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ON HYPERELLIPTICITY DEGREE OF BISYMMETRIC RIEMANN SURFACES ADMITTING A FIXED POINT FREE SYMMETRY

BY

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Abstract

We study pairs of commuting symmetries of a Riemann surface of genus $g \geq 2$, assuming that one of them is fixed point free. We find necessary and sufficient conditions for an integer p to be the degree of hyperellipticity of their product, being given the number of ovals and separabilities of the symmetries. In the last part of the paper, using well known formula on the number m of points fixed by the conformal involution of a Riemann surface, we find all possible values of m that can be attained for the product of our symmetries.

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$. By a *symmetry* of X we mean an antiholomorphic involution σ of X. By the classical result of Harnack, the set of fixed points of σ consists of at most $g+1$ disjoint simple closed curves, which are called *ovals*. If σ has $g+1-q$ ovals then we shall call it an $(M-q)$-*symmetry*, according to Natanzon's terminology from [9]. Furthermore, σ is called *separating* or *non-separating* if $X \setminus \mathrm{Fix}(\sigma)$ has two

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or one connected component respectively. The surface X is *p-hyperelliptic* if X admits a conformal involution ρ such that its orbit space X/ρ has genus p. Such an involution is called a *p-hyperelliptic involution*.

The study of symmetries of Riemann surfaces is important due to the categorical equivalence under which a compact, connected Riemann surface X corresponds to a smooth, complex, projective and irreducible algebraic curve $\mathcal{C}_X$. Furthermore, a Riemann surface X admits a symmetry σ if and only if the corresponding curve $\mathcal{C}_X$ has a real form $\mathcal{C}_X(\sigma)$ and two such symmetries give rise to the real forms non-isomorphic over the reals $\mathbb{R}$, if and only if they are not conjugate in the group $\mathrm{Aut}^{\pm}(X)$ of all, including antiholomorphic, automorphisms of X. Finally, the set $\mathrm{Fix}(\sigma)$ of points fixed by σ is homeomorphic to a smooth projective model of the corresponding real form $\mathcal{C}_X(\sigma)$ and in this paper we focus our attention on curves having two real forms one of which has no $\mathbb{R}$-rational points. The latter are known in the literature as the *purely imaginary curves* and they correspond to fixed point free symmetries of Riemann surfaces.

The aim of this paper is to solve some of the problems brought up by Bujalance and Costa in [2], which were also studied in [5, 8], for the case of at least one of the symmetries being fixed point free and fill in this way some of the gaps existing in the literature of the topic. In [8] we studied the necessary and sufficient conditions for an integer p to be the degree of hyperellipticity of the product of two $(M-q)$- and $(M-q')$-symmetries, with given separabilities, on a Riemann surface of genus $g \geq q+q'+1$. Here we assume that our symmetries commute and that $q' = g+1$, which means that one of the symmetries, say σ, is a fixed point free symmetry, hence in particular σ is a non-separating symmetry. Taking into account separability of the $(M-q)$-symmetry τ, we give necessary and sufficient conditions for an integer p to be the degree of hyperellipticity of $\sigma\tau$. Later, we remind the well known formula on the number m of points fixed by a p-hyperelliptic involution on a Riemann surface of genus g and, as a byproduct of our studies, we obtain necessary and sufficient conditions for an integer m to be the number of points fixed by $\sigma\tau$.

For our considerations we recall some of the results given by Izquierdo and Singerman in [6] and following their terminology, we shall say that a Riemann surface of genus g admits the pair $(0,t)_2$, if it admits a pair of commuting symmetries σ, τ where σ is fixed point free and τ has t ovals.

2. Preliminaries

We shall prove our results using theory of non-euclidean crystallographic groups (*NEC groups* in short), by which we mean discrete and cocompact subgroups of the group $\mathcal{G}$ of all, including orientation reversing, isometries of the hyperbolic plane $\mathcal{H}$. The algebraic structure of such a group Λ is coded in the signature:

$$s(\Lambda) = (h; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}), \qquad (1)$$

where the brackets $(n_{i1}, \ldots, n_{is_i})$ are called the *period cycles*, the integers n_{ij} are the *link periods*, m_i - *the proper periods* and finally h is the *orbit genus* of Λ.

