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ON PRIME SPECTRUMS OF 2-PRIMAL RINGS

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Abstract

A $2-$primal ring is one in which the prime radical is exactly the set of nilpotent elements. A ring is clean, provided every element is the sum of a unit and an idempotent. Keith Nicholson introduced clean rings in 1977 and proved the following: "Every clean ring is an exchange ring. Conversely, every exchange ring in which all idempotents are central, is clean." In this paper, we investigate some of the relationships among ring-theoretic properties and topological conditions, such as a 2-primal weakly exchange ring and its prime spectrum Spec(R).

1. Introduction

Throughout this paper, R denotes an associative ring with identity and Spec(R) (resp. Max(R)) denotes the set of all prime (resp. maximal) ideals of R. In addition, $\mathcal{P}(R)$, $\mathcal{J}(R)$ and $\mathcal{N}(R)$ are used to denote the prime radical, Jacobson radical and the set of all nilpotent elements of R, respectively.

From Birkenmeier, Heatherly and Lee [2], a ring R is called 2-primal if $\mathcal{P}(R) = \mathcal{N}(R)$. Every reduced rings are 2-primal, but the converse is not true. According to Crawley and Jonsson [4], a left R-module M is said to have the exchange property if, for every left R-module A and any two decompositions of A

$$A = M' \bigoplus N = \bigoplus_{i\in I} A_i,$$

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where $M' \cong M$, there exists submodules $A_i' \subseteq A_i$ such that

$$A = M' \bigoplus \Big(\bigoplus_{i \in I} A_i' \Big).$$

A left R-module M has the finite exchange property if the above condition is satisfied whenever the index set is finite. Warfield [14] called a ring R an exchange ring when ${}_RR$ has the finite exchange property. Nicholson [10] gave characterization of exchange rings: R is an exchange ring if and only if $R/\mathcal{J}(R)$ is an exchange ring and idempotents can be lifted modulo $\mathcal{J}(R)$, if and only if for any $a \in R$ there exists an idempotent $e \in Ra$ such that $1 - e \in R(1-a)$.

A ring R is said to be suitable [10] if for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in Ra$ and $1 - e \in R(1-a)$. Nicholson [10] proved that the suitable rings and the exchange rings are the same. We adopted the definition of a suitable ring instead of the definition of an exchange ring.

A ring is called a clean ring if every element is the sum of a unit and an idempotent. Nicholson [10] introduced clean rings in 1977 and proved: Every clean ring is an exchange ring. Conversely, every exchange ring in which all idempotents are central is clean. Lu and Yu [8] characterized clean rings by topological properties of their prime spectrums in the commutative case.

In this paper, we characterize weakly exchange rings by topological properties of their prime spectrums in non commutative case by proving the results that if R is a 2-primal ring, then $\mathrm{Spec}(R)$ is strongly zero-dimensional, if and only if $\overline{R} = R/\mathcal{P}(R)$ is a weakly exchange ring, if and only $\mathrm{Spec}(\overline{R})$ is strongly zero-dimensional.

We use $\overline{a}$ and $\overline{I}$ to denote $a + \mathcal{P}(R)$ and $I/\mathcal{P}(R)$, where $a \in R$ and I is an ideal of R containing $\mathcal{P}(R)$, respectively.

2. Preliminaries

In this section we recall basic definitions that are needed for our purpose.

An ideal P of a ring R is said to be prime (resp. completely prime) if for any $a, b \in R$, $aRb \subseteq P$ (resp. $ab \in P$) implies either $a \in P$ or $b \in P$. A ring R is called a strongly 2-primal ring [12] if $\mathcal{P}(R/I) = \mathcal{N}(R/I)$ for all proper ideal I of R, where the term proper means only $I \neq R$. Note that every strongly 2-primal is 2-primal. A ring is reduced provided it has no non zero nilpotent elements. An element $e \in R$ is said to be idempotent if $e^2 = e$. Let I be any ideal of R, we say that idempotents lift modulo I if every idempotents in R/I can be lifted to R (i.e., for any idempotent $a + I$ of R/I, there exists an idempotent e of R such that $a+I = e+I$). A ring R is called a *pm* ring if each prime ideal of R is contained in a unique maximal ideal of R.

