-\xi^3)^2| < 1 \,\&\, |\boldsymbol{\xi}_*-\boldsymbol{\xi}_0| > |\boldsymbol{\xi}_*-\boldsymbol{\xi}|\right\}, \\
&\mathcal{A}_3 \equiv \mathcal{D}_1 \setminus \Big(\mathcal{A}_1 \cup \mathcal{A}_2\Big).
\end{aligned}$$

For I_{11},

$$
\begin{aligned}
|I_{11}| \le& \int_{\mathcal{A}_1} \frac{1}{|\xi_*^1-\xi_0^1|}\frac{1}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|}\frac{1}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|}d\boldsymbol{\xi}_* \le \int_{\mathcal{A}_1} \frac{1}{|\xi_*^1-\xi_0^1|}\frac{1}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}d\boldsymbol{\xi}_*\\
\le& 2\int_{\left|\frac{x-\xi_0^1 t}{t}\right|}^1 \int_0^1\int_0^{2\pi}\frac{1}{z}\frac{r}{z^2+r^2}d\theta dr dz = \int_{\left|\frac{x-\xi_0^1 t}{t}\right|}^1 \frac{4\pi}{z}\int_0^1 \frac{r}{z^2+r^2}dr dz\\
\le& \int_{\left|\frac{x-\xi_0^1 t}{t}\right|}^1 4\pi\frac{|\log z|}{z}dz = 4\pi\left(\log\left|\frac{x-\xi_0^1 t}{t}\right|\right)^2,
\end{aligned}
$$

where we use the cylindrical coordinates:

$$
r=\sqrt{(\xi_*^2-\xi_0^2)^2+(\xi_*^3-\xi_0^3)^2},\ z=|\xi_*^1-\xi_0^1|,\ \theta=\tan^{-1}\frac{\xi_*^2-\xi_0^2}{\xi_*^3-\xi_0^3}.
$$

Next, we compute I_{12} under the following three different conditions: $2<|\xi^1-\xi_0^1|$, $1<|\xi^1-\xi_0^1|<2$, and $|\xi^1-\xi_0^1|<1$. For $|\xi^1-\xi_0^1|>2$, since $\mathcal{A}_2=\emptyset$, $I_{12}=0$. For $1<|\xi^1-\xi_0^1|<2$,

$$
|I_{12}| \le \int_{\mathcal{A}_2}\frac{2}{|\xi_*^1-\xi_0^1||\boldsymbol{\xi}_*-\boldsymbol{\xi}|}d\boldsymbol{\xi}_*. \tag{6.26}
$$

Without lost of generality, assume $1<\xi^1-\xi_0^1<2$. We consider the following cylindrical coordinates

$$
r=\sqrt{(\xi_*^2-\xi^2)^2+(\xi_*^3-\xi^3)^2},\ z=\xi_*^1-\xi_0^1,\ \theta=\tan^{-1}\frac{\xi_*^2-\xi^2}{\xi_*^3-\xi^3}.
$$

Thus,

$$
\begin{aligned}
|I_{12}| \le& \int_{\left|\frac{x-\xi_0^1 t}{t}\right|}^1\int_0^{4\pi}\int_0^1 \frac{2r}{|z|\times\left|\sqrt{r^2+(z-(\xi^1-\xi_0^1))^2}\right|}dr d\theta dz\\
\le& \int_{\left|\frac{x-\xi_0^1 t}{t}\right|}^1 \frac{2\pi}{|z|}dz \le 4\pi\left|\log\left|\frac{x-\xi_0^1 t}{t}\right|\right| \le 4\pi\left(1+\left|\log\left|\frac{x-\xi_0^1 t}{t}\right|\right|^2\right).
\end{aligned} \tag{6.27}
$$

For $a = |\xi^1 - \xi_0^1| \le 1$, we have

$$\begin{aligned}
|I_{12}| &\le \int_{\mathcal{A}_2} \frac{1}{|\xi_*^1 - \xi_0^1||\boldsymbol{\xi}_* - \boldsymbol{\xi}||\boldsymbol{\xi}_* - \boldsymbol{\xi}_0|} d\boldsymbol{\xi}_* \le \int_{\mathcal{A}_2} \frac{1}{|\xi_*^1 - \xi_0^1||\boldsymbol{\xi}_* - \boldsymbol{\xi}|^2} d\boldsymbol{\xi}_* \\
&\le \int_{\left|\frac{x-\xi_0^1 t}{t}\right|}^1 \int_0^{2\pi} \int_0^1 \frac{r}{|z|(r^2 + (z-a)^2)} dr d\theta dz \\
&\le \pi \int_{|\frac{x-\xi_0^1 t}{t}|}^1 \frac{\big|\log|z-a|\big|}{z} dz. \qquad (6.28)
\end{aligned}$$

We shall estimate the integral on $(|\frac{x-\xi_0^1 t}{t}|, \frac{a}{2})$, and $(\frac{a}{2}, 2a)$, $(2a, 1)$ separately.

In the first interval,

$$\begin{aligned}
\int_{|\frac{x-\xi_0^1 t}{t}|}^{\frac{a}{2}} \frac{\big|\log|z-a|\big|}{z} dz &\le (2 + |\log a|) \int_{|\frac{x-\xi_0^1 t}{t}|}^{\frac{a}{2}} \frac{1}{z} dz \\
&\le (2 + |\log a|) \big|\log|\frac{x-\xi_0^1 t}{t}|\big| \\
&\le C(1 + |\log a|^2 + \big|\log|\frac{x-\xi_0^1 t}{t}|\big|^2).
\end{aligned}$$

In the second interval, we let $\tau = \frac{z}{a}$. Thus,

$$\begin{aligned}
\int_{\frac{a}{2}}^{2a} \frac{\big|\log|z-a|\big|}{z} dz &= \int_{\frac{1}{2}}^{2} \frac{\big|\log|a(1-\tau)|\big|}{\tau} d\tau \\
&\le 2\log 2 |\log a| + \int_{\frac{1}{2}}^{2} \frac{\big|\log|1-\tau|\big|}{\tau} d\tau \\
&\le C(1 + |\log a|^2).
\end{aligned}$$

For the last interval,

$$\int_{2a}^1 \frac{\big|\log|z-a|\big|}{z} dz \le |\log a| \int_{2a}^1 \frac{1}{z} dz \le C|\log a|^2. \qquad (6.29)$$

Therefore, we can conclude

$$I_{12} = O(1)\left(1 + \left|\log\left|\xi^1 - \xi_0^1\right|\right|^2 + \left|\log\left|\frac{x - \xi_0^1 t}{t}\right|\right|^2\right). \qquad (6.30)$$

Similarly,

$$\begin{aligned}|I_{13}| &\le \int_{\mathcal{A}_3} \frac{1}{|\xi_*^1-\xi_0^1|} e^{-\frac{5}{144}|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2} e^{-\frac{5}{144}|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2} d\boldsymbol{\xi}_* \\ &= O(1)\int_{|\frac{x-\xi_0^1 t}{t}|}^1 \frac{1}{z}dz = O(1)\left(\big|\log|\frac{x-\xi_0^1 t}{t}|\big|\right) \\ &= O(1)\left(1+\big|\log|\frac{x-\xi_0^1 t}{t}|\big|^2\right). \qquad (6.31)\end{aligned}$$

In $\mathcal{D}_2$, note that $|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0| > |\xi_*^1-\xi_0^1| > 1$. Thus,

$$|I_2| \le \int_{\mathcal{D}_2} \frac{1}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|} e^{-\frac{5}{144}|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2} d\boldsymbol{\xi}_* = O(1). \qquad (6.32)$$

Combine all the inequalities above we can conclude (6.24). □

Apply the Duhamel principle to (6.5), we obtain a integral representation of h^2:

$$\mathsf{h}^2(x,t,\boldsymbol{\xi}) = \int_0^t e^{-\nu(\boldsymbol{\xi})(t-s)} \mathsf{K}(\mathsf{h}^1)(x-(t-s)\xi^1, s, \xi)ds. \qquad (6.33)$$

Lemma 6.8.

$$\mathsf{h}^2(x,t,\boldsymbol{\xi}) = O(1)e^{-\frac{\nu_0}{3}(t+|x|)} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{32}} (1+\big|\log|\xi^1-\xi_0^1|\big|^2). \qquad (6.34)$$

Proof. Using the estimate (6.24), we obtain

$$\begin{aligned}|\mathsf{h}^2(x,t,\boldsymbol{\xi})| &\le \int_0^t O(1)e^{-\nu(\boldsymbol{\xi})(t-s)} e^{-\frac{\nu_0}{2}(s+|x-\boldsymbol{\xi}_1(t-s)|)} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{32}} \\ &\quad\times\Bigg(1+\Big|\log|\xi^1-\xi_0^1|\Big|^2 + \left|\log\left|\frac{x-\xi_0^1 t+(\xi^1-\xi_0^1)s}{s}\right|\right|^2 \\ &\qquad \times\chi_{\left\{\left|\frac{x-\xi_0^1 t+(\xi^1-\xi_0^1)s}{s}\right|<1\right\}}\Bigg)ds \\ &\le e^{-\frac{\nu_0}{2}(t+|x|)} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{32}} \\ &\quad\times\int_0^t \left(1+\Big|\log|\xi^1-\xi_0^1|\Big|^2 + \left|\log\left|\frac{x-\xi_0^1 t+(\xi^1-\xi_0^1)s}{s}\right|\right|^2\right.\end{aligned}$$

$$\times\chi_{\left\{\left|\frac{x-\xi_0^1t+(\xi^1-\xi_0^1)s}{s}\right|<1\right\}}\Bigg)ds. \tag{6.35}$$

For simplicity, put $\beta=-(\xi^1-\xi_0^1)$, $\alpha=x-\xi^1t$. The problem is now reduced to the estimate of the following integral

$$\begin{aligned}R &= \int_0^t\left|\log\left|\frac{x-\xi_0^1t+(\xi^1-\xi_0^1)s}{s}\right|\right|^2\chi_{\left\{\left|\frac{x-\xi_0^1t+(\xi^1-\xi_0^1)s}{s}\right|<1\right\}}ds\\ &= \int_0^t\left|\log\left|\frac{\alpha-\beta s}{s}\right|\right|^2\chi_{\left\{\left|\frac{\alpha-\beta s}{s}\right|<1\right\}}ds. \end{aligned}\tag{6.36}$$

In order to prove the lemma, it is enough to show $R=O(1)t(1+\left|\log|\beta|\right|^2)$.

Since we are considering (6.36), in order that $\chi\neq0$, we require

$$-s<\alpha-\beta s<s. \tag{6.37}$$

Express (6.37) in seven different cases, classified by the signs of α,β, Figure 6:

$$\begin{cases} 1.\quad 0<s<\infty & \text{, for } \alpha=0,\ -1<\beta<1,\\ 2.\quad \frac{\alpha}{\beta+1}<s<\frac{\alpha}{\beta-1} & \text{, for } \alpha>0,\ \beta>1,\\ 3.\quad \frac{\alpha}{\beta+1}<s<\infty & \text{, for } \alpha>0,\ 0<\beta<1,\\ 4.\quad \frac{\alpha}{\beta+1}<s<\infty & \text{, for } \alpha>0,\ -1<\beta<0,\\ 5.\quad \frac{-\alpha}{-\beta+1}<s<\frac{-\alpha}{-\beta-1} & \text{, for } \alpha<0,\ \beta<-1,\\ 6.\quad \frac{-\alpha}{-\beta+1}<s<\infty & \text{, for } \alpha<0,\ -1<\beta<0,\\ 7.\quad \frac{-\alpha}{-\beta+1}<s<\infty & \text{, for } \alpha<0,\ 0<\beta<1. \end{cases}$$

For case 1, we can evaluate the integral (6.36) directly and obtain

$$R=\left|\log|\beta|\right|^2t.$$

Case 5, 6, and 7 are similar to case 2, 3, and 4 respectively, so, without lost of generality, we consider only case 2, 3, and 4.

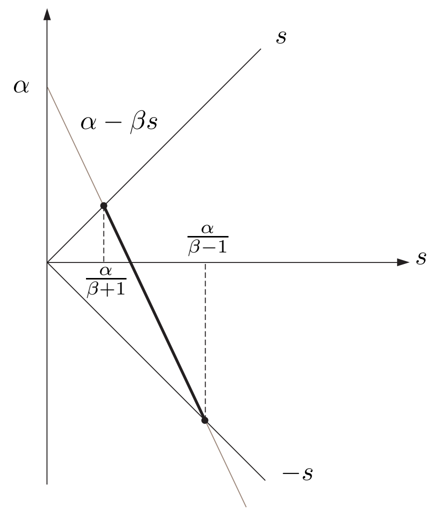


Figure 6: range of s.

For case 2, since $\beta > 1$, $\frac{\alpha}{2\beta} < \frac{\alpha}{\beta+1}$. Therefore,

$$\begin{aligned}
R \leq& \int_{\frac{\alpha}{2\beta}}^{t} \left|\log\left|\frac{\alpha-\beta s}{s}\right|\right|^2 ds \leq \int_{\frac{\alpha}{2\beta}}^{t} 2\Big(\log\beta\Big)^2 + 2\left|\log\left|\frac{\alpha}{\beta s}-1\right|\right|^2 ds \\
\leq& 2t\Big(\log\beta\Big)^2 + 2\left(\int_{\frac{\alpha}{2\beta}}^{\frac{2\alpha}{\beta}} \chi_{\left\{t>\frac{\alpha}{2\beta}\right\}} + \int_{\frac{2\alpha}{\beta}}^{t} \chi_{\left\{t>\frac{2\alpha}{\beta}\right\}}\right)\left|\log\left|\frac{\alpha}{\beta s}-1\right|\right|^2 ds \\
\leq& 2t\Big(\log\beta\Big)^2 + \chi_{\left\{t>\frac{\alpha}{2\beta}\right\}} \\
&\times\left[\frac{2\alpha}{\beta}\int_{\frac{1}{2}}^{2}\left(\log\left|\frac{1}{z}-1\right|\right)^2 dz + 2\int_{\frac{2\alpha}{\beta}}^{t}\left|\log\left|\frac{\alpha}{\beta s}-1\right|\right|^2 ds\right] \\
=& 2t\Big(\log\beta\Big)^2 + \chi_{\left\{t>\frac{\alpha}{2\beta}\right\}} \times\left(O(1)\frac{\alpha}{\beta} + 2t\Big(\log 2\Big)^2\right) = O(1)t(1+(\log\beta)^2).
\end{aligned}$$

For case 3, since $0 < \beta < 1$,

$$\begin{aligned}
R \;\leq&\; \int_{\frac{\alpha}{2}}^{t}\left(\log\left|\frac{\alpha}{s}-\beta\right|\right)^2 ds \\
=&\; \left(\int_{\frac{\alpha}{2}}^{\min\left\{t,\frac{\alpha}{2\beta}\right\}} + \int_{\frac{\alpha}{2\beta}}^{\frac{2\alpha}{\beta}} \chi_{\left\{t>\frac{\alpha}{2\beta}\right\}} + \int_{\frac{2\alpha}{\beta}}^{t} \chi_{\left\{t>\frac{2\alpha}{\beta}\right\}}\right)\left(\log\left|\frac{\alpha}{s}-\beta\right|\right)^2 ds
\end{aligned}$$

$$\le \int_{\frac{\alpha}{2}}^{\min\left\{t,\frac{\alpha}{2\beta}\right\}} \left(\log\left|\frac{\alpha}{s}-\beta\right|\right)^2 ds + O(1)t\Big(1+(\log\beta)^2\Big).$$

For $\frac{\alpha}{2} < s < \frac{\alpha}{2\beta}$, $0 < \beta < \frac{\alpha}{s} - \beta < 2$. Therefore,

$$\begin{aligned} R &\le \int_{\frac{\alpha}{2}}^{\min\left\{t,\frac{\alpha}{2\beta}\right\}} \left(\log\left|\frac{\alpha}{s}-\beta\right|\right)^2 ds + O(1)t\Big(1+(\log\beta)^2\Big) \\ &\le t\Big(2(\log 2)^2 + 2(\log\beta)^2\Big) + O(1)t\Big(1+(\log\beta)^2\Big) \\ &= O(1)t\Big(1+(\log\beta)^2\Big). \end{aligned}$$

For case 4,

$$\frac{\alpha}{1-|\beta|} < s \implies |\beta| < |\beta| + \frac{\alpha}{s} < 1.$$

Therefore,

$$R \le \int_{\frac{\alpha}{1-|\beta|}}^{t} \left|\log\left(|\beta| + \frac{\alpha}{s}\right)\right|^2 ds \le t\Big(\log|\beta|\Big)^2. \qquad \square$$

The next lemma shows that the source $\mathsf{K}(\mathsf{h}^2)$ for the next Picard iteration (6.5) is finite. That the source is gaussian in microscopic velocity is an interesting fact, which follows from the detailed estimates in the above lemma.

Lemma 6.9.

$$|\mathsf{K}(\mathsf{h}^2)(x,t,\boldsymbol{\xi})| \le Ce^{-\frac{\nu_0}{3}(t+|x|)}e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{128}}. \tag{6.38}$$

Proof. Applying the estimate for h^2, Lemma 6.8, and the estimate (6.25) for K, we obtain

$$\begin{aligned} |\mathsf{K}(\mathsf{h}^2)(x,t,\boldsymbol{\xi})| &= O(1)\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|} e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2}{9}} e^{-\frac{\nu_0}{3}(t+|x|)} e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}{32}} \\ &\quad \times \Big(1+\big|\log|\xi_*^1-\xi_0^1|\big|^2\Big)\, d\boldsymbol{\xi}_* \\ &= O(1)e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{128}} e^{-\frac{\nu_0}{3}(t+|x|)} \int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{\xi}_*-\boldsymbol{\xi}|} e^{-(\frac{1}{9}-\frac{1}{64})|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2} \\ &\quad \times (1+\big|\log|\xi_*^1-\xi_0^1|\big|^2) e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}{64}}\, d\boldsymbol{\xi}_*. \end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{\xi}_* - \boldsymbol{\xi}|} e^{-(\frac{1}{9}-\frac{1}{64})|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2} (1+\big|\log|\xi^1_* - \xi^1_0|\big|^2) e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}{64}} d\boldsymbol{\xi}_* \\
&\le \left(\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{\xi}_* - \boldsymbol{\xi}|^2} e^{-2(\frac{1}{9}-\frac{1}{64})|\boldsymbol{\xi}_*-\boldsymbol{\xi}|^2} d\boldsymbol{\xi}_*\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{\mathbb{R}^3} (1+\big|\log|\xi^1_* - \xi^1_0|\big|^2)^2 e^{-\frac{|\boldsymbol{\xi}_*-\boldsymbol{\xi}_0|^2}{32}} d\boldsymbol{\xi}_*\right)^{\frac{1}{2}} \le \infty.
\end{aligned}$$

This completes the proof of the lemma. □

6.3. Essential kinetic waves

With the estimate of the source $\mathsf{K}(\mathsf{h}^2)$ in Lemma 6.9, we now study the essential kinetic wave $\bar{\mathsf{h}}$, (6.8).

Lemma 6.10. *For any $\beta \ge 0$, there exists $D > 0$ such that*

$$\|\bar{\mathsf{h}}\|_{L^\infty_{\boldsymbol{\xi},\beta}} = O(1) e^{-\frac{\nu_1(|x|+t)}{D}}. \tag{6.39}$$

Proof. With the operator $\mathbb{O}^t_D$ of (6.2), we have by Duhamel principle that

$$\bar{\mathsf{h}} = \int_0^t \mathbb{O}^{t-s}_D \mathsf{K}\mathsf{h}^2(s) ds = \int_0^t \mathbb{O}^{t-s}_D O(1) e^{-\frac{\nu_0}{3}(s+|x|)} e^{-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|^2}{128}} ds,$$

where we have used Lemma 6.9 for $\mathsf{K}\mathsf{h}^2$. The estimate (6.39) now follows from Lemma 6.5 on the exponential decay in time and the hyperbolicity property of the operator $\mathbb{O}_D$. We omit the details. □

With this we have

$$\begin{cases} \mathsf{g}_t + \xi^1 \mathsf{g}_x + \nu(\boldsymbol{\xi})\mathsf{g} = \mathsf{K}\mathsf{g} + \mathsf{K}_1\bar{\mathsf{h}}, \\ \mathsf{g}(x,0,\boldsymbol{\xi}) = 0; \end{cases} \tag{6.40}$$

$$\mathsf{g} \equiv \mathbb{G} - [\mathsf{h}^0 + \mathsf{h}^1 + \mathsf{h}^2] - \bar{\mathsf{h}}. \tag{6.41}$$

We have the following estimate on the smoothness in microscopic velocity $\boldsymbol{\xi}$ for the source $\mathsf{K}_1\bar{\mathsf{h}}$.

Lemma 6.11. *For any fixed natural number $l \geq 0$,*

$$\|\partial_{\boldsymbol{\xi}}^l(\mathsf{K}_1\bar{\mathsf{h}})\|_{L_x^2(L_{\boldsymbol{\xi}}^2)} = O(1)e^{-\frac{\nu_1 t}{3D}}. \tag{6.42}$$

Proof. From Lemma 3.3, we have $\|\mathsf{K}_1\mathsf{h}\|_{H^i_{\boldsymbol{\xi}}} = O(1)\|\mathsf{h}\|_{L^2_{\boldsymbol{\xi}}}$ The lemma now follows from Lemma 6.10. □

For the singular waves g^j of (6.10), we have the following estimate.

Lemma 6.12. *For each fixed $\beta > 0$ and $j = 0, 1, \dots,$ there exists positive constants D_j such that*

$$\|\mathsf{g}^j\|_{L^\infty_{\boldsymbol{\xi},\beta}} = O(1)e^{-\frac{\nu_1(|x|+t)}{D_j}}. \tag{6.43}$$

Proof. This is proved by induction on j and uses Lemma 6.5. The procedure is similar to the proof of Lemma 6.10; details are omitted. □

We have, c.f. (6.9),

$$\begin{cases} (\mathbb{G} - \sum_{i=0}^2 \mathsf{h}^i - \bar{\mathsf{h}} - \sum_{i=0}^j \mathsf{g}^i)_t + \xi^1(\mathbb{G} - \sum_{i=0}^2 \mathsf{h}^i - \bar{\mathsf{h}} - \sum_{i=0}^j \mathsf{g}^i)_x \\ = \mathsf{L}(\mathbb{G} - \sum_{i=0}^2 \mathsf{h}^i - \bar{\mathsf{h}} - \sum_{i=0}^j \mathsf{g}^i) + \mathsf{K}\mathsf{g}^j, \\ (\mathbb{G} - \sum_{i=0}^2 \mathsf{h}^i - \bar{\mathsf{h}} - \sum_{i=0}^j \mathsf{g}^i)(x, 0, \boldsymbol{\xi}) = 0. \end{cases} \tag{6.44}$$

Using the notion of damped transport operator $\mathbb{S}$ and the integral operator K, the Picard iteration (6.10) has the representation:

$$\mathsf{g}^{j+1} = \int_0^t \mathbb{S}^{t-s_1}\mathsf{K}\mathsf{g}^j(s_1)ds_1 = \int_0^t \int_0^{s_1} \mathbb{S}^{t-s_1}\mathsf{K}\mathbb{S}^{s_1-s_2}\mathsf{K}\mathsf{g}^{j-1}(s_2)ds_2 ds_1.$$

and so inductively we have

$$\mathsf{g}^j = \int_0^t \int_0^{s_1} \cdots \int_0^{s_j} \mathbb{S}^{t-s_1}\mathsf{K}\mathbb{S}^{s_1-s_2}\mathsf{K}\mathbb{S}^{s_2-s_3}\mathsf{K}\cdots\mathbb{S}^{s_j-s_{j+1}}\mathsf{K}_1\bar{\mathsf{h}}(s_{j+1}) \, ds_{j+1}\cdots ds_1, \quad j = 0, 1, 2, \dots. \tag{6.45}$$

A major point is that the above repeated convolutions of the damped transport operator $\mathbb{S}$ and the integral operator K yield the smoothness in (x, t) as a consequence of the smoothness of the source $\mathsf{K}_1\bar{\mathsf{h}}$ in $\boldsymbol{\xi}$, (6.42). This is the Mixture Lemma we study in the next subsection.

6.4. Mixture Lemma

We rewrite the linearized Boltzmann equation as follows:

$$\mathsf{g}_t + \xi^1 \mathsf{g}_x + \nu(\xi)\mathsf{g} = \mathsf{K}\mathsf{g}. \tag{6.46}$$

This form indicates that there are two essential mixing mechanisms:

(1) The mixing mechanism in x is due to particles travelling in different velocity ξ^1. This is represented by the operator $\mathbb{S}^t$, the LHS of the equation, which represents the transport as well as part of the loss term in the collision operator.

(2) The mixing mechanism in ξ is due to the rest of the collision of particles. This is represented through the integral operator K.

We introduce a sequence of mixture operators $\mathbb{M}_i^t$ as follows.

Definition 6.13. For any $\mathsf{g}_0 \in L_x^2(L_\xi^2)$, k-th degree Mixture operator $\mathbb{M}_k^t$ is given as follows:

$$\mathbb{M}_k^t \mathsf{g}_0 \equiv \int_0^t \int_0^{s_1} \cdots \int_0^{s_{2k-1}} \mathbb{S}^{t-s_1}\mathsf{K}\mathbb{S}^{s_1-s_2}\mathsf{K}\mathbb{S}^{s_2-s_3}\mathsf{K}\cdots\mathbb{S}^{s_{2k-1}-s_{2k}}\mathsf{K}\mathbb{S}^{s_{2k}}\mathsf{g}_0 ds_{2k}\cdots ds_1. \tag{6.47}$$

The Picard iteration for solving the initial value problem $\partial_t \mathsf{h} + \xi^1\partial_x \mathsf{h} - L\mathsf{h} = 0$, $\mathsf{h}(x,0) \equiv \mathsf{h}_0(x)$, can be rewritten as

$$\begin{aligned} \mathsf{h}(x,t) &= \mathbb{S}^t \mathsf{h}_0(x) + \int_0^t \mathbb{S}^{t-s}\mathsf{K}\mathbb{S}^s \mathsf{h}_0 ds(x) + \mathbb{M}_1^t \mathsf{h}_0(x) \\ &\quad + \sum_{j=1}^{\infty} \left(\int_0^t \mathbb{S}^{t-s}\mathsf{K}\mathbb{M}_j^s \mathsf{h}_0 ds(x) + \int_0^t \int_0^{t_1} \mathbb{S}^{t-t_1}\mathsf{K}\mathbb{S}^{t_1-t_2}\mathsf{K}\mathbb{M}_j^{t_2}\mathsf{h}_0 dt_2 dt_1(x) \right). \end{aligned} \tag{6.48}$$

This series is an asymptotic expansion, useful for identifying the singular parts of the solution.

Remark 6.14. The *Mixture Lemma* states that the mixture of the two operators $\mathbb{S}$ and K in $\mathbb{M}_k^t$ transports the regularity in the microscopic velocity $\boldsymbol{\xi}$ to the regularity of the space time (x,t). The Mixture Lemma is similar in spirit as the well-known *Averaging Lemma*, [15], [16], [3], [27], (see also [40] for the gliding regularity for the non-dissipative Landau damping.) These two Lemmas have been introduced independently and used for different purposes.