A group Λ with signature (1) has the presentation with the following generators, called *canonical generators* :

$x_1, \ldots, x_r, e_i,\ c_{ij}, 1 \leq i \leq k, 0 \leq j \leq s_i$ and $a_1, b_1, \ldots, a_h, b_h$ if the sign is $+$ or $d_1, \ldots, d_h$ otherwise,

and relators:

$x_i^{m_i}, i = 1, \ldots, r,\ c_{ij-1}^2, c_{ij}^2, (c_{ij-1}c_{ij})^{n_{ij}}, c_{i0}e_i^{-1}c_{is_i}e_i,\ i = 1, \ldots, k, j = 1, \ldots, s_i$ and

$$x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_h b_h a_h^{-1} b_h^{-1} \text{ or } x_1 \ldots x_r e_1 \ldots e_k d_1^2 \ldots d_h^2,$$

according to whether the sign is $+$ or $-$. The elements x_i are elliptic transformations, a_i, b_i hyperbolic translations, d_i glide reflections and c_{ij} hyperbolic reflections. Reflections c_{ij-1}, c_{ij} are said to be *consecutive*. Every element of finite order in Λ is conjugate either to a canonical reflection or to a power of some canonical elliptic element x_i, or to a power of the product of two consecutive canonical reflections.

Now an abstract group with such a presentation can be realized as an NEC group Λ if and only if the value

$$2\pi\Big(\varepsilon h + k - 2 + \sum_{i=1}^{r}\Big(1 - \frac{1}{m_i}\Big) + \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{s_i}\Big(1 - \frac{1}{n_{ij}}\Big)\Big),$$

where $\varepsilon = 2$ or 1 according to the sign being $+$ or $-$, is positive. This value turns out to be the hyperbolic area $\mu(\Lambda)$ of an arbitrary fundamental region for such a group and we have the following Hurwitz-Riemann formula

$$[\Lambda : \Lambda'] = \mu(\Lambda')/\mu(\Lambda)$$

for a subgroup Λ' of finite index in an NEC group Λ.

Now NEC groups having no orientation reversing elements are classical Fuchsian groups. They have signatures $(g; +; [m_1, \ldots, m_r]; \{-\})$, which shall be abbreviated as $(g; m_1, \ldots, m_r)$. Given an NEC group Λ, the subgroup Λ^+ of Λ consisting of the orientation preserving elements is called the *canonical Fuchsian subgroup of* Λ and for a group with signature (1) it has, by [11], signature

$$(\varepsilon h + k - 1; m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots, n_{ks_k}). \tag{2}$$

A torsion free Fuchsian group Γ is called a *surface group* and it has signature $(g; -)$. In such a case $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g and conversely, each compact Riemann surface can be represented as such an orbit space for some Γ. Furthermore, given a Riemann surface so represented, a finite group G is a group of automorphisms of X if and only if $G = \Lambda/\Gamma$ for some NEC group Λ.

The following result from [4] is crucial for the paper

Theorem 2.1. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with the group G of all automorphisms of X, let $G = \Lambda/\Gamma$ for some NEC group Λ and let $\theta : \Lambda \to G$ be the canonical epimorphism. Then the number of ovals of a symmetry τ of X equals*

$$\sum [C(G, \theta(c_i)) \,:\, \theta(C(\Lambda, c_i))],$$

where the sum is taken over a set of representatives of all conjugacy classes of canonical reflections whose images under θ are conjugate to τ.

For a symmetry τ we shall denote by $\|\tau\|$ the number of its ovals. The index $w_i = [C(G, \theta(c_i)) \,:\, \theta(C(\Lambda, c_i))]$ will be called a *contribution* of c_i to $\|\theta(c_i)\|$.