A ring R is called π-regular if for every $x \in R$, there exists a natural number $n = n(x)$, depending on x, such that $x^n \in x^n R x^n$. A ring R is called right (left) weakly π-regular if for any $x \in R$, there exists a natural number n such that $x^n \in x^n R x^n R$ ($x^n \in R x^n R x^n$). A set $S \subseteq R$ is m-system if for any a, b $\in S$, there exists $r \in R$ such that $arb \in S$. For any ideal I of $R, \mathcal{P}(I) = \{a \in R \mid$ every m-system containing a meets $I\}$. Note that $\mathcal{P}(I) \subseteq \{a \in R \mid a^n \in I \text{ for some } n \geq 1\}$ and $\mathcal{P}(I)$ is the intersection of all prime ideals which contain I. An ideal I is called a radical ideal if $\mathcal{P}(I) = I$.

For any ideal I of R and $a \in R$, we set $V(a) = \{P \in \operatorname{Spec}(R) \mid a \in P\}$ and $V(I) = \{P \in \operatorname{Spec}(R) \mid I \subseteq P\}$. Hence the sets $V(I) = \bigcap_{a \in I} V(a)$, where I is the ideal of R, satisfy the axioms for the closed sets of a topology on $\operatorname{Spec}(R)$, called Zariski topology. Dually we set

$$\begin{aligned} U(a) &= \{P \in \operatorname{Spec}(R) \mid a \notin P\} \quad \text{and} \\ U(I) &= \{P \in \operatorname{Spec}(R) \mid I \nsubseteq P\}. \end{aligned}$$

We say that a space X is zero-dimensional if it has a base consisting of clopen sets, and is strongly zero-dimensional if for any closed set A and open set V containing A, there exists a clopen set U such that $A \subseteq U \subseteq V$. Note that these are equivalent to the concepts defined in McGovern [9] and Samei [11], respectively, for a Tychonoff space. Since a space satisfies T_1 if and only if every singleton set is closed, any strongly zero-dimensional space with T_1

is always zero-dimensional and converse holds for a compact T_1-space by Gillman and Jerision [6, Theorem 16.16], or being proven directly.

3. Prime spectrums of 2-primal rings

In this section, we introduce the notion of weakly exchange rings. We begin with the following definition.

Definition 3.1. A ring R is called a weakly exchange ring if for any $a \in R$, there exists an idempotent $e \in R$ such that $e \in RaR$ and $1-e \in R(1-a)R$.

Example 3.2.

(1) Let Q be the ring of all rational numbers and L is the set of all rational numbers with odd denominators. Then clearly L is a sub ring of Q. Define

$$R=R(Q,L)=\{(x_1,x_2,\ldots,x_n,s,s,\ldots)\mid n\geq 1, x_i\in Q, \text{for } 1\leq i\leq n, s\in L\}$$

with componentwise operations. It can be easily seen that R is a weakly exchange ring.

(2) Let $R=M_2(D)$, where D is a division ring. Then R is a weakly exchange ring. For, if $x=\begin{pmatrix}0&0\\0&0\end{pmatrix}\in R$, then choose $e=\begin{pmatrix}0&0\\0&0\end{pmatrix}$. It is clear that $e^2=e$, $e\in RxR$ and $1-e\in R(1-x)R$. If $x=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, then choose $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}$. Clearly $e^2=e, e=\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}x\begin{pmatrix}1&0\\0&1\end{pmatrix}\in RxR$. Since $1-x=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, we have $1-e=\begin{pmatrix}0&0\\0&0\end{pmatrix}\in R(1-x)R$. Therefore, assume that $x=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in R-\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\}$. Since $x\neq 0$, any one of a,b,c and d is non-zero. If $a\neq 0$, then choose $e=\begin{pmatrix}1&1\\0&0\end{pmatrix}$. It can be checked

that $e^2 = e$, $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix} x \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in$ RxR and $1 - e = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{1-a} & 0 \\ -\frac{1}{1-a} & 0 \end{pmatrix} \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$ $\begin{pmatrix} -\frac{1}{1-a} & 0 \\ \frac{1}{1-a} & 0 \end{pmatrix} (1-x) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R(1-x)R$. Similarly, we can prove that the same idempotent works for $b \neq 0$, $c \neq 0$ and $d \neq 0$. Thus R is a weakly exchange ring.