Lemma 6.15 (Mixture Lemma)**.** *There exist $C_k > 0$, $k = 1, 2, \dots,$ such that*

$$\left\|\partial_x^k \mathbb{M}_k^t g_0\right\|_{L_x^2(L_{\boldsymbol{\xi}}^2)} \le C_k e^{-\frac{\nu_0 t}{2}} \sum_{l=0}^{k} \left\|\partial_{\xi^1}^l g_0\right\|_{L_x^2(L_{\boldsymbol{\xi}}^2)}. \tag{6.49}$$

Proof. In order to explain the main ideas, we will go to some details for the cases of $k = 1$ and $k = 2$. The proof uses the characteristic method, [31], [26]. For $k = 1$, we write down the explicit form of the Mixture operator by spelling out the operator $\mathbb{S}^t \mathsf{h}(x,\boldsymbol{\xi}) = e^{-\nu(\boldsymbol{\xi})t}\mathsf{h}(x-\xi^1 t,\boldsymbol{\xi})$, (6.11):

$$\begin{aligned} \mathbb{M}_1^t g_0(x,\boldsymbol{\xi}) \;=\; & \int_0^t\int_0^{s_1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\nu(\boldsymbol{\xi})(t-s_1)-\nu(\boldsymbol{\xi}_1)(s_1-s_2)-\nu(\boldsymbol{\xi}_2)s_2} K(\boldsymbol{\xi},\boldsymbol{\xi}_1)\, K(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \\ & \cdot g_0(x-\xi^1(t-s_1)-\xi_1^1(s_1-s_2)-\xi_2^1 s_2,\boldsymbol{\xi}_2)\, d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_1 ds_2 ds_1. \end{aligned} \tag{6.50}$$

Set

$$\begin{aligned} A(\boldsymbol{\xi},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,t,s_1,s_2) \;&\equiv\; e^{-\nu(\boldsymbol{\xi})(t-s_1)-\nu(\boldsymbol{\xi}_1)(s_1-s_2)-\nu(\boldsymbol{\xi}_2)s_2} K(\boldsymbol{\xi},\boldsymbol{\xi}_1)\, K(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2), \\ z \;&\equiv\; x-\xi^1(t-s_1)-\xi_1^1(s_1-s_2)-\xi_2^1 s_2, \end{aligned} \tag{6.51}$$

and we have

$$\partial_x \mathbb{M}_1^t g_0(x,\boldsymbol{\xi}) = \int_0^t\int_0^{s_1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} A(\boldsymbol{\xi},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,t,s_1,s_2)\partial_x g_0(z,\boldsymbol{\xi}_2)\, d\boldsymbol{\xi}_2 d\boldsymbol{\xi}_1 ds_2 ds_1. \tag{6.52}$$

Next, we use the change of variables and the chain rule:

$$\begin{aligned} V_1 \;&\equiv\; \boldsymbol{\xi}-\boldsymbol{\xi}_1,\ V_2 \equiv \boldsymbol{\xi}_1-\boldsymbol{\xi}_2, \\ \partial_{V_1^1} g_0(z,\boldsymbol{\xi}_2) \;&=\; s_1\partial_x g_0(z,\boldsymbol{\xi}_2) - \partial_{\xi_2^1} g_0(z,\boldsymbol{\xi}_2). \end{aligned} \tag{6.53}$$

By the switching the order of differentiations and applying the integration by parts, we obtain

$$\begin{aligned}&\partial_x \mathbb{M}_1^t g_0(x,\boldsymbol{\xi})\\ &=\int_0^t\int_0^{s_1}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{s_1}\Big\{A(\boldsymbol{\xi},\boldsymbol{\xi}-V_1,\boldsymbol{\xi}-V_1-V_2,t,s_1,s_2)\partial_{\xi_2^1}g_0(z,\boldsymbol{\xi}_2)\\ &\quad -g_0(z,\boldsymbol{\xi}_2)\partial_{V_1^1}A(\boldsymbol{\xi},\boldsymbol{\xi}-V_1,\boldsymbol{\xi}-V_1-V_2,t,s_1,s_2)\Big\}\,dV_2dV_1ds_2ds_1. \quad (6.54)\end{aligned}$$

To estimate the L^2 norm of $\partial_x \mathbb{M}_1^t g_0(x,\boldsymbol{\xi})$, we only need the integrability of $\frac{1}{s_1}\left(A+\partial_{V_1^1}A\right)$. Note that, as the result of the above switching of the order of differentiation, it yields the integrability $\int_0^t\int_0^{s_1}\frac{1}{s_1}ds_2ds_1=t$. If we were to switch the differentiation with respect to x with the differentiation with respect to V_2^1, then we would get the non-integrable factor $\frac{1}{s_2}$, instead of $\frac{1}{s_1}$. This would implies that the characteristic method only works for the short time scale. On the other hand, we now have the partial differentiation of the function A. The second point is then to show the integrability of $\partial_{V_1^1}A$. We have from (6.51)

$$\partial_{V_1^1}A\sim e^{-\frac{3}{4}\nu_0 t}K(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\partial_{\xi_1^1}K(\boldsymbol{\xi},\boldsymbol{\xi}_1). \quad (6.55)$$

From the Hőlder inequality and Fubini Theorem, we conclude

$$\left\|\partial_x\mathbb{M}_1^t g_0\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}\le C_1 e^{-\frac{\nu_0 t}{2}}\left(\|g_0\|_{L^2_x(L^2_{\boldsymbol{\xi}})}+\left\|\partial_{\xi^1}g_0\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}\right). \quad (6.56)$$

For $k=2$, there are more changes of variables that have to be done in the right order. We have

$$\begin{aligned}&\partial_x^2\mathbb{M}_2^t g_0(x,\boldsymbol{\xi})\\ &=\int_0^t\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}B(\boldsymbol{\xi},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4,t,s_1,s_2,s_3,s_4)\\ &\quad\cdot\partial_x^2 g_0(w,\boldsymbol{\xi}_4)d\boldsymbol{\xi}_4d\boldsymbol{\xi}_3d\boldsymbol{\xi}_2d\boldsymbol{\xi}_1ds_4ds_3ds_2ds_1 \quad (6.57)\end{aligned}$$

where

$$\begin{aligned}&B(\boldsymbol{\xi},\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3,\boldsymbol{\xi}_4,t,s_1,s_2,s_3,s_4)\\ &\equiv K(\boldsymbol{\xi},\boldsymbol{\xi}_1)\,K(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)\,K(\boldsymbol{\xi}_2,\boldsymbol{\xi}_3)\,K(\boldsymbol{\xi}_3,\boldsymbol{\xi}_4)\end{aligned}$$

$$\cdot e^{-\nu(\boldsymbol{\xi})(t-s_1)-\nu(\boldsymbol{\xi}_1)(s_1-s_2)-\nu(\boldsymbol{\xi}_2)(s_2-s_3)-\nu(\boldsymbol{\xi}_3)(s_3-s_4)-\nu(\boldsymbol{\xi}_4)s_4} \tag{6.58}$$

$$w \equiv x-\xi^1(t-s_1)-\xi^1_1(s_1-s_2)-\xi^1_2(s_2-s_3)-\xi^1_3(s_3-s_4)-\xi^1_4 s_4. \tag{6.59}$$

Again, we change the variables and make the switching of the differentiations:

$$\begin{aligned}
&V_1=\boldsymbol{\xi}-\boldsymbol{\xi}_1, \qquad V_2=\boldsymbol{\xi}_1-\boldsymbol{\xi}_2,\ V_3=\boldsymbol{\xi}_2-\boldsymbol{\xi}_3,\ V_4=\boldsymbol{\xi}_3-\boldsymbol{\xi}_4,\\
&\partial_{V_1^1}g_0(w,\boldsymbol{\xi}_4) = s_1\partial_x g_0(w,\boldsymbol{\xi}_4)-\partial_{\xi_2^1}g_0(w,\boldsymbol{\xi}_4),\\
&\partial_{V_3^1}g_0(w,\boldsymbol{\xi}_4) = s_3\partial_x g_0(w,\boldsymbol{\xi}_4)-\partial_{\xi_2^1}g_0(w,\boldsymbol{\xi}_4),\\
&\partial_x^2 g_0 = \frac{1}{s_1s_3}\left\{\partial_{V_1^1}\partial_{V_3^1}g_0+\partial_{V_1^1}\partial_{\xi_2^1}g_0+\partial_{V_3^1}\partial_{\xi_2^1}g_0+\partial^2_{\xi_2^1}g_0\right\}.
\end{aligned} \tag{6.60}$$

The key observation is that we switch the second derivative with respect to x *evenly* to V_1^1 and V_3^1. There are two reasons for doing this. The first is that the resulting factor $\frac{1}{s_1s_3}$ is integrable. Another is that B keeps the integrability after integration by parts. We set $s_0\equiv t$, $s_5\equiv 0$ so that

$$\begin{aligned}
B =& \prod_{i=0}^{4}\left\{e^{-\nu\left(\boldsymbol{\xi}-\sum_{j=1}^{i}V_j\right)(s_i-s_{i+1})}\right\}\\
&\cdot\prod_{i=1}^{4}\left\{\frac{2}{\sqrt{2\pi}|V_i|}\exp\left[-\frac{(|\boldsymbol{\xi}-\sum_{j=1}^{i-1}V_j|^2-|\boldsymbol{\xi}-\sum_{j=1}^{i}V_j|^2)^2}{8|V_i|^2}-\frac{|V_i|^2}{8}\right]\right.\\
&\left.-\frac{|V_i|}{2}\exp\left(-\frac{|\boldsymbol{\xi}-\sum_{j=1}^{i-1}V_j|^2+|\boldsymbol{\xi}-\sum_{j=1}^{i}V_j|^2}{4}\right)\right\}.
\end{aligned} \tag{6.61}$$

With this and the boundedness of $|\nabla_{\boldsymbol{\xi}}\nu(\boldsymbol{\xi})|$, we have

$$\begin{aligned}
&\partial_{V_1^1}B\\
&= O(1)e^{-\frac{3}{4}\nu_0 t}\Big\{\left[\partial_{\xi_1^1}K(\epsilon\boldsymbol{\xi},\epsilon\boldsymbol{\xi}_1)\right]K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)K(\epsilon\boldsymbol{\xi}_3,\epsilon\boldsymbol{\xi}_4)\\
&\quad +K(\epsilon\boldsymbol{\xi},\epsilon\boldsymbol{\xi}_1)K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)K(\epsilon\boldsymbol{\xi}_3,\epsilon\boldsymbol{\xi}_4)\Big\},\\
&\partial_{V_3^1}B\\
&= O(1)e^{-\frac{3}{4}\nu_0 t}\Big\{K(\epsilon\boldsymbol{\xi},\epsilon\boldsymbol{\xi}_1)K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)\left[\partial_{\boldsymbol{\xi}_3^1}K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)\right]K(\epsilon\boldsymbol{\xi}_3,\epsilon\boldsymbol{\xi}_4)\\
&\quad +K(\epsilon\boldsymbol{\xi},\epsilon\boldsymbol{\xi}_1)K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)K(\epsilon\boldsymbol{\xi}_3,\epsilon\boldsymbol{\xi}_4)\Big\},\\
&\partial_{V_3^1}\partial_{V_1^1}B\\
&= O(1)e^{-\frac{3}{4}\nu_0 t}\Big\{\left[\partial_{\xi_1^1}K(\epsilon\boldsymbol{\xi},\epsilon\boldsymbol{\xi}_1)\right]K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)\left[\partial_{\xi_3^1}K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)\right]K(\epsilon\boldsymbol{\xi}_3,\epsilon\xi_4)
\end{aligned}$$

$$
\begin{aligned}
&+\left[\partial_{\xi_1^1} K(\epsilon\xi,\epsilon\boldsymbol{\xi}_1)\right] K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)\, K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)\, K(\epsilon\boldsymbol{\xi}_3,\epsilon\boldsymbol{\xi}_4)\\
&+K(\epsilon\boldsymbol{\xi},\epsilon\boldsymbol{\xi}_1)K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)\left[\partial_{\xi_3^1} K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)\right] K(\epsilon\boldsymbol{\xi}_3,\epsilon\boldsymbol{\xi}_4)\\
&+K(\epsilon\boldsymbol{\xi},\epsilon\boldsymbol{\xi}_1)K(\epsilon\boldsymbol{\xi}_1,\epsilon\boldsymbol{\xi}_2)\, K(\epsilon\boldsymbol{\xi}_2,\epsilon\boldsymbol{\xi}_3)\, K(\epsilon\boldsymbol{\xi}_3,\epsilon\boldsymbol{\xi}_4)\Big\}, \qquad (6.62)
\end{aligned}
$$

for some $0<\epsilon<1$. From these we obtain

$$
\begin{aligned}
\iiiint \partial_{V_1^1} B\, dV_4 dV_3 dV_2 dV_1 &= O(1)e^{-\frac{3}{4}\nu_0 t},\\
\iiiint \partial_{V_3^1} B\, dV_4 dV_3 dV_2 dV_1 &= O(1)e^{-\frac{3}{4}\nu_0 t}, \qquad (6.63)\\
\iiiint \partial_{V_3^1}\partial_{V_1^1} B\, dV_4 dV_3 dV_2 dV_1 &= O(1)e^{-\frac{3}{4}\nu_0 t}.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\partial_x^2 \mathbb{M}_2^t g_0(x,\boldsymbol{\xi})\\
&\quad=\int_0^t\int_0^{s_1}\int_0^{s_2}\int_0^{s_3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{s_1 s_3}\Big\{g_0\partial_{V_3^1}\partial_{V_1^1}B\\
&\qquad-\left(\partial_{\xi_2^1} g_0\right)\left(\partial_{V_1^1}B+\partial_{V_3^1}B\right)+B\partial^2_{\xi_2^1} g_0\Big\}\, dV_4dV_3dV_2dV_1ds_4ds_3ds_2ds_1. \qquad (6.64)
\end{aligned}
$$

Apply (6.63) to the above, we conclude from the Hőlder inequality and Fubini Theorem that

$$
\left\|\partial_x^2\mathbb{M}_2^t g_0\right\|_{L_x^2(L^2_{\boldsymbol{\xi}})}\le C_2 e^{-\frac{\nu_0 t}{2}}\Big(\|g_0\|_{L_x^2(L^2_{\boldsymbol{\xi}})}+\left\|\partial_{\xi^1} g_0\right\|_{L_x^2(L^2_{\xi})}+\left\|\partial^2_{\xi^1} g_0\right\|_{L_x^2(L^2_{\boldsymbol{\xi}})}\Big). \qquad (6.65)
$$

This completes the proof of the Mixture Lemma. □

Lemma 6.16. *The wave* g^j *of* (6.45) *satisfy, for some positive constant* D_k*,*

$$
\|\partial_x^l(\mathsf{g}^{2k})\|_{L_x^2(L^2_{\boldsymbol{\xi}})}=O(1)e^{-\frac{\nu_1 t}{D_k}},\ \ k=0,1,2\cdots,\ \ 0\le l\le k. \qquad (6.66)
$$

Proof. This is direct consequence of the smoothness of $\mathsf{K}_1\bar{\mathsf{h}}$ in microscopic velocity, Lemma 6.10 and the Mixture Lemma, (6.49). □

6.5. Global wave structure of the Green's function

We now put together the previous pieces of the waves in the Green's function to gain global understanding of the Green's function. There is a missing information on the pointwise description outside the finite Mach region. This missing information is obtained through the following process. First, the regularity resulting from Picard iterations through the Mixture Lemma allows use to apply the Sobolev calculus if we have pointwise information on the Sobolev norms. The pointwise description of these Sobolev norms is obtained by a weighted energy estimate in the proof of the following main theorem for the 1-D Green's function. In this theorem, we describe only the Green's function itself. We will be interested in the differential of the Green's function. For that we need to include other singular waves besides the kinetic-like waves.

Lemma 6.17. *The Green's function minus the singular waves satisfies*

$$\left\|\partial_x^l\left(\mathbb{G}-\sum_{j=0}^{2}\mathsf{h}_j-\bar{\mathsf{h}}-\sum_{j=0}^{2k}\mathsf{g}^j\right)\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}=O(1),\ \ k=0,1,2\cdots,\ \ 0\le l\le k. \tag{6.67}$$

Proof. The function $\mathbb{G}-\sum_{j=0}^{2}\mathsf{h}_j-\bar{\mathsf{h}}-\sum_{j=0}^{2k}\mathsf{g}^j$ satisfies the linearized Boltzmann equation with the source $\mathsf{K}\mathsf{g}^{2k}$, (6.44). From Lemma 6.16, we know the smoothness and exponential decay in time of the source. Thus the lemma follows from the boundedness of the solution operator for the linearized Boltzmann equation, (3.14), and the Duhamel's principle:

$$\begin{aligned}
&\left\|\partial_x^l\left(\mathbb{G}-\sum_{j=0}^{2}\mathsf{h}_j-\bar{\mathsf{h}}-\sum_{j=0}^{2k}\mathsf{g}^j\right)\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}=\left\|\int_0^t\mathbb{G}^{t-s}\mathsf{K}\partial_x^l(\mathsf{g}^{2j})(s)ds\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}\\
&\le\int_0^t\left\|\mathbb{G}^{t-s}\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}\times\|\mathsf{K}\|_{L^2_{\boldsymbol{\xi}}}\times\left\|\partial_x^l(\mathsf{g}^{2j})\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}ds\\
&=O(1)\int_0^t e^{-\frac{(t-s)}{C}}ds=O(1).
\end{aligned}\tag{6.68}$$

□

Theorem 6.18. *The Green's function*

$$\mathbb{G}=\mathbb{G}_F+\mathbb{G}_K+\mathbb{G}_R, \tag{6.69}$$

consists of the main fluid-like part:

$$\mathbb{G}_F \equiv \sum_{j=1}^{3} \frac{1}{\sqrt{4\pi A_j(t+1)}} e^{-\frac{(x-\lambda_j t)^2}{4A_j t}} \mathsf{E}_j \otimes \langle \mathsf{E}_j | , \tag{6.70}$$

where $(\lambda_j, \mathsf{E}_j)$, $j = 1, 2, 3$, *are Euler characteristics,* (4.14)*, and* A_j *is the Navier-Stokes dissipation parameters,* (5.13)*; the particle-like part:*

$$\mathbb{G}_K \equiv \mathsf{h}^0 + \mathsf{h}^1 + \mathsf{h}^2, \tag{6.71}$$

(6.20)*,* (6.21)*,* (6.22)*,* (6.33)*,* (6.34)*; and the remainder* $\mathbb{G}_R$*, which is a bounded function satisfying, for any fixed constants* $C_j > A_j$*,* $j = 1, 2, 3$*, and some positive constant* C*,*

$$\|\mathbb{G}_R\|_{L^2_{\boldsymbol{\xi}}} = O(1) \sum_{j=1}^{3} \frac{1}{t+1} e^{-\frac{(x-\lambda_j t)^2}{4C_j t}} + O(1) e^{-\frac{|x|+t}{C}}. \tag{6.72}$$

Moreover, the Green's function as an operator has the following properties:

$$\|\mathbb{G}\mathsf{P}_1\|_{L^2_{\boldsymbol{\xi}}}, \|\mathsf{P}_1\mathbb{G}\|_{L^2_{\boldsymbol{\xi}}} = O(1) \sum_{j=1}^{3} \frac{1}{t+1} e^{-\frac{(x-\lambda_j t)^2}{4C_j t}} + O(1) e^{-\frac{|x|+t}{C}} \tag{6.73}$$

$$\|\mathsf{P}_1\mathbb{G}\mathsf{P}_1\|_{L^2_{\boldsymbol{\xi}}} = O(1) \sum_{j=1}^{3} \frac{1}{(t+1)^{\frac{3}{2}}} e^{-\frac{(x-\lambda_j t)^2}{4C_j t}} + O(1) e^{-\frac{|x|+t}{C}}. \tag{6.74}$$

Proof. The study of the structure of the Green's function is done according to the following two cases:

Case 1. Within finite Mach region, $|x| \le \mathscr{M} t$, for some constant $\mathscr{M} > |\lambda_j|$, $j = 1, 2, 3$, and sufficiently large.

By (), $\|\mathbb{G} - \mathbb{G}_L\|_{L^2_x(L^2_{\boldsymbol{\xi}})}$ decays exponentially in time. As the essential kinetic waves $\mathsf{h}^j, j = 0, 1, 2$, and $\bar{\mathsf{h}}$, (6.20) Lemmas 6.6, Lemmas 6.8, Lemma 6.10, and g^j, $j = 0, 1, \ldots$, Lemma 6.12, also decay exponentially in time, we have

$$\left\| \mathbb{G} - \mathbb{G}_L - \sum_{j=0}^{2} \mathsf{h}^j - \bar{\mathsf{h}} - \sum_{j=0}^{2} \mathsf{g}^j \right\|_{L^2_x(L^2_{\boldsymbol{\xi}})} = O(1) e^{-Ct}. \tag{6.75}$$

The fluid-like wave $\mathbb{G}_L$ is smooth and bounded, Theorem 5.9. This fact and Lemma 6.17, applied here with $k=1$, yield

$$\left\|\mathbb{G}-\mathbb{G}_L-\sum_{j=0}^{2}\mathsf{h}^j-\bar{\mathsf{h}}-\sum_{j=0}^{2}\mathsf{g}^j\right\|_{H^1_x(L^2_{\boldsymbol{\xi}})}=O(1). \tag{6.76}$$

By Sobolev inequality, we have from (6.75) and (6.76) that for some constant $C>0$,

$$\left\|\mathbb{G}-\mathbb{G}_L-\sum_{j=0}^{2}\mathsf{h}^j-\bar{\mathsf{h}}-\sum_{j=0}^{2}\mathsf{g}^j\right\|_{L^\infty_x(L^2_{\boldsymbol{\xi}})}=O(1)e^{-\frac{t}{C}}. \tag{6.77}$$

Thus in the finite Mach region, for some constant $C_1>0$,

$$\left\|\mathbb{G}-\mathbb{G}_L-\sum_{j=0}^{2}\mathsf{h}^j-\bar{\mathsf{h}}-\sum_{j=0}^{2}\mathsf{g}^j\right\|_{L^2_{\boldsymbol{\xi}}}=O(1)e^{-\frac{t+|x|}{C_1}}. \tag{6.78}$$

The theorem then follows from Theorem 5.9 and Theorem 5.11.

Case 2. Outside finite Mach region, $|x|>\mathscr{M}t$.

Outside of the finite Mach region, (6.75) still holds. Therefore, it remains to obtain the pointwise estimate in the space variable x. The Green's function minus the essential kinetic waves satisfies, (6.44):

$$\begin{cases}\mathsf{R}_t+\xi^1\mathsf{R}_x=\mathsf{L}\mathsf{R}+\mathsf{S},\\ \mathsf{R}(x,0,\boldsymbol{\xi})=0,\\ \mathsf{R}\equiv\mathbb{G}-\mathbb{G}_K-\bar{\mathsf{h}}-\sum_{j=0}^{2}\mathsf{g}^j,\\ \mathsf{S}\equiv\mathsf{K}\mathsf{g}^2.\end{cases} \tag{6.79}$$

We apply the *weighted energy method* to this equation with a weight $w(x,t)\mathsf{R}$:

$$w(x,t)\equiv e^{\frac{|x|-Mt}{N}}, \tag{6.80}$$

for some large constants M,N to be chosen later. Multiply (6.79) with

$w(x,t)$ and integrate to obtain

$$\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}w(\mathsf{R},\mathsf{R})dx+\int_{\mathbb{R}}\frac{1}{2N}w(\mathsf{R},(M-\xi^1\frac{x}{|x|})\mathsf{R})+w(\mathsf{R},-\mathsf{L}\mathsf{R})dx=\int_{\mathbb{R}}w(\mathsf{R},\mathsf{S})dx. \tag{6.81}$$

There are terms on the left hand side of (6.81) that are of good sign:

$$\int_{\mathbb{R}}w[\frac{1}{2N}(\mathsf{R},M\mathsf{R})+(\mathsf{R},-\mathsf{L}\mathsf{R})]dx=\int_{\mathbb{R}}w[\frac{M}{2N}(\mathsf{R},\mathsf{R})dx+(\mathsf{R},-\mathsf{L}\mathsf{R})]dx.$$

From (3.23) and (3.24),

$$|\int_{\mathbb{R}}\frac{1}{N}w(\mathsf{R},-\xi^1\frac{x}{|x|}\mathsf{R})dx|\le C_2\int_{\mathbb{R}}\frac{1}{N}w[(\mathsf{R},\mathsf{R})+(-\mathsf{L}\mathsf{R},\mathsf{R})]dx.$$

Thus we may choose

$$\frac{M}{2N}=\frac{C_1}{N}=2,$$

and yield

$$\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}w(\mathsf{R},\mathsf{R})dx+\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})dx\le\int_{\mathbb{R}}w(\mathsf{R},\mathsf{S})dx. \tag{6.82}$$

We conclude from the Cauchy-Schwarz inequality and the estimate of the source S by (6.43) that

$$\frac{d}{dt}\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})dx+\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})dx=O(1)\int_{\mathbb{R}}w(\mathsf{S},\mathsf{S})dx=O(1)\int_{\mathbb{R}}we^{-\frac{\nu_1(|x|+t)}{D_2}}dx. \tag{6.83}$$

By further restriction of the choice of the weight function $w(x,t)$, (6.80),

$$\frac{\nu_1}{D_2}\ge\frac{2}{N},$$

we have

$$\frac{d}{dt}\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})dx+\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})dx=O(1)\int_{\mathbb{R}}e^{\frac{|x|-Mt}{N}}e^{-\frac{\nu_1(|x|+t)}{D_2}}dx=O(1)e^{-4t}, \tag{6.84}$$

and we obtain the desired energy estimate

$$\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})(x,t)dx+\int_0^t\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})(x,s)dxds\le\int_{\mathbb{R}}w(\mathsf{R},\mathsf{R})(x,0)dx+O(1)$$

$$= O(1). \tag{6.85}$$

Similarly, since the function R is in $H_x^1(L^2_{\boldsymbol{\xi}})$, (6.76), we may apply the above weighted energy method to the equation for $\partial_x \mathsf{R}$ and obtain

$$\int_{\mathbb{R}} w(\partial_x \mathsf{R}, \partial_x \mathsf{R})(x,t)dx = O(1). \tag{6.86}$$

By the Sobolev inequality and with $\mathscr{M}$ chosen sufficiently large, we have from (6.85) and (6.86) that, for some positive constant C,

$$\|\mathsf{R}\|_{L^2_{\boldsymbol{\xi}}} = O(1)e^{-\frac{|x|-Mt}{N}} = O(1)e^{-C(|x|+t)}, \text{ for } |x| > \mathscr{M}t. \tag{6.87}$$

This completes the proof of the theorem. □

7. 3-D Green's Function

In this section we consider the 3-D Green's function for the initial-value problem, (1.7):

$$\begin{cases} \mathbb{G}_t + \boldsymbol{\xi} \cdot \partial_{\boldsymbol{x}} \mathbb{G} = \mathsf{L}\mathbb{G} \text{ for } \boldsymbol{x} \in \mathbb{R}^3, \\ \mathbb{G}(\boldsymbol{x}, 0, \boldsymbol{\xi}; \boldsymbol{\xi}_0) = \delta(\boldsymbol{x})\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0). \end{cases} \tag{7.1}$$

Here $\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$ was denoted by $\delta^3(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$ in the last two sections for 1-D case to emphasize that it is a 3-dimensional delta function. Here both $\delta(\boldsymbol{x})$ and $\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0)$ are 3-dimensional delta functions. We will derive the explicit form of the Green's function and give pointwise estimate of the remaining terms. As in the 1-D case, we construct the essential kinetic waves and fluid-like waves separately. However, there are essential differences with the 1-D case. For instance, there is Huygens' principle for 3-D case. The complex analytic method for inverting the Fourier transform is also more sophisticated than the 1-D case.

7.1. Kinetic waves

As with the 1-dimensional case, we construct the kinetic waves using the Picard's iteration,c.f. (6.3), (6.5):

$$\begin{cases} \mathsf{h}^0_t + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}\mathsf{h}^0 + \nu(\boldsymbol{\xi})\mathsf{h}^0 = 0, \\ \mathsf{h}^0(\boldsymbol{x},0) = \delta(\boldsymbol{x})\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0); \end{cases} \tag{7.2}$$

$$\begin{cases} \mathsf{h}^j_t + \boldsymbol{\xi}\cdot\partial_{\boldsymbol{x}}\mathsf{h}^j + \nu(\boldsymbol{\xi})\mathsf{h}^j = \mathsf{K}\mathsf{h}^{j-1}, \\ \mathsf{h}^j(\boldsymbol{x},0) = 0,\ \ j = 1,2,\ldots. \end{cases} \tag{7.3}$$

The difference here is that the delta function $\delta(\boldsymbol{x}) = \delta^3(\boldsymbol{x})$ is now 3-dimensional. Meanwhile, there is a stronger dispersion for the 3-D case. As a consequence, the same number of iterations will result in a bounded source. The first term is the same as before, (6.20), an exponentially decaying delta function along the characteristic direction:

$$\mathsf{h}^0(\boldsymbol{x},t,\boldsymbol{\xi},\boldsymbol{\xi}_0) \equiv \mathbb{S}^t\delta(\boldsymbol{x})\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0) = e^{-\nu(\boldsymbol{\xi})t}\delta(\boldsymbol{x}-\boldsymbol{\xi}t)\delta(\boldsymbol{\xi}-\boldsymbol{\xi}_0). \tag{7.4}$$

The second term is:

$$\begin{aligned} &\mathsf{h}^1(\boldsymbol{x},t,\boldsymbol{\xi},\boldsymbol{\xi}_0) \\ &\equiv \int_0^t \mathbb{S}^{t-s}\mathsf{K}\mathsf{h}^0(\boldsymbol{x},s,\boldsymbol{\xi},\boldsymbol{\xi}_0)ds \\ &= \int_0^t e^{-\nu(\boldsymbol{\xi})(t-s)}\int_{\mathbb{R}^3} K(\boldsymbol{\xi},\boldsymbol{\xi}_*)h_0(\boldsymbol{x}-\boldsymbol{\xi}(t-s),s,\boldsymbol{\xi}_*,\boldsymbol{\xi}_0)d\boldsymbol{\xi}_*ds \\ &= \int_0^t\int_{\mathbb{R}^3} e^{-\nu(\boldsymbol{\xi})(t-s)-\nu(\boldsymbol{\xi}_*)s}K(\boldsymbol{\xi},\boldsymbol{\xi}_*)\delta(\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_*s)\delta(\boldsymbol{\xi}_*-\boldsymbol{\xi}_0)d\boldsymbol{\xi}_*ds \\ &= \int_0^t e^{-\nu(\boldsymbol{\xi})(t-s)-\nu(\boldsymbol{\xi}_0)s}K(\boldsymbol{\xi},\boldsymbol{\xi}_0)\delta(\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0s)ds \end{aligned} \tag{7.5}$$

which is still a generalized function, though the generalized part is of lower dimension. From direct calculations,

$$\begin{aligned} &\mathsf{h}^2(\boldsymbol{x},t,\boldsymbol{\xi},\boldsymbol{\xi}_0) \\ &\equiv \int_0^t \mathbb{S}^{t-s}\mathsf{K}\mathsf{h}^1(\boldsymbol{x},s,\boldsymbol{\xi},\boldsymbol{\xi}_0)ds \\ &= \int_0^t\int_{\mathbb{R}^3}\int_0^s e^{-\nu(\boldsymbol{\xi})(t-s)-\nu(\boldsymbol{\xi}_*)(s-\tau)-\nu(\boldsymbol{\xi}_0)\tau}K(\boldsymbol{\xi},\boldsymbol{\xi}_*)K(\boldsymbol{\xi}_*,\boldsymbol{\xi}_0) \\ &\quad\cdot\delta(\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_*(s-\tau)-\boldsymbol{\xi}_0\tau)d\tau d\boldsymbol{\xi}_*ds \end{aligned}$$

$$= \int_0^t \int_0^s \exp\left[-\nu(\boldsymbol{\xi})(t-s) - \nu(\frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau})(s-\tau) - \nu(\boldsymbol{\xi}_0)\tau\right]$$
$$\cdot K(\boldsymbol{\xi}, \frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau}) K(\frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau}, \boldsymbol{\xi}_0) \frac{1}{(s-\tau)^3} d\tau ds \quad (7.6)$$

From the explicit expression in (3.5),

$$K(\boldsymbol{\xi}, \boldsymbol{\xi}_*) = O(1)\frac{1}{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|}\exp\left(-\frac{|\boldsymbol{\xi}-\boldsymbol{\xi}_*|^2}{8}\right)$$

and hence

$$\begin{aligned} &K(\boldsymbol{\xi}, \frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau}) \\ &\quad = O(1)\frac{|s-\tau|}{|\boldsymbol{\xi}(t-\tau)+\boldsymbol{\xi}_0\tau-\boldsymbol{x}|}\exp\left(-\frac{|\boldsymbol{\xi}(t-\tau)+\boldsymbol{\xi}_0\tau-\boldsymbol{x}|^2}{8|s-\tau|^2}\right), \\ &K(\frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau}, \boldsymbol{\xi}_0) \\ &\quad = O(1)\frac{|s-\tau|}{|\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0 s|}\exp\left(-\frac{|\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0 s|^2}{8|s-\tau|^2}\right). \end{aligned}$$

For $(\boldsymbol{x}, t, \boldsymbol{\xi}, \boldsymbol{\xi}_0) \in \mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfying

$$\min_{0\le s\le t} |\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0 s| > 0,$$

we have

$$K(\boldsymbol{\xi}, \frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau}) K(\frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau}, \boldsymbol{\xi}_0) \frac{1}{(s-\tau)^3} = O(1),$$
$$0 \le \tau \le s \le t.$$

We denote

$$\mathcal{S} \equiv \{(\boldsymbol{x}, t, \boldsymbol{\xi}, \boldsymbol{\xi}_0) : 0 \le \frac{x_1-\xi^1 t}{\xi_0^1-\xi^1} = \frac{x_2-\xi^2 t}{\xi_0^2-\xi^2} = \frac{x_3-\xi^3 t}{\xi_0^3-\xi^3} \le t\}.$$

Then for $(\boldsymbol{x}, t, \boldsymbol{\xi}, \boldsymbol{\xi}_0) \in \left(\mathbb{R}^3 \times \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3\right) \backslash \mathcal{S}$, we have $|\mathsf{h}^2(\boldsymbol{x}, t, \boldsymbol{\xi}, \boldsymbol{\xi}_0)| < \infty$.