3. Existence of Given Pairs of Symmetries

Let X be a Riemann surface of genus $g \geq 2$, having a fixed point free symmetry σ and an $(M-q)$-symmetry τ and let us assume that these symmetries commute. Now $X = \mathcal{H}/\Gamma$, σ and τ are the images of orientation reversing elements of Λ and $G = \mathrm{Z}_2 \oplus \mathrm{Z}_2 = \langle \sigma, \tau \rangle = \Lambda/\Gamma$ for some Fuchsian surface group Γ being a normal subgroup of an NEC-group Λ with signature

$$(h; \pm; [2, \overset{r}{\ldots}, 2]; \{(-)^k\}). \tag{3}$$

Let $\theta : \Lambda \to G = \Lambda/\Gamma$ denote the canonical projection and let Γ_σ and Γ_τ be the inverse images of the groups generated by σ and τ, respectively. These are the subgroups of Λ of index 2. By (2) they have neither proper periods nor link periods and the separability of τ depends on the sign in signature of Γ_τ; for the sign $+$, symmetry τ is separating and for the sign $-$ it is a non-separating one. Furthermore, the number of ovals of τ equals the number of empty period cycles in Γ_τ. The product $\sigma\tau$ is a p-hyperelliptic involution, where p is the orbit genus of Λ^+. In this part of the paper we shall use the Lemma below, which follows easily from the Theorem 2.1

Lemma 3.1. *Let $G = \mathrm{Z}_2^2 = \Lambda/\Gamma$ be the group of automorphisms of a Riemann surface $X = \mathcal{H}/\Gamma$ generated by two symmetries σ, τ of which σ is fixed point free and let C be an empty period cycle of Λ. Then reflection c_i corresponding to C contributes to $\|\tau\|$ with 1 oval if $\theta(e_i) \neq 1$ and with 2 ovals otherwise.*

Proof. Let $\theta : \Lambda \to G$ denote the canonical epimorphism. The centralizer of any element in G has order 4. Since c_i and e_i belong to $C(\Lambda, c_i)$, we have that $w_i = 1$ if $\theta(e_i) \neq 1$ and $w_i = 2$ otherwise. $\square$

First we shall prove the following easy fact concerning existence of pairs $(0, g+1-q)_2$ on a Riemann surface of genus g. The result below can be found in [6], however there is an error in the Case 2 of the proof (see p. 13 of [6]), which we fix here.

Proposition 3.2.(see also Theorem 3.2, 4.8 and 4.9 in [6]) *There exists a Riemann surface of genus g admitting the pair $(0, g+1-q)_2$ if and only if q is even.*

Proof. Let us assume first that X is a Riemann surface of genus g, admitting a pair $(0, g+1-q)_2$. Now $\mathrm{Aut}(X) = G = \langle \sigma, \tau \rangle = Z_2^2 = \Lambda/\Gamma$ for some surface Fuchsian group Γ and an NEC group Λ with signature (3). Also, the canonical epimorphism $\theta : \Lambda \to G$ takes all the canonical reflections to τ and the canonical elliptic generators to $\sigma\tau$. Observe that an integer r has the same parity as the number k' of period cycles C_i for which $\theta(e_i) \neq 1$, as the relation $\theta(x_1 \dots x_r e_1 \dots e_k) = 1$ holds in G. Now, by Lemma 3.1, each of the empty period cycles C_i contributes with 1 or 2 ovals to symmetry τ, depending on if $\theta(e_i) \neq 1$ or not. Hence, the number of ovals $g+1-q$ has the same parity as k'. Moreover, by the Hurwitz-Riemann formula we have

$$\begin{aligned} \pi(g-1) &= \mu(\Lambda) \\ &= 2\pi(\varepsilon h - 2 + r/2 + k) \end{aligned}$$

and so $g = 2\varepsilon h + r - 3 + 2k$. This means that g has parity different than r. Hence g and $g+1-q$ have different parity and so q must be even.