Lemma 3.3. *Let R be a 2-primal ring. For any prime ideal P of R, there is a completely prime ideal Q of R such that $\mathcal{P}(R) \subseteq Q \subseteq P$.*

Proof. Assume that R is a 2-primal ring. Then $\overline{R} = R/\mathcal{P}(R)$ is reduced and hence every minimal prime ideal of $\overline{R}$ is completely prime by [12, Proposition 1.11], since any reduced ring is 2-primal. Clearly P is a prime ideal of R if and only if $\overline{P} = P/\mathcal{P}(R)$ is a prime ideal of $\overline{R}$. For every $P \in$Spec(R), there is a minimal prime ideal Q of R such that $\mathcal{P}(R) \subseteq Q \subseteq P$. We claim that Q is a completely prime ideal of R. Since Q is minimal prime such that $\mathcal{P}(R) \subseteq Q$, $\overline{Q} = Q/\mathcal{P}(R)$ is minimal prime in $\overline{R}$. Hence $\overline{Q}$ is completely prime in $\overline{R}$ and consequently Q is a completely prime ideal of R. □

The following is Theorem 3.5 of Zhang et al. [15]. But we prove it in a different way.

Lemma 3.4. *Let R be a 2-primal ring. Then*

(i) *$U(x) = V(1-x)$ and $V(x) = U(1-x)$ for any idempotent $\overline{x}$ in $\overline{R}$.*

(ii) *$U(e) = V(1-e)$ and $V(e) = U(1-e)$ for any idempotent e in R.*

Proof. (i) Let $\overline{x}$ be an idempotent in $\overline{R}$. It is clear that $V(1-x) \subseteq U(x)$. Let $P \in U(x)$. Then $x \notin P$. Suppose that $P \notin V(1-x)$. Since R is 2 primal, there is a completely prime ideal Q of R such that $\mathcal{P}(R) \subseteq Q \subseteq P$ by Lemma 3.3. Since $x \notin P$ and $1 - x \notin P, x^2 - x \notin Q$ and hence $x^2 - x \notin \mathcal{P}(R)$. This shows that $\overline{x}$ is not an idempotent in $\overline{R}$, a contradiction. Therefore $1-x \in P$. So $P \in V(1-x)$. Thus $U(x) = V(1-x)$. Similarly, we can prove that $V(x) = U(1-x)$.

(ii) It is clear that $V(1-e) \subseteq U(e)$ for any idempotent e of R. Let $P \in U(e)$. Then $e \notin P$. Suppose that $P \notin V(1-e)$. By the similar argument used in part (i), there is a completely prime ideal Q of R such that $e^2-e \notin Q$, i.e., $0 \notin Q$, a contradiction. Therefore $U(e) = V(1-e)$. Similarly, we can prove that $V(e) = U(1-e)$. □

Note that clop(Spec(R)) denotes the set of all clopen (ie., both open and closed) sets of Spec(R). Idempotents always lift modulo any nil ideal [1, Proposition 27.1], and 2-primality guarantees that $\mathcal{P}(R)$ is a nil ideal. The following lemma shows that every clopen subset of Spec(R) is from $V(e)$, where $e \in R$ is an idempotent as proved in [15, Theorem 3.6]. However, it is proved using a different approach in this paper.

Lemma 3.5.

(i) *Let R be a 2-primal ring. Then*

$$\text{clop}(\text{Spec}(R)) = \left\{V\left(x\right) \mid \overline{x} \textit{ is an idempotent in } \overline{R}\right\}$$

and

$$\text{clop}(\text{Spec}(\overline{R})) = \left\{V\left(\overline{x}\right) \mid \overline{x} \text{ is an idempotent in } \overline{R}\right\}.$$

(ii)

$$\text{clop}(\text{Spec}(R)) = \{V(e) \,|\, e \textit{ is an idempotent in } R\}.$$