Moreover, we have

$$\nu(\boldsymbol{\xi})(t-s) + \nu(\frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau})(s-\tau) + \nu(\boldsymbol{\xi}_0)\tau$$

$$
\begin{aligned}
&\geq \nu_0(1+|\boldsymbol{\xi}|)(t-s)+\nu_0(1+|\frac{\boldsymbol{x}-\boldsymbol{\xi}(t-s)-\boldsymbol{\xi}_0\tau}{s-\tau}|)(s-\tau)+\nu_0(1+|\boldsymbol{\xi}_0|)\tau \\
&\geq \nu_0\,(t+|\boldsymbol{x}|) \ \text{ for } \ (\boldsymbol{x},t,\boldsymbol{\xi},\boldsymbol{\xi}_0)\notin \mathcal{S}.
\end{aligned}
$$

Therefore, there exists a $C>0$ such that

$$
\left\|\mathsf{h}^2(\boldsymbol{x},t,\cdot,\boldsymbol{\xi}_0)\right\|_{L^2_{\boldsymbol{\xi}}} \leq O(1)e^{-\frac{(t+|\boldsymbol{x}|)}{C}}.
$$

This implies

$$
\left\|\mathsf{K}\mathsf{h}^2\right\|_{L^\infty_{\boldsymbol{\xi}}} \leq O(1)e^{-\frac{(t+|\boldsymbol{x}|)}{C}}
$$

and so

$$
\left\|\mathsf{h}^3(\boldsymbol{x},t,\cdot,\boldsymbol{\xi}_0)\right\|_{L^\infty_{\boldsymbol{\xi}}} \leq O(1)e^{-\frac{(t+|\boldsymbol{x}|)}{C}}.
$$

As with the 1-D case, we can stop at h^2 and construct $\bar{\mathsf{h}}$ as well as $\mathsf{g}^j,\ j=0,1,\ldots$. The Mixture Lemma can also be generalized easily to the present 3-D case. We omit the details.

7.2. Euler waves

Before we study the fluid-like waves for the Boltzmann equation, we consider the Euler waves. We use the conservative form of the linearized Euler equations (4.5):

$$
\begin{cases}
\rho_t+\nabla_{\boldsymbol{x}}\cdot\boldsymbol{m}=0,\\
\boldsymbol{m}_t+\frac{2}{3}\nabla_{\boldsymbol{x}}E=0,\\
E_t+\frac{5}{2}\nabla_{\boldsymbol{x}}\cdot\boldsymbol{m}=0.
\end{cases}
$$

This is the linearization with base state $\rho_0=1,\boldsymbol{v}_0=0,\theta_0=1$ and so the sound speed $\boldsymbol{c}$ is given as

$$
\boldsymbol{c}^2=\frac{5\theta_0}{3}=\frac{5}{3}.
$$

From this we derive the wave equation for ρ_t:

$$
\left(\frac{5}{2}\rho-E\right)_t \;=\; 0,
$$

$$(\rho_t)_{tt} = -\nabla_{\boldsymbol{x}} \cdot \left(\frac{2}{3}\nabla_{\boldsymbol{x}} E\right) = \nabla_{\boldsymbol{x}} \cdot \left(\frac{5}{3}\nabla_{\boldsymbol{x}}\rho\right) = \boldsymbol{c}^2\Delta_{\boldsymbol{x}}(\rho_t). \tag{7.7}$$

The wave equation can be solved explicitly by the Kirchhoff's formula and exhibits the *Huygens' principle* for the 3-D situation we consider here. With ρ_t thus determined, the other variables can also be explicitly constructed:

$$\begin{cases} \rho(\boldsymbol{x},t) = \rho(\boldsymbol{x},0) + \int_0^t \rho_t(\boldsymbol{x},s)ds, \\ E(\boldsymbol{x},t) = E(\boldsymbol{x},0) + \frac{5}{2}\int_0^t \rho_t(\boldsymbol{x},s)ds, \\ \boldsymbol{m}(\boldsymbol{x},t) = \boldsymbol{m}(\boldsymbol{x},0) + \frac{2}{3}\int_0^t \nabla_{\boldsymbol{x}} E(\boldsymbol{x},s)ds. \end{cases} \tag{7.8}$$

Note that, although ρ_t satisfies the Huygens' principle, the formula for the other variables involves time integration and so in general are not zero inside the acoustic cone and decay algebraically there. The viscous version is studied in Lemma 7.3 later.

To draw the comparison with the Boltzmann equation, we take Fourier transform of the Euler equations

$$\begin{pmatrix} \hat{\rho} \\ \hat{\boldsymbol{m}} \\ \hat{E} \end{pmatrix} + i \begin{pmatrix} 0 & \boldsymbol{\eta}^t & 0 \\ 0 & 0 & \frac{2}{3}\boldsymbol{\eta} \\ 0 & \frac{5}{2}\boldsymbol{\eta}^t & 0 \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{\boldsymbol{m}} \\ \hat{E} \end{pmatrix} = 0; \tag{7.9}$$

$$\begin{pmatrix} \hat{\rho}(\boldsymbol{\eta},t) \\ \hat{\boldsymbol{m}}(\boldsymbol{\eta},t) \\ \hat{E}(\boldsymbol{\eta},t) \end{pmatrix} = \begin{pmatrix} 1 & -\frac{i\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\boldsymbol{\eta}^t & \frac{2}{3\boldsymbol{c}^2}(-1+\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)) \\ 0 & I+\frac{\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1}{|\boldsymbol{\eta}|^2}\boldsymbol{\eta}\otimes\boldsymbol{\eta}^t & -\frac{2i\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{3\boldsymbol{c}|\boldsymbol{\eta}|}\boldsymbol{\eta} \\ 0 & -\frac{5i\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{2\boldsymbol{c}|\boldsymbol{\eta}|}\boldsymbol{\eta}^t & \cos(\boldsymbol{c}|\boldsymbol{\eta}|t) \end{pmatrix} \begin{pmatrix} \hat{\rho}(\boldsymbol{\eta},0) \\ \hat{\boldsymbol{m}}(\boldsymbol{\eta},0) \\ \hat{E}(\boldsymbol{\eta},0) \end{pmatrix}. \tag{7.10}$$

We therefore will be interested in the well-known inversion of the Fourier transform of the following types:

Theorem 7.1 (Kirchhoff)**.** *The inversion of the Fourier transform of* $\hat{\mathsf{g}}\hat{\mathsf{w}}$ *and* $\hat{\mathsf{g}}\hat{\mathsf{w}}_t$*, where*

$$\hat{\mathsf{w}} = \frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|},\ \ \hat{\mathsf{w}}_t = \cos(\boldsymbol{c}|\boldsymbol{\eta}|t),$$

are

$$\mathsf{w} * \mathsf{g}(\boldsymbol{x}) = \frac{t}{4\pi}\int\int_{|\boldsymbol{y}|=1} \mathsf{g}(\boldsymbol{x}+\boldsymbol{c}t\boldsymbol{y})d\boldsymbol{S}(\boldsymbol{y}), \tag{7.11}$$

$$\mathsf{w}_t * \mathsf{g}(\boldsymbol{x}) = \frac{1}{4\pi}\iint_{|\boldsymbol{y}|=1} \mathsf{g}(\boldsymbol{x}+\boldsymbol{c}t\boldsymbol{y})d\boldsymbol{S}(\boldsymbol{y}) + \frac{\boldsymbol{c}t}{4\pi}\iint_{|\boldsymbol{y}|=1} \nabla\mathsf{g}(\boldsymbol{x}+\boldsymbol{c}t\boldsymbol{y})\cdot\boldsymbol{y}d\boldsymbol{S}(\boldsymbol{y}). \tag{7.12}$$

The Kirchhoff's formula for the Green's function of the Euler equations involves the function g as $\delta-$ functions. For study of the dissipative Navier-Stokes and the Boltzmann equation, we consider g as the heat kernel. The following lemma gives estimates of the viscous version of Huygens' principle.

Lemma 7.2. *For any positive integer l,*

$$\left|\mathsf{w} * \frac{e^{-\frac{|\boldsymbol{x}|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}}\right| = O(1)\frac{e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{2C(t+1)}}}{(t+1)^{\frac{l}{2}}}, \tag{7.13}$$

$$\left|\mathsf{w}_t * \frac{e^{-\frac{|\boldsymbol{x}|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}}\right| = O(1)\frac{e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{2C(t+1)}}}{(t+1)^{\frac{l+1}{2}}}. \tag{7.14}$$

Proof. By the Kirchhoff formula (7.11),

$$J_1 \equiv \mathsf{w} * \frac{e^{-\frac{|\boldsymbol{x}|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}} = \frac{t}{4\pi}\iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}t\boldsymbol{y}|^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}} d\boldsymbol{S}(\boldsymbol{y}).$$

There are two cases.

Case 1. $||\boldsymbol{x}| - \boldsymbol{c}t| \le O(1)\sqrt{1+t}$.

$$\begin{aligned} J_1 &= O(1)(t+1)\iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-O(1)t|\boldsymbol{y}|^2}}{(t+1)^{\frac{l}{2}}} d\boldsymbol{S}(\boldsymbol{y}) = O(1)(t+1)\frac{1}{(t+1)^{\frac{l}{2}}}\left(\frac{1}{\sqrt{1+t}}\right)^2 \\ &= O(1)\frac{1}{(1+t)^{\frac{l}{2}}}. \end{aligned} \tag{7.15}$$

Case 2. $||\boldsymbol{x}| - \boldsymbol{c}t| \ge O(1)\sqrt{1+t}$.

In this case,

$$\min_{|\boldsymbol{y}|=1} |\boldsymbol{x} - \boldsymbol{c}t\boldsymbol{y}| = ||\boldsymbol{x}| - \boldsymbol{c}t|,$$

and so

$$e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}t\boldsymbol{y}|^2}{C(t+1)}} \le e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}t\boldsymbol{y}|^2}{2C(t+1)}-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{2C(t+1)}}; \tag{7.16}$$

and

$$\begin{aligned} J_1 &= (t+1)\iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}t\boldsymbol{y}|^2}{2C(t+1)}-\frac{|\boldsymbol{x}-\boldsymbol{c}t\boldsymbol{y}|^2}{2C(t+1)}}}{(t+1)^{\frac{l}{2}}} d\boldsymbol{S}(\boldsymbol{y}) \\ &\le (t+1)e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{2C(t+1)}} \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}t\boldsymbol{y}|^2}{2C(t+1)}}}{(t+1)^{\frac{l}{2}}} d\boldsymbol{S}(\boldsymbol{y}) \le \frac{e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{C(t+1)}}}{(t+1)^{\frac{l}{2}}}. \end{aligned} \tag{7.17}$$

This proves (7.13); (7.14) is shown similarly. □

The following will be needed for the viscous version of the procedure (7.8) in yielding the algebraic decaying rate inside the acoustic cone.

Lemma 7.3. *For* $|\boldsymbol{x}| < \boldsymbol{c}t$

$$\left| \int_0^t \left\{ \tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y}) \right\} d\tau \right| \le \frac{C}{(1+t)(|\boldsymbol{x}|+\sqrt{t+1})}; \tag{7.18}$$

and for $|\boldsymbol{x}| > \boldsymbol{c}t$

$$\left| \int_0^t \left\{ \tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y}) \right\} d\tau \right| \le \frac{Ce^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{2Ct}}}{(1+t)^2}. \tag{7.19}$$

Proof. From (7.16),

$$\begin{aligned} &\left| \int_0^t \left\{ \tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y}) \right\} d\tau \right| \\ &\le \left| \int_0^t \left\{ \tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{2Ct}} e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y}) \right\} d\tau \right| \end{aligned} \tag{7.20}$$

There are three cases.

Case 1. $|\boldsymbol{x}| \le O(1)\sqrt{1+t}$.

$$\begin{aligned}
&\left|\int_0^t \left\{\tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y})\right\} d\tau\right| \\
&\le \left|\left(\int_0^{\sqrt{1+t}} + \int_{\sqrt{1+t}}^t\right)\left\{\tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{2Ct}} e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y})\right\} d\tau\right| \\
&\le \int_0^{\sqrt{1+t}} \frac{\tau}{(1+t)^{5/2}} d\tau + O(1)\int_{\sqrt{1+t}}^t \frac{\tau}{(t+1)^{\frac{5}{2}}} \frac{te^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{\tau^2} d\tau = \frac{O(1)}{(1+t)^{\frac{3}{2}}}.
\end{aligned}$$

Case 2. $\sqrt{1+t} \le |\boldsymbol{x}| \le \boldsymbol{c}t + \sqrt{1+t}$.

$$\begin{aligned}
&\left|\int_0^t \left\{\tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y})\right\} d\tau\right| \\
&\le \left|\left(\int_0^{\sqrt{1+t}} + \int_{\sqrt{1+t}}^t\right)\left\{\tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{2Ct}} e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y})\right\} d\tau\right| \\
&\le \int_0^{\sqrt{1+t}} \frac{\tau e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{(1+t)^{5/2}} d\tau + O(1)\int_{\sqrt{1+t}}^t \frac{\tau}{(t+1)^{\frac{5}{2}}} \frac{te^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{\tau^2} d\tau = \frac{O(1)}{(1+t)|\boldsymbol{x}|}.
\end{aligned}$$

Case 3. $|\boldsymbol{x}| \ge \boldsymbol{c}t + \sqrt{1+t}$.

$$\begin{aligned}
&\left|\int_0^t \left\{\tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y})\right\} d\tau\right| \\
&\le \left|\left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)\left\{\tau \iint\limits_{|\boldsymbol{y}|=1} \frac{e^{-\frac{|\boldsymbol{x}-\boldsymbol{c}\tau\boldsymbol{y}|^2}{2Ct}} e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{(1+t)^{5/2}} d\boldsymbol{S}(\boldsymbol{y})\right\} d\tau\right| \\
&\le \int_0^{\frac{t}{2}} \frac{\tau e^{-\frac{\boldsymbol{c}^2 t}{4C}} e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{(1+t)^{5/2}} d\tau + O(1)\int_{\frac{t}{2}}^t \frac{\tau}{(t+1)^{\frac{5}{2}}} \frac{te^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}\tau)^2}{2Ct}}}{\tau^2} d\tau \\
&= \frac{O(1)e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{2Ct}}}{(1+t)^2}.
\end{aligned}$$

This completes the proof of the lemma. □

7.3. Spectrum near origin

Like the one-dimensional case, we have

Lemma 7.4. *Consider the spectrum* $\mathrm{Spec}\,(\boldsymbol{\eta})$ *of the operator* $-i\boldsymbol{\xi}\cdot\boldsymbol{\eta}+\mathsf{L}$, $\boldsymbol{\eta}\in\mathbb{R}^3$.

(I) *For any* $0<\delta\ll 1$, *there corresponds* $\tau=\tau(\delta)>0$ *such that*

(i) *For* $|\boldsymbol{\eta}|>\delta$,

$$\mathrm{Spec}\,(\boldsymbol{\eta})\subset\{z\in\mathbb{C}:\ \mathrm{Re}\,(z)<-\tau\}\,.$$

(ii) *For* $|\boldsymbol{\eta}|\le\delta$, *the spectrum within the region* $\{z\in\mathbb{C}:-\tau\le\mathrm{Re}\,(z)\}$ *consisting of exactly five eigenvalues* $\sigma_1(\boldsymbol{\eta})$, $\sigma_2(\boldsymbol{\eta})$, $\sigma_3(\boldsymbol{\eta})$, $\sigma_4(\boldsymbol{\eta})$, $\sigma_5(\boldsymbol{\eta})$:

$$\mathrm{Spec}\,(\boldsymbol{\eta})\cap\{z\in\mathbb{C}:-\tau\le\mathrm{Re}\,(z)\}=\{\sigma_1(\boldsymbol{\eta}),\sigma_2(\boldsymbol{\eta}),\sigma_3(\boldsymbol{\eta}),\sigma_4(\boldsymbol{\eta}),\sigma_5(\boldsymbol{\eta})\}\,.$$

(II) *For all* $0<\delta\ll 1$, *the semigroup* $e^{(-i\boldsymbol{\xi}\cdot\boldsymbol{\eta}+\mathsf{L})t}$ *can be decomposed as*

$$e^{(-i\boldsymbol{\xi}\cdot\boldsymbol{\eta}+\mathsf{L})t}=\Pi_\delta+\chi_{\{|\boldsymbol{\eta}|<\delta\}}\frac{1}{2\pi i}\oint_\Gamma e^{zt}\Big(z-(-i\boldsymbol{\xi}\cdot\boldsymbol{\eta}+\mathsf{L})\Big)^{-1}dz,\qquad(7.21)$$

where $\|\Pi_\delta\|_{L^2_{\boldsymbol{\xi}}}=O(1)e^{-a(\tau)t}$, $a(\tau)>0$ *depends on* τ *(and therefore on* δ*), and* Γ *can be any close curve that lies entirely on* $\{\mathrm{Re}\,z>-\tau\}$ *and that encloses the five eigenvalues* $\sigma_1(\boldsymbol{\eta}),\sigma_2(\boldsymbol{\eta}),\sigma_3(\boldsymbol{\eta}),\sigma_4(\boldsymbol{\eta}),\sigma_5(\boldsymbol{\eta})$.

We now compute the eigenvalues and eigenfunctions:

$$(-i\boldsymbol{\xi}\cdot\boldsymbol{\eta}+\mathsf{L})\psi_j(\boldsymbol{\eta})=\lambda_j(\boldsymbol{\eta})\psi_j(\boldsymbol{\eta}),\ \boldsymbol{\eta}\in\mathbb{R}^3.\qquad(7.22)$$

To simplify 7.22, we invoke a symmetric property of L. Consider the action of a three dimensional orthonormal transformation $\mathfrak{Q}\in O(3)$ on $L^2_{\boldsymbol{\xi}}$:

$$\begin{aligned}O(3)\times L^2_{\boldsymbol{\xi}}&\longrightarrow L^2_{\boldsymbol{\xi}}\\(\mathfrak{Q},\mathsf{f}(\boldsymbol{\xi}))&\longmapsto\mathsf{f}(\mathfrak{Q}\boldsymbol{\xi}).\end{aligned}\qquad(7.23)$$

Lemma 7.5. *The collision operator* Q *and the linearized collision operator* L *are invariant under orthonormal transformation. More precisely, for any*

$\mathfrak{Q} \in O(3)$:

$$\mathsf{Q}\Big(\mathfrak{Q}\mathsf{f}, \mathfrak{Q}\mathsf{g}\Big) = \mathfrak{Q}\mathsf{Q}(\mathsf{f}, \mathsf{g}), \ \ \textit{and}\ \mathsf{L}\mathfrak{Q}\,\mathsf{f} = \mathfrak{Q}\mathsf{L}\,\mathsf{g}.$$

Proof.

$$\begin{aligned}&\Big(\mathfrak{Q}\mathsf{Q}(\mathsf{f},\mathsf{g})\Big)(\boldsymbol{\xi}) = \mathsf{Q}(\mathsf{f},\mathsf{g})(\mathfrak{Q}\boldsymbol{\xi})\\ &= \int_{\bar{\boldsymbol{\xi}}\in\mathbb{R}^3}\int_{\bar{\Omega}\in S^2}\Big[\mathsf{f}((\mathfrak{Q}\boldsymbol{\xi})')\mathsf{g}(\bar{\boldsymbol{\xi}}_*') - \mathsf{f}(\mathfrak{Q}\boldsymbol{\xi})\mathsf{g}(\bar{\boldsymbol{\xi}}_*)\Big]B(\mathfrak{Q}\boldsymbol{\xi} - \bar{\boldsymbol{\xi}}_*, \bar{\Omega})d\bar{\Omega}d\bar{\boldsymbol{\xi}}_*.\end{aligned}$$

We change variables $\bar{\Omega} = \mathfrak{Q}\Omega$ and $\bar{\boldsymbol{\xi}}_* = \mathfrak{Q}\boldsymbol{\xi}_*$ and observe the following relations

$$\begin{cases}(\mathfrak{Q}\boldsymbol{\xi})' = \mathfrak{Q}\boldsymbol{\xi} - [(\mathfrak{Q}\boldsymbol{\xi} - \bar{\boldsymbol{\xi}}_*)\cdot\bar{\Omega}]\bar{\Omega} = \mathfrak{Q}\boldsymbol{\xi} - [(\mathfrak{Q}\boldsymbol{\xi} - \mathfrak{Q}\boldsymbol{\xi}_*)\cdot\mathfrak{Q}\Omega]\mathfrak{Q}\Omega\\ = \mathfrak{Q}\boldsymbol{\xi} - [(\boldsymbol{\xi}-\boldsymbol{\xi}_*)\cdot\Omega]\mathfrak{Q}\Omega = \mathfrak{Q}\boldsymbol{\xi}',\\ \bar{\boldsymbol{\xi}}_*' = \bar{\boldsymbol{\xi}}_* + [(\mathfrak{Q}\boldsymbol{\xi} - \bar{\boldsymbol{\xi}}_*)\cdot\bar{\Omega}]\bar{\Omega} = \mathfrak{Q}\boldsymbol{\xi}_* + [(\mathfrak{Q}\boldsymbol{\xi} - \mathfrak{Q}\boldsymbol{\xi}_*)\cdot\mathfrak{Q}\Omega]\mathfrak{Q}\Omega\\ = \mathfrak{Q}\boldsymbol{\xi}_* + [(\boldsymbol{\xi}-\boldsymbol{\xi}_*)\cdot\Omega]\mathfrak{Q}\Omega = \mathfrak{Q}\boldsymbol{\xi}_*',\end{cases}$$

Therefore,

$$\begin{aligned}\Big(\mathfrak{Q}\mathsf{Q}(\mathsf{f},\mathsf{g})\Big)(\boldsymbol{\xi}) &= \int_{\mathfrak{Q}\boldsymbol{\xi}\in\mathbb{R}^3}\int_{\mathfrak{Q}\Omega\in S^2}\Big[\mathsf{f}(\mathfrak{Q}\boldsymbol{\xi}')\mathsf{g}(\mathfrak{Q}\boldsymbol{\xi}_*') - \mathsf{f}(\mathfrak{Q}\boldsymbol{\xi})\mathsf{g}(\mathfrak{Q}\boldsymbol{\xi}_*)\Big]\\ &\qquad\times B(\mathfrak{Q}\boldsymbol{\xi} - \mathfrak{Q}\boldsymbol{\xi}_*, \mathfrak{Q}\Omega)d\mathfrak{Q}\Omega d\mathfrak{Q}\boldsymbol{\xi}_*\\ &= \int_{\boldsymbol{\xi}\in\mathbb{R}^3}\int_{\Omega\in S^2}\Big[\mathsf{f}(\mathfrak{Q}\boldsymbol{\xi}')\mathsf{g}(\mathfrak{Q}\boldsymbol{\xi}_*') - \mathsf{f}(\mathfrak{Q}\boldsymbol{\xi})\mathsf{g}(\mathfrak{Q}\boldsymbol{\xi}_*)\Big]\\ &\qquad\times B(\boldsymbol{\xi}-\boldsymbol{\xi}_*, \Omega)d\Omega d\boldsymbol{\xi}_* = \mathsf{Q}(\mathfrak{Q}\mathsf{f}, \mathfrak{Q}\mathsf{g})(\boldsymbol{\xi}).\end{aligned}$$

The invariance of L under Q can be proven similarly. We omit the details. □

Let $\mathfrak{g} \in O(3)$ be an orthonormal transformation that sends $\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}$ to $(1,0,0)$. Apply $\mathfrak{g}^{-1}$ to (7.22), we have, by Lemma 7.5,

$$\begin{aligned}&-i\left(\mathfrak{g}^{-1}\boldsymbol{\xi}\cdot\boldsymbol{\eta}\right)\mathfrak{g}^{-1}\psi_j + \mathfrak{g}^{-1}\mathsf{L}\psi_j = -i\Big(\mathfrak{g}\left(\mathfrak{g}^{-1}\boldsymbol{\xi}\right)\cdot\mathfrak{g}\boldsymbol{\eta} + \mathsf{L}\Big)\mathfrak{g}^{-1}\psi_j\\ &\quad= -i\Big(\boldsymbol{\xi}\cdot\mathfrak{g}\boldsymbol{\eta} + \mathsf{L}\Big)\mathfrak{g}^{-1}\psi_j = -i\Big(\xi^1|\boldsymbol{\eta}| + \mathsf{L}\Big)\mathfrak{g}^{-1}\psi_j = \lambda_j\mathfrak{g}^{-1}\psi_j.\end{aligned}\tag{7.24}$$

Therefore, the original equation (7.22) is reduced to

$$\Big(-i\xi^1|\boldsymbol{\eta}| + \mathsf{L}\Big)\mathsf{e}_j(|\boldsymbol{\eta}|) = \sigma_j(|\boldsymbol{\eta}|)\mathsf{e}_j(|\boldsymbol{\eta}|),\tag{7.25}$$

with $\lambda_j = \sigma_j$, $\psi_j = \mathfrak{g}\mathsf{e}_j$.

Apply Macro-Micro projection to the equation $(-i\xi^1|\boldsymbol{\eta}| + \mathsf{L})\mathsf{e}_j = \sigma_j \mathsf{e}_j$:

$$-i|\boldsymbol{\eta}|\mathsf{P}_0\xi^1\Big((\mathsf{P}_0\mathsf{e}_j) + (\mathsf{P}_1\mathsf{e}_j)\Big) = \sigma_j(\mathsf{P}_0\mathsf{e}_j), \tag{7.26a}$$

$$-i|\boldsymbol{\eta}|\mathsf{P}_1\xi^1(\mathsf{P}_0\mathsf{e}_j) - i|\boldsymbol{\eta}|\mathsf{P}_1\xi^1(\mathsf{P}_1\mathsf{e}_j) + \mathsf{L}(\mathsf{P}_1\mathsf{e}_j) = \sigma_j(\mathsf{P}_1\mathsf{e}_j). \tag{7.26b}$$

Set $i|\boldsymbol{\eta}| = \epsilon$ and $\sigma_j = \epsilon\gamma_j$. From (7.26b) we can solve $\mathsf{P}_1\mathsf{e}_j$ in terms of $\mathsf{P}_0\mathsf{e}_j$:

$$\mathsf{P}_1\mathsf{e}_j = \epsilon[\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_j]^{-1}\mathsf{P}_1\xi^1(\mathsf{P}_0\mathsf{e}_j). \tag{7.27}$$

Substituting this back to (7.26a), we obtain the following equation for $\mathsf{P}_0\mathsf{e}_j$:

$$\left(\mathsf{P}_0\xi^1 + \epsilon\mathsf{P}_0\xi^1\Big(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_j\Big)^{-1}\mathsf{P}_1\xi^1\right)(\mathsf{P}_0\mathsf{e}_j) = -\gamma_j(\mathsf{P}_0\mathsf{e}_j). \tag{7.28}$$

For notation simplicity, put

$$\mathscr{I}(\epsilon,\gamma) = \left(\mathsf{P}_0\xi^1 + \epsilon\mathsf{P}_0\xi^1\Big(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma\Big)^{-1}\mathsf{P}_1\xi^1\right).$$

When $\epsilon = 0$, (7.28) has degeneracy: $\gamma_2 = \gamma_4 = \gamma_5 = 0$, so implicit function theorem does not apply directly. However, as it turns out,

Lemma 7.6. *$(\mathscr{I}(\epsilon,\gamma)\mathsf{E}^1_j, \mathsf{E}^1_k) = 0$, if $j \neq k$ and $j,k = 4$ or 5, for any ϵ,γ. Consequently, under the basis $\{\mathsf{E}^1_1, \mathsf{E}^1_2, \mathsf{E}^1_3, \mathsf{E}^1_4, \mathsf{E}^1_5\}$, the matrix representation of $\mathscr{I}(\epsilon,\gamma)$ becomes:*

$$\left(\begin{array}{c|cc} 3\times 3 & \multicolumn{2}{c}{0} \\ \hline 0 & * & \\ & & * \end{array}\right).$$

Proof. Consider the reflection $\mathfrak{J}_2 \in O(3) : (\xi^1,\xi^2,\xi^3) \mapsto (\xi^1,-\xi^2,\xi^3)$. Clearly, $\mathsf{P}_0, \mathsf{P}_1, \mathsf{L}, \xi^1, \gamma_j, \epsilon$, as linear operators, commute with $\mathfrak{J}_2$. Therefore, $\mathscr{I}$ commutes with $\mathfrak{J}_2$. Consequently, since $\mathsf{E}^1_4 = \xi^2\sqrt{\mathsf{M}}$ is an odd function of ξ^2, $\mathscr{I}\mathsf{E}^1_4$ is also an odd function of ξ^2. Now that $\mathsf{E}^1_1, \mathsf{E}^1_2, \mathsf{E}^1_3, \mathsf{E}^1_5$ are even function of ξ^2,

$$(\mathscr{I}\mathsf{E}_j^1, \mathsf{E}_k^1) = \int_{\mathbb{R}^3} \mathscr{I}\mathsf{E}_j^1(\boldsymbol{\xi})\overline{\mathsf{E}_k^1(\boldsymbol{\xi})}d\boldsymbol{\xi} = 0, \text{ for } j = 4,\ k = 1,2,3,5. \tag{7.29}$$

The case $j = 5$ can be proven similarly. We omit the details. □

With the aid of Lemma 7.6, we can compute the eigenvalues and eigenfunctions. By (7.29), $\mathsf{P}_0\mathsf{e}_1, \mathsf{P}_0\mathsf{e}_2, \mathsf{P}_0\mathsf{e}_3 \in \left\langle \mathsf{E}_1^1, \mathsf{E}_2^1, \mathsf{E}_3^1 \right\rangle$ and $\mathsf{P}_0\mathsf{e}_4, \mathsf{P}_0\mathsf{e}_5 \in \left\langle \mathsf{E}_4^1, \mathsf{E}_5^1 \right\rangle$. Therefore, the problem of solving for $(\gamma_j, \mathsf{P}_0\mathsf{e}_j)$ for $j = 1,2,3$ is reduced to the 1-dimensional case.