The proof of sufficient condition can be found in [6], but we remind it here for the completeness of the proof. Assume first that g is odd and consider an NEC group Λ with signature

$$(1; -; [2, \overset{q}{\dots}, 2]; \{(-)^{(g+1-q)/2}\}).$$

Define an epimorphism $\theta : \Lambda \to G = Z_2 \oplus Z_2 = \langle \sigma, \tau \rangle$ by taking the canonical elliptic generators to $\sigma\tau$, canonical reflections to τ and generators e_i to 1. Then by the Hurwitz-Riemann formula for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ is a Riemann surface of genus g admitting a pair $(0, g+1-q)_2$ by Lemma 3.1. This also fixes an error in Case 2 on page 13 in [6].

Let now g be even. Consider an NEC group Λ with signature

$$(0; +; [2, \overset{q+2}{\dots}, 2]; \{(-)^{(g+2-q)/2}\})$$

and an epimorphism $\theta : \Lambda \to G = Z_2 \oplus Z_2 = \langle \sigma, \tau \rangle$ given by $\theta(e_1) = \theta(x_i) = \sigma\tau$ for all i, $\theta(c_i) = \tau$ for all i and $\theta(e_i) = 1$ for $i > 1$. Again by the Hurwitz-Riemann formula for $\Gamma = \ker\theta$, $X = \mathcal{H}/\Gamma$ is a Riemann surface of genus g admitting a pair $(0, g+1-q)_2$ by Lemma 3.1. $\square$

Observe that for $q = g+1$ from the above Proposition it follows that

Corollary 3.3. *A Riemann surface of genus g admits the pair of fixed point free commuting symmetries if and only if g is odd.*

4. Degree of Hyperellipticity of the Product

Recall, that by the degree of hyperellipticity of an involution ρ we understand the genus of the orbit space X/ρ. In [8] we gave the necessary and sufficient conditions for a Riemann surface of genus g to admit a pair of commuting symmetries with given numbers of ovals, separabilities and the degree of hyperellipticity of the product, assuming that both symmetries have ovals. Here we shall deal the case when at least one of the symmetries in the pair is a fixed point free symmetry.

The result below, which follows easily from [3], is crucial for the next part of the paper

Let Λ' be a normal subgroup of an NEC-group Λ. A canonical generator of Λ is *proper* (with respect to Λ') if it does not belong to Λ'. The elements of Λ expressable as a composition of proper generators of Λ' are the *words* of Λ (with respect to Λ'). With these notations we have.

Lemma 4.1.(c.f. Theorem 2.1.3) *Let us suppose that Λ' is a normal subgroup of Λ and $[\Lambda : \Lambda']$ is even. If Λ has sign $+$, then Λ' has sign $+$ if and only if no orientation reversing word belongs to Λ'. If Λ has the sign $-$, then Λ' has the sign $-$ if and only if either a glide reflection of the canonical generators of Λ or an orientation reversing word belongs to Λ'.*

Now we shall prove the following Lemma, concerning separability of the symmetry, which is the analogue of Lemma 3.2 from [8] for the case of one symmetry in the pair being fixed point free.

Lemma 4.2. *Let $X = \mathcal{H}/\Gamma$ be a Riemann surface of genus g, admitting a pair $(0, g+1-q)_2$. Let $G = \Lambda/\Gamma$ be the group generated by these symmetries and let $\theta : \Lambda \to G$ denote the canonical projection. Then, the following conditions hold:*

(1) *Symmetry τ is separating if and only if one of the following holds*
  - (a) $\mathrm{sgn}(\Lambda) = -$, $r = 0$, $\theta(e_i) = 1, \theta(d_j) = \sigma$ *for all i and j;*
  - (b) $\mathrm{sgn}(\Lambda) = +$, *and* $r > 0$ *or* $\theta(e_i) \neq 1$, *or* $\theta(a_j) \neq 1$, *or* $\theta(b_j) \neq 1$ *for some i or j.*