Proof. (i) Let $V(I) \in$ clop(Spec(R)), where I is an ideal of R. Then there exists an ideal J of R such that $V(I) \cap V(J) = \emptyset$ and $V(I) \cup V(J) = \text{Spec}(R)$. It can be seen that $I + J = R$ and $IJ \subseteq \mathcal{P}(R)$. So that there exist $a \in I$ and $1-a \in J$ such that $a(1-a) \in \mathcal{P}(R)$. Hence $\overline{a}$ is an idempotent in $\overline{R}$. Since $a \in I, V(I) \subseteq V(a)$. Let $P \in V(a)$. If $1-a \in P$, then $1 \in P$, a contradiction. So that $1-a \notin P$, but $1-a \in J$. This shows that $P \notin V(J)$ and consequently $P \in V(I)$. Therefore $V(I) = V(a)$. Thus clop(Spec(R)) $\subseteq \{V(x) \,|\, \overline{x}$ is an idempotent in $\overline{R}\}$. On the other hand, let $\bar{a}$ be an idempotent in $\overline{R}$. It is enough to prove that the complement of $V(a)$ is closed, i.e., $U(a)$ is closed. Since $U(a) = V(1-a)$, by Lemma 3.4 (i), $U(a)$ is closed. Thus we get clop(Spec(R)) $= \{V(x) \,|\, \bar{x}$ an idempotent in $\overline{R}\}$. Similarly, we can prove that clop(Spec(R)) $= \{V(\bar{x}) \,|\, \bar{x}$ is an idempotent in $\overline{R}\}$ by using Lemma

3.4(ii) and the fact that $\mathcal{P}(\overline{R}) = \bar{0}$, where $\bar{0} = \mathcal{P}(R)$, since R is 2-primal.

(ii) Let $V(I) \in \text{clop}(\text{Spec}(R))$, where I is an ideal of R. By the similar argument used in part (i), we get an idempotent $\bar{a}$ in $\overline{R}$ such that $a \in I$ and an ideal J of R such that $V(I) \cap V(J) = \emptyset, V(I) \cup V(J) = \text{Spec}(R), 1-a \in J$. Since idempotents lift modulo $\mathcal{P}(R)$, there exists an idempotent e of R such that $\bar{a} = \bar{e}$. Our claim is that $V(I) = V(e)$. Let $P \in V(e)$, then $a \in P$, because $\bar{a} = \bar{e}$. Hence $1 - a \notin P$, but $1 - a \in J$. So that $P \notin V(J)$ and consequently $P \in V(I)$. Clearly $V(I) \subseteq V(e)$ because $a \in I$. Thus

$$\text{clop}(\text{Spec}(R)) \subseteq \{V(e) \,|\, e \text{ is an idempotent in } R\}.$$

On the other hand, let e be an idempotent in R. It is enough to prove that the complement of $V(e)$ is closed, i.e., $U(e)$ is closed. Since $U(e) = V(1-e)$, by Lemma 3.4 (ii), $U(e)$ is closed. Thus clop(Spec(R))= $\{V(e) \,|\, e$ is an idempotent in $R\}$. ◻

Theorem 3.6. *Let R be a 2-primal ring. If R is π-regular, then* Spec(R) *is zero-dimensional.*

Proof. Assume that R is π-regular. Let $U(I)$ be any open set in Spec(R). For any $a \in I$, there exist a positive integer n and $b \in R$ such that $a^n = a^n b a^n$ because of π-regularness. Take $e = a^n b$, then e is an idempotent of R. We claim that $U(a) = U(e)$. Let $P \in U(a)$. Then $a \notin P$. Since R is π-regular, R is right weakly π-regular and hence $R/\mathcal{P}(R)$ is right weakly π-regular. So every prime ideal of R is maximal by [7, Lemma 6]. Hence every prime ideal of R is minimal by [3, Proposition 3.6]. From this we obtain every prime ideal of R is completely prime by [12, Proposition 1.11]. So that $a^n \notin P$. If $e \in P$, then $a^n b \in P$ and hence $b \in P$, which shows that $a^n = a^n b a^n \in P$, a contradiction and consequently $e \notin P$, $P \in U(e)$. Therefore $U(a) \subseteq U(e)$. It is clear that $U(e) \subseteq U(a)$. Thus $U(a) = U(e)$. Since $U(e) = V(1-e)$, by Lemma 3.4(ii), $U(a)$ is a clopen set. Again since $U(I) = \bigcup_{a \in I} U(a)$, the result follows. ◻

Lemma 3.7. *Let a and b be elements of a ring R such that $U(a) \subseteq U(b)$. Then there exists a positive integer n such that $a^n \in RbR$. In particular, if e is an idempotent with $V(e) \subseteq V(b)$, then $e \in RbR$.*

Proof. Suppose that $a^n \notin RbR$ for all n. Let $S = \{a, a^2, \ldots\}$. Then S is an m-system which contains a and does not intersect RbR. So that $a \notin \mathcal{P}(RbR)$. Since $\mathcal{P}(RbR)$ equals the intersection of all prime ideals which contain RbR, there exists $P \in \operatorname{Spec}(R)$ such that $a \notin P$ and $RbR \subseteq P$. Hence $P \in U(a)$ and $P \notin U(b)$, a contradiction to hypothesis. Therefore $a^n \in RbR$ for some n.

