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ACYCLIC LIST EDGE COLORING OF PLANAR GRAPHS

BY

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Abstract

A proper edge coloring of a graph is said to be *acyclic* if any cycle is colored with at least three colors. The *acyclic chromatic index*, denoted $a'(G)$, is the least number of colors required for an acyclic edge coloring of G. An *edge-list* L of a graph G is a mapping that assigns a finite set of positive integers to each edge of G. An acyclic edge coloring ϕ of G such that $\phi(e) \in L(e)$ for any $e \in E(G)$ is called an *acyclic L-edge coloring* of G. A graph G is said to be *acyclically k-edge choosable* if it has an acyclic L-edge coloring for any edge-list L that satisfies $|L(e)| \geqslant k$ for each edge e. The *acyclic list chromatic index* is the least integer k such that G is acyclically k-edge choosable. In [2, 3, 4, 7, 10, 11, 12], upper bounds for the acyclic chromatic index of several classes of planar graphs were obtained. In this paper, we generalize these results to the acyclic list chromatic index of planar graphs.

1. Introduction

Graphs considered in this paper are finite, without loops or multiple edges unless otherwise stated. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use $|G|$ and $\|G\|$ to denote the cardinalities of $V(G)$

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and $E(G)$, respectively. An *edge coloring* of G is an assignment of colors to the edges of G. In this paper, we always use some initial segment $[k] = \{1, 2, \ldots, k\}$ of positive integers to represent colors. An edge coloring is said to be *proper* if adjacent edges receive distinct colors. The least number of colors, denoted $\chi'(G)$, needed for a proper edge coloring of G is called the *chromatic index* of G. A proper edge coloring is said to be *acyclic* if any cycle is colored with at least 3 colors. The *acyclic chromatic index*, denoted $a'(G)$, is the least number of colors required for an acyclic edge coloring of G.

An *edge-list* L of G is a mapping that assigns a finite set of positive integers to each edge of G. Assume that f is a mapping from $E(G)$ to the set of nonnegative integers. An edge-list L is an *f-edge-list* if $|L(e)| \geqslant f(e)$ for each $e \in E(G)$. When f is the constant mapping $f_k(e) = k$ for every $e \in E(G)$, we also say that L is a k-edge-list. An acyclic edge coloring ϕ of G such that $\phi(e) \in L(e)$ for any $e \in E(G)$ is called an *acyclic L-edge coloring* of G. A graph G is said to be *acyclically f-edge choosable* if it has an acyclic L-edge coloring for any f-edge-list L. The *acyclic list chromatic index*, denoted $a'_{\text{list}}(G)$, is the least integer k such that G is acyclically f_k-edge choosable. We also say that G is acyclically k-edge choosable when it is acyclically f_k-edge choosable. Let $\Delta(G)$ denote the maximum degree of a vertex in G. Obviously, $\Delta(G) \leqslant \chi'(G) \leqslant a'(G) \leqslant a'_{\text{list}}(G)$.

We initiated the study of the list version of acyclic edge coloring in [8]. At the end of [8], the following conjecture was proposed.

Conjecture 1. *For any graph G, $a'_{\text{list}}(G) \leqslant \Delta(G) + 2$.*

This is the list version of the following outstanding conjecture about acyclic edge coloring independently given by Fiamčík [5] and Alon, Sudakov, and Zaks [1].

Conjecture 2. *For any graph G, $a'(G) \leqslant \Delta(G) + 2$.*

The organization of this paper is as follows. After this introduction section, the next section supplies auxiliary results that are needed to establish later results. Section 3 gives a general acyclic edge choosability result for a planar graph. In Sections 4 to 9, sufficient conditions on planar graphs are determined to achieve acyclic $(\Delta+13)$-edge choosability, acyclic $(\Delta+5)$-edge choosability, acyclic $(\Delta+3)$-edge choosability, acyclic $(\Delta+2)$-edge choosability, acyclic $(\Delta+1)$-edge choosability, and acyclic Δ-edge choosability.

2. Auxiliary Results

We have already established a sequence of useful lemmas for handling acyclic list edge coloring in [8]. Lemmas 3 to 10 in this paper are paraphrased from [8] without reproducing their proofs.

Let $d_G(v)$ denote the degree of a vertex v in the graph G. A vertex of degree k is called a *k-vertex* and a vertex of degree at most k is called a *k^--vertex*. A *leaf* is synonymous with a 1-vertex. For an edge $e = uv$ of the graph G, let $N_0(e)$ and $N_1(e)$ denote the sets $\{u, v\}$ and $\{x \mid xu \in E(G) \text{ or } xv \in E(G)\}$, respectively. For $i = 0$ and 1, let Δ_i^G be the mapping $\Delta_i^G(e) = \max\{d_G(x) \mid x \in N_i(e)\}$ for each $e \in E(G)$. We use Δ_i to denote Δ_i^G when no confusion arises.

Lemma 3. *If H is a subgraph of a graph G, then $a'_{\text{list}}(H) \leqslant a'_{\text{list}}(G)$.*

Lemma 4. *Assume that $G_1, G_2, \ldots, G_k$ are all the components of the graph G and f is a mapping from $E(G)$ to the nonnegative integers. Then G is acyclically f-edge choosable if and only if G_i is acyclically f-edge choosable for each i. In particular, $a'_{\text{list}}(G) = \max\{a'_{\text{list}}(G_1), a'_{\text{list}}(G_2), \ldots, a'_{\text{list}}(G_k)\}$.*

Lemma 5. *Let u be a leaf of the graph G such that the neighbor of u is u'. Suppose that L is an edge-list of G satisfying $|L(uu')| \geqslant d_G(u')$. If ϕ is an acyclic L-edge coloring of $G-u$, then ϕ can be extended to an acyclic L-edge coloring of G.*

Proposition 6. *If T is a tree, then T is acyclically Δ_0-edge choosable and $a'_{\text{list}}(T) = \Delta(T)$.*

Lemma 7. *Let G be a graph and w be a 2-vertex with neighbors v and x. Let L be an edge-list of G such that $|L(e)| \geqslant \Delta_0(e) + 1$ for each edge e containing w as an endpoint. Suppose that $G - w$ has an acyclic L-edge coloring ϕ. Then ϕ can be extended to an acyclic L-edge coloring of G if all of the following conditions hold.*

1. $d_G(v) + d_G(x) \leqslant |L(wx)| + 3$.
2. *If* $d_G(v) + d_G(x) = |L(wx)| + 3$, *then* v *and* x *are adjacent.*
3. *If* $d_G(v) + d_G(x) \geqslant |L(wx)| + 2$, *then* $d_G(u) + d_G(v) \leqslant |L(vw)| + 1$ *for some neighbor* $u \neq w$ *of* v.

Lemma 8. *Assume that* u *and* v *are two vertices of a graph* H *and* G *is obtained from* H *by adding a new path of length at least 4 joining* u *and* v *and* $\Delta(G) \geqslant 3$. *If* H *is acyclically* $\max\{\Delta_0, 3\}$*-edge choosable, then* G *is acyclically* $\max\{\Delta_0, 3\}$*-edge choosable, where* $\max\{\Delta_0, 3\}$ *denotes the mapping that takes the value* $\max\{\Delta_0(e), 3\}$ *on any edge* e *of* G.

Lemma 9. *Let* H *be a graph and* $uvwxy$ *be a path of* H *in which* v *and* x *are non-adjacent 2-vertices and* w *is their unique common neighbor. Let* G *be the graph obtained from* H *by adding a new edge joining* v *and* x. *Let* L *be an edge-list of* G *such that* $|L(e)| \geqslant \Delta_0(e) + 1$ *for* $e \in \{uv, vw, wx, xy\}$ *and* $|L(vx)| \geqslant \Delta_1(vx) + 1$. *Suppose that* H *has an acyclic* L*-edge coloring. Then* G *has an acyclic* L*-edge coloring if* $\max\{d_G(u), d_G(w), d_G(y)\} \leqslant 3$ *or at most one of* u *and* y *is of degree* $\Delta_1(vx)$ *in* G.

Lemma 10. *Let* H *be a graph with two non-adjacent vertices* v *and* x *of degree 2 such that each of them has a neighbor of degree at most 3. Let* G *be the graph obtained from* H *by adding a new edge joining* v *and* x. *Let* L *be an edge-list of* G *such that* $|L(e)| \geqslant \max\{\Delta_1(e) + 1, 5\}$ *for any edge* e *incident with* v *or* x. *If* H *has an acyclic* L*-edge coloring, so does* G.

Now we are going to establish further auxiliary lemmas for later use. We use the notation $C_\phi(v)$ to denote the set of colors assigned by a proper edge coloring ϕ to all the edges incident with v.

Lemma 11. *Let* G *be a graph and* u *be a* d*-vertex adjacent to one 2-vertex* u_1 *and* $(d-k-1)$ $(l+1)^-$*-vertices, for some* $1 \leqslant k \leqslant d-1$ *and* $l \geqslant 1$. *Let* L *be an edge-list of* G *such that* $|L(e)| \geqslant \Delta_0(e) + \max\{k, l\}$ *for each edge* e *containing* u_1 *as an endpoint. If* $G - u_1$ *has an acyclic* L*-edge coloring* ϕ, *then* ϕ *can be extended to an acyclic* L*-edge coloring of* G.