(2) *Symmetry τ is non-separating if and only if one of following holds*
   (a) $\mathrm{sgn}(\Lambda) = -,\ h > 1,\ \theta(d_i) \neq \theta(d_j)$ *for some* $i \neq j$;
   (b) $\mathrm{sgn}(\Lambda) = -,\ h \geq 1,\ \theta(d_i) = \theta(d_j) = \sigma$ *for all* i, j *and* $r > 0$;
   (c) $\mathrm{sgn}(\Lambda) = -,\ h \geq 1,\ \theta(d_i) = \theta(d_j) = \sigma$ *for all* i, j *and* $\theta(e_i) \neq 1$ *for some* i.

Proof. Let σ and τ be two commuting symmetries of a Riemann surface X and let λ and λ' be two orientation reversing elements of an NEC group Λ with signature (3) such that $\theta(\lambda) = \sigma$ and $\theta(\lambda') = \tau$.

Assume first that τ is separating. It is easy to see that if $\mathrm{sgn}(\Lambda) = -, h > 0$ then it must be $\theta(d_1) = \ldots = \theta(d_h) = \sigma,\ r = 0,\ \theta(e_i) = 1$ for all i. Indeed, otherwise we would have either a canonical glide reflection or an orientation reversing word λx or λe_i in Γ_τ and so τ would be non-separating. Now if $\mathrm{sgn}(\Lambda) = +$, there are no canonical glide reflections in Λ, hence there must be a canonical elliptic generator x, generator e_i, a_i or b_i for some i with nontrivial image under θ. Indeed, otherwise all the orientation reversing elements in Λ would be mapped to λ'.

Conversely, if any of conditions (a),(b) holds, then τ must be separating because no orientation reversing word belongs to Γ_τ.

Let now τ be non-separating. If none of the conditions required in theorem holds we have two possible cases: $\{\mathrm{sgn}(\Lambda) = +,\ r > 0$ or $\theta(e_i) \neq 1$ for some $i\}$ or $\{\mathrm{sgn}(\Lambda) = -,\ r = 0,\ \theta(e_i) = 1$ for all i and $\theta(d_i) = \theta(d_j)$ for all $i, j\}$. In both cases τ would be separating by the first part of the proof.

Conversely, if condition (a) holds, then τ must be non-separating as one of the canonical glide reflections belongs to Γ_τ. Similarly, if one of conditions (b), (c) holds, then τ must be non-separating as we have an orientation reversing word $d_1 x$ or $d_1 e_i$ in Γ_τ. $\square$

In the following Propositions we assume that σ and τ are commuting fixed point free and $(M - q)$- symmetries of a Riemann surface X of genus g. Observe that by Proposition 3.2 in such case q must be even. Let also $G = \langle \sigma, \tau \rangle = \Lambda/\Gamma$ for an NEC group Λ having signature (3) and $\sigma\tau$ be a p-hyperelliptic involution.

Proposition 4.3. *If τ is separating, then $q \leq g$, $(g - 1 - q)/2 \leq p \leq (g + 1)/2$. Furthermore, if g is even and $q = g$, then p is even.*

Proof. Obviously $q \leq g$ as τ has ovals. As the group Λ has signature of the form (3) and $p = \varepsilon h + k - 1$, by the Hurwitz-Riemann formula we get

$$\begin{aligned} \pi(g-1) &= \mu(\Lambda) \\ &= 2\pi(\varepsilon h - 2 + k + r/2) \\ &\geq 2\pi(p - 1 + r/2) \end{aligned}$$

which gives $p \leq (g+1)/2$ as $r \geq 0$. For the proof of the lower bound for p observe that by Lemma 3.1 we have $2k \geq g + 1 - q$. Hence $p = \varepsilon h + k - 1 \geq h + (g+1-q)/2 - 1 = h + (g-1-q)/2$. Observe now that if $q = g$, then τ has 1 oval and so $k = 1$ with $\theta(e_1) = \sigma\tau$ for the canonical epimorphism $\theta : \Lambda \to G$. Now, as τ is separating, by Lemma 4.2, we have that Λ has sign $+$. Therefore $p = 2h$ is even. □