For $j = 4, 5$, $\gamma_4(\epsilon), \gamma_5(\epsilon)$ can be solved from

$$\begin{aligned}(\mathscr{I}(\epsilon,\gamma_4)\mathsf{E}_4^1, \mathsf{E}_4^1) = \gamma_4, \quad \gamma_4|_{\epsilon=0} = 0,\\ (\mathscr{I}(\epsilon,\gamma_5)\mathsf{E}_5^1, \mathsf{E}_5^1) = \gamma_5, \quad \gamma_5|_{\epsilon=0} = 0.\end{aligned} \tag{7.30}$$

Consider $\mathfrak{C} \in O(3) : (\xi^1, \xi^2, \xi^3) \mapsto (\xi^1, \xi^3, \xi^2)$.

$$(\mathscr{I}\mathsf{E}_4, \mathsf{E}_4) = (\mathfrak{C}\mathscr{I}\mathsf{E}_4, \mathfrak{C}\mathsf{E}_4) = (\mathscr{I}\mathfrak{C}\mathsf{E}_4, \mathfrak{C}\mathsf{E}_4) = (\mathscr{I}\mathsf{E}_5, \mathsf{E}_5).$$

This, together with (7.30), imply $\gamma_4 = \gamma_5$. Since the eigenspace for $\gamma_4 = \gamma_5$ always degenerates with multiplicity 2, we gain extra freedom in specifying $\mathsf{e}_4, \mathsf{e}_5$. However, the choice must obey $[e_j, e_k] = \delta_{jk}$. As it turns out (see the proof of the lemma below), we can choose

$$\mathsf{P}_0\mathsf{e}_4 \in \left\langle \mathsf{E}_4^1 \right\rangle, \ \mathsf{P}_0\mathsf{e}_5 \in \left\langle \mathsf{E}_5^1 \right\rangle.$$

Lemma 7.7. *View ϵ as a real variable. For $|\epsilon| \ll 1$, $\gamma_j(\epsilon)$, $j = 1, \ldots, 5$, are real and analytic functions with*

$$\gamma_j(\epsilon) = -\lambda_j + A_j\epsilon + O(\epsilon^2), \ \textit{for } j = 1,2,3, \tag{7.31a}$$

$$\gamma_4(\epsilon) = \gamma_5(\epsilon) = A_4\epsilon + O(\epsilon^2), \tag{7.31b}$$

where A_j is the Navier-Stokes dissipation coefficients. Note that $A_4 = A_5$. Moreover,

$$\begin{aligned}\mathsf{P}_0\mathsf{e}_j(\epsilon) =&\beta_{j1}(\epsilon)\mathsf{E}_1^1 + \beta_{j2}(\epsilon)\mathsf{E}_2^1 + \beta_{j3}(\epsilon)\mathsf{E}_3^1, \ \textit{for } j = 1,2,3,\\ \mathsf{P}_0\mathsf{e}_4(\epsilon) =&\beta_{44}(\epsilon)\mathsf{E}_4^1,\\ \mathsf{P}_0\mathsf{e}_5(\epsilon) =&\beta_{55}(\epsilon)\mathsf{E}_5^1,\end{aligned}$$

where $\beta_{jk}(\epsilon)$ is a real and analytic function with

$$\beta_{jj} = 1 + O(\epsilon) \text{ for } j = 1,2,3,4,5,$$
$$\beta_{jk} = O(\epsilon), \text{ for } j \neq k.$$
$$\beta_{44} = \beta_{55}.$$

Proof. Since $(\mathscr{I}\mathsf{E}_j, \mathsf{E}_k)$ is real (ϵ is thought of as a real variable here), $\sigma_j(\epsilon)$, $\beta_{jk}(\epsilon)$ are real. (7.31b) follows at once after we differentiate (7.30).

It remains only to compute $\mathsf{P}_0\mathsf{e}_4, \mathsf{P}_0\mathsf{e}_5$. Consider the ansatz:

$$\mathsf{P}_0\mathsf{e}_4 = \beta_{44}\mathsf{E}_4^1,\ \mathsf{P}_0\mathsf{e}_5 = \beta_{55}\mathsf{E}_5^1,\ \beta_{44}, \beta_{55} \in \mathbb{R}. \tag{7.32}$$

Recall that $\mathsf{P}_1\mathsf{e}_j = \epsilon[\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_j]^{-1}\mathsf{P}_1\xi^1(\mathsf{P}_0\mathsf{e}_j)$, (7.27). Therefore $\mathsf{e}_4, \mathsf{e}_5$ are real and

$$[\mathsf{e}_4, \mathsf{e}_5] = (\mathsf{e}_4, \mathsf{e}_5) = \epsilon^2\beta_{44}\beta_{55}\Big(\Big(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_4\Big)^{-1}\mathsf{E}_4^1, \Big(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_5\Big)^{-1}\mathsf{E}_5^1\Big).$$

This is zero, since $(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_4)^{-1}\mathsf{E}_4^1$ is odd in ξ^2 while $(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_5)^{-1}\mathsf{E}_5^1$ is even in ξ^2. The validity of the ansatz (7.32) is justified. Subsequently, β_{44}, β_{55} can be solved from the normalization condition $[\mathsf{e}_j, \mathsf{e}_j] = (\mathsf{e}_j, \mathsf{e}_j) = 1$:

$$\beta_{44} = \frac{1}{1 + \Big\|\epsilon\Big(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_4\Big)^{-1}\mathsf{E}_4^1\Big\|_{L^2_{\boldsymbol{\xi}}}^2} = \frac{1}{1 + \Big\|\epsilon\Big(\mathsf{L} - \epsilon\mathsf{P}_1\xi^1 - \epsilon\gamma_5\Big)^{-1}\mathsf{E}_5^1\Big\|_{L^2_{\boldsymbol{\xi}}}^2}$$
$$= \beta_{55} = 1 + O(\epsilon^2). \qquad \square$$

We continue with more detailed analysis of (σ_j, e_j). Consider the reflection $\mathfrak{J}_1 : (\xi^1, \xi^2, \xi^2) \mapsto (-\xi^1, \xi^2, \xi^3)$. Note that

$$\begin{aligned}(-\xi^1\epsilon + \mathsf{L})\mathfrak{J}_1\mathsf{e}_j(-\epsilon) &= \mathfrak{J}_1(\xi^1\epsilon + \mathsf{L})\mathsf{e}_j(-\epsilon) = \mathfrak{J}_1(-\xi^1(-\epsilon) + \mathsf{L})\mathsf{e}_j(-\epsilon)\\ &= \sigma_j(-\epsilon)\mathfrak{J}_1\mathsf{e}_j(-\epsilon) = \epsilon\big(-\gamma_j(-\epsilon)\big)\mathfrak{J}_1\mathsf{e}_j(-\epsilon).\end{aligned}$$

Thus$(\epsilon(-\gamma_j(-\epsilon)), \mathfrak{J}_1\mathsf{e}_j(-\epsilon))$ is an eigenvalue-eigenfunction pair. Note also that, for $j = 1,2,3$, $\mathsf{P}_0\mathfrak{J}_1\mathsf{e}_j = \mathfrak{J}_1\mathsf{P}_0\mathsf{e}_j \in \big\langle \mathsf{E}_1^1, \mathsf{E}_2^1, \mathsf{E}_3^1\big\rangle$, and this implies $-\gamma_1(-\epsilon), -\gamma_2(-\epsilon), -\gamma_3(-\epsilon) \in \{\gamma_1(\epsilon), \gamma_2(\epsilon), \gamma_3(\epsilon)\}$. Since $\gamma_1(\epsilon), \gamma_2(\epsilon), \gamma_3(\epsilon)$ are distinct (for small ϵ), by the fact that $\gamma_1(0) = -\mathbf{c} = -\gamma_3(0), \gamma_2(0) = 0$

and by continuity, we conclude

$$\gamma_1(\epsilon) = -\gamma_3(-\epsilon), \gamma_2(-\epsilon) = -\gamma_2(\epsilon),$$
$$\mathfrak{J}_1 \mathsf{e}_1(-\epsilon) = \mathsf{e}_3(\epsilon), \mathfrak{J}_1 \mathsf{e}_2(-\epsilon) = \mathsf{e}_2(\epsilon),$$

for all ϵ small.

For $\gamma_4 = \gamma_5$, a similar argument yields $\gamma_4(-\epsilon) = -\gamma_4(-\epsilon)$. As for the eigenvectors, $\mathsf{P}_0\mathfrak{J}_1\mathsf{e}_4(-\epsilon), \mathsf{P}_0\mathfrak{J}_1\mathsf{e}_5(-\epsilon) \in \left\langle \mathsf{E}_4^1, \mathsf{E}_5^1 \right\rangle$. Moreover, $\mathsf{P}_0\mathfrak{J}_1\mathsf{e}_4(-\epsilon)$ is odd in ξ^2. Therefore, $\mathsf{P}_0\mathfrak{J}_1\mathsf{e}_4(-\epsilon) \in \left\langle \mathsf{E}_4^1 \right\rangle$ and $\mathfrak{J}_1\mathsf{e}_4(-\epsilon) = \mathsf{e}_4(\epsilon)$. Similarly, $\mathfrak{J}_1\mathsf{e}_5(-\epsilon) = \mathsf{e}_5(\epsilon)$.

Summarizing, we have

$$\sigma_1(-\epsilon) = \sigma_3(\epsilon), \tag{7.33a}$$
$$\sigma_j(-\epsilon) = \sigma_j(\epsilon), \text{ for } j = 2, 4, 5, \tag{7.33b}$$
$$\mathfrak{J}_1\mathsf{e}_1(-\epsilon) = \mathsf{e}_3(\epsilon), \tag{7.33c}$$
$$\mathfrak{J}_1\mathsf{e}_j(-\epsilon) = \mathsf{e}_j(\epsilon), \text{ for } j = 2, 4, 5. \tag{7.33d}$$

Now we switch back to the variable $i|\boldsymbol{\eta}| = \epsilon$. Although so far we have been working with real ϵ, since σ_j, e_j are analytic in ϵ, (7.33) holds for all complex ϵ, as long as $|\epsilon| \ll 1$. For simplicity, set

$$\mathscr{L}_j = 1 + \Big(\mathsf{L} - i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta} - \lambda_j(\boldsymbol{\eta})\Big)^{-1}\big(i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}\big). \tag{7.34}$$

Lemma 7.8. *For $\boldsymbol{\eta} \in \mathbb{R}^3$ with $|\boldsymbol{\eta}| \ll 1$,*

$$\begin{aligned}
\lambda_1(\boldsymbol{\eta}) &= -\, i|\boldsymbol{\eta}|\Big(\boldsymbol{c} + \Lambda(|\boldsymbol{\eta}|^2)\Big) - A_1|\boldsymbol{\eta}|^2 + O(|\boldsymbol{\eta}|^4),\\
\lambda_2(\boldsymbol{\eta}) &= -\, A_2\boldsymbol{\eta}^2 + O(|\boldsymbol{\eta}|^4)\\
\lambda_3(\boldsymbol{\eta}) &= i|\boldsymbol{\eta}|\Big(\boldsymbol{c} + \Lambda(|\boldsymbol{\eta}|^2)\Big) - A_1|\boldsymbol{\eta}|^2 + O(|\boldsymbol{\eta}|^4),\\
\lambda_4(\boldsymbol{\eta}) &= -\, A_4|\boldsymbol{\eta}|^2 + O(|\boldsymbol{\eta}|^4),\\
\lambda_5(\boldsymbol{\eta}) &= -\, A_4|\boldsymbol{\eta}|^2 + O(|\boldsymbol{\eta}|^4),\\
A_j &= -\, (\mathsf{P}_1\xi^1\mathsf{E}_j^1, \mathsf{L}^{-1}\mathsf{P}_1\xi^1\mathsf{E}_j^1),
\end{aligned} \tag{7.35}$$

for some analytic function $\Lambda : \mathbb{R} \to \mathbb{R}$. Furthermore, there exists analytic *functions* $\mathsf{a}^j_{k,l}:\mathbb{R}\to\mathbb{R}$ *such that*

$$\begin{aligned}
\psi_1(\boldsymbol{\eta}) &= \mathscr{L}_1\Big[\Big(\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2)\Big)\chi_0 \\
&\quad+\big(\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2)\big)\sum_{k=1}^3\frac{\eta^k\chi_k}{|\boldsymbol{\eta}|} \\
&\quad+\Big(\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2)\Big)\chi_4\Big], \qquad (7.36a)\\
\psi_2(\boldsymbol{\eta}) &= \mathscr{L}_2\Big[\mathsf{a}^0_{2,1}(|\boldsymbol{\eta}|^2)\chi_0+i\mathsf{a}^1_{2,2}(|\boldsymbol{\eta}|^2)\sum_{j=1}^3\eta^j\chi_j+\mathsf{a}^4_{2,1}(|\boldsymbol{\eta}|^2)\chi_4\Big], \qquad (7.36b)\\
\psi_3(\boldsymbol{\eta}) &= \mathscr{L}_3\Big[\Big(\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2)\Big)\chi_0 \\
&\quad-\big(\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2)\big)\sum_{k=1}^3\frac{\eta^k\chi_k}{|\boldsymbol{\eta}|} \\
&\quad+\Big(\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2)\Big)\chi_4\Big], \qquad (7.36c)\\
\psi_4(\boldsymbol{\eta}) &= \mathscr{L}_4\Big[\mathsf{a}^2_{4,1}(|\boldsymbol{\eta}|^2)\mathfrak{g}\chi_2\Big], \qquad (7.36d)\\
\psi_5(\boldsymbol{\eta}) &= \mathscr{L}_4\Big[\mathsf{a}^2_{4,1}(|\boldsymbol{\eta}|^2)\mathfrak{g}\chi_3\Big], \qquad (7.36e)
\end{aligned}$$

with

$$\mathsf{a}^0_{1,1}(0)=\sqrt{\frac{3}{10}},\quad \mathsf{a}^1_{1,1}(0)=-\sqrt{\frac{1}{2}},\quad \mathsf{a}^4_{1,1}(0)=\sqrt{\frac{1}{5}},\quad a^0_{2,1}(0)=-\sqrt{\frac{2}{5}},$$
$$\mathsf{a}^4_{2,1}(0)=\sqrt{\frac{3}{5}},\quad \mathsf{a}^2_{4,1}(0)=1.$$

Proof. From (7.31a), we have

$$\begin{aligned}
\sigma_1(i|\boldsymbol{\eta}|) &= i|\boldsymbol{\eta}|\Big(-\mathbf{c}+A_1(i|\boldsymbol{\eta}|)+\sum_{k=1}^\infty(-|\boldsymbol{\eta}|^2)^k\frac{1}{(2k)!}\frac{d^{2k}\gamma_1}{d\epsilon^{2k}}(0) \\
&\qquad+i|\boldsymbol{\eta}|\sum_{k=1}^\infty(-|\boldsymbol{\eta}|^2)^k\frac{1}{(2k+1)!}\frac{d^{2k+1}\gamma_1}{d\epsilon^{2k+1}}(0)\Big) \\
&\equiv i|\boldsymbol{\eta}|\Big(-\mathbf{c}+A_1(i|\boldsymbol{\eta}|)-\Lambda(|\boldsymbol{\eta}|^2) \\
&\qquad+i|\boldsymbol{\eta}|\sum_{k=1}^\infty(-|\boldsymbol{\eta}|^2)^k\frac{1}{(2k+1)!}\frac{d^{2k+1}\gamma_1}{d\epsilon^{2k+1}}(0)\Big) \\
&= -i|\boldsymbol{\eta}|\left(\mathbf{c}+\Lambda(|\boldsymbol{\eta}|^2)\right)-A_1|\boldsymbol{\eta}|^2+O(|\boldsymbol{\eta}|^4).
\end{aligned}$$

By (7.33),

$$\begin{aligned}\sigma_3(i|\boldsymbol{\eta}|) &= \sigma_1(-i|\boldsymbol{\eta}|)\\ &= -i|\boldsymbol{\eta}|\Big(-\mathbf{c}-A_1(i|\boldsymbol{\eta}|)-\Lambda(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|\sum_{k=1}^{\infty}(-|\boldsymbol{\eta}|^2)^k\frac{1}{(2k+1)!}\frac{d^{2k+1}\gamma_1}{d\epsilon^{2k+1}}(0)\Big)\\ &= i|\boldsymbol{\eta}|\left(\mathbf{c}+\Lambda(|\boldsymbol{\eta}|^2)\right)-A_1|\boldsymbol{\eta}|^2+O(|\boldsymbol{\eta}|^4).\end{aligned}$$

Similarly, for $j=2,4,5$, by (7.33),

$$\begin{aligned}\sigma_j(i|\boldsymbol{\eta}|) =& i|\boldsymbol{\eta}|\left(A_j(i|\boldsymbol{\eta}|)+i|\boldsymbol{\eta}|\sum_{k=1}^{\infty}(-|\boldsymbol{\eta}|^2)^k\frac{1}{(2k+1)!}\frac{d^{2k+1}\gamma_j}{d\epsilon^{2k+1}}(0)\right)\\ =& -A_j|\boldsymbol{\eta}|^2+O(|\boldsymbol{\eta}|^4).\end{aligned}$$

This proves (7.35). (7.36) follows by similar computations. We omit the details. Note that

$$\begin{aligned}\mathfrak{g}\,\mathsf{P}_1\xi^1|\boldsymbol{\eta}| = \mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}\,\mathfrak{g} \Longrightarrow \mathfrak{g}&\left(1+\Big(\mathsf{L}-i\mathsf{P}_1\xi^1|\boldsymbol{\eta}|-\lambda_j(\boldsymbol{\eta})\Big)^{-1}\Big(i\mathsf{P}_1\xi^1|\boldsymbol{\eta}|\Big)\right)\\ &=\left(1+\Big(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}-\lambda_j(\boldsymbol{\eta})\Big)^{-1}\Big(i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}\Big)\right)\mathfrak{g},\\ \mathfrak{g}\Big(\xi^1\sqrt{\mathsf{M}}\Big) = (\mathfrak{g}\boldsymbol{\xi})^1\sqrt{\mathsf{M}} &= \sum_{k=1}^{3}\frac{\xi^k\eta^k}{|\boldsymbol{\eta}|}\sqrt{\mathsf{M}}. \qquad\square\end{aligned}$$

7.4. Fluid-like waves, I

This and next subsections are for the study of the fluid-like waves. For the 3-D case considered here, there is a geometrically richer class of fluid-like waves as hinted by the study of the Euler waves. In this subsection, we will explicitly constructed the *leading fluid-like waves*. The next subsection will be concerned with the remaining fluid-like waves, which decay at faster rate than the leading fluid-like waves considered here.

As the one dimensional case, we have

$$\frac{1}{2\pi i}\oint_{\Gamma} e^{zt}\Big(z-(-i\boldsymbol{\eta}\cdot\boldsymbol{\xi}+\mathsf{L})\Big)^{-1}dz = \sum_{j=1}^{5} e^{\lambda_j t}\psi_j\otimes[\psi_j|.$$

This implies

$$\mathbb{G}=\frac{1}{\sqrt{2\pi}}\mathscr{F}^{-1}\circ\Pi_\delta+\sum_{j=1}^{5}\frac{1}{(2\pi)^3}\iiint_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}+\lambda_j(\eta)t}\psi_j\otimes[\psi_j|d\eta,$$

with

$$\left\|\mathscr{F}^{-1}\circ\Pi_\delta\right\|_{L^2_x(L^2_{\boldsymbol{\xi}})}=O(1)e^{-\frac{t}{C(\delta)}}.$$

We again define the fluid-like waves in the Green's function as:

$$\begin{aligned}
&\mathbb{G}_L(\boldsymbol{x},t)\\
&\equiv\sum_{j=1}^{5}\frac{1}{(2\pi)^3}\iiint_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}+\lambda_j(\boldsymbol{\eta})t}\psi_j(\boldsymbol{\eta})\otimes[\psi_j(\boldsymbol{\eta})|d\boldsymbol{\eta}\\
&=\iiint_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}\Big[e^{\lambda_1(\boldsymbol{\eta})t}\psi_1(\boldsymbol{\eta})\otimes[\psi_1(\boldsymbol{\eta})|+e^{\lambda_3(\boldsymbol{\eta})t}\psi_3(\boldsymbol{\eta})\otimes[\psi_3(\boldsymbol{\eta})|\Big]d\boldsymbol{\eta}\\
&\quad+\iiint_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}e^{\lambda_2(\boldsymbol{\eta})t}\psi_2(\boldsymbol{\eta})\otimes[\psi_2(\boldsymbol{\eta})|d\boldsymbol{\eta}\\
&\quad+\iiint_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}e^{\lambda_4(\boldsymbol{\eta})t}\Big[\psi_4(\boldsymbol{\eta})\otimes[\psi_4(\boldsymbol{\eta})|+\psi_5(\boldsymbol{\eta})\otimes[\psi_5(\boldsymbol{\eta})|\Big]d\boldsymbol{\eta}. \qquad (7.37)
\end{aligned}$$

We arrange the pairing in (7.37) to define the following pairings:

$$\hat{\mathbb{G}}_L(\boldsymbol{\eta},t)=\hat{\mathbb{G}}_{\mathfrak{H}}(\boldsymbol{\eta},t)+\hat{\mathbb{G}}_{\mathfrak{C}}(\boldsymbol{\eta},t)+\hat{\mathbb{G}}_{\mathfrak{R}}(\boldsymbol{\eta},t)+\hat{\mathbb{G}}_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{\eta},t)+\hat{\mathbb{G}}_{\mathfrak{P}\mathfrak{R}_2}(\boldsymbol{\eta},t), \quad (7.38)$$

where

Huygens Pairing

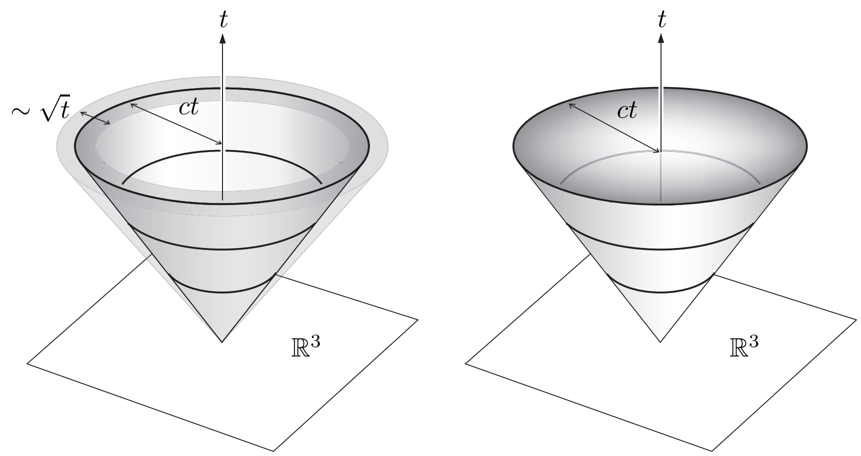
$$\begin{aligned}
\hat{\mathbb{G}}_{\mathfrak{H}}(\boldsymbol{\eta},t)&=\sum_{j=1,3}e^{\lambda_j(\boldsymbol{\eta})}\psi_j(\boldsymbol{\eta})\otimes[\psi_j(\boldsymbol{\eta})|\\
&\quad-\sum_{j=1,3}e^{\lambda_j(\boldsymbol{\eta})}\mathscr{L}_j\mathsf{P}_0^m\psi_j(\boldsymbol{\eta})\otimes[\mathscr{L}_j\mathsf{P}_0^m\psi_j(\boldsymbol{\eta})|,
\end{aligned}$$

Contact Pairing,

$$\hat{\mathbb{G}}_{\mathfrak{C}}(\boldsymbol{\eta},t)=e^{\lambda_2(\boldsymbol{\eta})}\psi_2(\boldsymbol{\eta})\otimes[\psi_2(\boldsymbol{\eta})|,$$

Rotational Pairing,

$$\hat{\mathbb{G}}_{\mathfrak{R}}(\boldsymbol{\eta},t)=e^{\lambda_4(\boldsymbol{\eta})}\bigg(\sum_{j=4,5}\psi_j(\boldsymbol{\eta})\otimes[\psi_j(\boldsymbol{\eta})|$$



(a) Huygens wave. (b) Solid Huygens wave.

Figure 7: Huygens wave and solid Huygens wave.

$$+\sum_{j=1,3} \mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta}) \otimes [\mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta})| \Big),$$

1st Riesz Pairing,

$$\begin{aligned} \hat{\mathbb{G}}_{\mathfrak{PR}_1}(\boldsymbol{\eta}, t) &= \sum_{j=1,3} e^{\lambda_j(\boldsymbol{\eta})} \mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta}) \otimes [\mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta})| \\ &\quad - e^{-A_1(|\boldsymbol{\eta}|^2)} \sum_{j=1,3} \mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta}) \otimes [\mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta})|, \end{aligned}$$

2nd Riesz Pairing,

$$\hat{\mathbb{G}}_{\mathfrak{PR}_2}(\boldsymbol{\eta}, t) = (e^{-A_1(|\boldsymbol{\eta}|^2)} - e^{\lambda_4(\boldsymbol{\eta})}) \sum_{j=1,3} \mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta}) \otimes [\mathscr{L}_j \mathsf{P}_0^m \psi_j(\boldsymbol{\eta})|, \tag{7.39}$$

We define the *leading fluid-like waves* to be those induced by the quadratic terms, in the Fourier variable $\boldsymbol{\eta}$, in the eigenvalues and the constant term in the eigenfunctions. This is made definite in the proof of the following theorem, (7.44). There are richer wave patterns for the 3-D case. The *Huygens wave* for the Euler equations is now a dissipative version $\mathbb{G}^0_{\mathfrak{H}}$, with essential support of width $\sqrt{t}$ around the acoustic cone, Figure . As we have seen for the Euler equations, there are time integrations of the waves around the acoustic cone. These appear in $\mathbb{G}^0_{\mathfrak{PR}_1} + \mathbb{G}^0_{\mathfrak{PR}_2}$, Figure . The *contact, thermal waves* $\mathbb{G}^0_{\mathfrak{C}}$ as well as the *rotational waves* $\mathbb{G}^0_{\mathfrak{R}}$ are supported as the heat kernel, Figure 8.

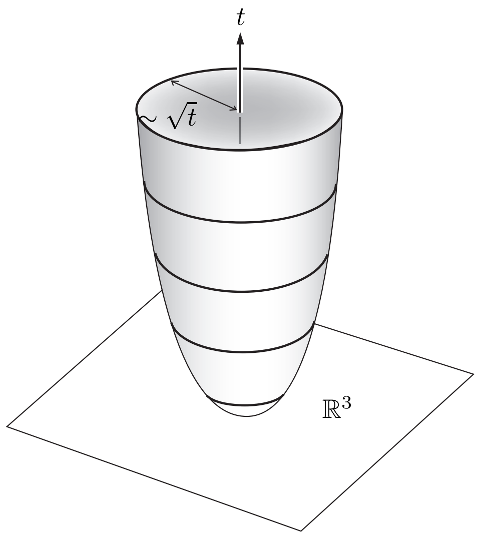


Figure 8: Diffuse wave.