Proof. Assume that the neighbors of u are $u_1, \ldots, u_d$, where $u_2, \ldots, u_{d-k}$ have degree at most $l+1$. Let $w_1 \neq u$ be the second neighbor of u_1.

We may assume that $\phi(uu_i) = i$ for $2 \leqslant i \leqslant d$. We color u_1w_1 with some $j \in L(u_1w_1) - (C_\phi(w_1) \cup \{d-k+1, \ldots, d\})$. If $j \notin \{2, \ldots, d-k\}$, then we color uu_1 with some element in $L(uu_1) - (C_\phi(u) \cup \{j\})$. If $j \in \{2, \ldots, d-k\}$, then we color uu_1 with some element in $L(uu_1) - (C_\phi(u) \cup C_\phi(u_j))$. $\square$

We have the following by letting k be $d-1$ in the above lemma.

Corollary 12. *Let G be a graph and u_1 be a 2-vertex adjacent to a d-vertex for some $d \geqslant 1$. Let L be an edge-list of G such that $|L(e)| \geqslant \Delta_0(e) + d - 1$ for each edge e containing u_1 as an endpoint. If $G - u_1$ has an acyclic L-edge coloring ϕ, then ϕ can be extended to an acyclic L-edge coloring of G.*

Lemma 13. *Let G be a graph and v be a k-vertex with neighbors $v_1, v_2, \ldots, v_k$ such that $k \geqslant 3$ and $2 \leqslant d_G(v_1) \leqslant \cdots \leqslant d_G(v_k)$. Let L be an edge-list of G such that $|L(e)| \geqslant \Delta_1(e) + \max\{d_G(v_{k-1}) + \cdots + d_G(v_2) - k + 2, d_G(v_{k-2}) + \cdots + d_G(v_1) - k + 3\}$ for each edge e containing v as an endpoint. If $G - v$ has an acyclic L-edge coloring ϕ, then ϕ can be extended to an acyclic L-edge coloring of G.*

Proof. We color the edges adjacent to v according to the ordering $vv_k, vv_{k-1}, \ldots, vv_1$. Since $|C_\phi(v_k) \cup \cdots \cup C_\phi(v_3) \cup C_\phi(v_2)| \leqslant d_G(v_k) + d_G(v_{k-1}) + \cdots + d_G(v_2) - k + 1 \leqslant \Delta_1(vv_k) + d_G(v_{k-1}) + \cdots + d_G(v_2) - k + 1$, we can color vv_k with some $c_k \in L(vv_k) - (C_\phi(v_k) \cup \cdots \cup C_\phi(v_3) \cup C_\phi(v_2))$. Since $|C_\phi(v_{k-1}) \cup \cdots \cup C_\phi(v_2) \cup C_\phi(v_1) \cup \{c_k\}| \leqslant \Delta_1(vv_{k-1}) + d_G(v_{k-2}) + \cdots + d_G(v_1) - k + 2$, we can color vv_{k-1} with some $c_{k-1} \in L(vv_{k-1}) - (C_\phi(v_{k-1}) \cup \cdots \cup C_\phi(v_1) \cup \{c_k\})$.

For $2 \leqslant i \leqslant k-2$, $|C_\phi(v_i) \cup \cdots \cup C_\phi(v_1) \cup \{c_{i+1}, \ldots, c_k\}| \leqslant d_G(v_i) + \cdots + d_G(v_1) - i + k - i \leqslant \Delta_1(vv_i) + d_G(v_{i-1}) + \cdots + d_G(v_1) - i + d_G(v_{i+1}) + \cdots + d_G(v_{k-2}) - (k-i-2) + 2 \leqslant \Delta_1(vv_i) + d_G(v_{k-2}) + \cdots + d_G(v_1) - k + 2$. We color vv_i with $c_i \in L(vv_i) - (C_\phi(v_i) \cup C_\phi(v_{i-1}) \cup \cdots \cup C_\phi(u_1) \cup \{c_{i+1}, \ldots, c_k\})$. If $c_k \notin C_\phi(v_1)$, we color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup \{c_k, \ldots, c_2\})$. If $c_k \in C_\phi(v_1)$, since $|C_\phi(v_1) \cup C_\phi(v_k) \cup \{c_k, \ldots, c_2\}| \leqslant d_G(v_k) + d_G(v_1) + k - 4 \leqslant \Delta_1(vv_1) + d_G(v_1) + k - 4 \leqslant \Delta_1(vv_1) + d_G(v_{k-2}) + \cdots + d_G(v_1) - k + 2$, we can color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_k) \cup \{c_k, \ldots, c_2\})$. $\square$

3. A general upper bound

In this and subsequent sections, we will employ results proved in [2, 3, 4, 6, 7, 10, 11, 12] to establish our theorems about acyclic edge choosability. Some statements of these lemmas have been adapted to suit our proofs.

Lemma 14. ([6]) *Let G be a planar graph. Then there exists a vertex v whose k neighbors $v_1, v_2, \ldots, v_k$ satisfy $d_G(v_1) \leqslant \cdots \leqslant d_G(v_k)$ and one of the following statements holds.*

(A1) $k \leqslant 2$;

(A2) $k = 3$ *with* $d_G(v_1) \leqslant 11$;

(A3) $k = 4$ *with* $d_G(v_1) \leqslant 7$ *and* $d_G(v_2) \leqslant 11$;

(A4) $k = 5$ *with* $d_G(v_1) \leqslant 6$, $d_G(v_2) \leqslant 7$ *and* $d_G(v_3) \leqslant 11$.

Theorem 15. *If G is a planar graph, then G is acyclically* $\max\{2\Delta_1 - 2, \Delta_1 + 22\}$*-edge choosable.*

Proof. The proof is by induction on the number of vertices. The theorem is trivially true for the induction basis of a single vertex graph.

By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a $\max\{2\Delta_1 - 2, \Delta_1 + 22\}$-edge-list of G. By Lemma 14, there exists a vertex v whose k neighbors $v_1, v_2, \ldots, v_k$ satisfy $d_G(v_1) \leqslant \cdots \leqslant d_G(v_k)$ and we have the following cases to discuss.

For **(A1)**, we may assume that $k = 2$ in view of Lemma 5. Let $G' = G - v$ if $v_1v_2 \in E(G)$. By the induction hypothesis, G' has an acyclic L-edge coloring ϕ. Since $|C_\phi(v_1) \cup C_\phi(v_2)| \leqslant 2d_G(v_2) - 3 \leqslant 2\Delta_1(vv_2) - 3$ and $L(vv_2) \geqslant 2\Delta_1(vv_2) - 2$, $L(vv_2) - (C_\phi(v_1) \cup C_\phi(v_2))$ is nonempty. We color vv_2 with some $s \in L(vv_2) - (C_\phi(v_1) \cup C_\phi(v_2))$ and vv_1 with some $t \in L(vv_1) - (C_\phi(v_1) \cup \{s\})$.

If v_1 and v_2 are non-adjacent, let $G' = (G - v) + v_1v_2$. Let i be the index such that $\Delta_1(vv_i) = \max\{\Delta_1(vv_1), \Delta_1(vv_2)\}$. Define a $\max\{2\Delta_1 - 2, \Delta_1 + 22\}$-edge-list L' of G' by letting $L'(v_1v_2) = L(vv_i)$ and $L'(e) = L(e)$ for $e \neq v_1v_2$. By the induction hypothesis, G' has an acyclic L'-edge coloring ψ. Let ϕ be the acyclic L-edge coloring of $G - v$ such that $\phi(e) = \psi(e)$ for each edge e. We color vv_i with $s = \psi(v_1v_2)$ and color the other edge vv_{3-i} with some $t \in L(vv_{3-i}) - (C_\phi(v_{3-i}) \cup \{s\})$.

For **(A2)**, let $G' = G - v$ if $v_2v_3 \in E(G)$. By the induction hypothesis, G' has an acyclic L-edge coloring ϕ. We color vv_3 with some $s \in L(vv_3) - (C_\phi(v_2) \cup C_\phi(v_3))$, vv_2 with some $t \in L(vv_2) - (C_\phi(v_1) \cup C_\phi(v_2) \cup \{s\})$, and vv_1 with some $p \in L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_3) \cup \{s, t\})$.

If v_2 and v_3 are non-adjacent, let $G' = (G-v)+v_2v_3$. Let i be the index such that $\Delta_1(vv_i) = \max\{\Delta_1(vv_2), \Delta_1(vv_3)\}$. Define a $\max\{2\Delta_1 - 2, \Delta_1 + 22\}$-edge-list L' of G' by letting $L'(v_2v_3) = L(vv_i)$ and $L'(e) = L(e)$ for $e \neq v_2v_3$. By the induction hypothesis, G' has an acyclic L'-edge coloring ψ. Let ϕ be the acyclic L-edge coloring of $G - v$ such that $\phi(e) = \psi(e)$ for each edge e. We color vv_i with $s = \psi(v_2v_3)$ and vv_{5-i} with some $t \in L(vv_{5-i}) - (C_\phi(v_1) \cup C_\phi(v_{5-i}) \cup \{s\})$, and then color vv_1 with some $p \in L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_i) \cup \{s, t\})$.