Proposition 4.4. *If τ is non-separating, then $0 < q \leq g$ and $(g+1-q)/2 \leq p \leq (g+1)/2$.*

Proof. Clearly q must hold condition $0 < q \leq g$ as τ has ovals and is non-separating. The upper bound for p can be found in exactly the same way as in the previous Proposition. To obtain the lower bound observe that because τ is non-separating, from part (2) in Lemma 4.2 it follows that $\mathrm{sgn}(\Lambda) = -$, $h \geq 1$. As before, we get $p = h + k - 1 \geq (g+1-q)/2$ which proves the Proposition. □

The next Theorems show that every value of p, within the ranges given in the previous Propositions, can be attained. In order to simplify each theorem's proof we shall assume that an epimorphism $\theta : \Lambda \to G$ will be defined on canonical reflections $c_1, \ldots, c_k$ of Λ sending them to τ as by (3) there are no nonempty period cycles in Λ. Observe also that it must be $\theta(x_i) = \sigma\tau$ for arbitrary canonical elliptic generator x_i of Λ. Unless directly stated otherwise, we take $\theta(d_i) = \sigma$ for any canonical glide reflection and $\theta(a_i) = \theta(b_i) = 1$ for all the canonical hyperbolic generators.

Theorem 4.5. *Given $g \geq 2$, q and p such that $q \leq g$ is even and $(g-1-q)/2 \leq p \leq (g+1)/2$ with p even for $q = g$, there exists a Riemann surface of genus g having a fixed point free symmetry and a separating $(M-q)$-symmetry, which commute and whose product is a p-hyperelliptic involution.*

Proof. Let first g be odd, $p \equiv (g-1-q)/2 \bmod 2$ and consider an NEC-group Λ with signature

$$(h; +; [2, \overset{g+1-2p}{\ldots}, 2]; \{(-)^k\}), \tag{4}$$

where $k = (g+1-q)/2$, $h = (p-(g-1-q)/2)/2$. Then for $p < (g+1)/2$ we have $r > 0$ and an epimorphism $\theta : \Lambda \to G$ defined by $\theta(e_i) = 1$ for all i gives rise to a configuration of fixed point free symmetry and separating $(M-q)$-symmetry whose product is a p-hyperelliptic involution by Lemmata 3.1 and 4.2. Now if $p = (g+1)/2$ then $h > 0$ and we take $\theta(a_1) = \theta(b_1) = \sigma\tau$ and also in such case we obtain the configuration in question.

Let now g be odd and $p \not\equiv (g-1-q)/2 \bmod 2$ and consider an NEC-group Λ with signature (4) where $k = (g+3-q)/2$, $h = (p-(g+1-q)/2)/2$. Then an epimorphism $\theta : \Lambda \to G$ defined by $\theta(e_1) = \theta(e_2) = \sigma\tau$, $\theta(e_i) = 1$ for all $i > 2$ gives rise to a configuration of symmetries we looked for, by Lemmata 3.1 and 4.2.

Assume now that g is even and $p \equiv (g-q)/2 \bmod 2$. Again, take an NEC group Λ with signature (4) where $k = (g+2-q)/2$, $h = (p-(g-q)/2)/2$. Then an epimorphism $\theta : \Lambda \to G$ defined by $\theta(e_1) = \sigma\tau$, $\theta(e_i) = 1$ for all $i > 1$ as above by Lemmata 3.1 and 4.2 gives rise to a configuration of fixed point free symmetry and separating $(M-q)$-symmetry whose product is a p-hyperelliptic involution.