Now suppose that $e \notin RbR$, then $e^n \notin RbR$ for all n, since e is an idempotent of R. Hence the result follows from the above. □

The following is Lemma 2.5 of Lu et al. [8].

Lemma 3.8. *For a space X, the following statements are equivalent:*

(i) *The space X is strongly zero-dimensional;*

(ii) *Any two disjoint closed sets are separated by clopen sets, i.e., if A, B are disjoint closed sets, then there exist disjoint clopen sets, C_1, C_2 such that $A \subseteq C_1$ and $B \subseteq C_2$;*

(iii) *If U_1, U_2 are open sets covering X, then there exist clopen sets C_1, C_2 such that $C_i \subseteq U_i, i = 1, 2$, $C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = X$.*

Proof. Straight forward. □

Lemma 3.9. *Let X and Y be two spaces and $f : X \to Y$ a homeomorphic function. If X is strongly zero-dimensional, then Y is strongly zero-dimensional.*

Proof. Assume that X is strongly zero-dimensional. Let U_1 and U_2 be two open sets which cover Y. Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open sets which cover X. Since X is strongly zero-dimensional, there exist clopen sets C_1 and C_2 such that $C_1 \subseteq f^{-1}(U_1)$, $C_2 \subseteq f^{-1}(U_2), C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = X$. Hence it can be easily verified that $f(C_1) \subseteq U_1$, $f(C_2) \subseteq U_2$, $f(C_1) \cap f(C_2) = \emptyset$ and $f(C_1) \cup f(C_2) = Y$. It is enough to prove that $f(C_i)(i = 1, 2)$ is clopen. Since C_1, C_2 are open and f is an open map, $f(C_1), f(C_2)$ are open. Again since $f(C_1)^C = f(C_2), f(C_1)$ is closed, where $f(C_1)^C$ is the complement of $f(C_1)$. Thus $f(C_1)$ is clopen. Similarly, we can prove that $f(C_2)$ is clopen. Thus, by Lemma 3.8 (iii), Y is strongly zero-dimensional. □

The main result is now ready to be obtained.

Theorem 3.10. *Let R be a 2-primal ring. Then the following statements are equivalent:*

(i) $\mathrm{Spec}(R)$ *is strongly zero-dimensional;*

(ii) $\overline{R}$ *is a weakly exchange ring;*

(iii) $\mathrm{Spec}(\overline{R})$ *is strongly zero-dimensional.*

Proof. (i) implies (ii):

Assume that $\mathrm{Spec}(R)$ is strongly zero-dimensional and a is any element of R. Since $U(a) \cup U(1-a) = \mathrm{Spec}(R)$, there exist clopen sets C_1, C_2 such that $C_1 \subseteq U(1-a)$ and $C_2 \subseteq U(a), C_1 \cap C_2 = \emptyset, C_1 \cup C_2 = \mathrm{Spec}(R)$ by Lemma 3.8 (iii). Hence there exist idempotent $\bar{e}_1$ and $\bar{e}_2$ in $\bar{R}$ such that $C_1 = V(e_1)$ and $C_2 = V(e_2)$ by Lemma 3.5 (i). By Lemma 3.4 (i), we obtain $V(e_1) = U(1-e_1)$. Again since $V(e_2) = V(e_1)^C, V(e_2) = U(e_1)$. Hence $U(1-e_1) \subseteq U(1-a)$ and $U(e_1) \subseteq U(a)$. From Lemma 3.7, we have $e_1^n \in RaR$ and $(1-e_1)^m \in R(1-a)R$ for some positive integers n and m. Hence $\bar{e}_1^n \in \bar{R}\bar{a}\bar{R}$ and $(\bar{1}-\bar{e}_1)^n \in \bar{R}(\bar{1}-\bar{a})\bar{R}$. Since $\bar{e}_1$ is an idempotent in $\bar{R}$, $\bar{e}_1 \in \bar{R}\bar{a}\bar{R}$ and $\bar{1}-\bar{e}_1 \in \bar{R}(\bar{1}-\bar{a})\bar{R}$. Thus $\bar{R}$ is a weakly exchange ring.