For **(A3)**, let $G' = G - v$ if $v_3v_4 \in E(G)$. By the induction hypothesis, G' has an acyclic L-edge coloring ϕ. We color vv_4 with some $s \in L(vv_4) - (C_\phi(v_3) \cup C_\phi(v_4))$. Since $|C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup \{s\}| \leqslant \Delta_1(vv_3) + 16$, we can color vv_3 with some $t \in L(vv_3) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup \{s\})$. Since $|C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_4) \cup \{s, t\}| \leqslant d_G(v_4) + 17 \leqslant \Delta_1(vv_2) + 17$, we can color vv_2 with some $p \in L(vv_2) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_4) \cup \{s, t\})$. Since $|C_\phi(v_1) \cup C_\phi(v_4) \cup \{s, t, p\}| \leqslant \Delta_1(vv_1) + 8$, we can color vv_1 with some $q \in L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_4) \cup \{s, t, p\})$.

If v_3 and v_4 are non-adjacent, let $G' = (G - v) + v_3v_4$. Let i be the index such that $\Delta_1(vv_i) = \max\{\Delta_1(vv_3), \Delta_1(vv_4)\}$ for some $i = 3, 4$. Define a $\max\{2\Delta_1 - 2, \Delta_1 + 22\}$-edge-list L' of G' such that $L'(v_3v_4) = L(vv_i)$ and $L'(e) = L(e)$ for $e \neq v_3v_4$. By the induction hypothesis, G' has an acyclic L'-edge coloring ψ. Let ϕ be the acyclic L-edge coloring of $G - v$ such that $\phi(e) = \psi(e)$ for each edge e. We color vv_i with $s = \psi(v_3v_4)$, vv_{7-i} with some $t \in L(vv_{7-i}) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_{7-i}) \cup \{s\})$, vv_2 with some $p \in L(vv_2) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_i) \cup \{s, t\})$, and then vv_1 with some $q \in L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_i) \cup \{s, t, p\})$.

For **(A4)**, let $G' = G - v$ if $v_4v_5 \in E(G)$. By the induction hypothesis, G' has an acyclic L-edge coloring ϕ. We color vv_5 with some $s \in L(vv_5) - (C_\phi(v_4) \cup C_\phi(v_5))$. Since $|C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{s\}| \leqslant \Delta_1(vv_4) + 21$, we can color vv_4 with some $t \in L(vv_4) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup C_\phi(v_4) \cup \{s\})$. If $s \notin C_\phi(v_1)$, we color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup \{s, r\})$. Otherwise, we can color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3) \cup C_\phi(v_5) \cup \{s, t\})$ since $|C_\phi(v_1) \cup$

$C_\phi(v_2)\cup C_\phi(v_3)\cup C_\phi(v_5)\cup\{s,t\}| \leqslant d_G(v_5)+21 \leqslant \Delta_1(vv_1)+21$. In both cases, we denote the color of vv_1 by p. Since $|C_\phi(v_2)\cup C_\phi(v_3)\cup C_\phi(v_5)\cup\{s,t,p\}| \leqslant \Delta_1(vv_2)+18$, we can color vv_2 with some $q \in L(vv_2)-(C_\phi(v_2)\cup C_\phi(v_3)\cup C_\phi(v_5)\cup\{s,t,p\})$. Since $|C_\phi(v_3)\cup C_\phi(v_5)\cup\{s,t,p,q\}| \leqslant \Delta_1(vv_3)+13$, we can color vv_3 with some $r \in L(vv_3)-(C_\phi(v_3)\cup C_\phi(v_5)\cup\{s,t,p,q\})$.

If v_4 and v_5 are non-adjacent, let $G' = (G-v)+v_4v_5$. Let i be the index such that $\Delta_1(vv_i) = \max\{\Delta_1(vv_4), \Delta_1(vv_5)\}$ for some $i = 4, 5$. Define a $\max\{2\Delta_1-2, \Delta_1+22\}$-edge-list L' of G' such that $L'(v_4v_5) = L(vv_i)$ and $L'(e) = L(e)$ for $e \neq v_4v_5$. By the induction hypothesis, G' has an acyclic L'-edge coloring ψ. Let ϕ be the acyclic L-edge coloring of $G-v$ such that $\phi(e) = \psi(e)$ for each edge e. We color vv_i with $s = \psi(v_4v_5)$ and vv_{9-i} with some $t \in L(vv_{9-i})-(C_\phi(v_1)\cup C_\phi(v_2)\cup C_\phi(v_3)\cup C_\phi(v_{9-i})\cup\{s\})$. If $s \notin C_\phi(v_1)$, we color vv_1 with some element in $L(vv_1)-(C_\phi(v_1)\cup C_\phi(v_2)\cup C_\phi(v_3)\cup\{s,t\})$. Otherwise, we color vv_1 with some element in $L(vv_1)-(C_\phi(v_1)\cup C_\phi(v_2)\cup C_\phi(v_3)\cup C_\phi(v_i)\cup\{s,t\})$. In both cases, we denote the color of vv_1 by p. We color vv_2 with some $q \in L(vv_2)-(C_\phi(v_2)\cup C_\phi(v_3)\cup C_\phi(v_i)\cup\{s,t,p\})$ and vv_3 with some $r \in L(vv_3)-(C_\phi(v_3)\cup C_\phi(v_i)\cup\{s,t,p,q\})$.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically $\max\{2\Delta_1-2, \Delta_1+22\}$-edge choosable. □

Corollary 16. *If G is a planar graph, then $a'_{\text{list}}(G) \leqslant \max\{2\Delta(G)-2, \Delta(G)+22\}$.*

4. A Sufficient Condition for Acyclic $(\Delta+13)$-edge Choosability

Lemma 17. ([2]) *If G is a planar graph without cycles of length 4, and $\delta(G) \geqslant 2$, then G contains at least one of the following configurations.*

- (B1) *a 2-vertex v adjacent to one 9^--vertex;*
- (B2) *a 3-vertex v adjacent to two 9^--vertices;*
- (B3) *a 4-vertex v adjacent to three 7^--vertices;*
- (B4) *a d-vertex v adjacent to one 2-vertex u_1 and $(d-6)$ 4^--vertices, where $d \geqslant 10$;*
- (B5) *a triangle $v_1v_2v_3$ with $d_G(v_1) = 3$ and $d_G(v_2) = 3$.*

Remark. The original statements of Lemmas 17 and 29 in [2] assumed 2-connectedness of the planar graph G. However, an examination of their proofs in [2] shows the results still hold for planar graphs with $\delta(G) \geqslant 2$.

Theorem 18. *If G is a planar graph without cycles of length 4, then G is acyclically $(\Delta_1 + 13)$-edge choosable.*

Proof. The proof is by induction on the number of vertices plus the number of edges. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a $(\Delta_1 + 13)$-edge-list of G. By Lemma 17, we have five cases to discuss.

By the induction hypothesis, each of the graphs $G - v$ in cases **(B1)**, **(B2)**, and **(B3)**, the graph $G - u_1$ in case **(B4)**, and the graph $G - v_1v_2$ in case **(B5)** has an acyclic L-edge coloring ϕ. Using Corollary 12, Lemma 13 twice, Lemma 11, and Lemma 9, respectively, G has an acyclic L-edge coloring.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically $(\Delta_1 + 13)$-edge choosable. □

Corollary 19. *If G is a planar graph without cycles of length 4, then $a'_{\text{list}}(G) \leqslant \Delta(G) + 13$.*

5. Sufficient Conditions for Acyclic $(\Delta + 5)$-edge Choosability

Lemma 20. ([4]) *Let G be a graph with $\delta(G) \geqslant 2$. If G satisfies $\|G\| < 2|G|$, then G contains at least one of the following configurations.*

- (C1) *a 2-vertex v adjacent to one 5^--vertices;*
- (C2) *a 3-vertex v adjacent to two 5^--vertices;*
- (C3) *a 6-vertex v adjacent to five 3^--vertices;*
- (C4) *a 7-vertex v adjacent to seven 3^--vertices;*
- (C5) *a d-vertex v adjacent to one 2-vertex u_1 and $(d-4)$ 3^--vertices, where $d \geqslant 4$.*

The *maximum average degree* $\mathrm{mad}(G)$ of a graph G is defined to be $\max\{2\|H\|/|H| \mid H \text{ is a subgraph of } G\}$.

Theorem 21. *If G is a graph with* $\mathrm{mad}(G) < 4$, *then G is acyclically* $(\Delta_1 + 5)$*-edge choosable.*

Proof. The proof is by induction on the number of vertices plus the number of edges. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a $(\Delta_1 + 5)$-edge-list of G. By Lemma 20, we have five cases to discuss.

For **(C1)** and **(C2)**, the graph $G - v$ has an acyclic L-edge coloring ϕ by the induction hypothesis. Using Corollary 12 and Lemma 13, G has an acyclic L-edge coloring.

For **(C3)**, assume that the neighbors of v are $u_1, \dots, u_6$ where $u_1, \dots, u_5$ are 3^--vertices.