Theorem 7.9. *The leading fluid-like waves are* $\mathbb{G}^0 = \mathbb{G}^0_{\mathfrak{H}} + \mathbb{G}^0_{\mathfrak{C}} + \mathbb{G}^0_{\mathfrak{R}} + \mathbb{G}^0_{\mathfrak{PR}_1} + \mathbb{G}^0_{\mathfrak{PR}_2}$

$$\begin{aligned}
\mathbb{G}^0_{\mathfrak{H}}(\boldsymbol{x},t) &= \frac{\boldsymbol{c}t}{4\pi}\iint\limits_{|\boldsymbol{y}|=1} H_1(\boldsymbol{x}+\boldsymbol{c}t\boldsymbol{y})d\boldsymbol{S}(\boldsymbol{y}) + \frac{1}{4\pi}\iint\limits_{|\boldsymbol{y}|=1} H_2(\boldsymbol{x}+\boldsymbol{c}t\boldsymbol{y})d\boldsymbol{S}(\boldsymbol{y}) \\
&\quad + \frac{\boldsymbol{c}t}{4\pi}\iint\limits_{|\boldsymbol{y}|=1} \nabla H_2(\boldsymbol{x}+\boldsymbol{c}t\boldsymbol{y})\cdot\boldsymbol{y}d\boldsymbol{S}(\boldsymbol{y}), \\
H_1 &= \frac{1}{\sqrt{15}}\sum_{j=1}^{3}((4\pi A_1(1+t))^{-3/2}e^{-\frac{|\boldsymbol{x}|^2}{4A_1t}})_{x^j}\Big(\xi^j\sqrt{\mathsf{M}}\otimes\Big[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\Big| \\
&\quad +|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes\Big[\xi^j\sqrt{\mathsf{M}}\Big|\Big), \\
H_2 &= \frac{1}{15}(4\pi A_1(1+t))^{-3/2}e^{-\frac{|\boldsymbol{x}|^2}{4A_1t}}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes\Big[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\Big|, \\
\mathbb{G}^0_{\mathfrak{C}}(\boldsymbol{x},t) &= \frac{1}{10}(4\pi A_2(1+t))^{-3/2}e^{-\frac{|\boldsymbol{x}|^2}{4A_2t}}(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}\otimes\Big[(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}\Big|, \\
\mathbb{G}^0_{\mathfrak{PR}_1}(\boldsymbol{x},t) &= \int_0^t \frac{\boldsymbol{c}^2\tau}{4\pi}\iint\limits_{|\boldsymbol{y}|=1} H_3(\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y})d\boldsymbol{S}(\boldsymbol{y})d\tau \\
H_3 &= \sum_{j,k=1}^{3}((4\pi A_1(1+t))^{-3/2}e^{-\frac{|\boldsymbol{x}|^2}{4A_1t}})_{x^jx^k}\xi^j\sqrt{\mathsf{M}}\otimes\Big[\xi^k\sqrt{\mathsf{M}}\Big|,
\end{aligned}$$

$$\begin{aligned}
\mathbb{G}^0_{\mathfrak{R}}(\boldsymbol{x},t) &= (4\pi A_4(1+t))^{-3/2}e^{-\frac{|\boldsymbol{x}|^2}{4A_4t}}\sum_{j=1}^{3}\xi^j\sqrt{\mathsf{M}}\otimes\Big[\xi^j\sqrt{\mathsf{M}}\Big|,\\
\mathbb{G}^0_{\mathfrak{PR}_2}(\boldsymbol{x},t) &= (A_1-A_4)\sum_{j,k=1}^{3}\\
&\quad\Big(\int_0^t\frac{\exp\{-\frac{|\boldsymbol{x}|^2}{4(\max\{A_1,A_4\}t+|A_1-A_4|\tau)}\}}{(4\pi(\max\{A_1,A_4\}(1+t)+|A_1-A_4|\tau))^{3/2}}d\tau\Big)_{x^jx^k}\xi^j\sqrt{\mathsf{M}}\\
&\quad\otimes\Big[\xi^k\sqrt{\mathsf{M}}\Big|,
\end{aligned}\tag{7.40}$$

and

$$\begin{cases}
\|\mathbb{G}^0_{\mathfrak{H}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}\le C(1+t)^{-2}e^{\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{Ct}},\\
\|\mathbb{G}^0_{\mathfrak{C}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}},\|\mathbb{G}^0_{\mathfrak{R}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}},\|\mathbb{G}^0_{\mathfrak{PR}_2}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}\le C(1+t)^{-\frac{3}{2}}e^{-\frac{|\boldsymbol{x}|^2}{Ct}},\\
\|\mathbb{G}^0_{\mathfrak{PR}_1}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}\le C\Big((1+t)^{-2}e^{\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{Ct}}+(1+t)^{-\frac{3}{2}}e^{-\frac{|\boldsymbol{x}|^2}{Ct}}\\
+\chi_{|\boldsymbol{x}|\le\boldsymbol{c}t}(1+t)^{-\frac{3}{2}}(1+\frac{|\boldsymbol{x}|^2}{t})^{-\frac{3}{2}}\Big).
\end{cases}\tag{7.41}$$

Proof. From (7.35) and (7.36), we can see that the leading term of $\hat{\mathbb{G}}_L(\boldsymbol{\eta},t)$ is

$$\begin{aligned}
&e^{-i\boldsymbol{c}|\boldsymbol{\eta}|t-A_1|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_1\otimes[\mathfrak{g}\mathsf{E}^1_1|+e^{-A_2|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_2\otimes[\mathfrak{g}\mathsf{E}^1_2|+e^{i\boldsymbol{c}|\boldsymbol{\eta}|t-A_1|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_3\otimes[\mathfrak{g}\mathsf{E}^1_3|\\
&\quad+e^{-A_4|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_4\otimes[\mathfrak{g}\mathsf{E}^1_4|+e^{-A_4|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_5\otimes[\mathfrak{g}\mathsf{E}^1_5|\\
&\quad\equiv\ \mathcal{G}_1+\mathcal{G}_2+\mathcal{G}_3+\mathcal{G}_4+\mathcal{G}_5.
\end{aligned}$$

Direct computations yield

$$\mathcal{G}_2\equiv e^{-A_2|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_2\otimes[\mathfrak{g}\mathsf{E}^1_2|=e^{-A_2|\boldsymbol{\eta}|^2t}\frac{1}{10}(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}\otimes[(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}|\equiv\hat{\mathbb{G}}^1_{\mathfrak{C}}.$$

For $\mathcal{G}_1+\mathcal{G}_3$, we use symmetry of (λ_1,ψ_1) and (λ_3,ψ_3) to calculate

$$\begin{aligned}
&\mathcal{G}_1+\mathcal{G}_3\\
&\equiv\ e^{-i\boldsymbol{c}|\boldsymbol{\eta}|t-A_1|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_1\otimes[\mathfrak{g}\mathsf{E}^1_1|+e^{i\boldsymbol{c}|\boldsymbol{\eta}|t-A_1|\boldsymbol{\eta}|^2t}\mathfrak{g}\mathsf{E}^1_3\otimes[\mathfrak{g}\mathsf{E}^1_3|\\
&=\ e^{-i\boldsymbol{c}|\boldsymbol{\eta}|t-A_1|\boldsymbol{\eta}|^2t}(\frac{1}{\sqrt{30}}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}-\frac{1}{\sqrt{2}}\sum_{j=1}^{3}\frac{\eta^j}{|\boldsymbol{\eta}|}\xi^j\sqrt{\mathsf{M}})
\end{aligned}$$

$$
\begin{aligned}
&\otimes[(\frac{1}{\sqrt{30}}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}} - \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\frac{\eta^j}{|\boldsymbol{\eta}|}\xi^j\sqrt{\mathsf{M}})| \\
&+e^{i\boldsymbol{c}|\boldsymbol{\eta}|t - A_1|\boldsymbol{\eta}|^2 t}(\frac{1}{\sqrt{30}}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}} + \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\frac{\eta^j}{|\boldsymbol{\eta}|}\xi^j\sqrt{\mathsf{M}}) \\
&\otimes[(\frac{1}{\sqrt{30}}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}} + \frac{1}{\sqrt{2}}\sum_{j=1}^{3}\frac{\eta^j}{|\boldsymbol{\eta}|}\xi^j\sqrt{\mathsf{M}})| \\
= \;& e^{-A_1|\boldsymbol{\eta}|^2 t}(e^{-i\boldsymbol{c}|\boldsymbol{\eta}|t} + e^{i\boldsymbol{c}|\boldsymbol{\eta}|t})\Big[\frac{1}{30}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}} \otimes [|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}| \\
&+\frac{1}{2}\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}|\Big] \\
&-e^{-A_1|\boldsymbol{\eta}|^2 t}(e^{-i\boldsymbol{c}|\boldsymbol{\eta}|t} - e^{i\boldsymbol{c}|\boldsymbol{\eta}|t})\frac{1}{2\sqrt{15}}\sum_{j=1}^{3}\frac{\eta^j}{|\boldsymbol{\eta}|}\Big[\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}| \\
&+|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\Big] \\
= \;& e^{-A_1|\boldsymbol{\eta}|^2 t}\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\Big[\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}| + \sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}|\Big] \\
&+e^{-A_1|\boldsymbol{\eta}|^2 t}\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j=1}^{3}\frac{i\boldsymbol{c}\eta^j}{\sqrt{15}}\Big[\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}| + |\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\Big] \\
\equiv \;& \hat{\mathbb{G}}^1_{\mathfrak{H}} + e^{-A_1|\boldsymbol{\eta}|^2 t}\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}|.
\end{aligned}
$$

In order to estimate $\mathcal{G}_4+\mathcal{G}_5$, we recall that $\mathfrak{g}\boldsymbol{\eta}/|\boldsymbol{\eta}| = (1,0,0)$. Let $\{\boldsymbol{\eta}/|\boldsymbol{\eta}|, \boldsymbol{\alpha}, \boldsymbol{\beta}\}$ be a basis for $\mathbb{R}^3$ satisfying

$$
\mathfrak{g}\begin{cases} \boldsymbol{\eta}/|\boldsymbol{\eta}|, \\ \boldsymbol{\alpha}, \\ \boldsymbol{\beta}, \end{cases} \longrightarrow \begin{cases} (1,0,0), \\ (0,1,0), \\ (0,0,1), \end{cases} \tag{7.42}
$$

Using the above relation, the operator $\mathfrak{g}\mathsf{E}^1_4 \otimes [\mathfrak{g}\mathsf{E}^1_4| + \mathfrak{g}\mathsf{E}^1_5 \otimes [\mathfrak{g}\mathsf{E}^1_5|$ can be rewritten as

$$
\begin{aligned}
&\Big[\mathfrak{g}\mathsf{E}^1_4 \otimes [\mathfrak{g}\mathsf{E}^1_4| + \mathfrak{g}\mathsf{E}^1_5 \otimes [\mathfrak{g}\mathsf{E}^1_5|\Big]h \\
&= \mathfrak{g}(\xi^2\sqrt{\mathsf{M}})\int \mathfrak{g}(\xi^2_*\sqrt{\mathsf{M}})h(\boldsymbol{\xi}_*)d\boldsymbol{\xi}_* + \mathfrak{g}(\xi^3\sqrt{\mathsf{M}})\int \mathfrak{g}(\xi^3_*\sqrt{\mathsf{M}_*})h(\boldsymbol{\xi}_*)d\boldsymbol{\xi}_*
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{\xi}\cdot\boldsymbol{\alpha}\sqrt{\mathsf{M}}\int \boldsymbol{\xi}_*\cdot\boldsymbol{\alpha}\sqrt{\mathsf{M}_*}h(\boldsymbol{\xi}_*)d\boldsymbol{\xi}_* + \boldsymbol{\xi}\cdot\boldsymbol{\beta}\sqrt{\mathsf{M}}\int \boldsymbol{\xi}_*\cdot\boldsymbol{\beta}\sqrt{\mathsf{M}_*}h(\boldsymbol{\xi}_*)d\boldsymbol{\xi}_* \\
&= \boldsymbol{\xi}\sqrt{\mathsf{M}}\cdot\Big[\int[(\boldsymbol{\xi}_*\cdot\boldsymbol{\alpha})\boldsymbol{\alpha}+(\boldsymbol{\xi}_*\cdot\boldsymbol{\beta})\boldsymbol{\beta}]\sqrt{\mathsf{M}_*}h(\boldsymbol{\xi}_*)d\boldsymbol{\xi}_*\Big] \\
&= \boldsymbol{\xi}\sqrt{\mathsf{M}}\cdot\Big[\int[\boldsymbol{\xi}_*-(\boldsymbol{\xi}_*\cdot\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|})\frac{\boldsymbol{\eta}}{|\boldsymbol{\eta}|}]\sqrt{\mathsf{M}_*}h(\boldsymbol{\xi}_*)d\boldsymbol{\xi}_*\Big]. \qquad (7.43)
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\mathcal{G}_4+\mathcal{G}_5 &= e^{-A_4|\boldsymbol{\eta}|^2t}\Big[\sum_{j=1}^{3}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|-\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}|\Big] \\
&\equiv \hat{\mathbb{G}}^1_{\mathfrak{R}} - e^{-A_4|\boldsymbol{\eta}|^2t}\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}|.
\end{aligned}
$$

The two terms not yet given names are

$$
\begin{aligned}
&e^{-A_1|\boldsymbol{\eta}|^2t}\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}| \\
&\quad -e^{-A_4|\boldsymbol{\eta}|^2t}\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}| \\
&= e^{-A_1|\boldsymbol{\eta}|^2t}(\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}| \\
&\quad +(e^{-A_1|\boldsymbol{\eta}|^2t}-e^{-A_4|\boldsymbol{\eta}|^2t})\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}| \equiv \hat{\mathbb{G}}^1_{\mathfrak{PR}_1}+\hat{\mathbb{G}}^1_{\mathfrak{PR}_2}.
\end{aligned}
$$

Thus we have the following named terms:

$$
\begin{cases}
\hat{\mathbb{G}}^1_{\mathfrak{H}} = e^{-A_1|\boldsymbol{\eta}|^2t}\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|, \\
+e^{-A_1|\boldsymbol{\eta}|^2t}\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j=1}^{3}\frac{i\boldsymbol{c}\eta^j}{\sqrt{15}}\Big[\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|+|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\Big], \\
\hat{\mathbb{G}}^1_{\mathfrak{C}} = e^{-A_2|\boldsymbol{\eta}|^2t}\frac{1}{10}(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}\otimes[(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}|, \\
\hat{\mathbb{G}}^1_{\mathfrak{R}} = e^{-A_4|\boldsymbol{\eta}|^2t}\sum_{j=1}^{3}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|, \\
\hat{\mathbb{G}}^1_{\mathfrak{PR}_2} = (e^{-A_1|\boldsymbol{\eta}|^2t}-e^{-A_4|\boldsymbol{\eta}|^2t})\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}|, \\
\hat{\mathbb{G}}^1_{\mathfrak{PR}_1} = e^{-A_1|\boldsymbol{\eta}|^2t}(\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\xi^j\sqrt{\mathsf{M}}\otimes[\xi^k\sqrt{\mathsf{M}}|.
\end{cases}
\qquad (7.44)
$$

To extract the leading waves (7.40) it is necessary to compute the inverse Fourier transform of (7.44) explicitly. We carry the calculation for $\mathbb{G}^0_{\mathfrak{C}}$.

$$\begin{aligned}
\mathscr{F}^{-1}_{\boldsymbol{\eta}\to\boldsymbol{x}}\{\mathcal{G}_2\} &\equiv \frac{1}{(2\pi)^3}\int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} e^{-A_2|\boldsymbol{\eta}|^2 t}\mathfrak{g}\mathsf{E}^1_2\otimes[\mathfrak{g}\mathsf{E}^1_2| d\boldsymbol{\eta} \\
&= \frac{1}{(2\pi)^3}\int_{[-\frac{\delta}{2},\frac{\delta}{2}]^3} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} e^{-A_2|\boldsymbol{\eta}|^2 t}\mathsf{E}^1_2\otimes[\mathsf{E}^1_2| d\boldsymbol{\eta} + \mathcal{O}(e^{-\frac{t}{C}}) \\
&= \prod_{j=1}^{3}\Big(\frac{1}{2\pi}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} e^{ix^j\cdot\eta^j} e^{-A_2|\eta^j|^2 t} d\eta^j\Big)\mathsf{E}^1_2\otimes[\mathsf{E}^1_2| + \mathcal{O}(e^{-\frac{t}{C}}).
\end{aligned}$$

We now introduce contours $\Gamma^j_1+\Gamma^j_2+\Gamma^j_3$

$$\begin{aligned}
&\Gamma^j_1 = \{\eta^j : \mathrm{Re}(\eta^j) = -\frac{\delta}{2}, \mathrm{Im}(\eta^j) \text{ lies between 0 and } \frac{ix^j}{2A_2 t}\}, \\
&\Gamma^j_2 = \{\eta^j :\leq -\frac{\delta}{2} \leq \mathrm{Re}(\eta^j) \leq \frac{\delta}{2}, \mathrm{Im}(\eta^j) = \frac{ix^j}{2A_2 t}\}, \\
&\Gamma^j_3 = \{\eta^j : \mathrm{Re}(\eta^j) = \frac{\delta}{2}, \mathrm{Im}(\eta^j) \text{ lies between 0 and } \frac{ix^j}{2A_2 t}\},
\end{aligned} \tag{7.45}$$

to move the path $[-\frac{\delta}{2},\frac{\delta}{2}]$ to $\Gamma^j_1+\Gamma^j_2+\Gamma^j_3$, then use that $\mathrm{Re}\{e^{ix^j\cdot\eta^j}e^{-A_2|\eta^j|^2 t}\} \leq -A_2\delta^2 t$ on both Γ^j_1 and Γ^j_3

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} e^{ix^j\cdot\eta^j} e^{-A_2|\eta^j|^2 t} d\eta^j \\
&\quad= \frac{1}{2\pi}\int_{\Gamma^j_2} e^{ix^j\cdot\eta^j} e^{-A_2|\eta^j|^2 t} d\eta^j + O(1)e^{-\frac{t}{C}} \\
&\quad= \frac{1}{\sqrt{4\pi A_2(1+t)}} e^{-\frac{|x^j|^2}{4A_2 t}} + O(1)\frac{1}{1+t}e^{-\frac{|x^j|^2}{4A_2 t}} + O(1)e^{-\frac{t}{C}}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{1}{(2\pi)^3}\int_{|\boldsymbol{\eta}|<\delta}\mathcal{G}_2 d\boldsymbol{\eta} \;&=\; (4\pi A_2(1+t))^{-\frac{3}{2}} e^{-\frac{|\boldsymbol{x}|^2}{4A_2 t}}\frac{1}{10}(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}\otimes[(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}| \\
&\quad+\mathcal{O}(1)\frac{e^{-\frac{|x^j|^2}{4A_2 t}}}{(1+t)^2} + \mathcal{O}(e^{-\frac{t}{C}}).
\end{aligned}$$

The Fourier inversion of $\hat{\mathbb{G}}^1_{\mathfrak{H}}$ and $\hat{\mathbb{G}}^1_{\mathfrak{P}\mathfrak{R}_1}$ are done using the Kirchhoff formulas in Theorem 7.1. Unlike the heat kernel, our domain of integration is $|\boldsymbol{\eta}| < \delta$, which doesn't yield the singularity at $t=0$. A slight generalization of these formulas of the following form is used:

$$\frac{\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1}{|\boldsymbol{\eta}|^2}=-\boldsymbol{c}^2\int_0^t\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|\tau)}{\boldsymbol{c}|\boldsymbol{\eta}|}d\tau,$$

and so the inverse Fourier of

$$\frac{\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1}{|\boldsymbol{\eta}|^2}\hat{\mathrm{g}}$$

is

$$-\boldsymbol{c}^2\int_0^t\frac{\tau}{4\pi}\int\int_{|\boldsymbol{y}|=1}\mathrm{g}(\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y})d\boldsymbol{S}(\boldsymbol{y}).$$

Note also that the term η^j corresponds to $\partial/\partial x_j$. We omit the details. This establishes the explicit form of the leading fluid-like waves (7.40).

It remains to study the estimates (7.41). The second one is easy to see. We use Theorem 7.1 and Lemma 7.2 to obtain the first one. We consider the last Riesz leading wave for small time and large time scale, i.e. $t\le 1$ and $t\ge 1$.

Case 1 $t\le 1$ We consider two subcase $|\boldsymbol{x}|\le \boldsymbol{c}t^{\frac{1}{2}}$ and $|\boldsymbol{x}|\ge \boldsymbol{c}t^{\frac{1}{2}}$

For $|\boldsymbol{x}|\le \boldsymbol{c}t^{\frac{1}{2}}$,note that $|\boldsymbol{x}|^2/t$ is bounded above and so $1\le Ce^{-|\boldsymbol{x}|^2/t}$, and, for some positive constant C

$$\begin{aligned}\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}} &\le C\int_0^t\frac{\boldsymbol{c}^2\tau}{(1+t)^{3/2}t}\iint\limits_{|\boldsymbol{y}|=1}e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{8A_1t}}d\boldsymbol{S}(\boldsymbol{y})d\tau\\ &\le C(1+t)^{-3/2}\le C(1+t)^{-3/2}e^{-\frac{|\boldsymbol{x}|^2}{Ct}}.\end{aligned}$$

For $|\boldsymbol{x}|\ge \boldsymbol{c}t^{\frac{1}{2}}$, we have

$$\begin{aligned}\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}} &\le C\int_0^t\frac{\boldsymbol{c}^2\tau}{(1+t)^{3/2}t}\iint\limits_{|\boldsymbol{y}|=1}e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{8A_1t}}d\boldsymbol{S}(\boldsymbol{y})d\tau\\ &\le Ce^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1t}}\int_0^t\frac{\boldsymbol{c}^2\tau}{(1+t)^{3/2}t}\iint\limits_{|\boldsymbol{y}|=1}e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{16A_1t}}d\boldsymbol{S}(\boldsymbol{y})d\tau\end{aligned}$$

$$\leq\ C(1+t)^{-3/2}e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1t}} \leq C(1+t)^{-2}e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1t}}.$$

Case 2 $t\geq 1$ We have three cases $|\boldsymbol{x}|\leq \boldsymbol{c}t^{\frac{1}{2}}$, $\boldsymbol{c}t^{\frac{1}{2}}\leq|\boldsymbol{x}|\leq\boldsymbol{c}t$ and $|\boldsymbol{x}|\geq\boldsymbol{c}t$. The case $|\boldsymbol{x}|\leq\boldsymbol{c}t^{\frac{1}{2}}$ is dealt with the same way as above. For $\boldsymbol{c}t^{\frac{1}{2}}\leq|\boldsymbol{x}|\leq\boldsymbol{c}t$, in order to directly calculate the x^j and x^k derivatives in $\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{x},t)$ we first observe that

$$\int_0^t \frac{\mathbf{c}^2\tau}{4\pi(4\pi A_1(1+t))^{3/2}}\iint\limits_{|\boldsymbol{y}|=1} e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{4A_1t}}d\boldsymbol{S_y}d\tau = Ct^{\frac{1}{2}}\int_{|\boldsymbol{y}|\leq 1}\frac{1}{|\boldsymbol{y}|}e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}t\boldsymbol{y}|^2}{4A_1t}}d\boldsymbol{y},$$

where C is agenetic constant independent of $(\boldsymbol{x},\boldsymbol{\xi},t)$. We now adopt the spherical coordinate and let $\boldsymbol{x}$ to be the polar axis.

$$\begin{aligned}\int_{|\boldsymbol{y}|\leq 1}\frac{1}{|\boldsymbol{y}|}e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{4A_1t}}d\boldsymbol{y} \;&=\; C\int_0^1\int_0^{2\pi}\int_0^{\pi} re^{-\frac{|\boldsymbol{x}|^2+2\boldsymbol{c}t|\boldsymbol{x}|r\cos\theta+\boldsymbol{c}^2t^2r^2}{4A_1t}}\sin\theta d\theta d\phi dr\\ &=\; C\frac{1}{|\boldsymbol{x}|}\int_0^1\Big(e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}tr)^2}{4A_1t}}-e^{-\frac{(|\boldsymbol{x}|+\boldsymbol{c}tr)^2}{4A_1t}}\Big)dr. \qquad (7.46)\end{aligned}$$

Here we change a variable again to obtain

$$\begin{aligned}&\int_0^1\Big(e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}tr)^2}{4A_1t}}-e^{-\frac{(|\boldsymbol{x}|+\boldsymbol{c}tr)^2}{4A_1t}}\Big)dr\\ &=\;\frac{1}{\boldsymbol{c}t}\Big(\int_{|\boldsymbol{x}|-\boldsymbol{c}t}^{|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds-\int_{|\boldsymbol{x}|}^{\boldsymbol{c}t+|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds\Big)\\ &=\;\frac{1}{\boldsymbol{c}t}\Big(2\int_0^{|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds-\int_{\boldsymbol{c}t-|\boldsymbol{x}|}^{\boldsymbol{c}t+|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds\Big). \qquad (7.47)\end{aligned}$$

We now are ready to take derivatives with respect to spatial variables

$$\begin{aligned}&\frac{\partial^2}{\partial x^j\partial x^k}\Big(\frac{1}{|\boldsymbol{x}|}\Big(2\int_0^{|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds-\int_{\boldsymbol{c}t-|\boldsymbol{x}|}^{\boldsymbol{c}t+|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds\Big)\Big)\\ &=\;\Big(-\frac{\delta_{jk}}{|\boldsymbol{x}|^3}+\frac{3x^jx^k}{|\boldsymbol{x}|^5}\Big)\Big(2\int_0^{|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds-\int_{\boldsymbol{c}t-|\boldsymbol{x}|}^{\boldsymbol{c}t+|\boldsymbol{x}|}e^{-\frac{s^2}{4A_1t}}ds\Big)\\ &\quad+\Big(\frac{\delta_{jk}}{|\boldsymbol{x}|^2}-\frac{3x^jx^k}{|\boldsymbol{x}|^4}\Big)\Big(2e^{-\frac{|\boldsymbol{x}|^2}{4A_1t}}-e^{-\frac{(\boldsymbol{c}t+|\boldsymbol{x}|)^2}{4A_1t}}\Big)\\ &\quad-\Big(\frac{\delta_{jk}}{|\boldsymbol{x}|^2}-\frac{3x^jx^k}{|\boldsymbol{x}|^4}\Big)e^{-\frac{(\boldsymbol{c}t-|\boldsymbol{x}|)^2}{4A_1t}}-\frac{x^jx^k}{|\boldsymbol{x}|^3}\Big(\frac{|\boldsymbol{x}|}{2A_1t}e^{-\frac{|\boldsymbol{x}|^2}{4A_1t}}-\frac{\boldsymbol{c}t+|\boldsymbol{x}|}{2A_1t}e^{-\frac{(\boldsymbol{c}t+|\boldsymbol{x}|)^2}{4A_1t}}\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{x^j x^k}{|\boldsymbol{x}|^3}\frac{|\boldsymbol{x}|-\boldsymbol{c}t}{2A_1 t}e^{-\frac{(\boldsymbol{c}t-|\boldsymbol{x}|)^2}{4A_1 t}} \\
\equiv\ & \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3+\mathcal{A}_4+\mathcal{A}_5.
\end{aligned}
$$

it is easy to see that $|\mathcal{A}_1|$, $|\mathcal{A}_2|$ and $|\mathcal{A}_4| \le Ct^{1/2}|\boldsymbol{x}|^{-3}$ and for $|\boldsymbol{x}| \ge \boldsymbol{c}t^{-\frac{1}{2}}$

$$
\frac{1}{|\boldsymbol{x}|^3} \le (1+\boldsymbol{c}^2)^3\Big(\boldsymbol{c}^2 t+\boldsymbol{c}^2|\boldsymbol{x}|^2\Big)^{-\frac{3}{2}} \le (\frac{1+\boldsymbol{c}^2}{\boldsymbol{c}^2})^3 t^{-\frac{3}{2}}(1+\frac{|\boldsymbol{x}|^2}{t})^{-\frac{3}{2}}.
$$

For $|\mathcal{A}_3|$ and $|\mathcal{A}_5|$ we can estimate

$$
\begin{cases}
(|\boldsymbol{x}|^{-2}+t^{-\frac{1}{2}}|\boldsymbol{x}|^{-1})e^{-\frac{(\boldsymbol{c}t-|\boldsymbol{x}|)^2}{4A_1 t}} \le Ct^{-1}e^{-\frac{\boldsymbol{c}^2t^2}{32A_1 t}}e^{-\frac{(\boldsymbol{c}t-|\boldsymbol{x}|)^2}{8A_1 t}}, & |\boldsymbol{x}| \le \boldsymbol{c}t/2; \\
(|\boldsymbol{x}|^{-2}+t^{-\frac{1}{2}}|\boldsymbol{x}|^{-1})e^{-\frac{(\boldsymbol{c}t-|\boldsymbol{x}|)^2}{4A_1(1+t)}} \le Ct^{-\frac{3}{2}}e^{-\frac{(\boldsymbol{c}t-|\boldsymbol{x}|)^2}{4A_1 t}}, & |\boldsymbol{x}| \ge \boldsymbol{c}t/2.
\end{cases}
$$

We combine estimates of $\mathcal{A}_i$ to conclude that

$$
\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}\|_{L^2_{\boldsymbol{\xi}}} \le C\Big(t^{-\frac{3}{2}}(1+\frac{|\boldsymbol{x}|^2}{t})^{-\frac{3}{2}}+t^{-2}e^{-\frac{(\boldsymbol{c}t-|\boldsymbol{x}|)^2}{8A_1 t}}\Big).
$$

For $|\boldsymbol{x}| \ge \boldsymbol{c}t$,

$$
\begin{aligned}
\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}\|_{L^2_{\boldsymbol{\xi}}} &\le C\int_0^t \frac{\boldsymbol{c}^2\tau}{(1+t)^{3/2}t}\iint\limits_{|\boldsymbol{y}|=1} e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{8A_1 t}}d\boldsymbol{S}(\boldsymbol{y})d\tau \\
&\le Ce^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1 t}}\int_0^t \frac{\boldsymbol{c}^2\tau}{(1+t)^{3/2}t}\iint\limits_{|\boldsymbol{y}|=1} e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{16A_1 t}}d\boldsymbol{S}(\boldsymbol{y})d\tau.
\end{aligned}
$$

We break the time integral domain into two parts

$$
\begin{aligned}
&\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}} \\
&\le Ce^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1 t}}\Big\{\int_0^{t^{1/2}}+\int_{t^{1/2}}^t\Big\}\frac{\boldsymbol{c}^2\tau}{(1+t)^{3/2}t}\iint\limits_{|\boldsymbol{y}|=1} e^{-\frac{|\boldsymbol{x}+\boldsymbol{c}\tau\boldsymbol{y}|^2}{16A_1 t}}d\boldsymbol{S}(\boldsymbol{y})d\tau \\
&\le Ce^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1 t}}\Big(\int_0^{t^{1/2}}\frac{\tau}{(1+t)^{3/2}t}d\tau+\int_{t^{1/2}}^t \frac{\tau}{(1+t)^{3/2}t}\frac{t}{\tau|\boldsymbol{x}|}e^{-\frac{(\boldsymbol{x}-\boldsymbol{c}\tau)^2}{16A_1 t}}d\tau\Big),
\end{aligned}
$$

where we use (7.46) and (7.47) and simple calculation to yield

$$
\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}} \le C(1+t)^{-2}e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1 t}}
$$

$$+C(1+t)^{-\frac{3}{2}}|\boldsymbol{x}|^{-1}e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1t}}\int_{t^{1/2}}^{t}e^{-\frac{(\boldsymbol{x}-\boldsymbol{c}\tau)^2}{16A_1t}}d\tau$$

$$\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}\|_{L^2_{\boldsymbol{\xi}}} \le C(1+t)^{-2}e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1t}}+C(1+t)^{-1}|\boldsymbol{x}|^{-1}e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1t}}.$$

From $|\boldsymbol{x}|\ge \boldsymbol{c}t$ equivalently $|\boldsymbol{x}|^{-1}\le(\boldsymbol{c}t)^{-1}$ and so

$$\|\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}\|_{L^2_{\boldsymbol{\xi}}} \le C(1+t)^{-2}e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{16A_1t}}, \quad \text{for} \quad |\boldsymbol{x}|\ge \boldsymbol{c}t.$$

This completes the study of leading waves. □

7.5. Fluid-like waves, II

To study the remaining fluid-like waves, we first study the analyticity of the pairings (7.39).