(ii) implies (iii):
Assume that $\bar{R}$ is a weakly exchange ring. Let $\bar{I}$ and $\bar{J}$ be two ideals in $\bar{R}$ such that $U(\bar{I})U(\bar{J}) = \mathrm{Spec}(\bar{R})$. Observe that $\bar{I} + \bar{J} = \bar{R}$. So there exists $\bar{a} \in \bar{I}$ such that $\bar{1} - \bar{a} \in \bar{J}$. Since $\bar{R}$ is a weakly exchange ring, there exists an idempotent $\bar{x} \in \bar{R}$ such that $\bar{x} \in \bar{R}\bar{a}\bar{R}$ and $\bar{1} - \bar{x} \in R(\bar{1} - \bar{a})\bar{R}$.

It is clear that $U(\bar{x}) \subseteq U(\bar{I}), U(\bar{1}-\bar{x}) \subseteq U(\bar{J})$ $, U(\bar{x}) \cup U(\bar{1}-\bar{x}) =$ $\mathrm{Spec}(\bar{R})$, and by Lemma 3.3, $U(\bar{x}) \cap U(\bar{1}-\bar{x}) = \emptyset$. Since $U(\bar{x}) = V(\bar{1}-\bar{x})$ and $U(\bar{1}-\bar{x}) = V(\bar{x}), U(\bar{x})$ and $U(\bar{1}-\bar{x})$ are clopen sets of $\bar{R}$ by Lemma 3.5 (i). Thus, by Lemma 3.8 (iii), $\mathrm{Spec}(R)$ is strongly zero-dimensional .

(iii) implies (i):

Since $\mathrm{Spec}(R)$ is homeomorphic to $\mathrm{Spec}(\bar{R})$, by Lemma 3.9, $\mathrm{Spec}(R)$ is strongly zero-dimensional. □

Lemma 3.11. *Let X be compact T_1-space. Then X is strongly zero-dimensional space if and only if X is zero-dimensional.*

Proof. Any strongly zero-dimensional space with T_1 is always zero-dimensional and the converse holds for a compact T_1-space by [6, Theorem 16.16]. $\square$

Although the proof of (i) $\Leftrightarrow$ (ii) part in the following theorem is almost similar to that of Theorem 3.10, we have given to avoid difficulties. Note that $\mathrm{Max}(R) \subseteq \mathrm{Spec}(R)$, so we may assume that $\mathrm{Max}(R)$ is a subspace of $\mathrm{Spec}(R)$.

Theorem 3.12. *Let R be a 2-primal ring. Then the following statements are equivalent:*

(i) *R is a weakly exchange ring;*

(ii) $\mathrm{Spec}(R)$ *is strongly zero-dimensional;*

(iii) *R is a pm ring and* $\mathrm{Max}(R)$ *is zero-dimensional;*

(iv) *R is a pm ring and* $\mathrm{Max}(R)$ *is strongly zero-dimensional.*

Proof. (i) $\Leftrightarrow$ (ii):
Assume that $\mathrm{Spec}(R)$ is strongly zero dimensional and a is any element of R. Since $U(a) \cup U(1-a) = \mathrm{Spec}(R)$, there exists clopen sets C_1, C_2 such that $C_1 \subseteq U(1-a), C_2 \subseteq U(a), C_1 \cap C_2 = \emptyset$ and $C_1 \cup C_2 = \mathrm{Spec}(R)$ by Lemma 3.8 (iii). Hence there exist idempotents e_1 and e_2 such that $C_1 = V(e_1)$ and $C_2 = V(e_2)$ by Lemma 3.5 (ii). As in the proof of Theorem 3.10, we get $e_1^n \in RaR$ and $(1-e_1)^m \in R(1-a)R$. Since e_1 is an idempotent of $R, e_1 \in RaR$ and $1-e_1 \in R(1-a)R$. Thus R is a weakly exchange ring.
Conversly, assume that R is a weakly exchange ring. Let I, J be ideals such that $U(I) \cup U(J) = \mathrm{Spec}(R)$. Observing that $I + J = R$, so there exists an idempotent e of R such that $e \in RaR$ and $1-e \in R(1-a)R$. It is clear that $U(e) \subseteq U(I), U(1-e) \subseteq U(J), U(e) \cup U(1-e) = \mathrm{Spec}(R)$ and by Lemma 3.3, $U(e) \cap U(1-e) = \emptyset$. Thus $\mathrm{Spec}(R)$ is strongly zero-dimensional by Lemma 3.8 (iii).
(iii) $\Leftrightarrow$ (iv):
Since $\mathrm{Max}(R)$ is a compact T_1-space, the results follow from Lemma 3.11.