The graph $G - vu_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis.

Subcase 3.1. $C_\phi(u_1) \cap C_\phi(v) = \emptyset$.

We color vu_1 with some element in $L(vu_1) - (C_\phi(u_1) \cup C_\phi(v))$.

Subcase 3.2. $|C_\phi(u_1) \cap C_\phi(v)| = 1$ and $\phi(vu_6) \notin C_\phi(u_1)$.

Let $2 \leqslant i \leqslant 5$ be the index such that $\phi(vu_i) \in C_\phi(u_1)$. We color vu_1 with some element in $L(vu_1) - (C_\phi(u_1) \cup C_\phi(v) \cup C_\phi(u_i))$.

Subcase 3.3. $|C_\phi(u_1) \cap C_\phi(v)| = 1$ and $\phi(vu_6) \in C_\phi(u_1)$.

If $L(vu_1) \neq C_\phi(v) \cup C_\phi(u_1) \cup C_\phi(u_6)$, we color vu_1 with some element in $L(vu_1) - (C_\phi(v) \cup C_\phi(u_1) \cup C_\phi(u_6))$. We may assume that $L(vu_1) = C_\phi(v) \cup C_\phi(u_1) \cup C_\phi(u_6)$, $C_\phi(v) \cap C_\phi(u_6) = \{\phi(vu_6)\}$, and $d_G(u_6) = \Delta_1(vu_1)$.

If $L(vu_6) \neq C_\phi(v) \cup C_\phi(u_1) \cup C_\phi(u_6)$, we re-color vu_6 with some element in $L(vu_6) - (C_\phi(v) \cup C_\phi(u_1) \cup C_\phi(u_6))$ and this subcase is reduced to Subcase 3.1. We may assume that $L(vu_6) = C_\phi(v) \cup C_\phi(u_6) \cup C_\phi(u_1)$.

If $\phi(vu_6) \notin C_\phi(u_r)$ for a certain $2 \leqslant r \leqslant 5$, we re-color vu_6 with $\phi(vu_r)$. If $C_\phi(u_r) \cap C_\phi(v) = \{\phi(vu_r)\}$, we re-color vu_r with some element in $L(vu_r) - (C_\phi(u_r) \cup C_\phi(v))$. If $C_\phi(u_r) \cap C_\phi(v) = \{\phi(vu_r), \phi(vu_s)\}$ for some $2 \leqslant s \leqslant 5$, we re-color vu_r with some element in $L(vu_r) - (C_\phi(v) \cup C_\phi(u_r) \cup C_\phi(u_s))$. If $C_\phi(u_r) \cap C_\phi(v) = \{\phi(vu_r), \phi(vu_s), \phi(vu_t)\}$ for some $2 \leqslant s, t \leqslant 5$, we re-color vu_r with some element in $L(vu_r) - (C_\phi(v) \cup C_\phi(u_r) \cup C_\phi(u_s) \cup C_\phi(u_t))$. And these three subcases are reduced to Subcases 3.1 or 3.2.

We may assume that $\phi(vu_6) \in C_\phi(u_j)$ for each $2 \leqslant j \leqslant 5$. We re-color vu_6 with $\phi(vu_2)$ and re-color vu_2 with some element in $L(vu_2) - (C_\phi(v) \cup C_\phi(u_2) \cup C_\phi(u_3) \cup C_\phi(u_4) \cup C_\phi(u_5))$. And this subcase is reduced to Subcases 3.1 or 3.2.

Subcase 3.4. $|C_\phi(u_1) \cap C_\phi(v)| = 2$ and $\phi(vu_6) \notin C_\phi(u_1)$.

We may assume that $C_\phi(u_1) = \{\phi(vu_j), \phi(vu_k)\}$ for $2 \leqslant j, k \leqslant 5$. We color vu_1 with some element in $L(vu_1) - (C_\phi(u_1) \cup C_\phi(v) \cup C_\phi(u_j) \cup C_\phi(u_k))$.

Subcase 3.5. $|C_\phi(u_1) \cap C_\phi(v)| = 2$ and $\phi(vu_6) \in C_\phi(u_1)$.

We may assume that $C_\phi(u_1) = \{\phi(vu_s), \phi(vu_6)\}$ for some $2 \leqslant s \leqslant 5$.

Let t be an element in $L(vu_1) - (C_\phi(v) \cup C_\phi(u_6))$. If $t \notin C_\phi(u_s)$, we color vu_1 with t. We may assume that $L(vu_1) \subseteq C_\phi(v) \cup C_\phi(u_6) \cup C_\phi(u_s)$ and $|C_\phi(u_s) \cap C_\phi(v)| \leqslant 2$.

If $C_\phi(u_s) \cap C_\phi(v) = \{\phi(vu_s)\}$, we re-color vu_s with some element in $L(vu_s) - (C_\phi(u_s) \cup C_\phi(v))$ and this subcase is reduced to Subcase 3.3.

If $C_\phi(u_s) \cap C_\phi(v) = \{\phi(vu_s), \phi(vu_r)\}$ for some $2 \leqslant r \leqslant 5$, we re-color vu_s with some element in $L(vu_s) - (C_\phi(u_s) \cup C_\phi(u_r) \cup C_\phi(v))$ and this subcase is reduced to Subcase 3.3.

We may assume that $C_\phi(u_s) \cap C_\phi(v) = \{\phi(vu_s), \phi(vu_6)\}$. If $L(vu_s) \neq C_\phi(u_s) \cup C_\phi(v) \cup C_\phi(u_6)$, we re-color vu_s with some element in $L(u_s) - (C_\phi(u_s) \cup C_\phi(v) \cup C_\phi(u_6))$ and this subcase is reduced to Subcase 3.3. We may assume that $L(vu_s) = C_\phi(u_s) \cup C_\phi(v) \cup C_\phi(u_6)$, $d_G(u_6) = \Delta_1(vu_s)$, and $C_\phi(v) \cap C_\phi(u_6) = \{\phi(vu_6)\}$. We re-color vu_6 with some element in $L(vu_6) - (C_\phi(v) \cup C_\phi(u_6))$ and this subcase is reduced to Subcase 3.2.

For **(C4)**, assume that the neighbors of v are the 3^--vertices $u_1, \ldots, u_7$.

The graph $G - vu_7$ has an acyclic L-edge coloring ϕ by the induction hypothesis. Assume that $\phi(vu_i) = i$, for each $1 \leqslant i \leqslant 6$. If $C_\phi(u_7) \cap \{1, \ldots, 6\} = \emptyset$, we color vu_7 with some element in $L(vu_7) - (C_\phi(u_7) \cup \{1, \ldots, 6\})$. If $C_\phi(u_7) \cap \{1, \ldots, 6\} = \{k\}$, we color vu_7 with some element in $L(vu_7) - (C_\phi(u_7) \cup C_\phi(u_k) \cup \{1, \ldots, 6\})$. If $C_\phi(u_7) \cap \{1, \ldots, 6\} = \{k, l\}$, we color vu_7 with some element in $L(vu_7) - (C_\phi(u_7) \cup C_\phi(u_k) \cup C_\phi(u_l) \cup \{1, \ldots, 6\})$.

For **(C5)**, The graph $G - u_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Lemma 11, G has an acyclic L-edge coloring.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically (Δ_1+5)-edge choosable. $\square$

Corollary 22. *If G is a graph with $\mathrm{mad}(G)<4$, then $a'_{\mathrm{list}}(G) \leqslant \Delta(G)+5$.*

The following is a folklore result (cf. [9]). The *girth* $g(G)$ of a graph G is the length of a shortest cycle in G. If G is a planar graph with $g(G) \geqslant g$, then $\mathrm{mad}(G) < \frac{2g}{g-2}$.

Corollary 23. *If G is a triangle-free planar graph, then $a'_{\mathrm{list}}(G) \leqslant \Delta(G)+5$.*

6. A Sufficient Condition for Acyclic $(\Delta+3)$-edge Choosability

Lemma 24. ([10]) *Let G be a planar graph with $\delta(G) \geqslant 2$ and $g(G)>3$. If any two cycles of length 4 are vertex-disjoint, then G contains at least one of the following configurations.*

(D1) *a 2-vertex v adjacent to one 4^--vertex;*

(D2) *a 3-vertex v adjacent to two 3-vertices;*

(D3) *a d-vertex v adjacent to $(d-3)$ 2-vertices, where $d \geqslant 5$;*

(D4) *a 4-vertex v adjacent to three 3-vertices u, x, and y;*

(D5) *a face $f=v_1v_2v_3v_4$ with $d_G(v_1)=2$ and $d_G(v_2)=5$.*

Theorem 25. *If G is a planar graph with $g(G)>3$ such that any two cycles of length 4 are vertex-disjoint, then G is acyclically (Δ_1+3)-edge choosable.*

Proof. The proof is by induction on the number of vertices plus the number of edges. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a (Δ_1+3)-edge-list of G. By Lemma 24, we have the following cases to discuss.

For **(D1)** and **(D2)**, the graph $G-v$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Corollary 12 and Lemma 13, G has an acyclic L-edge coloring.