Proposition 7.10. *There exist operators $\mathscr{H}_j$, $\mathscr{H}_{jk}$, $\mathscr{C}_j$, $\mathscr{R}_{jk}$, $\mathscr{R}^0_{jk}$, $\mathscr{P}_{jkl}$, $\mathscr{P}^1_{jk}$, $\mathscr{P}^2_{jk}$, which are bounded operators in $L^2_{\boldsymbol{\xi}}$ and analytic in $\boldsymbol{\eta}$ so that*

$$\begin{aligned}
\hat{\mathbb{G}}_{\mathfrak{H}}(\boldsymbol{\eta},t) = e^{-A_1(|\boldsymbol{\eta}|^2)t}\Big[&\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\Big\{\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\\
&\times\Big(\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|+\sum_{j=1}^{3}\eta^j\mathscr{H}_j\Big)\\
&+\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big(\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3}i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\\
&+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{jk}\Big)\Big\}\\
&+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big\{\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\Big(\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3}i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\\
&+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{jk}\Big)\\
&+\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\boldsymbol{c}^2|\boldsymbol{\eta}|^2\Big(\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|+\sum_{j=1}^{3}\eta^j\mathscr{H}_j\Big)\Big\}\Big]
\end{aligned}$$

$$\hat{\mathbb{G}}_{\mathfrak{C}}(\boldsymbol{\eta},t) = e^{-A_2(|\boldsymbol{\eta}|^2)t}\Big(\frac{1}{10}(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}\otimes[(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}|+|\boldsymbol{\eta}|^2\mathscr{C}_0$$

$$+\sum_{j=1}^{3}\eta^j\mathscr{C}_j\Big)$$

$$\hat{\mathbb{G}}_{\mathfrak{R}}(\boldsymbol{\eta},t)=e^{-A_4(|\boldsymbol{\eta}|^2)t}\Big[\sum_{j=1}^{3}(\chi_j\otimes[\chi_j|+\sum_{k=1}^{3}\eta^k\mathscr{R}_{jk})+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{R}^0_{jk}\Big]$$

$$\begin{aligned}\hat{\mathbb{G}}_{\mathfrak{PR}_1}(\boldsymbol{\eta},t)=e^{-A_1(|\boldsymbol{\eta}|^2)t}\Big[&\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)(\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)-1)\\&\times\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big(\chi_j\otimes[\chi_k|+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)\\&+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big\{-\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{|\boldsymbol{\eta}|}\\&\times\sum_{j,k=1}^{3}\boldsymbol{c}\eta^j\eta^k\Big(\chi_j\otimes[\chi_k|+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)\\&+\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{P}^2_{jk}\Big\}\\&+\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{P}^2_{jk}\Big],\end{aligned}$$

$$\begin{aligned}\hat{\mathbb{G}}_{\mathfrak{PR}_2}(\boldsymbol{\eta},t)=&(e^{-A_1(|\boldsymbol{\eta}|^2)t}-e^{-A_4(|\boldsymbol{\eta}|^2)t})\\&\times\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big[\chi_j\otimes[\chi_k|+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big]\end{aligned}\tag{7.48}$$

Proof. **Huygens Pairing** $\hat{\mathbb{G}}_{\mathfrak{H}}(\boldsymbol{\eta},t)$

First, we consider the macro part of Huygens pairing using (7.35), (7.36),

$$\begin{aligned}&\hat{\mathbb{G}}_{\mathfrak{H}_1}(\boldsymbol{\eta},t)\\&=e^{-A_1(|\boldsymbol{\eta}|^2)t}2\cos(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)\Big[(\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)^2-|\boldsymbol{\eta}|^2\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2)^2)\chi_0\otimes[\chi_0|\\&+(\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)^2-|\boldsymbol{\eta}|^2\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2)^2)\chi_4\otimes[\chi_4|+(\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)\\&-\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2)\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2))(\chi_0\otimes[\chi_4|+\chi_4\otimes[\chi_0|)+(\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)^2\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2)^2\\&+\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)^2\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2)^2)\sum_{j=1}^{3}\eta^j(\chi_0\otimes[\chi_j|+\chi_j\otimes[\chi_0|)\end{aligned}$$

$$+(\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)^2\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2)^2+\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)^2\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2)^2)\sum_{j=1}^{3}\eta^j(\chi_4\otimes[\chi_j|+\chi_j\otimes[\chi_4|)\Big]$$
$$-e^{-A_1(|\boldsymbol{\eta}|^2)t}\frac{2i\sin(|\boldsymbol{\eta}|A^1_1(|\boldsymbol{\eta}|^2)t)}{|\boldsymbol{\eta}|}\Big[-2i|\boldsymbol{\eta}|^2\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2)\chi_0\otimes[\chi_0|$$
$$-2i|\boldsymbol{\eta}|^2\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2)\chi_4\otimes[\chi_4|-|\boldsymbol{\eta}|^2(\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2)$$
$$+\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2))(\chi_0\otimes[\chi_4|+\chi_4\otimes[\chi_0|)+(\mathsf{a}^0_{1,1}(|\boldsymbol{\eta}|^2)\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)$$
$$+|\boldsymbol{\eta}|^2\mathsf{a}^0_{1,2}(|\boldsymbol{\eta}|^2)\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2))\sum_{j=1}^{3}\eta^j(\chi_0\otimes[\chi_j|+\chi_j\otimes[\chi_0|)$$
$$+(\mathsf{a}^4_{1,1}(|\boldsymbol{\eta}|^2)\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)$$
$$+|\boldsymbol{\eta}|^2\mathsf{a}^4_{1,2}(|\boldsymbol{\eta}|^2)\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2))\sum_{j=1}^{3}\eta^j(\chi_4\otimes[\chi_j|+\chi_j\otimes[\chi_4|)\Big]$$

Evaluation of $\mathsf{a}^k_{1,j}(|\boldsymbol{\eta}|^2)$ at $\boldsymbol{\eta}=0$ in (7.36) gives

$$\begin{aligned}\hat{\mathbb{G}}_{\mathfrak{H}_1}(\boldsymbol{\eta}) \;=\;& e^{-A_1(|\boldsymbol{\eta}|^2)t}\cos(|\boldsymbol{\eta}|A^1_1(|\boldsymbol{\eta}|^2)t)\Big(\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|\\&+|\boldsymbol{\eta}|^2\mathscr{H}_{00}+\sum_{j=1}^{3}\eta^j\mathscr{H}_{0j}\Big)\\&+e^{-A_1(|\boldsymbol{\eta}|^2)t}\frac{\sin(|\boldsymbol{\eta}|A^1_1(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big(\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3}i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\\&+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)+|\boldsymbol{\eta}|^2\mathscr{H}_{04}\Big),\end{aligned}$$

where $\mathscr{H}_{0j}$ are analytic in $\boldsymbol{\eta}$ for $j=0,1,2,3,4$. We now consider one of the remaining micro parts

$$\begin{aligned}\hat{\mathbb{G}}_{\mathfrak{H}_1}(\boldsymbol{\eta},t)=&e^{\lambda_1(\boldsymbol{\eta})t}(\mathsf{P}_0\psi_1(\boldsymbol{\eta})\otimes[\mathsf{P}_1\psi_1(\boldsymbol{\eta})|-\mathsf{P}^m_0\psi_1(\boldsymbol{\eta})\otimes[(\mathscr{L}_1-1)\mathsf{P}^m_0\psi_1(\boldsymbol{\eta})|)\\&+e^{\lambda_3(\boldsymbol{\eta})t}(\mathsf{P}_0\psi_3(\boldsymbol{\eta})\otimes[\mathsf{P}_1\psi_3(\boldsymbol{\eta})|-\mathsf{P}^m_0\psi_3(\boldsymbol{\eta})\otimes[(\mathscr{L}_3-1)\mathsf{P}^m_0\psi_3(\boldsymbol{\eta})|)\end{aligned}$$

and note that $\mathsf{P}_1\psi_1$ and $\mathsf{P}_1\psi_3$ are related by

$$\begin{cases}\mathsf{P}_1\psi_1(\boldsymbol{\eta})=(\mathscr{L}_1-1)\mathsf{P}_0\psi_1(\boldsymbol{\eta})\\=(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|A^1_1(|\boldsymbol{\eta}|^2))^{-1}(\mathsf{P}_1 i\boldsymbol{\xi}\cdot\boldsymbol{\eta})\mathsf{P}_0\psi_1(\boldsymbol{\eta}),\\\mathsf{P}_1\psi_3(\boldsymbol{\eta})=(\mathscr{L}_3-1)\mathsf{P}_0\psi_3(\boldsymbol{\eta})\\=(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|A^1_1(|\boldsymbol{\eta}|^2))^{-1}(\mathsf{P}_1 i\boldsymbol{\xi}\cdot\boldsymbol{\eta})\mathsf{P}_0\psi_3(\boldsymbol{\eta}).\end{cases}\tag{7.49}$$

Here we expand the operator $(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)+i||A_1^1(|\boldsymbol{\eta}|^2))^{-1},\ |\boldsymbol{\eta}|\ll 1,$ as follows

$$\begin{aligned}
&\frac{1}{\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)\pm i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)}\\
&= \frac{1}{(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2))(1+\frac{\pm i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)}{\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)})}\\
&= \frac{1}{(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2))}\sum_{j=0}^{\infty}\Big(\frac{\mp i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)}{\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)}\Big)^j\\
&= \frac{1\mp\frac{i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)}{\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)}}{(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2))}\sum_{j=0}^{\infty}(-1)^j\Big(\frac{|\boldsymbol{\eta}|^2A_1^1(|\boldsymbol{\eta}|^2)^2}{(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2))^2}\Big)^j\\
&= \frac{1\mp\frac{i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)}{\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)}}{(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2))(1+\frac{|\boldsymbol{\eta}|^2A_1^1(|\boldsymbol{\eta}|^2)^2}{(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2))^2})}
\end{aligned}$$

Thus $(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)\pm i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2))^{-1}$ can be expressed by the summation of two operators, both analytic in $\boldsymbol{\eta}$ and $\mathsf{P}_0\psi_1$ and $\mathsf{P}_0\psi_3$ are related

$$\begin{aligned}
\mathscr{L}_1-1 &= \mathscr{M}_1(\boldsymbol{\eta})+|\boldsymbol{\eta}|\mathscr{M}_2(\boldsymbol{\eta}),\\
\mathscr{L}_3-1 &= \mathscr{M}_1(\boldsymbol{\eta})-|\boldsymbol{\eta}|\mathscr{M}_2(\boldsymbol{\eta}),\\
\mathsf{P}_0\psi_1(\boldsymbol{\eta}) &= (\mathsf{a}_{1,1}^0(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^0(|\boldsymbol{\eta}|^2))\chi_0+(\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2))\\
&\quad\times\sum_{k=1}^{3}\frac{\eta^k\chi_k}{|\boldsymbol{\eta}|}+(\mathsf{a}_{1,1}^4(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^4(|\boldsymbol{\eta}|^2))\chi_4\\
&\equiv H_1+|\boldsymbol{\eta}|(H_2+\frac{H_4}{|\boldsymbol{\eta}|^2}),\\
\mathsf{P}_0\psi_3(\boldsymbol{\eta}) &= (\mathsf{a}_{1,1}^0(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^0(|\boldsymbol{\eta}|^2))\chi_0-(\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2))\\
&\quad\times\sum_{k=1}^{3}\frac{\eta^k\chi_k}{|\boldsymbol{\eta}|}+(\mathsf{a}_{1,1}^4(|\boldsymbol{\eta}|^2)-i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^4(|\boldsymbol{\eta}|^2))\chi_4\\
&\equiv H_1-|\boldsymbol{\eta}|(H_2+\frac{H_4}{|\boldsymbol{\eta}|^2}),\\
\mathsf{P}_0^m\psi_1(\boldsymbol{\eta}) &= (\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2))\sum_{k=1}^{3}\frac{\eta^k\chi_k}{|\boldsymbol{\eta}|}\equiv H_3+\frac{H_4}{|\boldsymbol{\eta}|},
\end{aligned}$$

$$\mathsf{P}_0^m\psi_3(\boldsymbol{\eta}) \;=\; -(\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2) - i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2))\sum_{k=1}^{3}\frac{\eta^k\chi_k}{|\boldsymbol{\eta}|} \equiv H_3 - \frac{H_4}{|\boldsymbol{\eta}|}.$$

Thus we may rewrite

$$\begin{aligned}
&\hat{\mathbb{G}}_{\mathfrak{H}_1}(\boldsymbol{\eta},t)\\
&= e^{-A_1(|\boldsymbol{\eta}|^2)t}2\cos(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)\Big[(H_1-H_3)\otimes[\mathscr{M}_1H_1\\
&\quad+\mathscr{M}_2(|\boldsymbol{\eta}|^2H_2+H_4)|+H_2\otimes[|\boldsymbol{\eta}|^2(\mathscr{M}_2H_1+\mathscr{M}_1H_2)+\mathscr{M}_1H_4|\\
&\quad+H_3\otimes[\mathscr{M}_1(H_1-H_3)+|\boldsymbol{\eta}|^2\mathscr{M}_2H_2|+H_4\otimes[\mathscr{M}_1H_2-\mathscr{M}_2(H_1-H_3)|\Big]\\
&\quad-e^{-A_1(|\boldsymbol{\eta}|^2)t}\frac{2i\sin(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)}{|\boldsymbol{\eta}|}\Big[(H_1-H_3)\otimes[|\boldsymbol{\eta}|^2(\mathscr{M}_2H_1+\mathscr{M}_1H_2)\\
&\quad+H_4|+|\boldsymbol{\eta}|^2H_2\otimes[\mathscr{M}_1H_1+\mathscr{M}_2(|\boldsymbol{\eta}|^2H_2+H_4)|\\
&\quad+|\boldsymbol{\eta}|^2H_3\otimes[\mathscr{M}_1H_2-\mathscr{M}_1(H_1-H_3)|+H_4\otimes[\mathscr{M}_1(H_1-H_3)+|\boldsymbol{\eta}|^2\mathscr{M}_2H_2|\Big]\\
&= e^{-A_1(|\boldsymbol{\eta}|^2)t}2\cos(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)\Big[|\boldsymbol{\eta}|^2\mathscr{H}_{10}^1+\sum_{j=1}^{3}\eta^j\mathscr{H}_{1j}\Big]\\
&\quad-e^{-A_1(|\boldsymbol{\eta}|^2)t}\frac{\sin(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big[|\boldsymbol{\eta}|^2\mathscr{H}_{10}^2+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{1jk}\Big],
\end{aligned}$$

where $\mathscr{H}_{01}^l$, $\mathscr{H}_{1j}$ and $\mathscr{H}_{1jk}$ is analytic in $\boldsymbol{\eta}$ for $l=1,2$ $j,k=1,2,3$. We now sum them up to obtain

$$\begin{aligned}
\hat{\mathbb{G}}_{\mathfrak{H}}(\boldsymbol{\eta},t) &= e^{-A_1(|\boldsymbol{\eta}|^2)t}\cos(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)\Big(\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|+\sum_{j=1}^{3}\eta^j\mathscr{H}_j\Big)\\
&\quad+e^{-A_1(|\boldsymbol{\eta}|^2)t}\frac{\sin(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big(\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3}i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\\
&\quad+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{jk}\Big),
\end{aligned}$$

then trigonometric equality gives the result for the Huygens pairing.

Contact pairing $\hat{\mathbb{G}}_{\mathfrak{C}}(\boldsymbol{\eta},t)$

We recall that from (7.36b)

$$\begin{aligned}\psi_2(\boldsymbol{\eta}) &= \mathscr{L}_2\mathsf{P}_0\psi_2(\boldsymbol{\eta})\\ &= \Big(1+[\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_2(|\boldsymbol{\eta}|^2)]^{-1}i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}\Big)(\mathsf{a}^0_{2,1}(|\boldsymbol{\eta}|^2)\chi_0\\ &\quad +i\mathsf{a}^1_{2,2}(|\boldsymbol{\eta}|^2)\sum_{j=1}^{3}\eta^j\chi_j+\mathsf{a}^4_{2,1}(|\boldsymbol{\eta}|^2)\chi_4),\end{aligned}$$

and that the micro part is zero when $\boldsymbol{\eta}=0$. Thus we can calculate contact pairing as follows:

$$\begin{aligned}&\hat{\mathbb{G}}_{\mathfrak{C}}(\boldsymbol{x},t)\\ &= e^{-A_2(|\boldsymbol{\eta}|^2)t}\psi_2(\boldsymbol{\eta})\otimes[\psi_2(\boldsymbol{\eta})|\\ &= e^{-A_2(|\boldsymbol{\eta}|^2)t}\Big(\mathsf{P}_0\psi_2(\boldsymbol{\eta})\otimes[\mathsf{P}_0\psi_2(\boldsymbol{\eta})|+\mathsf{P}_0\psi_2(\boldsymbol{\eta})\otimes[(\mathscr{L}_2-1)\mathsf{P}_0\psi_2(\boldsymbol{\eta})|\\ &\quad +(\mathscr{L}_2-1)\mathsf{P}_0\psi_2(\boldsymbol{\eta})\otimes[\psi_2(\boldsymbol{\eta})|\Big)\\ &= e^{-A_2(|\boldsymbol{\eta}|^2)t}\Big(\frac{1}{10}(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}\otimes[(|\boldsymbol{\xi}|^2-5)\sqrt{\mathsf{M}}|+|\boldsymbol{\eta}|^2\mathscr{C}_0+\sum_{j=1}^{3}\eta^j\mathscr{C}_j\Big),\end{aligned}$$

for some $\mathscr{C}_j$ analytic in $\boldsymbol{\eta}$.

1nd Riesz Pairing $\hat{\mathbb{G}}_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{\eta},t)$

From (7.35), (7.36a), (7.36c) and (7.39) we have

$$\begin{aligned}\hat{\mathbb{G}}_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{\eta},t) &= e^{-A_1(|\boldsymbol{\eta}|^2)t-i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t}\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\otimes[\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})|\\ &\quad -e^{-A_1(|\boldsymbol{\eta}|^2)t}\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\otimes[\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})|\\ &\quad +e^{-A_1(|\boldsymbol{\eta}|^2)t+i|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t}\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\otimes[\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})|\\ &\quad -e^{-A_1(|\boldsymbol{\eta}|^2)t}\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\otimes[\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})|\\ &= e^{-A_1(|\boldsymbol{\eta}|^2)t}(\cos(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)-1)\Big[\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\otimes[\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\\ &\quad +\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\otimes[\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\Big]\\ &\quad -e^{-A_1(|\boldsymbol{\eta}|^2)t}\sin(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)\Big[\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\otimes[\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\\ &\quad -\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\otimes[\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\Big],\end{aligned}$$

$$
\begin{aligned}
= \ & e^{-A_1(|\boldsymbol{\eta}|^2)t} 2(\cos(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)-1) \\
& \times\Big[((1+\mathscr{M}_1)i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2)+|\boldsymbol{\eta}|\mathscr{M}_2\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)) \\
& \times\sum_{j=1}^3\frac{\eta^j}{|\boldsymbol{\eta}|}\chi_j\otimes[((1+\mathscr{M}_1)i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2)+|\boldsymbol{\eta}|\mathscr{M}_2\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2))\sum_{k=1}^3\frac{\eta^k}{|\boldsymbol{\eta}|}\chi_k| \\
& +((1+\mathscr{M}_1)\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|^2\mathscr{M}_2\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2)) \\
& \times\sum_{j=1}^3\frac{\eta^j}{|\boldsymbol{\eta}|}\chi_j\otimes[((1+\mathscr{M}_1)\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|^2\mathscr{M}_2\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2))\sum_{k=1}^3\frac{\eta^k}{|\boldsymbol{\eta}|}\chi_k|\Big] \\
& +e^{-A_1(|\boldsymbol{\eta}|^2)t}2\sin(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t) \\
& \times\Big[((1+\mathscr{M}_1)i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2)+|\boldsymbol{\eta}|\mathscr{M}_2\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)) \\
& \times\sum_{j=1}^3\frac{\eta^j}{|\boldsymbol{\eta}|}\chi_j\otimes[((1+\mathscr{M}_1)\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|^2\mathscr{M}_2\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2))\sum_{k=1}^3\frac{\eta^k}{|\boldsymbol{\eta}|}\chi_k| \\
& +((1+\mathscr{M}_1)\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|^2\mathscr{M}_2\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2)) \\
& \times\sum_{j=1}^3\frac{\eta^j}{|\boldsymbol{\eta}|}\chi_j\otimes[((1+\mathscr{M}_1)i|\boldsymbol{\eta}|\mathsf{a}_{1,2}^1(|\boldsymbol{\eta}|^2) \\
& +|\boldsymbol{\eta}|\mathscr{M}_2\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2))\sum_{k=1}^3\frac{\eta^k}{|\boldsymbol{\eta}|}\chi_k|\Big],
\end{aligned}
$$

and that $\mathsf{a}_{1,1}^1(0)=\sqrt{1/2}$ and the microscopic parts containing one more $\boldsymbol{\eta}$. Thus we obtain

$$
\begin{aligned}
\hat{\mathbb{G}}_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{\eta},t) = \ & e^{-A_1(|\boldsymbol{\eta}|^2)t}(\cos(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)-1) \\
& \times\sum_{j,k=1}^3\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big[\chi_j\otimes[\chi_k|+\sum_{l=1}^3\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}_{jk}^1\Big] \\
& +e^{-A_1(|\boldsymbol{\eta}|^2)t}\frac{\sin(|\boldsymbol{\eta}|A_1^1(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j,k=1}^3\eta^j\eta^k\mathscr{P}_{jk}^2,
\end{aligned}
$$

for some $\mathscr{P}_{jkl}$ $\mathscr{P}_{jk}^1$ and $\mathscr{P}_{jk}^2$ analytic in $\boldsymbol{\eta}$. Here we have used the trigonometric equalities as in the Huygens pairing case.

2nd Riesz Pairing $\hat{\mathbb{G}}_{\mathfrak{P}\mathfrak{R}_2}(\boldsymbol{\eta},t)$

$$\begin{aligned}
\hat{\mathbb{G}}_{\mathfrak{P}\mathfrak{R}_2}(\boldsymbol{\eta},t) &= (e^{-A_1(|\boldsymbol{\eta}|^2)t}-e^{-A_4(|\boldsymbol{\eta}|^2)t}) \\
&\quad\times\Big[((1+\mathscr{M}_1)i|\boldsymbol{\eta}|\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2)+|\boldsymbol{\eta}|\mathscr{M}_2\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)) \\
&\quad\times\sum_{j=1}^3\frac{\eta^j}{|\boldsymbol{\eta}|}\chi_j\otimes[((1+\mathscr{M}_1)i|\boldsymbol{\eta}|\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2)+|\boldsymbol{\eta}|\mathscr{M}_2\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)) \\
&\quad\times\sum_{k=1}^3\frac{\eta^k}{|\boldsymbol{\eta}|}\chi_k|+((1+\mathscr{M}_1)\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|^2\mathscr{M}_2\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2)) \\
&\quad\times\sum_{j=1}^3\frac{\eta^j}{|\boldsymbol{\eta}|}\chi_j\otimes[((1+\mathscr{M}_1)\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)+i|\boldsymbol{\eta}|^2\mathscr{M}_2\mathsf{a}^1_{1,2}(|\boldsymbol{\eta}|^2))\sum_{k=1}^3\frac{\eta^k}{|\boldsymbol{\eta}|}\chi_k|\Big] \\
&= (e^{-A_1(|\boldsymbol{\eta}|^2)t}-e^{-A_4(|\boldsymbol{\eta}|^2)t}) \\
&\quad\times\sum_{j,k=1}^3\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big[\chi_j\otimes[\chi_k|+\sum_{l=1}^3\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big].
\end{aligned}$$

Rotational pairing $\hat{\mathbb{G}}_{\mathfrak{R}}(\boldsymbol{\eta},t)$

We have

$$\begin{aligned}
\hat{\mathbb{G}}_{\mathfrak{R}}(\boldsymbol{\eta},t) \;=\; & e^{-A_4(|\boldsymbol{\eta}|^2)t}\Big(\psi_4(\boldsymbol{\eta})\otimes[\psi_4(\boldsymbol{\eta})|+\psi_5(\boldsymbol{\eta})\otimes[\psi_5(\boldsymbol{\eta})| \\
&+\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\otimes[\mathscr{L}_1\mathsf{P}_0\psi_1(\boldsymbol{\eta})|+\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\otimes[\mathscr{L}_3\mathsf{P}_0\psi_3(\boldsymbol{\eta})|\Big).
\end{aligned}$$

First, we recall that from the above cases

$$\begin{aligned}
&\mathscr{L}_1\mathsf{P}_0^m\psi_1(\boldsymbol{\eta})\otimes[\mathscr{L}_1\mathsf{P}_0\psi_1(\boldsymbol{\eta})|+\mathscr{L}_3\mathsf{P}_0^m\psi_3(\boldsymbol{\eta})\otimes[\mathscr{L}_3\mathsf{P}_0\psi_3(\boldsymbol{\eta})| \\
&\quad-\mathsf{a}^1_{1,1}(|\boldsymbol{\eta}|^2)^2\sum_{j,k=1}^3\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}(1+\mathscr{M}_1)\chi_j\otimes[(1+\mathscr{M}_1)\chi_k|.
\end{aligned}$$

We now use (7.36d) and (7.36e)

$$\begin{cases}\mathsf{P}_0\psi_4(\boldsymbol{\eta})=\mathsf{a}^2_{4,1}(|\boldsymbol{\eta}|^2)\mathfrak{g}\chi_2,\\ \mathsf{P}_0\psi_5(\boldsymbol{\eta})=\mathsf{a}^2_{4,1}(|\boldsymbol{\eta}|^2)\mathfrak{g}\chi_3.\end{cases}$$

to yield

$$\psi_4(\boldsymbol{\eta})\otimes[\psi_4(\boldsymbol{\eta})|+\psi_5(\boldsymbol{\eta})\otimes[\psi_5(\boldsymbol{\eta})|$$

$$= \mathsf{a}_{4,1}^2(|\boldsymbol{\eta}|^2)^2\Big(\mathscr{L}_4\mathfrak{g}\chi_2\otimes[\mathscr{L}_4\mathfrak{g}\chi_2|+\mathscr{L}_4\mathfrak{g}\chi_3\otimes[\mathscr{L}_4\mathfrak{g}\chi_3|)\Big)$$

$$\equiv \mathsf{a}_{4,1}^2(|\boldsymbol{\eta}|^2)^2(\sum_{j=1}^3\mathscr{L}_4\chi_j\otimes[\mathscr{L}_4\chi_j|-\sum_{j,k=1}^3\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\mathscr{L}_4\chi_j\otimes[\mathscr{L}_4\chi_k|),$$

Here our calculations are similarly to (7.43) and we have used $\mathfrak{g}(\boldsymbol{\eta}/|\boldsymbol{\eta}|,\boldsymbol{\alpha},\boldsymbol{\beta})=((1,0,0),(0,1,0),(0,0,1))$. We now observe that, for small $\boldsymbol{\eta}$,

$$\begin{aligned}&\mathscr{L}_4-(1+\mathscr{M}_1)\\&=\Big[\Big(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_4(|\boldsymbol{\eta}|^2)\Big)^{-1}\\&\quad-\Big(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2)\Big)^{-1}\Big(1+\frac{|\boldsymbol{\eta}|^2A_1^1(|\boldsymbol{\eta}|^2)^2}{(\mathsf{L}-i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}+A_1(|\boldsymbol{\eta}|^2))^2}\Big)^{-1}\Big](i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta})\\&=|\boldsymbol{\eta}|^2\mathscr{N}(i\mathsf{P}_1\boldsymbol{\xi}\cdot\boldsymbol{\eta}),\end{aligned}$$

for some $\mathscr{N}$ analytic in $\boldsymbol{\eta}$. Thus we have

$$\begin{aligned}&\sum_{j,k=1}^3\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big[\mathsf{a}_{4,1}^2(|\boldsymbol{\eta}|^2)^2\mathscr{L}_4\chi_j\otimes[\mathscr{L}_4\chi_k|-\mathsf{a}_{1,1}^1(|\boldsymbol{\eta}|^2)^2(1+\mathscr{M}_1)\chi_j\otimes[(1+\mathscr{M}_1)\chi_k|\Big]\\&=\sum_{j,k=1}^3\eta^j\eta^k\mathscr{R}_{jk}^0,\end{aligned}$$

and conclude that

$$\hat{\mathbb{G}}_{\mathfrak{R}}(\boldsymbol{\eta},t)=e^{-A_4(|\boldsymbol{\eta}|^2)t}\Big[\sum_{j=1}^3(\chi_j\otimes[\chi_j|+\sum_{k=1}^3\eta^k\mathscr{R}_{jk})+\sum_{j,k=1}^3\eta^j\eta^k\mathscr{R}_{jk}^0\Big].$$

This completes the proof of the proposition. ☐

With the analyticity property, we adopt complex analysis technique to estimate the pairings of (7.39). We state first a general lemma.