(ii) $\Leftrightarrow$ (iv):

If $\mathrm{Spec}(R)$ is strongly zero-dimensional, then $\mathrm{Spec}(R)$ is normal by Lemma 3.8 (ii). Hence R is *pm* and also $\mathrm{Max}(R)$ is a continuous retract of $\mathrm{Spec}(R)$ by [13, Theorem 2.3]. Therefore the results follow from the fact that the function $\mu : \mathrm{Spec}(R) \rightarrow \mathrm{Max}(R)$ given by sending each prime ideal

P to the unique maximal ideal containing it is a continuous closed map [5, Theorem 1.2]. $\square$

Since the strongly 2-primal rings are 2-primal, we have the following corollary.

Corollary 3.13. *Let R be a strongly 2-primal ring. If* $\mathrm{Spec}(R)$ *is zero-dimensional, then R is a weakly exchange ring.*

Proof. Assume that $\mathrm{Spec}(R)$ is zero-dimensional and a is any element of R. Then $U(a)$ is the union of some clopen sets $\{U(I_\lambda)/\lambda \in \Lambda\}$, where each I_λ is an ideal. Let $I = \sum I_\lambda$. Then $a \in \mathcal{P}(I)$, for otherwise, there is a prime ideal P containing I and $a \notin P$, which is impossible. So that $a^n \in I$ for some positive integer n. Let $\lambda_1, \lambda_2, \cdots, \lambda_k \in \Lambda$ such that $a^n \in I_{\lambda_1} + I_{\lambda_2} + \cdots + I_{\lambda_k}$, then $U(a^n) \subseteq \bigcup\limits_{i=1}^{k} U(I_{\lambda_i}) \subseteq U(a)$. Since R is strongly 2-primal, every prime ideal of R is completely prime [12, Proposition 1.13], and so for any $P \in \mathrm{Spec}(R)$ such that $P \in U(a)$ implies $P \in U(a^n)$. Therefore $U(a) = \bigcup\limits_{i=1}^{k} U(I_{\lambda_i})$. Thus $U(a)$ is a clopen set for all $a \in R$. Hence $\mathrm{Spec}(R)$ is a T_1-space by [12, Theorem 4.2]. But always $\mathrm{Spec}(R)$ is compact. This shows that $\mathrm{Spec}(R)$ is strongly zero-dimensional by Lemma 3.11. Thus R is a weakly exchange ring by Theorem 3.12. $\square$

Corollary 3.14. *Let R be a strongly 2-primal ring. Then R is a weakly exchange ring if and only if idempotents lift modulo I for any radical ideal I of R.*

Proof. Assume that R is a weakly exchange ring. Then idempotents in R/I can be lifted to every left ideal I of R by [10, Corollary 1.3]. In particular, idempotents in R/I can be lifted to R for every radical ideal I of R.
Conversely, let A_1 and A_2 be two disjoint closed sets in $\mathrm{Spec}(R)$. Take $I_1 = \bigcap\limits_{P \in A_1} P$ and $I_2 = \bigcap\limits_{Q \in A_2} Q$. Then I_1, I_2 are radical ideals, because R is strongly 2-primal. Since $A_1 \cap A_2 = \emptyset, I_1 + I_2 = R$. Choose $a \in I_1$ and $b \in I_2$ such that $a + b = 1$, then $ab = a(1-a) \in I_1 \cap I_2$. So $\bar{a}$ is an idempotent in $R/(I_1 \cap I_2)$. Since $I_1 \cap I_2$ is a radical ideal, there exists an idempotent $e \in R$ such that $e - a \in I_1 \cap I_2$ by hypothesis. Since $a \in I_1, e \in I_1$ and again since $b \in I_2, 1 - e \in I_2$. Therefore $V(I_1) \subseteq V(e)$ and $V(I_2) \subseteq V(1-e)$. Since

$V(e)$ and $V(1-e)$ are clopen sets, Spec(R) is strongly zero-dimensional by Lemma 3.8 (ii). Thus R is a weakly exchange ring by Theorem 3.12. $\square$

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