For **(D3)**, let the neighbors of v be $u_1,\ldots,u_d$, where $u_1,\ldots,u_{d-3}$ are 2-vertices and adjacent to $w_1,\ldots,w_{d-3}$, respectively. The graph $G-u_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Lemma 11, G has an acyclic L-edge coloring.

For **(D4)**, let $N_G(v) = \{u, x, y, w\}$, $N_G(u) = \{v, u_1, u_2\}$, $N_G(x) = \{v, x_1, x_2\}$, and $N_G(y) = \{v, y_1, y_2\}$. The graph $G' = G - uv$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We may assume that $\phi(vx) = 1$, $\phi(vy) = 2$, and $\phi(vw) = 3$.

Subcase 4.1. $C_\phi(u) \cap \{1, 2, 3\} = \emptyset$.

We color uv with some $j \in L(uv) - (C_\phi(u) \cup \{1, 2, 3\})$.

Subcase 4.2. $C_\phi(u) \cap \{1, 2, 3\}$ is $\{1\}$ or $\{2\}$.

It is sufficient to assume that $\phi(uu_1) = 1$ and $\phi(uu_2) = 4$. We color uv with some $j \in L(uv) - (C_\phi(x) \cup \{2, 3, 4\})$.

Subcase 4.3. $C_\phi(u) \cap \{1, 2, 3\} = \{3\}$.

We may assume that $\phi(uu_1) = 3$ and $\phi(uu_2) = 4$. If $L(uv) \neq C_\phi(u_1) \cup \{1, 2, 4\}$, then we color uv with some $j \in L(uv) - (C_\phi(u_1) \cup \{1, 2, 4\})$. Hence, we may assume that $4 \notin C_\phi(u_1)$ and $L(uv) = C_\phi(u_1) \cup \{1, 2, 4\}$. If $L(uu_1) \neq C_\phi(u_1) \cup \{1, 2, 4\}$, then we re-color uu_1 with some $j \in L(uu_1) - (C_\phi(u_1) \cup \{1, 2, 4\})$ and this subcase is reduced to Subcase 4.1. Otherwise, $L(uu_1) = C_\phi(u_1) \cup \{1, 2, 4\}$. We re-color uu_1 with 1 and this subcase is reduced to Subcase 4.2.

Subcase 4.4. $C_\phi(u) = \{1, 2\}$.

If $C_\phi(x) \cap \{2, 3\} = \emptyset$, then we re-color vx with some $j \in L(vx) - \{1, 2, 3, \phi(xx_1), \phi(xx_2)\}$ and this subcase is reduced to Subcase 4.2. Otherwise, $|C_\phi(x) \cup C_\phi(y) \cup \{3\}| \leqslant 6$. We color uv with some $k \in L(uv) - (C_\phi(x) \cup C_\phi(y) \cup \{3\})$.

Subcase 4.5. $C_\phi(u)$ is $\{1, 3\}$ or $\{2, 3\}$.

It is sufficient to assume that $C_\phi(u) = \{1, 3\}$. If $C_\phi(x) \cap \{2, 3\} = \emptyset$, then we re-color vx with some $j \in L(vx) - \{1, 2, 3, \phi(xx_1), \phi(xx_2)\}$ and this subcase is reduced to Subcase 4.3. If $C_\phi(w) \cap \{1, 2\} = \emptyset$, then we re-color vw with some $k \in L(vw) - (C_\phi(w) \cup \{1, 2\})$ and this subcase is reduced to Subcase 4.2. We may assume that $C_\phi(x) \cap \{2, 3\} \neq \emptyset$ and $C_\phi(w) \cap \{1, 2\} \neq \emptyset$. Hence, $|C_\phi(x) \cup C_\phi(w) \cup \{2\}| \leqslant d_G(w) + 2 \leqslant \Delta_1(uv) + 2$. We color uv with some $l \in L(uv) - (C_\phi(x) \cup C_\phi(w) \cup \{2\})$.

For **(D5)**, let $N_G(v_2) = \{v_1, v_3, x_1, x_2, x_3\}$. The graph $G' = G - v_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We may assume that $\phi(v_2x_1) = 1$, $\phi(v_2x_2) = 2$, $\phi(v_2x_3) = 3$, and $\phi(v_2v_3) = 4$. If $L(v_1v_4) - (C_\phi(v_4) \cup \{1,2,3,4\}) \neq \emptyset$, then we color v_1v_4 with some $j \in L(v_1v_4)-(C_\phi(v_4)\cup\{1,2,3,4\})$ and v_1v_2 with some $k \in L(v_1v_2)-\{1,2,3,4,j\}$. We may assume that $\{1,4\} \cap C_\phi(v_4) = \emptyset$ and $L(v_1v_4) = C_\phi(v_4) \cup \{1,2,3,4\}$.

Subcase 5.1. $1 \in C_\phi(v_3)$.

We color v_1v_4 with 4 and v_1v_2 with some $j \in L(v_1v_2)-(C_\phi(v_3)\cup\{2,3\})$.

Subcase 5.2. $1 \notin C_\phi(v_3)$.

We color v_1v_4 with 1 and v_1v_2 with some $k \in L(v_1v_2) - ((C_\phi(v_4) - \phi(v_3v_4)) \cup \{1,2,3,4\})$.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically $(\Delta_1 + 3)$-edge choosable. □

Corollary 26. *If G is a planar graph with $g(G) > 3$ such that any two cycles of length 4 are vertex-disjoint, then $a'_{\text{list}}(G) \leqslant \Delta(G) + 3$.*

7. Sufficient Conditions for Acyclic $(\Delta + 2)$-edge Choosability

Lemma 27. ([7]) *Let G be a planar graph with $\delta(G) \geqslant 2$. If $g(G) \geqslant 5$, then G contains at least one of the following configurations.*

- (E1) *a 2-vertex v adjacent to one 3^--vertex;*
- (E2) *a 3-vertex v adjacent to two 3-vertices u and w;*
- (E3) *a d-vertex adjacent to one 2-vertex u and $(d-3)$ 3^--vertices, where $d \geqslant 4$;*
- (E4) *a face $f = v_1v_2v_3v_4v_5$ with $d_G(v_1) = d_G(v_4) = 2$, $d_G(v_2) = d_G(v_3) = 4$, and $d_G(v_5) = 5$.*

Lemma 28. ([10]) *Let G be a planar graph with $\delta(G) \geqslant 2$. If any two cycles of length 4 are edge-disjoint and there are no cycles of length 3 or 5, then G contains at least one of* (E1), (E2), (E3) *of Lemma* 27, *or the following configuration.*

(E5) *a face $f = v_1v_2v_3v_4$ with $d_G(v_1) = 2$ and $d_G(v_2) = 4$.*

Lemma 29. ([2]) *Let G be a planar graph with $\delta(G) \geqslant 2$ containing no cycles of length 4, 6, 8, and 9. Then G contains at least one of* (E1), (E2), (E3) *of Lemma* 27, *or one of the following configurations.*

(E6) *a triangle $f = v_1v_2v_3$ with $d_G(v_1) = 3$ and $d_G(v_2) = 3$;*

(E7) *a triangle $f = v_1v_2v_3$ with $d_G(v_1) = 4$, $d_G(v_2) \leqslant 4$, and one of v_1's neighbors not incident with f is of degree* 2*;*

(E8) *a triangle $f = v_1v_2v_3$ with $d_G(v_1) = 2$ and $d_G(v_2) = 4$.*

Lemma 30. ([11]) *Let G be a planar graph with $\delta(G) \geqslant 2$ containing no cycles of length 4 or 5. If any two cycles of length 3 are vertex-disjoint or there are no cycles of length from 6 to 8, then G contains at least one of* (E1), (E3) *of Lemma* 27, *or* (E6) *of Lemma* 29, *or* (E8) *of Lemma* 29.

Theorem 31. *If G is a planar graph such that any of the following conditions holds, then G is acyclically $(\Delta_1 + 2)$-edge choosable.*

1. $g(G) \geqslant 5$*;*
2. *any two cycles of length* 4 *are edge-disjoint and there are no cycles of length* 3 *or* 5*;*
3. *there are no cycles of length* 4, 6, 8, *and* 9*;*
4. *there are no cycles of length from* 4 *to* 8*;*
5. *any two cycles of length* 3 *are vertex-disjoint and there are no cycles of length* 4 *or* 5.

Proof. The proof is by induction on the number of vertices plus the number of edges. The theorem is trivially true for the induction basis of a single vertex graph. By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a (Δ_1+2)-edge-list of G. By Lemmas 27 to 30, we have the following cases to discuss.

For **(E1)**, the graph $G - v$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Corollary 12, G has an acyclic L-edge coloring.

For **(E2)**, assume that $N_G(v) - \{u, w\} = \{x\}$, $N_G(u) - \{v\} = \{u_1, u_2\}$, and $N_G(w) - \{v\} = \{w_1, w_2\}$. If $\Delta_1(uv) = 3$, then all neighbors of u have degree at most 3. The graph $G - uv$ has an acyclic L-edge coloring by the induction hypothesis. By Lemma 10, G has an acyclic L-edge coloring.