Lemma 7.11. *Suppose that $f(\boldsymbol{\eta},t)=O(1)e^{O(|\boldsymbol{\eta}|^4)t}$ is analytic in $\boldsymbol{\eta}$ for $|\boldsymbol{\eta}|<\delta\ll 1$. Then in the region of $|\boldsymbol{x}|<(\mathscr{M}+1)\boldsymbol{c}t$, $\mathscr{M}$ any given positive constant, there exists a constant C such that the following inequality holds:*

$$\int_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}e^{-A|\boldsymbol{\eta}|^2t}\boldsymbol{\eta}^\alpha f(\boldsymbol{\eta},t)d\boldsymbol{\eta}\le C\Big((1+t)^{-\frac{3+|\alpha|}{2}}e^{-\frac{|\boldsymbol{x}|^2}{Ct}}+e^{-t/C}\Big).$$

Proof. We choose an orthogonal transformation $O_{\boldsymbol{x}}$ which maps $\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ to $(1,0,0)$, i.e.,

$$O_{\boldsymbol{x}}\frac{\boldsymbol{x}}{|\boldsymbol{x}|}=(1,0,0). \tag{7.50}$$

We change variables $\boldsymbol{\xi}\to\boldsymbol{\zeta}=O_{\boldsymbol{x}}\boldsymbol{\xi}$, then by virtue of the orthogonality of the transformation,

$$\boldsymbol{x}\cdot\boldsymbol{\xi}=O_{\boldsymbol{x}}\boldsymbol{x}\cdot O_{\boldsymbol{x}}\boldsymbol{\xi}=|\boldsymbol{x}|(1,0,0)\cdot\boldsymbol{\zeta}=|\boldsymbol{x}|\zeta^1,\quad |\boldsymbol{\xi}|=|\boldsymbol{\zeta}|,\quad \text{and}\quad \xi^i=(O_{\boldsymbol{x}}^{-1}\boldsymbol{\zeta})^i. \tag{7.51}$$

Thus we may rewrite

$$\begin{aligned}&\int_{|\boldsymbol{\eta}|<\delta}e^{i|\boldsymbol{x}|\eta^1}e^{-A|\boldsymbol{\eta}|^2t}\boldsymbol{\eta}^\alpha f(\boldsymbol{\eta},t)d\boldsymbol{\eta}\\ &\quad=\int_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}e^{-A|\boldsymbol{\eta}|^2t}(O_{\boldsymbol{x}}^{-1}\boldsymbol{\eta})^\alpha f(O_x^{-1}\boldsymbol{\eta},t)d\boldsymbol{\eta}\end{aligned}$$

We will prove this lemma only in the case of $\alpha=0$. The extra factor η^i, $i=1,2,3$ result in the extra time-decaying factor $t^{-\frac{1}{2}}$; details are omitted. We first divide the integral into two region $\{|\boldsymbol{\eta}|<\delta\}=\mathbb{B}\cup(\mathbb{B}^c\cap\{|\boldsymbol{\eta}|<\delta\})\equiv D_1\cup D_2$ where

$$\mathbb{B}\equiv[-\frac{\delta}{2},\frac{\delta}{2}]\times[-\frac{\delta}{2},\frac{\delta}{2}]\times[-\frac{\delta}{2},\frac{\delta}{2}].$$

On D_2, $|\boldsymbol{\eta}|>\frac{\delta}{2}$ and so

$$\int_{D_2}e^{i|\boldsymbol{x}|\eta^1}e^{-A|\boldsymbol{\eta}|^2t}f(O_{\boldsymbol{x}}^{-1}\boldsymbol{\eta},t)d\boldsymbol{\eta}\le Ce^{-t/C}. \tag{7.52}$$

To estimate the integration on D_1 part, we use analyticity of $f(O_{\boldsymbol{x}}^{-1}\boldsymbol{\eta},t)$ to equate the integral with respect to η^1 on $[-\frac{\delta}{2},\frac{\delta}{2}]$ with the integral on $\Gamma(-\frac{\delta}{2},\frac{\delta}{2},\frac{|\boldsymbol{x}|}{tM})$,

$$\begin{aligned}\Gamma(-\frac{\delta}{2},\frac{\delta}{2},\frac{|\boldsymbol{x}|}{tM})&=\Gamma_1\cup\Gamma_2\cup\Gamma_3\\ &=\{\eta^1:Re(\eta^1)=-\frac{\delta}{2},\quad 0<Im(\eta)<\frac{|\boldsymbol{x}|}{tM}\}\\ &\cup\{\eta^1:-\frac{\delta}{2}<Re(\eta^1)<\frac{\delta}{2},\quad Im(\eta^1)=\frac{|\boldsymbol{x}|}{tM}.\}\end{aligned}$$

$$\cup \{\eta^1 : Re(\eta^1) = \frac{\delta}{2}, \quad 0 < Im(\eta^1) < \frac{|\boldsymbol{x}|}{tM}\},$$
$$\mathbb{B}^2 = \{(\eta^2, \eta^3) \in \mathbb{R}^2 | |\eta^2|, |\eta^3| < \frac{\kappa}{2}\},$$

here we take M large enough such that on the contour we have the growth rate

$$|f(O_{\boldsymbol{x}}^{-1}\boldsymbol{\eta}, t)| \leq O(1)e^{o(1)|\eta|^2}.$$

We now break the domain of integration into two parts

$$\begin{aligned}
&\int_{D_1} e^{i|\boldsymbol{x}|\eta^1} e^{-A|\boldsymbol{\eta}|^2 t} f(O_{\boldsymbol{x}}^{-1}\boldsymbol{\eta}, t) d\boldsymbol{\eta} \\
&\quad = \int_{\mathbb{B}^2} \int_{\Gamma_1 \cup \Gamma_3} e^{i|\boldsymbol{x}|\eta^1} e^{-A|\boldsymbol{\eta}|^2 t} f(\boldsymbol{\eta}, t) d\eta^1 d\eta^2 d\eta^3 \\
&\qquad + \int_{\mathbb{B}^2} \int_{\Gamma_2} e^{i|\boldsymbol{x}|\eta^1} e^{-A|\boldsymbol{\eta}|^2 t} f(\boldsymbol{\eta}, t) d\eta^1 d\eta^2 d\eta^3 = \mathcal{J}_1 + \mathcal{J}_2.
\end{aligned}$$

Note that $|\boldsymbol{\eta}| > \frac{\delta}{2}$ on Γ_1 and Γ_3, and

$$\mathcal{J}_1 = \int_{\mathbb{B}^2} \int_{\Gamma^1 \cup \Gamma^3} e^{i|\boldsymbol{x}|\eta^1} e^{-A|\boldsymbol{\eta}|^2 t} f(\boldsymbol{\eta}, t) d\boldsymbol{\eta} \leq C e^{-t/C}. \tag{7.53}$$

We now estimate the integral over Γ_2

$$\begin{aligned}
\mathcal{J}_2 &\equiv \int_{\mathbb{B}^2} \int_{\Gamma_2} e^{i|\boldsymbol{x}|\eta^1} e^{-A|\boldsymbol{\eta}|^2 t + o(1)|\boldsymbol{\eta}|^2 t} d\eta^1 d\eta^2 d\eta^3 \\
&= e^{-\frac{|\boldsymbol{x}|^2}{4At}} \int_{\mathbb{B}^2} \int_{\Gamma_2} e^{-At|\eta^1 - \frac{i|\boldsymbol{x}|}{2At}| - At|\eta^2|^2 - At|\eta^3|^2 + o(1)|\eta|^2 t} d\eta^1 d\eta^2 d\eta^3 \\
&= e^{-\frac{|\boldsymbol{x}|^2}{4At}} \int_{\mathbb{B}^2} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} e^{-At|u + \frac{i|\boldsymbol{x}|}{Mt} - \frac{i|\boldsymbol{x}|}{2At}| - At|\eta^2|^2 - At|\eta^3|^2 + o(1)(|u + \frac{i|\boldsymbol{x}|}{Mt}|^2 + |\eta^2|^2 + |\eta^3|^2)t} \\
&\qquad du d\eta^2 d\eta^3 \\
&= e^{-\frac{|\boldsymbol{x}|^2}{4At}(1 - (1 - \frac{2A}{M})^2)} \\
&\quad \int_{\mathbb{B}^2} \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} e^{-Au^2 t - 2Aut(\frac{i|\boldsymbol{x}|}{Mt} - \frac{i|\boldsymbol{x}|}{2At}) - At|\eta^2|^2 - At|\eta^3|^2 + o(1)(|u + \frac{i|\boldsymbol{x}|}{Mt}|^2 + |\eta^2|^2 + |\eta^3|^2)t} \\
&\qquad du d\eta^2 d\eta^3,
\end{aligned}$$

and so

$$\begin{aligned}|\mathcal{J}_2| \;\le\; & Ce^{-\frac{|\boldsymbol{x}|^2}{Ct}}\int_{\mathbb{B}^2}\int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} e^{-\frac{A}{2}u^2t-\frac{At}{2}|\eta^2|^2-\frac{At}{2}|\eta^3|^2}dud\eta^2d\eta^3\\ \le\; & C\frac{1}{(1+t)^{\frac{3}{2}}}e^{-\frac{|\boldsymbol{x}|^2}{Ct}},\end{aligned}\tag{7.54}$$

We combine (7.52), (7.53) and (7.54) to complete the proof. □

Theorem 7.12. *For any given Mach number $\mathscr{M}>1$, there exists $C>0$ such that for $|\boldsymbol{x}|\le(\mathscr{M}+1)\boldsymbol{c}t$,*

$$\begin{aligned}&\|\mathbb{G}_{\mathfrak{H}}(\boldsymbol{x},t)-\mathbb{G}^0_{\mathfrak{H}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}\le C\Big[\frac{e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{Ct}}}{(1+t)^{5/2}}+e^{-t/C}\Big],\\ &\|\mathbb{G}_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{x},t)-\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_1}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}\le C\Big[\frac{e^{-\frac{(|\boldsymbol{x}|-\boldsymbol{c}t)^2}{Ct}}}{(1+t)^{5/2}}+\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^{5/2}}\\ &\qquad+\chi_{|\boldsymbol{x}|\le\boldsymbol{c}t}(1+t)^{-2}(1+\frac{|\boldsymbol{x}|^2}{t})^{-\frac{3}{2}}+e^{-t/C}\Big],\\ &\|\mathbb{G}_{\mathfrak{C}}(\boldsymbol{x},t)-\mathbb{G}^0_{\mathfrak{C}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}},\|\mathbb{G}_{\mathfrak{P}\mathfrak{R}_2}(\boldsymbol{x},t)-\mathbb{G}^0_{\mathfrak{P}\mathfrak{R}_2}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}},\|\mathbb{G}_{\mathfrak{R}}(\boldsymbol{x},t)-\mathbb{G}^0_{\mathfrak{R}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}\\ &\quad\le C\Big[\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^2}+e^{-t/C}\Big].\end{aligned}\tag{7.55}$$

Proof. It is enough to provide the proof for $\|\mathbb{G}_{\mathfrak{P}}(\boldsymbol{x},t)-\mathbb{G}^1_{\mathfrak{P}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}$ for any pairing $\mathfrak{P}$ since we have showed the remainder of $\mathbb{G}^1_{\mathfrak{P}}$ decay faster when the explicit leading fluid-like waves $\mathbb{G}^0_{\mathfrak{P}}(\boldsymbol{x},t)$ has been taken away in Theorem 7.9. The Huygens wave are calculated using Theorem 7.1 and Lemma 7.2:

$$\|\mathbb{G}_{\mathfrak{H}}(\boldsymbol{x},t)-\mathbb{G}^1_{\mathfrak{H}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}=\Big\|\frac{1}{(2\pi)^3}\int_{|\boldsymbol{\eta}|<\delta/2}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}(\hat{\mathbb{G}}_{\mathfrak{H}}(\boldsymbol{\eta},t)-\hat{\mathbb{G}}^1_{\mathfrak{H}}(\boldsymbol{\eta},t))d\boldsymbol{\eta}\Big\|_{L^2_{\boldsymbol{\xi}}}.$$

From Proposition 7.10,

$$\begin{aligned}&\hat{\mathbb{G}}_{\mathfrak{H}}(\boldsymbol{\eta},t)-\hat{\mathbb{G}}^1_{\mathfrak{H}}(\boldsymbol{\eta},t)\\ &=\;e^{-A_1|\boldsymbol{\eta}|^2t}(e^{O(|\boldsymbol{\eta}|^4)t}-1)\Big[\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\Big\{\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\\ &\quad\times\Big(\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|+\sum_{j=1}^{3}\eta^j\mathscr{H}_j\Big)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big(\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3} i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)\\
&+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{jk}\Big)\Big\}+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big\{\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\\
&\times\Big(\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3} i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{jk}\Big)\\
&+\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\boldsymbol{c}^2|\boldsymbol{\eta}|^2\Big(\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|+\sum_{j=1}^{3}\eta^j\mathscr{H}_j\Big)\Big\}\Big]\\
&+e^{-A_1|\boldsymbol{\eta}|^2t}\Big[\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\Big\{(\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)-1)\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|\\
&+\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\sum_{j=1}^{3}\eta^j\mathscr{H}_j\\
&+\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big(\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3} i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\\
&+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)+\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{jk}\Big)\Big\}\\
&+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big\{(\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)-1)\frac{\boldsymbol{c}}{\sqrt{15}}\sum_{j=1}^{3} i\eta^j(|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[\xi^j\sqrt{\mathsf{M}}|\\
&+\xi^j\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|)+\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{H}_{jk}\\
&+\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\boldsymbol{c}^2|\boldsymbol{\eta}|^2\Big(\frac{1}{15}|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}\otimes[|\boldsymbol{\xi}|^2\sqrt{\mathsf{M}}|+\sum_{j=1}^{3}\eta^j\mathscr{H}_j\Big)\Big\}\Big]\\
\equiv\ &\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)O(|\boldsymbol{\eta}|^4)te^{-A_1|\boldsymbol{\eta}|^2t}\mathcal{H}_1+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}O(|\boldsymbol{\eta}|^4)te^{-A_1|\boldsymbol{\eta}|^2t}\mathcal{H}_2\\
&+\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)e^{-A_1|\boldsymbol{\eta}|^2t}\mathcal{H}_3+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}e^{-A_1|\boldsymbol{\eta}|^2t}\mathcal{H}_4.
\end{aligned}
$$

We have

$$
\Big\|\frac{1}{(2\pi)^3}\int_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}e^{-A_1|\boldsymbol{\eta}|^2t}\mathcal{H}_j d\boldsymbol{\eta}\Big\|_{L^2_{\boldsymbol{\xi}}}
$$

$$= \left\| \mathsf{w} * \frac{1}{(2\pi)^3} \int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_j d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}},$$

$$\left\| \frac{1}{(2\pi)^3} \int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} \cos(\boldsymbol{c}|\boldsymbol{\eta}|t) e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_j d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}}$$

$$= \left\| \mathsf{w}_t * \frac{1}{(2\pi)^3} \int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_j d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}}.$$

To apply Theorem 7.1 and Lemma 7.2 we only need to estimate $\left\| \int_{|\boldsymbol{\eta}|<\delta/2} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_j d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}}$. Note that

$$|\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)|, \quad \left| \frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{|\boldsymbol{\eta}|} \right| \leq C e^{O(|\boldsymbol{\eta}|^3)t},$$

and that $\mathcal{H}_1$ and $\mathcal{H}_2$ have extra $O(|\boldsymbol{\eta}|^4)t$. These induce extra $(1+t)^{-1}$ decay by Lemma 7.11:

$$\left\| \int_{|\boldsymbol{\eta}|<\delta/2} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} O(|\boldsymbol{\eta}|^4) t e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_1 d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}} \leq C \Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^{5/2}} + e^{-t/C} \Big),$$

$$\left\| \int_{|\boldsymbol{\eta}|<\delta/2} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} O(|\boldsymbol{\eta}|^4) t e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_2 d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}} \leq C \Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^{3}} + e^{-t/C} \Big)$$

Similarly, for $\mathcal{H}_3$ and $\mathcal{H}_4$ we use the following equality

$$\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t) - 1 = |\boldsymbol{\eta}|^2 \Lambda(|\boldsymbol{\eta}|^2) \int_0^t \frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)\tau)}{|\boldsymbol{\eta}|} d\tau,$$

to obtain

$$\left\| \int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} O(|\boldsymbol{\eta}|^4) t e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_3 d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}} \leq C \Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{C(1+t)}}}{t^2} + e^{-t/C} \Big),$$

$$\left\| \int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}} O(|\boldsymbol{\eta}|^4) t e^{-A_1|\boldsymbol{\eta}|^2 t} \mathcal{H}_4 d\boldsymbol{\eta} \right\|_{L^2_{\boldsymbol{\xi}}} \leq C \Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{C(1+t)}}}{t^{5/2}} + e^{-t/C} \Big).$$

We now consider 1st Riesz wave.

$$\hat{\mathbb{G}}_{\mathfrak{PR}_1}(\boldsymbol{\eta}, t) - \hat{\mathbb{G}}^1_{\mathfrak{PR}_1}(\boldsymbol{\eta}, t)$$
$$= e^{-A_1|\boldsymbol{\eta}|^2 t} (e^{O(|\boldsymbol{\eta}|^4)t} - 1) \Big[(\cos(\boldsymbol{c}|\boldsymbol{\eta}|t) \cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t) - 1)$$

$$
\begin{aligned}
&\times\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big(\chi_j\otimes[\chi_k|+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)\\
&+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big\{-\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{|\boldsymbol{\eta}|}\sum_{j,k=1}^{3}\boldsymbol{c}\eta^j\eta^k\Big(\chi_j\otimes[\chi_k|+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}\\
&+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)+\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{P}^2_{jk}\Big\}\\
&+\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{P}^2_{jk}\Big]\\
&+e^{-A_1|\boldsymbol{\eta}|^2t}\Big[\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\Big\{(\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)-1)\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big(\chi_j\otimes[\chi_k|\\
&+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{P}^2_{jk}\Big\}\\
&+(\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big(\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)\\
&+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\Big\{-\frac{\sin(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)}{|\boldsymbol{\eta}|}\sum_{j,k=1}^{3}\boldsymbol{c}\eta^j\eta^k\Big(\chi_j\otimes[\chi_k|\\
&+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)+\cos(|\boldsymbol{\eta}|\Lambda(|\boldsymbol{\eta}|^2)t)\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{P}^2_{jk}\Big\}\Big]\\
\equiv\ &O(|\boldsymbol{\eta}|^4)te^{-A_1|\boldsymbol{\eta}|^2t}\Big(\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\mathcal{P}_1+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\mathcal{P}_2\\
&-\sum_{j,k=1}^{3}\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big(\chi_j\otimes[\chi_k|+\sum_{l=1}^{3}\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)\Big)\\
&+e^{-A_1|\boldsymbol{\eta}|^2t}\Big(\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)\mathcal{P}_3+\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|t)}{\boldsymbol{c}|\boldsymbol{\eta}|}\mathcal{P}_4+(\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\mathcal{P}_5\\
&-\sum_{j,k=1}^{3}\eta^j\eta^k\mathscr{P}^1_{jk}\Big).
\end{aligned}
$$

We now use Theorem 7.1 and Lemma 7.2 and Lemma 7.11 to obtain

$$
\Big\|\int_{|\boldsymbol{\eta}|<\delta}e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}O(|\boldsymbol{\eta}|^4)te^{-A_1|\boldsymbol{\eta}|^2t}\mathcal{P}_jd\boldsymbol{\eta}\Big\|_{L^2_{\boldsymbol{\xi}}}
$$

$$
\begin{aligned}
&\le C\Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^{5/2}}+e^{-t/C}\Big),\ \text{for}\ \ j=1,2,\\
&\Big\|\int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}e^{-A_1|\boldsymbol{\eta}|^2t}\mathcal{P}_j d\boldsymbol{\eta}\Big\|_{L^2_{\boldsymbol{\xi}}}\le C\Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^{5/2}}+e^{-t/C}\Big),\ \text{for}\ j=3,4,\\
&\Big\|\int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}O(|\boldsymbol{\eta}|^4)te^{-A_1|\boldsymbol{\eta}|^2t}\sum_{j,k=1}^3\frac{\eta^j\eta^k}{|\boldsymbol{\eta}|^2}\Big(\chi_j\otimes[\chi_k|\\
&\qquad +\sum_{l=1}^3\eta^l\mathscr{P}_{jkl}+|\boldsymbol{\eta}|^2\mathscr{P}^1_{jk}\Big)d\boldsymbol{\eta}\Big\|_{L^2_{\boldsymbol{\xi}}}\le C\Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^{5/2}}+e^{-t/C}\Big),\\
&\Big\|\int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}e^{-A_1|\boldsymbol{\eta}|^2t}\sum_{j,k=1}^3\eta^j\eta^k\mathscr{P}^1_{jk}\Big]d\boldsymbol{\eta}\Big\|_{L^2_{\boldsymbol{\xi}}}\le C\Big(\frac{e^{-\frac{|\boldsymbol{x}|^2}{Ct}}}{(1+t)^{5/2}}+e^{-t/C}\Big).
\end{aligned}
$$

To calculate $\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\mathcal{P}_5$, we write

$$
\begin{aligned}
\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\mathcal{P}_5 \ &=\ (\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\sum_{j,k,l=1}^3\frac{\eta^j\eta^k\eta^l}{|\boldsymbol{\eta}|^2}\mathscr{P}_{jkl}\\
&=\int_0^t\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|\tau)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j,k,l=1}^3\boldsymbol{c}^2\eta^j\eta^k\eta^l\mathscr{P}_{jkl}d\tau,
\end{aligned}
$$

and so the inverse Fourier transform of $\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\mathcal{P}_5$ satisfies

$$
\begin{aligned}
&\Big\|\int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}e^{-A_1|\boldsymbol{\eta}|^2t}\cos(\boldsymbol{c}|\boldsymbol{\eta}|t)-1)\mathcal{P}_5d\boldsymbol{\eta}\Big\|_{L^2_{\boldsymbol{\xi}}}\\
&=\Big\|\int_0^t\int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}\frac{\sin(\boldsymbol{c}|\boldsymbol{\eta}|\tau)}{\boldsymbol{c}|\boldsymbol{\eta}|}\sum_{j,k,l=1}^3\boldsymbol{c}^2\eta^j\eta^k\eta^l\mathscr{P}_{jkl}d\boldsymbol{\eta}d\tau\Big\|_{L^2_{\boldsymbol{\xi}}}\\
&=\Big\|\int_0^t\mathsf{w}(x,\tau)*\int_{|\boldsymbol{\eta}|<\delta} e^{i\boldsymbol{x}\cdot\boldsymbol{\eta}}\sum_{j,k,l=1}^3\boldsymbol{c}^2\eta^j\eta^k\eta^l\mathscr{P}_{jkl}d\boldsymbol{\eta}d\tau\Big\|_{L^2_{\boldsymbol{\xi}}}.
\end{aligned}
$$

The above has one more $\boldsymbol{\eta}$ factor comparing to $\hat{\mathbb{G}}^1_{\mathfrak{PR}_1}(\boldsymbol{\eta},t)$ which induces extra $(1+t)^{-1/2}$ decay. We can apply similar argument to $\|\mathbb{G}_{\mathfrak{C}}(\boldsymbol{x},t)-\mathbb{G}^1_{\mathfrak{C}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}$, $\|\mathbb{G}_{\mathfrak{R}}(\boldsymbol{x},t)-\mathbb{G}^1_{\mathfrak{R}}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}$ and $\|\mathbb{G}_{\mathfrak{PR}_2}(\boldsymbol{x},t)-\mathbb{G}^1_{\mathfrak{PR}_2}(\boldsymbol{x},t)\|_{L^2_{\boldsymbol{\xi}}}$ to complete the proof. □

7.6. Global wave pattern for Green's function

With the construction of essential kinetic waves and the fluid-like waves in the finite Mach region, the global structure of the Green's function can now be studied following the basic procedure of the 1-D case in the last two sections. The Mixture Lemma for the 3-D case is a straightforward generalization of the 1-D case; and similarly for the weighted energy method. We will not elaborate this and only state the result.

Theorem 7.13. *The Green's function,* (7.1),

$$\mathbb{G}=\mathbb{G}^0+\mathbb{G}_K+\mathbb{G}_R, \tag{7.56}$$

consists of the leading fluid-like waves $\mathbb{G}^0$ *of Theorem 7.9, the essential kinetic waves* $\mathbb{G}_K=\mathsf{h}^0+\mathsf{h}^1+\mathsf{h}^2$ *given by* (7.4), (7.5)(7.6), *and the remainder* $\mathbb{G}_R$ *satisfying*

$$\|\mathbb{G}_R\|_{L^2_{\boldsymbol{\xi}}}=O(1)\Big[\frac{e^{-\frac{(|\boldsymbol{x}|-ct)^2}{Ct}}}{(1+t)^{5/2}}+\frac{1}{(1+t)^2(1+\frac{|\boldsymbol{x}|^2}{t})^{\frac{3}{2}}}\chi_{\{|\mathbf{x}|<ct\}}+\frac{e^{-\frac{|\boldsymbol{x}|^2}{C(1+t)}}}{(1+t)^2}+e^{-(|x|+t)/C}\Big]. \tag{7.57}$$

Moreover,

$$\begin{aligned}\|\mathbb{G}\mathsf{P}_1\|_{L^2_{\boldsymbol{\xi}}}+\mathsf{P}_1\|\mathbb{G}\|_{L^2_{\boldsymbol{\xi}}} &= O(1)\Big[\frac{e^{-\frac{(|\boldsymbol{x}|-ct)^2}{Ct}}}{(1+t)^{5/2}}+\frac{1}{(1+t)^2(1+\frac{|\boldsymbol{x}|^2}{t})^{\frac{3}{2}}}\chi_{\{|\mathbf{x}|<ct\}}\\ &\quad+\frac{e^{-\frac{|\boldsymbol{x}|^2}{C(1+t)}}}{(1+t)^2}+e^{-(|x|+t)/C}\Big];\end{aligned} \tag{7.58}$$

$$\begin{aligned}\|\mathsf{P}_1\mathbb{G}\mathsf{P}_1\|_{L^2_{\boldsymbol{\xi}}} &= O(1)(t+1)^{-1/2}\Big[\frac{e^{-\frac{(|\boldsymbol{x}|-ct)^2}{Ct}}}{(1+t)^{5/2}}+\frac{1}{(1+t)^2(1+\frac{|\boldsymbol{x}|^2}{t})^{\frac{3}{2}}}\chi_{\{|\mathbf{x}|<ct\}}\\ &\quad+\frac{e^{-\frac{|\boldsymbol{x}|^2}{C(1+t)}}}{(1+t)^2}+e^{-(|x|+t)/C}\Big].\end{aligned} \tag{7.59}$$

Remark 7.14. The Boltzmann equation and the Navier-Stokes equations share some similarity, for instance, in the large-time behavior of fluid-like waves. On the other hand, the Boltzmann equation is semi-linear hyperbolic; while the Navier-Stokes equations are hyperbolic-parabolic. Thus the small time behaviors are quite different. In particular, the Green's function for the linear Navier-Stokes equations contains the singularity of the heat kernel type; while that for the Boltzmann equation does not. Thus in the above theorem we have expression like $(t+1)^{-\alpha}$ instead of $t^{-\alpha}$. This is seen analytically as follows: For the hyperbolic-parabolic systems, the inversion of the Fourier transform is done for the Fourier variables in the whole space, as the spectrum is known there. For the Boltzmann equation, the spectrum is known explicitly only near origin. Meanwhile, it is possible to identify the singular waves, the essential kinetic waves. The Green's function minus these singular waves can be as smooth as needed by the Mixture Lemma. Thus the singularity near the initial time is not of the heat kernel type.

8. Initial-Boundary Value Problem

In this section we use the 1-dimensional Green's function $\mathbb{G}(x,t)$, (5.2), Theorem 6.18, for the *initial value problem* to study the pointwise structure of the solution for the *initial-boundary value problem*:

$$\begin{cases} \mathsf{g}_t + \xi^1 \mathsf{g}_x = \mathsf{L}\mathsf{g}, \ x > 0 \\ \mathsf{g}(x,0) = \mathsf{I}_0(x), \\ \mathsf{g}(0,t,\boldsymbol{\xi})|_{\xi^1>0} = \mathsf{b}_+(t,\boldsymbol{\xi}). \end{cases} \tag{8.1}$$

The Boltzmann equation is semilinear hyperbolic with characteristic lines

$$\frac{d}{dt}x(t) = \xi^1.$$

The characteristic lines pointing toward the interior region $x > 0$ from the boundary $x = 0$ correspond to $\xi^1 > 0$. Thus the boundary data b_+ is defined only for positive ξ^1, c.f. [11], [17], [33].