Let finally g be even and $p \not\equiv (g-q)/2 \bmod 2$ and let Λ be an NEC group with signature (4) where $k = (g+4-q)/2$, $h = (p-(g+2-q)/2)/2$. Then an epimorphism $\theta : \Lambda \to G$ defined by $\theta(e_i) = \sigma\tau$ for $i \leq 3$ and $\theta(e_i) = 1$ for all $i > 3$ again leads to a configuration of fixed point free symmetry and separating $(M-q)$-symmetry whose product is a p-hyperelliptic involution by Lemmata 3.1 and 4.2. □

Theorem 4.6. *Given $g \geq 2$, q and p such that $0 < q \leq g$ is even and $(g+1-q)/2 \leq p \leq (g+1)/2$, there exists a Riemann surface of genus g having a fixed point free symmetry and a non-separating $(M-q)$-symmetry, which commute and whose product is a p-hyperelliptic involution.*

Proof. Let first g be odd and consider an NEC group Λ with signature

$$(h; -; [2, \overset{g+1-2p}{\ldots}, 2]; \{(-)^k\}) \tag{5}$$

where $h = p - (g-1-q)/2$ and $k = (g+1-q)/2$. Now for $p < (g+1)/2$ we have $r > 0$ and an epimorphism $\theta : \Lambda \to G$ defined by $\theta(e_i) = 1$ for all i gives rise to a configuration of fixed point free symmetry σ and non-separating $(M-q)$-symmetry τ whose product is a p-hyperelliptic involution by Lemmata 3.1 and 4.2. For $p = (g+1)/2$ we have $h \geq 2$ as $q \geq 2$, hence we may take epimorphism θ to be defined in the same way on all generators except $\theta(d_1) = \tau$. Again, we obtain the configuration in question.

Let now g be even and take an NEC group Λ with signature (5) for $h = p - (g-q)/2$ and $k = (g+2-q)/2$. An epimorphism defined as $\theta(e_1) = \sigma\tau$ and $\theta(e_i) = 1$ for $i > 1$ as above gives rise to the configuration we looked for. □

Recall that the number m of points fixed by the p-hyperelliptic product $\sigma\tau$ equals $2g+2-4p$. Using this formula we easily obtain all possible values of m that might be realized as the number of points fixed by $\sigma\tau$.

Corollary 4.7. *If g is even, then $m \equiv 2 \bmod 4$, if g is odd then $m \equiv 0 \bmod 4$. Moreover any integer $\mu \leq 2g+2$ holding $\mu \equiv 2 \bmod 4$ for even g or $\mu \equiv 0 \bmod 4$ for g odd can be realized as the number of points fixed by the product of two commuting symmetries with one being fixed point fee.*

Proof. For the proof of this Corollary we shall use the equality $m = 2g+2-4p$ for p being the degree of hyperellipticity of $\sigma\tau$. If $g = 2a$ then we get $m = 4(a-p)+2$ whence for $g = 2a+1$ we get $m = 4(a-p+1)$.

Conversely, assume first that g is even and $\mu \equiv 2 \bmod 4$ that is $\mu = 2g+2-4a$ for some a. Consider an NEC-group Λ with signature

$$(a; -; [2, \overset{g+1-2a}{\ldots}, 2]; \{(-)\})$$

and an epimorphism $\theta : \Lambda \to G = \mathrm{Z}_2^2 = \langle \sigma, \tau \rangle$ which takes all the canonical glide reflections (if there are any) to σ, the canonical elliptic elements and the generator e to $\sigma\tau$ and the canonical reflection to τ. This gives rise to configuration of symmetries, whose product has μ fixed points.

Now if g is odd and $\mu = 2g+2-4a$ for some a, consider an NEC group with the same signature as before and an epimorphism defined as above except generator e, for which we take $\theta(e) = 1$. □

Observe, that in particular, on a Riemann surface of odd genus g it is possible to construct a configuration of commuting fixed point free symmetries, whose product does not have fixed points. This is made by taking a group Λ with signature $((g-1)/2+2; -; [-]; \{-\})$ and an epimorphism $\theta : \Lambda \to G = \langle \sigma, \tau \rangle$ which takes the canonical glide reflections alternatively to σ and τ.

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