We may assume that $\Delta_1(uv) \geqslant 4$. The graph $G - v$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We color vx with some $j \in L(vx) - (C_\phi(x) \cup C_\phi(u))$. If $j \notin C_\phi(w)$, we color vw with some element in $L(vw) - (C_\phi(w) \cup \{j\})$. If $j \in C_\phi(w)$, we color vw with some element in $L(vw) - (C_\phi(w) \cup C_\phi(x))$. Denote the color of vw by k. If $k \notin C_\phi(u)$, we color uv with some element in $L(uv) - (C_\phi(u) \cup \{j, k\})$. If $k \in C_\phi(u)$, we color uv with some element in $L(uv) - (C_\phi(u) \cup C_\phi(w) \cup \{j\})$.

For **(E3)**, the graph $G - u_1$ has an acyclic L-edge coloring by the induction hypothesis. By Lemma 11, G has an acyclic L-edge coloring.

For **(E4)**, assume that $N_G(v_2) = \{v_1, v_3, x_1, x_2\}$ and $N_G(v_3) = \{v_2, v_4, y_1, y_2\}$. The graph $G - v_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We may assume that $\phi(v_2v_3) = 1$, $\phi(v_2x_1) = 2$, and $\phi(v_2x_2) = 3$. We color v_1v_5 with some $j \in L(v_1v_5) - (C_\phi(v_5) \cup \{2, 3\})$. If $j \neq 1$, then we color v_1v_2 with some $k \in L(v_1v_2) - \{1, 2, 3, j\}$. If $j = 1$, then we color v_1v_2 with some $p \in L(v_1v_2) - \{2, 3, \phi(v_3y_1), \phi(v_3y_2)\}$.

For **(E5)**, assume that $N_G(v_2) = \{v_1, v_3, x_1, x_2\}$. The graph $G - v_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We may assume that $\phi(v_2x_1) = 1$, $\phi(v_2x_2) = 2$, and $\phi(v_2v_3) = 3$. If $L(v_1v_4) - (C_\phi(v_4) \cup \{1, 2, 3\}) \neq \emptyset$, then we color v_1v_4 with some $j \in L(v_1v_4) - (C_\phi(v_4) \cup \{1, 2, 3\})$ and color v_1v_2 with some $k \in L(v_1v_2) - \{1, 2, 3, j\}$. We may assume that $\{1, 3\} \cap C_\phi(v_4) = \emptyset$ and $L(v_1v_4) = C_\phi(v_4) \cup \{1, 2, 3\}$.

Subcase 5.1. $1 \in C_\phi(v_3)$.

We color v_1v_4 with 3 and v_1v_2 with some $j \in L(v_1v_2) - (C_\phi(v_3) \cup \{2\})$.

Subcase 5.2. $1 \notin C_\phi(v_3)$.

We color v_1v_4 with 1 and v_1v_2 with some $k \in L(v_1v_2) - ((C_\phi(v_4) - \phi(v_3v_4)) \cup \{1, 2, 3\})$.

For **(E6)**, the graph $G - v_1v_2$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Lemma 9, G has an acyclic L-edge coloring.

For **(E7)**, assume that $N_G(v_1) = \{u, v_2, v_3, v_4\}$ and $N_G(u) = \{v_1, u'\}$. The graph $G - u$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We may assume that $\phi(v_1v_2) = 2$, $\phi(v_1v_3) = 3$, and $\phi(v_1v_4) = 4$. If $\phi(v_1v_2) \notin C_\phi(v_3)$, then we color uu' with some $j \in L(uu') - (C_\phi(u') \cup \{3, 4\})$. If $j \neq 2$, then we color uv_1 with some $k \in L(uv_1) - (C_\phi(v_1) \cup \{j\})$. If $j = 2$, then we color uv_1 with some $l \in L(uv_1) - (\{3, 4\} \cup (C_\phi(v_2) - \{\phi(v_2v_3)\}))$. If

$\phi(v_1v_2) \in C_\phi(v_3)$, then we color uu' with some $p \in L(uu') - (C_\phi(u') \cup \{2,4\})$. If $p \neq 3$, then we color uv_1 with some $q \in L(uv_1) - (C_\phi(v_1) \cup \{j\})$. If $p = 3$, then we color uv_1 with some $r \in L(uv_1) - (\{4\} \cup C_\phi(v_3))$.

For **(E8)**, the graph $G - v_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Lemma 7, G has an acyclic L-edge coloring.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically $(\Delta_1 + 2)$-edge choosable. $\square$

Corollary 32. *If G is a planar graph such that any of the five conditions of Theorem* 31 *holds, then $a'_{\text{list}}(G) \leqslant \Delta(G) + 2$.*

8. Sufficient Conditions for Acyclic $(\Delta + 1)$-edge Choosability

A vertex of degree 3 is called *weak* if it has a neighbor of degree 2.

Lemma 33. ([2]) *Let G be a graph such that* $\mathrm{mad}(G) < 3$ *and $\delta(G) \geqslant 2$, then G contains at least one of the following configurations.*

- (F1) *a 2-vertex v adjacent to one 2-vertex;*
- (F2) *a 3-vertex v adjacent to one 2-vertex u and one 3^--vertex w;*
- (F3) *a d-vertex v adjacent to $(d-1)$ 2-vertices, where $d \geqslant 4$;*
- (F4) *a 4-vertex v adjacent to one 2-vertex v_1 and three weak 3-vertices v_2, v_3, v_4;*
- (F5) *a 4-vertex v adjacent to two 2-vertices v_1, v_2 and one weak 3-vertex v_3.*

Theorem 34. *If G is a graph with* $\mathrm{mad}(G) < 3$, *then G is acyclically $(\Delta_1 + 1)$-edge choosable.*

Proof. The proof is by induction on the number of vertices. If $|G| \leqslant 4$, then G is not K_4, hence is acyclically $(\Delta_1 + 1)$-edge choosable. By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a $(\Delta_1 + 1)$-edge-list of G. By Lemma 33, we have the following cases to discuss.

For **(F1)**, the graph $G - u$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Corollary 12, G has an acyclic L-edge coloring.

For **(F2)**, if $d_G(w) = 3$, assume that $N_G(v) - \{u, w\} = \{x\}$, $N_G(u) - \{v\} = \{u'\}$, and $N_G(w) - \{v\} = \{w_1, w_2\}$. We may assume that $d_G(x) \geqslant d_G(w)$. The graph $G' = G - u$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We may assume that $\phi(vw) = 1$ and $\phi(vx) = 2$. We color uu' with some $j \in L(uu') - (C_\phi(u') \cup \{2\})$. If $j \neq 1$, we color uv with some element in $L(uv) - \{j, 1, 2\}$. We may assume that $j = 1$. By an (a, b)-colored path we mean a path whose edges are colored only by the two colors a and b. Let S be the set of positive integers s such that there exists a $(1, s)$-colored path in ϕ between u' and w avoiding v. If $L(uv) - (S \cup \{1, 2\})$ is nonempty, we color uv with some element in $L(uv) - (S \cup \{1, 2\})$. Otherwise, $d_G(x) = d_G(u') = 3$ and we may assume that $C_\phi(w) = C_\phi(u') = \{1, 3, 4\}$ and $L(uv) = \{1, 2, 3, 4\}$. If $L(uu') \neq \{1, 2, 3, 4\}$, we re-color uu' with some $j' \in L(uu') - \{1, 2, 3, 4\}$ and color uv with some element in $L(uv) - \{j', 1, 2\}$. We may assume that $L(uu') = \{1, 2, 3, 4\}$. Similarly, we may assume that $L(vw) = \{1, 2, 3, 4\}$. If $i \notin C_\phi(x)$ for some $i \in \{3, 4\}$, we re-color uu' with 2 and color uv with i. Otherwise, $C_\phi(x) = \{2, 3, 4\}$. If $L(vx) \neq \{1, 2, 3, 4\}$, then we re-color vx with some $p \in L(vx) - \{1, 2, 3, 4\}$ and color uv with 2. We may assume that $L(vx) = \{1, 2, 3, 4\}$. Since there is a $(1, 3)$-path between w and u', no $(1, 3)$-path between x and u' exists. We re-color vw with 2, vx with 1, and color uv with 3.

The case $d_G(w) \leqslant 2$ is similar.

For **(F3)**, assume that v_1 is a 2-vertices adjacent to v. The graph $G - v_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. By Lemma 11, G has an acyclic L-edge coloring.

For **(F4)**, assume that $N_G(v_1) = \{v, w_1\}$ and w_i is a neighbor of v_i of degree 2 for $i = 2, 3, 4$. The graph $G' = G - v_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. We color v_1w_1 with some $j \in L(v_1w_1) - C_\phi(w_1)$. If $j \notin C_\phi(v)$, then we color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup \{j\})$. Otherwise, we may assume that $j = \phi(vv_2) = 2$, $\phi(vv_3) = 3$, and $\phi(vv_4) = 4$. Let w_2' be the neighbor of v_2 distinct from v and v_2.