As the boundary $x = 0$ is stationary, we will see that it is important to specify the sign of the Euler characteristics λ_j, $j = 1, 2, 3$, of the base Maxwellian $\mathsf{M} = \mathsf{M}_{[1,u,\theta]}$ of the linearization. One of the main purposes is to understand the boundary effect on the wave propagation. For this, we choose a localized initial value:

$$\begin{cases} supp(\mathsf{I}_0) \subset [-1,1], \\ sup_x \|\mathsf{I}_0(x,\cdot)\|_{L^\infty_{\xi,3/2}} \leq 1, \end{cases} \tag{8.2}$$

and the imposed boundary data tends to a constant state exponentially in time:

$$\sup_{\xi^1>0}(1+|\xi|)^4|\mathsf{b}_+| \leq e^{-\gamma t/D} \text{ for some } D > 3/\min_{\lambda_i>0}\lambda_i. \tag{8.3}$$

Here, the Mach number of the Maxwellian state $\mathsf{M}_{[1,u,\theta]}$ is assumed to satisfy $|u|/\sqrt{5\theta/3} \neq 0,1$; and b_+ is a given boundary data for $\xi^1 > 0, t > 0$ with sufficient decay rate $\sup_{\xi^1>0}(1+|\xi|)^3|\mathsf{b}_+(t,\xi)| < \infty$ for all $t > 0$. When all the Euler characteristics are of the same sign, one can use the energy method; otherwise, there is no straightforward energy method for the study of the initial-boundary value problem (8.1). The energy method yields the existence of the solutions. Our aim is to study the pointwise behavior of solutions without the same sign property of the Euler characteristics. We will design a series of approximations through two approximate solution operators.

8.1. Two approximate solution operators

If the *full* boundary values $\mathsf{g}(0,t) = \mathsf{b}(t,\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathbb{R}^3$ is known, then there is the solution formula

$$\mathsf{g}(x,t) = \int_0^\infty \mathbb{G}(x-y,t)\mathsf{I}_0(y)dy + \int_0^t \mathbb{G}(x,t-\sigma)\xi^1\mathsf{b}(\cdot,\sigma)d\sigma. \tag{8.4}$$

We introduce the first approximate solution operator using (8.4):

$$\begin{cases} \mathbb{H}[\mathsf{I}_0,\mathsf{b}](x,t): \text{ a solution of linearized Boltzmann equation in } x>0, \\ \mathbb{H}[\mathsf{I}_0,\mathsf{b}](x,t) \equiv \displaystyle\int_0^\infty \mathbb{G}(x-y,t)\mathsf{I}_0(y)dy + \int_0^t \mathbb{G}(x,t-\sigma)\xi^1\mathsf{b}(\cdot,\sigma)d\sigma. \end{cases} \tag{8.5}$$

Remark 8.1. As the boundary value should only be posted for $\xi^1 > 0$, the function $\mathsf{g}(x,t) = \mathbb{H}[\mathsf{I}_0,\mathsf{b}](x,t)$ satisfies only the first two equations of (8.1) and not the third equation for the boundary values. In other words, in general, $\mathsf{g}(0,t,\boldsymbol{\xi}) \neq \mathsf{b}(t,\boldsymbol{\xi})$, even for $\xi^1 > 0$. Nevertheless, with the explicit expression of the Green's function, Theorem 6.18, the formula (8.4) offers pointwise expression and will be useful for our analysis of the initial-boundary value problem (8.1).

The second approximation solution operator is by the damped equation with physical boundary data b_+

$$\begin{cases} \mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+](x,t): \text{ a solution of damped Boltzmann equation in } x\geq 0,\\ \mathsf{h}(x,t)\equiv \mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+](x,t),\\ \begin{cases} \mathsf{h}_t+\xi^1\mathsf{h}_x=\mathsf{L}\mathsf{h}-\gamma\mathsf{B}_+\mathsf{h},\\ \mathsf{h}|_{x=0,\xi^+>0}=\mathsf{b}_+,\\ \mathsf{h}(x,0)=\mathsf{I}_0(x). \end{cases} \end{cases} \tag{8.6}$$

Here, the Euler waves propagating toward the gas region $x > 0$ is damped through the upwind Euler projection operator B_+, (4.21). We will also write

$$\mathbb{B}^\gamma[\mathsf{I}_0,\mathsf{b}_+](t)\equiv \mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+](0,t). \tag{8.7}$$

Remark 8.2. The operator $\mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+]$ does satisfy the physical boundary value

$$\mathbb{B}^\gamma[\mathsf{I}_0,\mathsf{b}_+](t,\boldsymbol{\xi})|_{\xi^1>0}=\mathsf{b}_+(t,\boldsymbol{\xi}).$$

On the other hand, $\mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+]$ is not an exact solution operator for the original Boltzmann equation. Nevertheless, as an approximate operator with the damped coefficient γ taken to be small, the operator is accurate in restoring the full boundary values as the waves being damped are leaving the boundary. In fact, in the iterations below, this operator is used mainly for the purpose of restoring the full boundary values. Moreover, as the Euler waves with positive speed are damped, we can use the weighted energy method to estimate the solutions of (8.6).

We now design an iterated scheme based on the above two operators for the construction of the original initial-boundary value problem (8.1).

Consider the iterations:

$$\begin{cases} \mathsf{q}_1(x,t) = \mathbb{I}^\gamma[\mathsf{I}_0, \mathsf{b}_+](x,t), \\ \mathsf{b}_1(t) = \mathsf{q}_1(0,t), \\ \mathsf{g}_1(x,t) = \mathbb{H}[\mathsf{I}_0, \mathsf{b}_1](x,t), \\ \mathsf{d}_{1,+}(t) \equiv -(\mathsf{g}_1(0,t) - \mathsf{b}_+(t))|_{\xi^1>0}; \\ \text{and, for } n \geq 2, \\ \quad \mathsf{q}_n(x,t) = \mathbb{I}^\gamma[0, \mathsf{d}_{n-1,+}](x,t), \\ \quad \mathsf{b}_n(t) = \mathsf{q}_n(0,t), \\ \quad \mathsf{g}_n(x,t) = \mathbb{H}[0, \mathsf{b}_n](x,t), \\ \quad \mathsf{d}_{n,+}(t) = -(\mathsf{g}_n(0,t) - \mathsf{d}_{n-1,+})|_{\xi^1>0}. \end{cases} \tag{8.8}$$

As noted above, the functions $\mathsf{g}_n(x,t)$ are solutions of the Boltzmann equation, but the boundary condition is not satisfied in general. However, the discrepancy of the boundary values will shown to decrease and, if the series $\sum_{n=1}^\infty \mathsf{g}_n(x,t)$ converges, then it is easy to see that it converges to the solution of the initial-boundary value problem (8.1).

8.2. Weighted energy method

The initial-boundary value problem for the damped Boltzmann equation with $0 < \gamma \ll 1$,

$$\begin{cases} \mathsf{q}_t + \xi^1 \mathsf{q}_x = \mathsf{L}\mathsf{q} - \gamma \mathsf{B}_+ \xi^1 \mathsf{q}, \ x > 0, \\ \mathsf{q}|_{x=0,\xi^1>0} = \mathsf{q}_+(t)|_{\xi^1>0}, \\ \mathsf{q}|_{t=0} = \mathsf{u}_0, \end{cases} \tag{8.9}$$

is studied by the weighted energy method. Integrate the first equation in (8.9) taken inner product with $2\mathsf{q}$ and multiplied with the weight $w(x) = e^{\gamma x}$:

$$\begin{aligned} &\frac{d}{dt}\int_0^\infty e^{\gamma x}(\mathsf{q},\mathsf{q})dx + \int_0^\infty \gamma e^{\gamma x}(\mathsf{q}, 2\mathsf{B}_+\xi^1\mathsf{q} - \xi^1\mathsf{q})dx - (\mathsf{q}(0,t), \xi^1\mathsf{q}(0,t)) \\ &= \int_0^\infty e^{\gamma x}(2\mathsf{q}, \mathsf{L}\mathsf{q}). \end{aligned} \tag{8.10}$$

We now study the second integral. Decompose the solution into

$$\mathsf{q}=\mathsf{P}_0\mathsf{q}+\mathsf{P}_1\mathsf{q}=\mathsf{q}_0+\mathsf{q}_1=\sum_{j=1}^{3}(\mathsf{q},\mathsf{E}_j)\mathsf{E}_j+\mathsf{q}_1=\sum_{j=1}^{3}q_{0j}\mathsf{E}_j+\mathsf{q}_1.$$

The macro part of the integrant for the second integral above is

$$\begin{aligned}(\mathsf{q}_0,2\mathsf{B}_+\xi^1\mathsf{q}_0-\xi^1\mathsf{q}_0) &= 2\sum_{\lambda_j>0}\lambda_j(q_{0j})^2-\sum_{j=1}^{3}\lambda_j(q_{0j})^2\\ &\geq \sum_{j=1}^{3}|\lambda_j|(q_{0j})^2\geq\Lambda(\mathsf{q}_0,\mathsf{q}_0),\end{aligned}\tag{8.11}$$

where

$$\Lambda\equiv\min_{\lambda_j>0}\lambda_j.$$

The estimate for the micro part is provided by the term $(2\mathsf{q},\mathsf{L}\mathsf{q})$. Thus we have control of the energy of both parts by using (3.23) and (3.24) and conclude that, for sufficiently small γ c.f. (6.81),

$$\begin{aligned}&\frac{d}{dt}\int_0^\infty e^{\gamma x}(\mathsf{q},\mathsf{q})dx+\frac{2\gamma}{3}\Lambda\int_0^\infty e^{\gamma x}(\mathsf{q},\mathsf{q})dx-2(\mathsf{q}(0,t),\xi^1\mathsf{q}(0,t))_-\\ &\leq 2(\mathsf{q}_+,\xi^1\mathsf{q}_+)_++2\int_0^\infty e^{\gamma x}(\mathsf{u}_0,\mathsf{u}_0)dx,\end{aligned}\tag{8.12}$$

Here,

$$(\mathsf{f},\mathsf{h})|_\pm\equiv\int_{\xi\in\mathbb{R}^3,\pm\xi^1>0}f(\xi)\mathsf{h}(\xi)d\xi.$$

This leads to the estimate

$$\begin{aligned}&\int_0^\infty e^{\gamma x}(\mathsf{q},\mathsf{q})dx|_\tau-\int_0^\tau e^{-\frac{\gamma}{2}\Lambda(\tau-t)}2(\mathsf{q}(0,t),\xi^1\mathsf{q}(0,t))_-dt\\ &\leq\int_0^\tau e^{-\frac{\gamma}{2}\Lambda(\tau-t)}2(\mathsf{q}(0,t),\xi^1\mathsf{q}(0,t))_+dt+e^{-\frac{\gamma}{2}\Lambda\tau}2\int_0^\infty e^{\gamma x}(\mathsf{u}_0,\mathsf{u}_0)dx.\end{aligned}\tag{8.13}$$

By the above estimates, one has

$$\int_0^\infty e^{\frac{\gamma}{2}\Lambda t}2(\mathsf{b}_1(0,t),\xi^1\mathsf{b}_1(0,t))_-dt\ \leq\ O(1)\frac{1}{\gamma\Lambda},\tag{8.14}$$

$$\int_0^\infty e^{\gamma x}(\mathsf{q},\mathsf{q})dx|_\tau \ \le\ O(1)\frac{e^{-\gamma\tau/D}}{\gamma}, \qquad (8.15)$$

where D is given in (8.3).

Remark 8.3. The operator $\mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+]$ is used in the iterations (8.8) to restore the full boundary values $\mathsf{b}(t)$ from the given values $\mathsf{b}_+(t)$. From (8.13) we have

$$\begin{aligned}&\int_0^t e^{-\frac{\gamma}{2}\Lambda(t-s)}(\mathsf{b}(s),|\xi^1|\mathsf{b}(s))ds\\ &\le\ 2\int_0^t e^{-\frac{\gamma}{2}\Lambda(t-s)}(\mathsf{b}_+(s),|\xi^1|\mathsf{b}_+(s))ds+4e^{-\frac{\gamma}{2}\Lambda\tau}\int_0^\infty e^{\gamma x}(\mathsf{I}_0,\mathsf{I}_0)dx, \quad (8.16)\end{aligned}$$

$$\mathsf{b}(t)\equiv\mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+](0,t).$$

Thus the weighted energy method yields the estimate for the full boundary values $\mathsf{b}(t)$ in terms of the boundary data b_+ and the initial data I_0. This turns out to be sufficient for further analysis.

8.3. Pointwise estimates

For the operator $\mathbb{I}^\gamma[0,\mathsf{d}_{n-1,+}](x,t)$ in (8.8), we need an estimate for the function of the form $\mathbb{H}[\mathsf{I}_0,\mathsf{b}]|_{x=0}$ for $\xi^1>0$. In carrying out the estimate for $\mathbb{H}[0,\mathsf{b}_n](x,t)$ in the iteration (8.8), what we have from the previous iteration is the boundary estimate of the form (8.16). We carry out these estimates in this subsection. For this, we need to make essential use of the decomposition of the Green's function into fluid-like, essential kinetic parts, and the remaining parts, $\mathbb{G}=\mathbb{G}_F+\mathbb{G}_K+\mathbb{G}_R$, (7.56), (6.71), Theorem 5.11, (8.14), and (8.15).

The first term $\mathsf{b}_1(0,t)=\mathbb{I}^\gamma[\mathsf{I}_0,\mathsf{b}_+](0,t)$ in the iteration is estimated by the weighted energy method and shown to satisfies estimate of the form (8.14). We then study the next term in the iteration. The component h^0 in (6.71), $\mathbb{G}_K=\mathsf{h}^0+\mathsf{h}^1+\mathsf{h}^2$, contributes $\int_0^t\mathsf{h}^0\xi^1\mathsf{b}_1d\sigma$ to $\mathbb{H}[\mathsf{I}_0,\mathsf{b}_1]$ which has pointwise estimates when b_1 has pointwise structure for $\xi^1>0$, with the same sign as x. This is due to the delta-function in h^0. The component $\int_0^t(\mathsf{h}^1+\mathsf{h}^2)\xi^1\mathsf{b}_1d\sigma$ gives pointwise structure for all x because of the regularizing effect from K in constructing h^1 and h^2. Thus we conclude that

$\int_0^t \mathbb{G}_K(x,t-\sigma)\xi^1 \mathsf{b}_1 d\sigma$ has pointwise estimate given that the boundary values b_1 are given with estimate of the form (8.14):

$$\begin{aligned}&\| \int_0^t \mathbb{G}_K(x,t-\sigma)\xi^1 \mathsf{b}_+(t-\sigma)d\sigma\|_{L^2_\xi} \\ &\le O(1)\frac{1}{\sqrt{\gamma}}e^{-(|x|+\gamma t)/C_0} \text{ for } x>0 \text{ and some } C_0>0. \end{aligned} \tag{8.17}$$

The detailed calculations are straightforward and omitted.

By the pointwise structure of the fluid-like wave component in (6.70) and the remainder component in (6.72) together with (8.14), one has that

$$\begin{aligned}&\left\| \int_0^t \mathbb{G}_F(x,t-\sigma)\xi^1 \mathsf{b}_1(\sigma)d\sigma \right\|_{L^2_\xi} \\ &= \left\| \int_0^t \mathbb{G}_F(x,t-\sigma)e^{-\gamma\Lambda\sigma/4}e^{\gamma\Lambda\sigma/4}\xi^1 \mathsf{b}_1(\sigma)d\sigma \right\|_{L^2_\xi} \\ &\le O(1)\left(\int_0^t \|\mathbb{G}_F(x,t-\sigma)\|^2_{L^2_\xi}e^{-\gamma\Lambda\sigma/2}d\sigma\right)^{\frac{1}{2}} \\ &\quad \times \left(\int_0^t e^{\gamma\Lambda\sigma/2}(\mathsf{b}_1(\sigma),|\xi^1|\mathsf{b}_1(\sigma))d\sigma\right)^{\frac{1}{2}} \\ &\le O(1)\frac{1}{\gamma}\left(\sum_{\lambda_j>0}\frac{e^{-\frac{(x-\lambda_j t)^2}{C_0 t}}}{\sqrt{1+t}} + O(1)e^{-\gamma\frac{|x|+t}{C_0}}\right) \end{aligned} \tag{8.18}$$

for some $C_0>0$. The estimates in (8.17) and (8.18) give the pointwise estimate of the operator $\mathbb{H}[\mathsf{I}_0,\mathsf{b}_1]$.

For the convergence of the iterations, we will need more detailed structure of the Green's function stated in Theorem 5.11. We can also construct the Green's function, $\mathbb{G}_D$ with similar property for the damped equation $\mathsf{h}_t+\xi^1\mathsf{h}_x-\mathsf{L}\mathsf{h}=-\gamma\mathsf{B}_+\xi^1\mathsf{h}$ in a whole space problem.

The estimates of $\mathsf{b}_{1|\xi^1>0}-\mathsf{b}_{0+}|_{\xi^1>0}$.

We know the full values of the function $\mathsf{b}_1(t)=\mathsf{q}(0,t)$ of the solution of an initial-boundary problem of the damped Boltzmann equation:

$$\begin{cases} \mathsf{q}_t + \xi^1 \mathsf{q}_x = \mathsf{L}\mathsf{q} - \gamma \mathsf{B}_+ \xi^1 \mathsf{q}, \ x > 0, \\ \mathsf{q}|_{x=0,\xi^1>0} = \mathsf{b}_+(t), \\ \mathsf{q}|_{t=0} = \mathsf{I}_0. \end{cases} \tag{8.19}$$

By the Green's identity we have in turn the following representation of the boundary data $\mathsf{b}_+|_{\xi^1>0}$:

$$\begin{aligned} \mathsf{b}_+|_{\xi^1>0} \ =\ & \left(\int_0^\infty \mathbb{G}(-y,t)\mathsf{I}_0(y)dy + \int_0^t \mathbb{G}(0,t-\sigma)\xi^1 \mathsf{q}(0,\sigma)d\sigma \right)\Big|_{\xi^1>0} \\ & -\gamma \int_0^t \int_0^\infty \mathbb{G}(-y,t-\sigma)\xi^1 \mathsf{B}_+ \xi^1 \mathsf{q} dy d\sigma \Big|_{\xi^1>0}. \end{aligned} \tag{8.20}$$

By the definitions of $\mathbb{H}[\mathsf{I}_0, \mathsf{b}_1]$ and that $\mathsf{b}_1(t) = \mathsf{q}(0,t)$, one has that

$$\begin{aligned} \mathsf{d}_{1,+} \ &=\ \mathsf{b}_+|_{\xi^1>0} - \mathbb{H}[\mathsf{I}_0, \mathsf{b}_1](0,t)|_{\xi^1>0} \\ &=\ -\gamma \left(\int_0^t \int_0^\infty \mathbb{G}(-y,t-\sigma)\mathsf{B}_+ \xi^1 \mathsf{q}(y,s) dy d\sigma \right)_{|\xi^1>0}. \end{aligned} \tag{8.21}$$

The combination of the energy estimate and separation scale of the Green's function is used in the next step:

$$\begin{aligned} & \gamma \left\| \int_0^t \int_0^\infty \mathbb{G}(-y,t-\sigma)\mathsf{B}_+ \xi^1 \mathsf{q}(y,s) dy d\sigma \right\|_{L^2_\xi} \\ & =\ \gamma \left\| \int_0^t \int_0^\infty \mathbb{G}(-y,t-\sigma) e^{-\gamma x/2} \mathsf{B}_+ \xi^1 \mathsf{q}(y,s) e^{\gamma x/2} dy d\sigma \right\|_{L^2_\xi} \\ & \le\ O(1)\gamma \int_0^t \left(\int_0^\infty \left(\sum_{\lambda_j<0} \frac{e^{-\frac{(-y-\lambda_j(t-\sigma))^2}{C_0(t-\sigma)}}}{(t-\sigma+1)^2} \right.\right. \\ & \qquad \left.\left. + \sum_{\lambda_j>0} \frac{e^{-\frac{(-y-\lambda_j(t-\sigma))^2}{C_0(t-\sigma)}}}{(t-\sigma+1)} + e^{-\frac{|x-y|+t-\sigma}{C_0}} \right) e^{-\gamma y} dy \right)^{1/2} \\ & \qquad \times \left(\int_0^\infty e^{\gamma y} (\mathsf{q},\mathsf{q}) dy \right)^{1/2} d\sigma \end{aligned}$$

$$
\begin{aligned}
&\leq O(1)\gamma\int_0^t\left(\frac{e^{-K_0(t-\sigma)\gamma}}{(t-\sigma+1)^{3/2}}+e^{-\Lambda(t-\sigma)}+e^{-\frac{(t-\sigma)}{C_0}}\right)^{1/2}\frac{e^{-\gamma\sigma/D}}{\sqrt{\gamma}}d\sigma\\
&= O(1)\gamma^{1/4}e^{-\gamma t/D}. \qquad (8.22)
\end{aligned}
$$

Here, we have used the fact that $\mathbb{G}(x,t)\mathsf{B}_+$ gains an extra decaying factor 0f $1/\sqrt{t+1}$ in the fluid-like wave component located around to $|x-\lambda_j t|\leq O(1)\sqrt{t}$ for $\lambda_j<0$. (8.22) gives the estimate of d_{0+} in $\|\cdot\|_{L^2_\xi,+}$. We need to improve it into a pointwise estimate in $\xi^1>0$. For this, one uses the Green's function $\mathbb{G}^\gamma_D(x,t)$ for the damped Boltzmann equation $(\partial_t+\xi^1\partial_x-\mathsf{L}+\gamma\mathsf{B}_+\xi^1)\mathsf{h}=0$ for a whole space problem and the estimate (8.14), and the representation

$$
\mathsf{q}(x,t)=\int_0^\infty \mathbb{G}^\gamma_D(x-y,t)\mathsf{u}_0(y)dy+\int_0^t \mathbb{G}^\gamma_D(x,t-\sigma)\xi^1\mathsf{b}_1(\sigma)d\sigma
$$

to yield a pointwise $\|\mathsf{q}(x,t)\|_{L^2_\xi}$ estimate in (x,t). With this estimate, one can interpret the function

$$
\mathsf{D}(x,t)\equiv\int_0^t\int_0^\infty \mathbb{G}(x-y,t-\sigma)\mathsf{B}_+\xi^1\mathsf{q}(y,\sigma)dyd\sigma
$$

as the solution of the following initial value problem with a given inhomogeneous term γ Heaviside $(x)\mathsf{B}_+\xi^1\mathsf{q}$ with pointwise structure in (x,t,ξ):

$$
\begin{cases}(\partial_t+\xi^1\partial_x-\mathsf{L})\mathsf{D}=\gamma\ \text{Heaviside}\,(x)\mathsf{B}_+\xi^1\mathsf{q},\\ \mathsf{D}(x,0)\equiv 0.\end{cases}
$$

where Heaviside (x) is the Heaviside function. With the estimate on $\|\int_0^t\int_0^\infty \mathbb{G}(x-y,t-\sigma)\mathsf{B}_+\xi^1\mathsf{q}(y,\sigma)dyd\sigma\|_{L^2_\xi}$, by a standard bootstrap procedure, one obtains estimate of the form:

$$
\sup_{\xi^1>0}(1+|\xi|)^4|\mathsf{d}_{1,+}(t,\xi)|\leq O(1)\gamma^{1/4}e^{-\gamma t/D}. \qquad (8.23)
$$

The estimate (8.23) is compared to that for the given boundary data b_+ in (8.3). Then, one can repeat the procedure from (8.9) by replacing b_+ with

$\mathsf{d}_{1,+}$ and l_0 with 0. We therefor show that the iteration scheme satisfies

$$\begin{cases} \sup_{\xi^1>0}(1+|\xi|)^4\mathsf{d}_{n,+}(t) \le C_1\gamma^{n/5}e^{-\gamma t/D}, \\ \|\mathsf{g}_n(x,t)\|_{L^2_\xi} \le C_1\gamma^{-1+\frac{n-1}{5}}\left(\sum_{\lambda_j>0}\frac{e^{-\frac{(x-\lambda_j t)^2}{C_0 t}}}{\sqrt{1+t}} + O(1)e^{-\gamma\frac{|x|+t}{C_0}}\right) \end{cases} \tag{8.24}$$

for some $C_1 > 0$. This yields the convergence of the iteration scheme and the series $\mathsf{g}(x,t) = \sum_{j=1}^\infty \mathsf{g}_j$ solves the initial boundary value problem (8.1) with

$$\|\mathsf{g}(x,t)\|_{L^2_\xi} \le O(1)C_1\gamma^{-1}\left(\sum_{\lambda_j>0}\frac{e^{-\frac{(x-\lambda_j t)^2}{C_0 t}}}{\sqrt{1+t}} + O(1)e^{-\gamma\frac{|x|+t}{C_0}}\right). \tag{8.25}$$

This structure of $\mathsf{g}(x,t)$ and that of the $\mathbb{G}(x,t)$ are used to generate the solution of an given initial boundary value problem with any $y > 0$:

$$\begin{cases} (\partial_t + \xi^1\partial_x - \mathsf{L})\mathsf{h} = 0, \\ \mathsf{h}(0,t)|_{\xi^1>0} = 0, \\ \mathsf{h}(x,0) = 0 \text{ for } |x-y| \ge 1, \\ \|\mathsf{h}(\cdot,0)\|_{L^\infty_x(L^\infty_{\xi,4})} \le 1. \end{cases} \tag{8.26}$$

The solution satisfies, for $x, t > 0$,

$$\begin{aligned} \|\mathsf{h}(x,t)\| \;\le\; & O(1)\left(\sum_{j=1}^3\frac{e^{-\frac{(x-y-\lambda_j t)^2}{C_0(1+t)}}}{\sqrt{1+t}} + e^{-\frac{|x-y|+t}{C_0}}\right) \\ & + O(1)\gamma^{-1}\int_0^t\sum_{j,i=1}^3\frac{e^{-\frac{(x-\lambda_j(t-\sigma))^2}{C_0(t-\sigma+1)}}e^{-\frac{(y+\lambda_i\sigma))^2}{C_0(\sigma+1)}}}{\sqrt{(t-\sigma+1)(\sigma+1)}}d\sigma. \end{aligned} \tag{8.27}$$

For the case $\lambda_1 < \lambda_2 < 0 < \lambda_3$, the solution has the essential support as depicted in Figure 9. The Green's function $\mathbb{G}_b$ for the initial-boundary value problem

$$\begin{cases} (\partial_t + \xi^1 \partial_x - \mathsf{L})\mathbb{G}_b = 0, \\ \mathbb{G}_b(0,t)|_{\xi^1>0} = 0, \\ \mathbb{G}_b(x,0) = \delta(x-x_0)\delta^3(\boldsymbol{\xi}-\boldsymbol{\xi}_0),\ x_0>0, \end{cases} \tag{8.28}$$

can also be constructed. Because of the boundary, the Green's function is not translational invariant in x and is of the general form $\mathbb{G}_b(x,x_0,t,\boldsymbol{\xi};\boldsymbol{\xi}_0)$. The procedure in the preceding sections of constructing the essential kinetic and fluid-like waves are needed. The construction of the fluid-like waves have been outlined above. The construction of essential kinetic waves are similar. We conclude with the form of the Green's function, again for the case of $\lambda_1 < \lambda_2 < 0 < \lambda_3$:

$$\begin{cases} \mathbb{G}_b = \mathbb{G}_b^F + \mathbb{G}_b^K + \mathbb{G}_b^R, \\ \mathbb{G}_b^K = \mathsf{h}^0 + \mathsf{h}^1 + \mathsf{h}^2, \\ \mathsf{h}^0 = e^{-\nu(\boldsymbol{\xi})t}\delta(x-x_0-\xi^1 t)\delta^3(\boldsymbol{\xi}-\boldsymbol{\xi}_0), \\ \mathsf{h}^1 = \begin{cases} 0,\ \text{for } (x-x_0-\xi_0^1 t)(\xi^1 t - x + x_0) < 0, \\ 0,\ \text{for } (x-x_0-\xi_0^1 t)(\xi^1 t - x + x_0) > 0, \\ \qquad -x_0/\xi_0^1 + x/\xi^1 < t,\ \xi^1>0,\ \xi_0^1<0, \\ \frac{e^{-nu(\boldsymbol{\xi})(t-t_1)-nu(\boldsymbol{\xi}_0)t_1}\mathsf{K}(\boldsymbol{\xi}-\boldsymbol{\xi}_0)}{\xi^1-\xi_0^1},\ \text{otherwise}, \\ \qquad t_1 \equiv t-(x_-x_0-\xi_0^1 t)/(\xi^1-\xi_0^1), \end{cases} \\ |\mathsf{h}^2| = O(1)e^{-\nu_0(t+|x-x_0|)/3}\frac{1+|\boldsymbol{\xi}-\boldsymbol{\xi}_0|}{|\boldsymbol{\xi}-\boldsymbol{\xi}_0|}(1+|\boldsymbol{\xi}|)^{-1}, \\ \|\mathbb{G}_b^F\|_{L^2_{\boldsymbol{\xi}}} = O(1)[\sum_{j=1}^{3}(1+t)^{-1/2}e^{-\frac{(x-\lambda_j t-x_0)^2}{Ct}} \\ +\sum_{j=1}^{2}[(t+1)(1+x_0)]^{-1/2}e^{-\frac{(x-\lambda_3 t-\lambda_3 x_0/\lambda_j)^2}{Ct}}], \\ \|\mathbb{G}_b^R\|_{L^2_{\boldsymbol{\xi}}} = O(1)e^{-\frac{(|x|+|x_0|+t)}{C}}. \end{cases} \tag{8.29}$$

Moreover, the algebraic decay rates are higher when the Green's function acts on the micro part of the initial data.

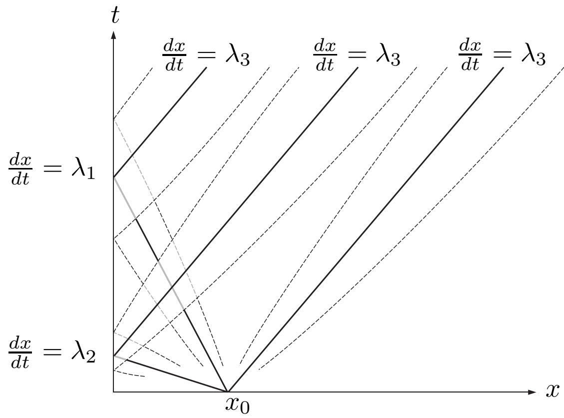


Figure 9: Solution for the case $\lambda_1 < \lambda_2 < 0 < \lambda_3$.

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