If $L(vv_1) - (C_\phi(v) \cup C_\phi(w_1)) \neq \emptyset$ or $L(vv_1) - (C_\phi(v) \cup C_\phi(v_2)) \neq \emptyset$, then we color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup C_\phi(w_1))$ or $L(vv_1) - (C_\phi(v) \cup C_\phi(v_2))$. Otherwise, $|C_\phi(v) \cup C_\phi(v_2)| = \Delta_1(vv_1) + 1 = 5$. In particular, $d_G(w_1) \leqslant 4$ and 3 or 4 cannot belong to $\phi(w_1)$. We may assume that $\phi(v_2w_2) = 1$, $\phi(v_2w_2') = 5$, and $L(vv_1) = \{1, 2, 3, 4, 5\}$. We can find some $k \in L(v_1w_1) - (C_\phi(w_1) \cup \{2\})$. If $k \notin \{3, 4\}$, then we re-color

v_1w_1 with k and color vv_1 with some element in $L(vv_1) - \{2,3,4,k\}$. Now we assume that $k \in \{3,4\}$. If $|C_\phi(v) \cup C_\phi(v_k)| < 5$, then we re-color v_1w_1 with k and color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup C_\phi(v_k))$. Otherwise, $|C_\phi(v) \cup C_\phi(v_k)| = 5$ and $2 \notin C_\phi(v_k)$. Let x_2 be the neighbor of w_2 different from v_2. If $\phi(w_2x_2) \neq 2$, then we color vv_1 with 1. Otherwise, $\phi(w_2x_2) = 2$. If $L(v_2w_2) - \{1,2,3,4,5\} \neq \emptyset$, then we re-color v_2w_2 with some element in $L(v_2w_2) - \{1,2,3,4,5\}$ and color vv_1 with 1. Otherwise, $L(v_2w_2) = \{1,2,3,4,5\}$. We re-color v_2w_2 with k and color vv_1 with 1. Since $2 \notin C_\phi(v_k)$, the resulted edge coloring is acyclic.

For **(F5)**, assume that $N_G(v) = \{v_1,v_2,v_3,v_4\}$, $N_G(v_1) = \{v,w_1\}$, $N_G(v_2) = \{v,w_2\}$, and $N_G(v_3) = \{v,w_3,w_3'\}$, where w_3 is a 2-vertex. The graph $G' = G - v_1$ has an acyclic L-edge coloring ϕ by the induction hypothesis. If $L(v_1w_1) - (C_\phi(w_1) \cup \{\phi(vv_3), \phi(vv_4)\})$ is not empty, then we color v_1w_1 with some $j \in L(v_1w_1) - (C_\phi(w_1) \cup \{\phi(vv_3), \phi(vv_4)\})$. If $j \neq \phi(vv_2)$, then we color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup \{j\})$. If $j = \phi(vv_2)$, then we color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup C_\phi(v_2))$. Otherwise, we may assume that $\phi(vv_2) = 2$, $\phi(vv_3) = 3$, $\phi(vv_4) = 4$, and $L(v_1w_1) = C_\phi(w_1) \cup \{3,4\}$. We color v_1w_1 with 3.

If $L(vv_1) - (C_\phi(v) \cup C_\phi(w_1)) \neq \emptyset$ or $L(vv_1) - (C_\phi(v) \cup C_\phi(v_3)) \neq \emptyset$, then we color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup C_\phi(w_1))$ or $L(vv_1) - (C_\phi(v) \cup C_\phi(v_3))$. Otherwise, $|C_\phi(v) \cup C_\phi(v_3)| = \Delta_1(vv_1) + 1 = 5$. In particular, $d_G(w_1) \leqslant 4$ and $d_G(v_4) \leqslant 4$. If $3 \notin C_\phi(w_3)$, then we color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup \{\phi(v_3w_3)\})$. We may assume that $C_\phi(w_3) = \{1,3\}$, $\phi(v_3w_3') = 5$, $L(vv_1) = \{1,2,3,4,5\}$, and $\{1,5\} \subseteq C_\phi(w_1)$. If $1 \notin C_\phi(v_4)$, then we re-color v_1w_1 with 4 and color vv_1 with 1. If $5 \notin C_\phi(v_4)$, then we re-color v_1w_1 with 4 and color vv_1 with 5. We may assume that $\{1,5\} \subseteq C_\phi(v_4)$.

If $L(v_3w_3) \neq \{1,2,3,4,5\}$, then we re-color v_3w_3 with some element in $L(v_3w_3) - \{1,2,3,4,5\}$ and color vv_1 with 1. We may assume that $L(v_3w_3) = \{1,2,3,4,5\}$.

If $C_\phi(v_2) \neq \{2,3\}$, then we re-color v_3w_3 with 2 and color vv_1 with 1. We may assume that $C_\phi(v_2) = \{2,3\}$.

If $L(vv_2) \neq \{1,2,3,4,5\}$, then we re-color v_3w_3 with 2, vv_2 with some element in $L(vv_2) - \{1,2,3,4,5\}$ and color vv_1 with 1. We may assume that $L(vv_2) = \{1,2,3,4,5\}$.

If $3 \in C_\phi(v_4)$, let s be an element in $L(vv_4) - \{1,3,4,5\}$. If $s \neq 2$, then we re-color vv_4 with s, v_1w_1 with 4, and color vv_1 with 1. If $s = 2$, then we re-color vv_4 with 2, v_1w_1 with 4, vv_2 with 4, and color vv_1 with 1.

If $3 \notin C_\phi(v_4)$, then we re-color v_3w_3 with 4 and color vv_1 with 1.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically $(\Delta_1 + 1)$-edge choosable. $\square$

Corollary 35. *If G is a graph with* $\mathrm{mad}(G) < 3$*, then* $a'_{\mathrm{list}}(G) \leqslant \Delta(G) + 1$.

Lemma 36. ([12]) *Let G be a planar graph with $\delta(G) \geqslant 2$. If $g(G) \geqslant 5$ and $\Delta(G) \geqslant 11$, then G contains at least one of* (F1), (F2), (F3) *of Lemma* 33*, or one of the following configurations.*

- (F6) *a* 3*-vertex v adjacent to one* 2*-vertex v_1 and the other neighbors have degree at least* 4*;*
- (F7) *a* 3*-vertex v such that the sum of degrees of its neighbors is at most* $\Delta(G) + 3$*;*
- (F8) *a d-vertex v adjacent to one* 2*-vertex v_1 and $d - 1$ $(\Delta(G) + 2 - d)^-$-vertices, where $d \geqslant 4$.*

Theorem 37. *If G is a planar graph with $g(G) \geqslant 5$ and $\Delta(G) \geqslant 11$, then G is acyclically* $\min\{\Delta_0 + 2\Delta_1 - 2, \Delta(G) + 1\}$*-edge choosable.*

Proof. The proof is by induction on the number of vertices plus the number of edges. The theorem is trivially true for the induction basis of $K_{1,11}$. By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a $\min\{\Delta_0 + 2\Delta_1 - 2, \Delta(G) + 1\}$-edge-list of G. By Lemma 36, we have eight cases to discuss. For each cases, a subgraph G' is obtained by deleting a suitable vertex or a suitable edge. If $\Delta(G') = \Delta(G)$, then G' has an acyclic L-edge coloring ϕ by the induction hypothesis. If $\Delta(G') < \Delta(G)$, then G' has an acyclic L-edge coloring ϕ by Theorem 31.

For **(F1)**, **(F2)**, and **(F3)**, we can use the same arguments in the proof of Theorem 34.

For **(F6)**, assume that $N_G(v) = \{v_1, v_2, v_3\}$ and $N_G(v_1) = \{v, w_1\}$. Let G' be the subgraph $G - v_1$. Assume that $\phi(vv_2) = 2$ and $\phi(vv_3) = 3$.

If $L(v_1w_1) \neq C_\phi(w_1)\cup\{2,3\}$, then we color v_1w_1 with some $t \in L(v_1w_1) - (C_\phi(w_1) \cup \{2,3\})$ and vv_1 with some element in $L(vv_1) - \{t,2,3\}$. We may assume that $L(v_1w_1) = C_\phi(w_1) \cup \{2,3\}$.

Let S_i, $i \in \{2,3\}$, be the set of positive integers s such that there exists an (i,s)-colored path in ϕ between w_1 and v avoiding v_{5-i}. If $L(vv_1) \neq S_i \cup \{2,3\}$ for some i, then we color v_1w_1 with i and vv_1 with some element in $L(vv_1)-(S_i\cup\{2,3\})$. We may assume that $S_2 = S_3 = C_\phi(w_1) = C_\phi(v_2) - \{2\} = C_\phi(v_3) - \{3\}$, $3 \notin C_\phi(v_2)$, and $2 \notin C_\phi(v_3)$.

If $L(vv_j) \neq C_\phi(w_1) \cup \{2,3\}$ for some $j \in \{2,3\}$, we re-color vv_j with some $p \in L(vv_j) - (C_\phi(w_1) \cup \{2,3\})$, color v_1w_1 with j, and color vv_1 with some element in $L(vv_1) - \{2,3,p\}$. We may assume that $L(vv_2) = L(vv_3) = C_\phi(w_1) \cup \{2,3\}$. Let r be an element in $C_\phi(w_1)$. Since there is a $(2,r)$-path between w_1 and v_2 and $2 \notin C_\phi(v_3)$, there is no $(2,r)$-path between w_1 and v_3. We re-color vv_2 with 3, vv_3 with 2, and color v_1w_1 with 2, vv_1 with r.

For **(F7)**, assume that $N_G(v) = \{v_1,v_2,v_3\}$. By **(F2)** and **(F6)**, we may assume that $3 \leqslant d_G(v_i) \leqslant \Delta(G) - 3$ for each i. Let G' be the subgraph $G - vv_1$. Assume that $\phi(vv_2) = 2$ and $\phi(vv_3) = 3$.

Subcase 7.1. $|C_\phi(v_1) \cap \{2,3\}| = 0$.

We color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup \{2,3\})$.

Subcase 7.2. $|C_\phi(v_1) \cap \{2,3\}| = 1$.

Let s be the index such that $C_\phi(v_1) \cap \{2,3\} = \{s\}$. We color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_s) \cup \{5-s\})$.

Subcase 7.3. $|C_\phi(v_1) \cap \{2,3\}| = 2$.

We color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup C_\phi(v_2) \cup C_\phi(v_3))$.

For **(F8)**, assume that $N_G(v_1) = \{v,w_1\}$. Let G' be the subgraph $G - vv_1$.

Subcase 8.1. $\phi(v_1w_1) \notin C_\phi(v)$.

We color vv_1 with some element in $L(vv_1) - (C_\phi(v_1) \cup C_\phi(v))$.

Subcase 8.2. $\phi(v_1w_1) \in C_\phi(v)$.

Let r be the index such that $\phi(vv_r) = \phi(v_1w_1)$. We color vv_1 with some element in $L(vv_1) - (C_\phi(v) \cup C_\phi(v_r))$.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically $\min\{\Delta_0 + 2\Delta_1 - 2, \Delta(G) + 1\}$-edge choosable. □

Corollary 38. *A planar graph G satisfies $a'_{\text{list}}(G) \leqslant \Delta(G) + 1$ if either $g(G) \geqslant 6$ or $g(G) \geqslant 5$ and $\Delta(G) \geqslant 11$.*

9. Sufficient Conditions for Acyclic Δ-edge Choosability

Lemma 39. ([3]) *Let G be a planar graph with $\delta(G) \geqslant 2$. Suppose that any of the following conditions holds.*

1. $\Delta(G) \geqslant 8$ *and* $g(G) \geqslant 7$;
2. $\Delta(G) \geqslant 6$ *and* $g(G) \geqslant 8$;
3. $\Delta(G) \geqslant 5$ *and* $g(G) \geqslant 9$;
4. $\Delta(G) \geqslant 4$ *and* $g(G) \geqslant 10$;
5. $\Delta(G) \geqslant 3$ *and* $g(G) \geqslant 14$.

Then G contains at least one of the following configurations.

(G1) *a 2-vertex u with neighbors v, w such that $d_G(v) + d_G(w) \leqslant \Delta(G) + 1$;*

(G2) *a d-vertex u adjacent to d 2-vertex, where $d \leqslant \Delta(G) - 1$;*

(G3) *a path $w_1v_1uv_2w_2$ with $d_G(w_1) = d_G(v_1) = d_G(v_2) = 2$, $d_G(u) = \Delta(G)$, and $d_G(w_2) < \Delta(G)$.*

Theorem 40. *If G is a planar graph such that any of the five conditions in Lemma 39 holds, then G is acyclically $\min\{\Delta_0 + \Delta_1 - 1, \Delta(G)\}$-edge choosable.*

Proof. The proof is by induction on the number of vertices. The theorem is true for the induction basis, which consists of the following graphs, respectively: $K_{1,8}$, $K_{1,6}$, $K_{1,5}$, $K_{1,4}$, and $K_{1,3}$.

By Lemma 5, we may assume that $\delta(G) \geqslant 2$. Let L be a $\min\{\Delta_0 + \Delta_1 - 1, \Delta(G)\}$-edge-list of G. By Lemma 39, we have the following cases to discuss. For each cases, a subgraph G' is obtained by deleting a suitable vertex. If $\Delta(G') = \Delta(G)$, then G' has an acyclic L-edge coloring ϕ by the induction hypothesis. If $\Delta(G') < \Delta(G)$, then G' has an acyclic L-edge coloring ϕ by Theorem 34.

For **(G1)**, let G' be the subgraph $G-u$. Since $|C_\phi(v)\cup C_\phi(w)|\leqslant d_G(v)+d_G(w)-2\leqslant \min\{\Delta_0(uv)+\Delta_1(uv)-2,\Delta(G)-1\}$, we color uv with some $k\in L(uv)-(C_\phi(v)\cup C_\phi(w))$. Since $d_G(v)\geqslant\delta(G)\geqslant 2$, we have $|C_\phi(w)\cup\{k\}|\leqslant d_G(w)\leqslant\min\{\Delta_0(uw)+\Delta_1(uw)-2,\Delta(G)-1\}$. We color uw with some element in $L(uw)-(C_\phi(w)\cup\{k\})$.

For **(G2)**, let v be one neighbor of u and w be the other neighbor of v. Let G' be the subgraph $G-v$. We color vw with some element $j\in L(vw)-C_\phi(w)$. If $j\notin C_\phi(u)$, then we color uv with some element in $L(uv)-(C_\phi(u)\cup\{j\})$. If $j=\phi(ux)\in C_\phi(u)$ for some neighbor x of u, then we color uv with some element in $L(uv)-(C_\phi(u)\cup C_\phi(x))$.

For **(G3)**, assume that the neighbors of u are v_1, v_2, …, $v_{\Delta(G)}$. Let G' be the subgraph $G-v_1$. We color w_1v_1 with some element $k\in L(w_1v_1)-(C_\phi(w_1)\cup\{\phi(uv_3),\dots,\phi(uv_{\Delta(G)})\})$. If $k\neq\phi(uv_2)$ and $L(uv_1)\neq C_\phi(u)\cup\{k\}$, then we color uv_1 with some element in $L(uv_1)-(C_\phi(u)\cup\{k\})$. If $k\neq\phi(uv_2)$ and $L(uv_1)=C_\phi(u)\cup\{k\}$, then we color uv_1 with k and re-color w_1v_1 with some element in $L(w_1v_1)-(C_\phi(w_1)\cup\{k\})$. We may assume that $k=\phi(uv_2)$ and $L(w_1v_1)=C_\phi(w_1)\cup C_\phi(u)$. Let j be some element in $L(uv_1)-C_\phi(u)$. If $\phi(uv_2)\notin C_\phi(w_2)$, or $j\notin C_\phi(w_1)$, or $j\notin C_\phi(v_2)$, then we color uv_1 with j. We may assume that $L(uv_1)=C_\phi(w_1)\cup C_\phi(u)=C_\phi(u)\cup\{\phi(v_2w_2)\}$ and $\phi(uv_2)\in C_\phi(w_2)$. If $L(uv_2)\neq C_\phi(u)\cup\{\phi(v_2w_2)\}$, then we re-color uv_2 with some element in $L(uv_2)-(C_\phi(u)\cup\{\phi(v_2w_2)\})$ and color uv_1 with $\phi(v_2w_2)$. We may assume that $L(uv_2)=C_\phi(u)\cup\{\phi(v_2w_2)\}$. We re-color v_2w_2 with some $\alpha\in L(v_2w_2)-C_\phi(w_2)$ and uv_2 with $\phi(v_2w_2)$. We color uv_1 with $\phi(uv_2)$ and re-color v_1w_1 with some element in $L(v_1w_1)-\{\phi(v_2w_2),\phi(uv_2)\}$.

In every case, G has an acyclic L-edge coloring. Therefore, G is acyclically $\min\{\Delta_0+\Delta_1-1,\Delta(G)\}$-edge choosable. □

Corollary 41. *If G is a planar graph such that any of the five conditions in Lemma* 39 *holds, then $a'_{\rm list}(G)=\Delta(G)$.*

For planar graphs with girth at least 16, we have the following stronger result.

Theorem 42. *If G is a planar graph with $g(G)\geqslant 16$, then G is acyclically $\max\{\Delta_0,3\}$-edge choosable.*

Proof. The proof is by induction on the number of vertices. If $|G| \leqslant 16$, then G is a forest or a cycle on sixteen vertices. Hence, G is acyclically $\max\{\Delta_0, 3\}$-edge choosable. By Lemma 5, we may assume that $\delta(G) \geqslant 2$. It is known ([7]) that, if a planar graph G satisfies $\delta(G) \geqslant 2$ and $g(G) \geqslant 16$, then G has a 2-vertex adjacent to two 2-vertices. Hence, there exists a path $uvwxy$ such that $d_G(v) = d_G(w) = d_G(x) = 2$. Let L be a $\max\{\Delta_0, 3\}$-edge-list of G. By the induction hypothesis, $G - \{v, w, x\}$ is acyclically $\max\{\Delta_0, 3\}$-edge choosable and $G - \{v, w, x\}$ has an acyclic L-edge coloring. By Lemma 8, G is acyclically $\max\{\Delta_0, 3\}$-edge choosable. $\square$

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